Categorification of acyclic cluster algebras: an introduction

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To Murray Gerstenhaber and Jim Stasheff

Summary. This is a concise introduction to Fomin-Zelevinsky's cluster algebras and their links with the representation theory of quivers in the acyclic case. We review the definition cluster algebras (geometric, without coefficients), construct the cluster category and present the bijection between cluster variables and rigid indecomposable objects of the cluster category.

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1 Introduction

1.1 Context

Cluster algebras were invented by S. Fomin and A. Zelevinsky [27] in the spring of the year 2000 in a project whose aim it was to develop a combinatorial approach to the results obtained by G. Lusztig concerning total positivity in algebraic groups [54] on the one hand and canonical bases in quantum groups [53] on the other hand (let us stress that canonical bases were discovered independently and simultaneously by M. Kashiwara [45]). Despite great progress during the last few years [28] [8] [31], we are still relatively far from these initial aims. Presently, the best results on the link between cluster algebras and canonical bases are probably those of C. Geiss, B. Leclerc and J. Schröer [38] [39] [35] but even they cannot construct canonical bases from

cluster variables for the moment. Despite these difficulties, the theory of cluster algebras has witnessed spectacular growth thanks notably to the many links that have been discovered with a wide range of subjects including

- Poisson geometry [40] [41] ...,
- integrable systems [30] ...,
- higher Teichmüller spaces [20] [22] [23] [21] ...,
- combinatorics and the study of combinatorial polyhedra like the Stasheff associahedra [18] [17] [52] [25] [58] [26] ...,
- commutative and non commutative algebraic geometry, in particular the study of stability conditions in the sense of Bridgeland [10] [9], Calabi-Yau algebras [42], Donaldson-Thomas invariants [66] [49] [50] [51] ...,
- and last not least the representation theory of quivers and finite-dimensional algebras, cf. for example the surveys [4] [63] [62].

We refer to the introductory papers [69] [29] [68] [70] [71] and to the cluster algebras portal [24] for more information on cluster algebras and their links with other parts of mathematics.

The link between cluster algebras and quiver representations follows the spirit of categorification: One tries to interpret cluster algebras as combinatorial (perhaps K-theoretic) invariants associated with categories of representations. Thanks to the rich structure of these categories, one can then hope to prove results on cluster algebras which seem beyond the scope of the purely combinatorial methods. It turns out that the link becomes especially beautiful if we use a *triangulated category* constructed from the category of quiver representations, the so-called cluster category.

In this brief survey, we will review the definition of cluster algebras and Fomin-Zelevinsky's classification theorem for cluster-finite cluster algebras [28]. We will then recall some basic notions on the representations of a quiver without oriented cycles, introduce the cluster category and describe its link with the cluster algebra.
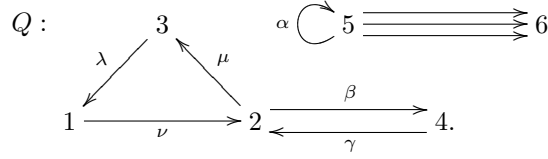
2 Cluster algebras

The cluster algebras we will be interested in are associated with antisymmetric matrices with integer coefficients. Instead of using matrices, we will use quivers (without loops and 2-cycles), since they are easy to visualize and well-suited to our later purposes.

2.1 Quivers

Let us recall that a *quiver* Q is an oriented graph. Thus, it is a quadruple given by a set Q_0 (the set of vertices), a set Q_1 (the set of arrows) and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ which take an arrow to its source respectively

its target. Our quivers are 'abstract graphs' but in practice we draw them as in this example:

$$Q : \qquad 3 \qquad \alpha \circlearrowright 5 \rightrightarrows 6$$
$$\lambda \swarrow \quad \nwarrow \mu$$
$$1 \xrightarrow[\nu]{} 2 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 4.$$

A *loop* in a quiver Q is an arrow α whose source coincides with its target; a 2-*cycle* is a pair of distinct arrows $\beta \neq \gamma$ such that the source of β equals the target of γ and vice versa. It is clear how to define 3-*cycles*, *connected components* .... A quiver is *finite* if both, its set of vertices and its set of arrows, are finite.

2.2 Seeds and mutations

Fix an integer $n \geq 1$. A *seed* is a pair (R, u), where

a) R is a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$;
b) u is a free generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of fractions of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in n indeterminates.

Notice that in the quiver R of a seed, all arrows between any two given vertices point in the same direction (since R does not have 2-cycles). Let (R, u) be a seed and k a vertex of R. The *mutation* $\mu_k(R, u)$ of (R, u) at k is the seed (R', u'), where

a) R' is obtained from R as follows:
   1) reverse all arrows incident with k;
   2) for all vertices $i \neq j$ distinct from k, modify the number of arrows between i and j as follows:

R	R'
$i \xrightarrow{r} j$ $\quad s \searrow \quad \nearrow t$ $\quad k$	$i \xrightarrow{r+st} j$ $\quad s \nwarrow \quad \swarrow t$ $\quad k$
$i \xrightarrow{r} j$ $\quad s \nwarrow \quad \swarrow t$ $\quad k$	$i \xrightarrow{r-st} j$ $\quad s \searrow \quad \nearrow t$ $\quad k$

where r, s, t are non negative integers, an arrow $i \xrightarrow{l} j$ with $l \geq 0$ means that l arrows go from i to j and an arrow $i \xrightarrow{l} j$ with $l \leq 0$ means that $-l$ arrows go from j to i.

b) u' is obtained from u by replacing the element u_k with

$$u'_k = \frac{1}{u_k} \left(\prod_{\text{arrows } i \to k} u_i + \prod_{\text{arrows } k \to j} u_j \right). \tag{1}$$
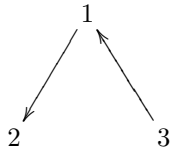
In the *exchange relation* (1), if there are no arrows from i with target k, the product is taken over the empty set and equals 1. It is not hard to see that $\mu_k(R, u)$ is indeed a seed and that μ_k is an involution: we have $\mu_k(\mu_k(R, u)) = (R, u)$.
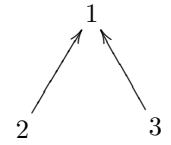
2.3 Examples of mutations

Let R be the cyclic quiver

$$\tag{2}$$

and $u = \{x_1, x_2, x_3\}$. If we mutate at $k = 1$, we obtain the quiver

and the set of fractions given by $u'_1 = (x_2 + x_3)/x_1$, $u'_2 = u_2 = x_2$ and $u'_3 = u_3 = x_3$. Now, if we mutate again at 1, we obtain the original seed. This is a general fact: Mutation at k is an involution. If, on the other hand, we mutate (R', u') at 2, we obtain the quiver
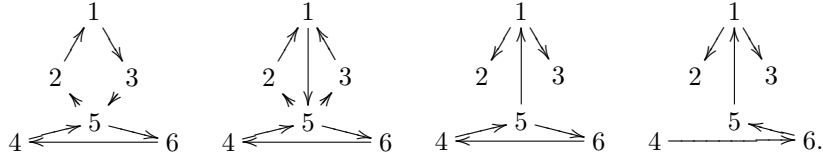

and the set u'' given by $u''_1 = u'_1 = (x_2 + x_3)/x_1$, $u''_2 = \frac{x_1 + x_2 + x_3}{x_1 x_2}$ and $u''_3 = u'_3 = x_3$.

Let us consider the following, more complicated quiver glued together from four 3-cycles:

$$\tag{3}$$

If we successively perform mutations at the vertices 5, 3, 1 and 6, we obtain the sequence of quivers (we use [46])


Notice that the last quiver no longer has any oriented cycles and is in fact an orientation of the Dynkin diagram of type D_6. The sequence of new fractions appearing in these steps is

$$u_5' = \frac{x_3 x_4 + x_2 x_6}{x_5} , \qquad u_3' = \frac{x_3 x_4 + x_1 x_5 + x_2 x_6}{x_3 x_5} ,$$

$$u_1' = \frac{x_2 x_3 x_4 + x_3^2 x_4 + x_1 x_2 x_5 + x_2^2 x_6 + x_2 x_3 x_6}{x_1 x_3 x_5} , \qquad u_6' = \frac{x_3 x_4 + x_4 x_5 + x_2 x_6}{x_5 x_6} .$$

It is remarkable that all the denominators appearing here are monomials and that all the coefficients in the numerators are positive.

Finally, let us consider the quiver

(4)
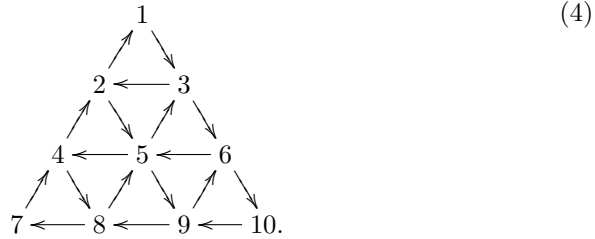
One can show [48] that it is impossible to transform it into a quiver without oriented cycles by a finite sequence of mutations. However, its mutation class (the set of all quivers obtained from it by iterated mutations) contains many quivers with just one oriented cycle, for example
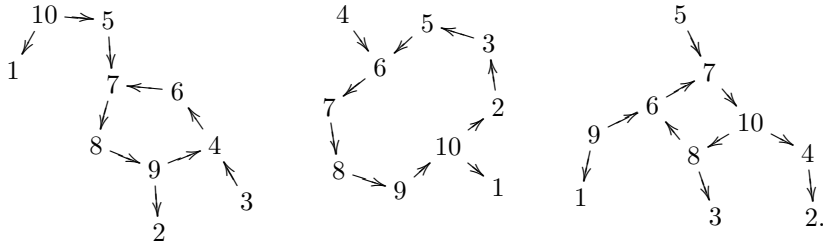

In fact, in this example, the mutation class is finite and it can be completely computed using, for example, [46]: It consists of 5739 quivers up to isomorphism. The above quivers are members of the mutation class containing relatively few arrows. The initial quiver is the unique member of its mutation

class with the largest number of arrows. Here are some other quivers in the mutation class with a relatively large number of arrows:
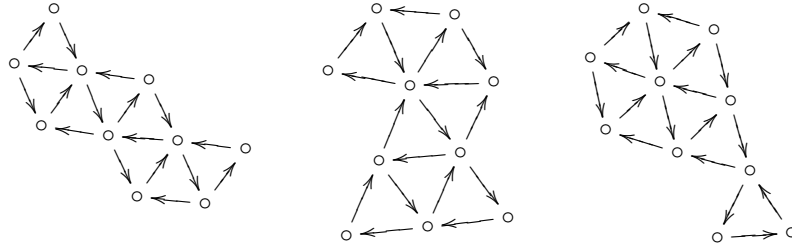
Only 84 among the 5739 quivers in the mutation class contain double arrows (and none contain arrows of multiplicity ≥ 3). Here is a typical example
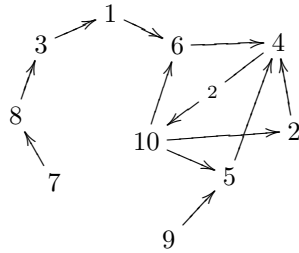
The quivers (2), (3) and (4) are part of a family which appears in the study of the cluster algebra structure on the coordinate algebra of the subgroup of upper unitriangular matrices in $SL_n(\mathbb{C})$, *cf.* [39]. The study of coordinate algebras on varieties associated with reductive algebraic groups (in particular, double Bruhat cells) has provided a major impetus for the development of cluster algebras, *cf.* [8].

2.4 Definition of cluster algebras

Let Q be a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Consider the seed (Q, x) consisting of Q and the set x formed by the variables $x_1, \ldots, x_n$. Following [27] we define

- the *clusters with respect to* Q to be the sets u appearing in seeds (R, u) obtained from (Q, x) by iterated mutation,
- the *cluster variables* for Q to be the elements of all clusters,
- the *cluster algebra* $\mathcal{A}_Q$ to be the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

Thus the cluster algebra consists of all $\mathbb{Q}$-linear combinations of monomials in the cluster variables. It is useful to define another combinatorial object associated with this recursive construction: The *exchange graph* associated with Q is the graph whose vertices are the seeds modulo simultaneous renumbering of the vertices and the associated cluster variables and whose edges correspond to mutations.

2.5 The example A_3

Let us consider the quiver

$$Q : \quad 1 \longrightarrow 2 \longrightarrow 3$$

obtained by endowing the Dynkin diagram A_3 with a linear orientation. By applying the recursive construction to the initial seed (Q, x) one finds exactly fourteen seeds (modulo simultaneous renumbering of vertices and cluster variables). These are the vertices of the exchange graph, which is isomorphic to the third Stasheff associahedron [65] [18]:
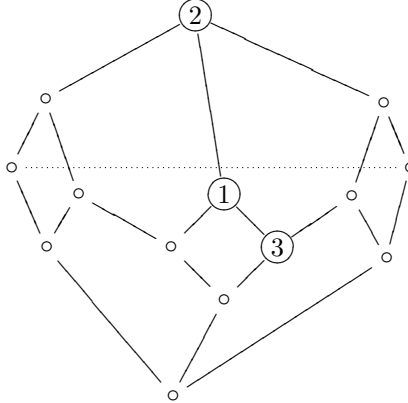
The vertex labeled 1 corresponds to (Q, x), the vertex 2 to $\mu_2(Q, x)$, which is given by

$$1 \overset{\frown}{\longleftarrow} 2 \longleftarrow 3 \ , \ \{x_1, \frac{x_1 + x_3}{x_2}, x_3\} \ ,$$

and the vertex 3 to $\mu_1(Q, x)$, which is given by

$$1 \longleftarrow 2 \longrightarrow 3 \ , \ \{\frac{1 + x_3}{x_1}, x_2, x_3\}.$$

We find a total of 9 cluster variables, namely

$$x_1 \ , \ x_2 \ , \ x_3, \frac{1 + x_2}{x_1} \ , \ \frac{x_1 + x_3 + x_2 x_3}{x_1 x_2} \ , \ \frac{x_1 + x_1 x_2 + x_3 + x_2 x_3}{x_1 x_2 x_3} \ ,$$
$$\frac{x_1 + x_3}{x_2} \ , \ \frac{x_1 + x_1 x_2 + x_3}{x_2 x_3} \ , \ \frac{1 + x_2}{x_3} \ .$$

Again we observe that all denominators are monomials. Notice also that $9 = 3 + 6$ and that 3 is the rank of the root system associated with A_3 and 6 its number of positive roots. Moreover, if we look at the denominators of the non trivial cluster variables (those other than x_1, x_2, x_3), we see a natural bijection with the positive roots

$$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3, \alpha_3$$

of the root system of A_3, where α_1, α_2, α_3 denote the three simple roots.

2.6 Cluster algebras with finitely many cluster variables

The phenomena observed in the above example are explained by the following key theorem:

Theorem 1 (Fomin-Zelevinsky [28]). *Let Q be a finite connected quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Let $\mathcal{A}_Q$ be the associated cluster algebra.*

a) *All cluster variables are Laurent polynomials, i.e. their denominators are monomials.*
b) *The number of cluster variables is finite iff Q is mutation equivalent to an orientation of a simply laced Dynkin diagram Δ. In this case, Δ is unique and the non trivial cluster variables are in bijection with the positive roots of Δ; namely, if we denote the simple roots by $\alpha_1, \ldots, \alpha_n$, then for each positive root $\sum d_i \alpha_i$, there is a unique non trivial cluster variable whose denominator is $\prod x_i^{d_i}$.*

3 Categorification

We refer to the books [64] [33] [2] and [1] for a wealth of information on the representation theory of quivers and finite-dimensional algebras. Here, we will only need very basic notions.

Let Q be a finite quiver without oriented cycles. For example, Q can be an orientation of a simply laced Dynkin diagram or the quiver

$$1 \xrightarrow{\quad \alpha \quad 2 \quad \beta \quad} 3.$$
$$\xrightarrow[\gamma]{}$$
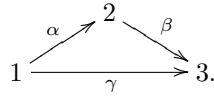
Let k be an algebraically closed field. Recall that a *representation of Q* is a diagram of finite-dimensional vector spaces of the shape given by Q. Thus, in the above example, a representation of Q is a (not necessarily commutative) diagram

$$V_1 \xrightarrow{\quad V_\alpha \quad V_2 \quad V_\beta \quad} V_3$$
$$\xrightarrow[V_\gamma]{}$$
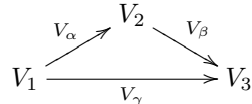
formed by three finite-dimensional vector spaces and three linear maps. A *morphism of representations* is a morphism of diagrams. We thus obtain the *category of representations* $\mathsf{rep}(Q)$. Notice that it is an abelian category (since it is a category of diagrams in an abelian category, that of finite-dimensional vector spaces): Sums, kernels and cokernels in the category $\mathsf{rep}(Q)$ are computed componentwise. We denote by $\mathcal{D}_Q$ its bounded derived category. Thus, the objects of $\mathcal{D}_Q$ are the bounded complexes

$$\cdots \longrightarrow V^p \xrightarrow{\ d\ } V^{p+1} \longrightarrow \cdots$$

of representations and its morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms (=morphisms inducing isomorphisms in homology). The category $\mathcal{D}_Q$ is still an additive category (direct sums are given by direct sums of complexes) but it is almost never abelian. In fact, it is abelian if and only if Q does not have any arrows. But it is always *triangulated*. This means that $\mathcal{D}_Q$ is additive and endowed with

a) a *suspension functor* $\Sigma : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$, namely the functor taking a complex V to $V[1]$, where $V[1]^p = V^{p+1}$ for all $p \in \mathbb{Z}$ and $d_{V[1]} = -d_V$;
b) a class of *triangles*, namely the sequences

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma U$$

which are 'induced' from short exact sequences of complexes.

The triangulated category $\mathcal{D}_Q$ admits a *Serre functor*, i.e. an autoequivalence $S : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$ which makes the Serre duality formula true: We have

$$D\,\mathsf{Hom}(X,Y) \xrightarrow{\sim} \mathsf{Hom}(Y, SX)$$

bifunctorially in X, Y belonging to $\mathcal{D}_Q$, where D denotes the duality functor $\mathsf{Hom}_k(?, k)$ over the ground field k. The *cluster category* is defined as the *orbit category*

$$\mathcal{C}_Q = \mathcal{D}_Q/(S^{-1} \circ \Sigma^2)^{\mathbb{Z}}$$

of $\mathcal{D}_Q$ under the action of the cyclic group generated by the automorphism $S^{-1} \circ \Sigma^2$. Thus, its objects are the same as those of $\mathcal{D}_Q$ and its morphisms are defined by

$$\mathsf{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}_Q}(X, (S^{-1} \circ \Sigma^2)^p Y).$$

One can show [47] that $\mathcal{C}_Q$ admits a structure of triangulated category such that the projection functor $\mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor (in general, the orbit category of a triangulated category under the action of an automorphism group is no longer triangulated). It is not hard to see that the cluster category has finite-dimensional morphism spaces, and that it admits a Serre functor induced by that of the derived category. The definition of the cluster category then immediately yields an isomorphism

$$S \xrightarrow{\sim} \Sigma^2$$

and this means that $\mathcal{C}_Q$ is 2-Calabi-Yau: A k-linear triangulated category with finite-dimensional morphism spaces is d-Calabi-Yau if it admits a Serre functor S and if S is isomorphic to Σ^d (the dth power of the suspension functor) as a triangle functor. The definition of the cluster category is due to Buan-Marsh-Reineke-Reiten-Todorov [5] (for arbitrary Q without oriented cycles)

and, independently and with a very different, more geometric description, to
Caldero-Chapoton-Schiffler [13] (for Q of type A_n).

To state the close relationship between the cluster category $\mathcal{C}_Q$ and the
cluster algebra $\mathcal{A}_Q$, we need some notation: For two objects L and M of $\mathcal{C}_Q$,
we write

$$\mathsf{Ext}^1(L, M) = \mathsf{Hom}_{\mathcal{C}_Q}(L, \Sigma M).$$

Notice that it follows from the Calabi-Yau property that we have a canonical
isomorphism

$$\mathsf{Ext}^1(L, M) \xrightarrow{\sim} D\,\mathsf{Ext}^1(M, L).$$

An object L of $\mathcal{C}_Q$ is *rigid* if we have $\mathsf{Ext}^1(L, L) = 0$. It is *indecomposable* if
it is non zero and in each decomposition $L = L_1 \oplus L_2$, we have $L_1 = 0$ or
$L_2 = 0$.

Theorem 2 ([15]). *Let Q be a finite quiver without oriented cycles with ver-
tex set $\{1, \ldots, n\}$.*

a) *There is an explicit bijection $L \mapsto X_L$ from the set of isomorphism classes
    of rigid indecomposables of the cluster category $\mathcal{C}_Q$ onto the set of cluster
    variables of the cluster algebra $\mathcal{A}_Q$.*
b) *Under this bijection, the clusters correspond exactly to the* cluster-tilting
    subsets, i.e. the sets $T_1, \ldots, T_n$ of rigid indecomposables such that

$$Ext^1(T_i, T_j) = 0$$

   for all i, j.
c) *If L and M are rigid indecomposables such that the space $\mathsf{Ext}^1(L, M)$ is
    one-dimensional, then we have the generalized exchange relation*

$$X_L = \frac{X_B + X_{B'}}{X_M} \tag{5}$$

   where B and B' are the middle terms of 'the' non split triangles

$$L \longrightarrow B \longrightarrow M \longrightarrow \Sigma L \quad and \quad M \longrightarrow B' \longrightarrow L \longrightarrow \Sigma M$$

   and we define

$$X_B = \prod_{i=1}^{s} X_{B_i}\,,$$

   where $B = B_1 \oplus \cdots \oplus B_s$ is a decomposition into indecomposables.

The relation (5) in part c) of the theorem can be generalized to the case
where the extension group is of higher dimension, cf. [14] [43] [67]. One can
show using [7] that relation (5) generalizes the exchange relation (1) which
appeared in the definition of the mutation.

The proof of the theorem builds on work by many authors notably Buan-
Marsh-Reiten-Todorov [11], Buan-Marsh-Reiten [6], Buan-Marsh-Reineke-
Reiten-Todorov [5], Marsh-Reineke-Zelevinsky [57], ... and especially on

Caldero-Chapoton's explicit formula for X_L proved in [12] for orientations of simply laced Dynkin diagrams. We include the formula below. Another crucial ingredient of the proof is the Calabi-Yau property of the cluster category. An alternative proof of part c) was given by A. Hubery [43] for quivers whose underlying graph is an extended simply laced Dynkin diagram.

The theorem does shed new light on cluster algebras. In particular, we have the following

Corollary 1 (Qin [61], Nakajima [59]). *Suppose that Q does not have oriented cycles. Then all cluster variables of $\mathcal{A}_Q$ belong to $\mathbb{N}[x_1^\pm, \ldots, x_n^\pm]$.*

This settles a conjecture of Fomin-Zelevinsky [27] in the case of cluster algebras associated with acyclic quivers. The proof is based on Lusztig's [56] and in this sense it does not quite live up to the hopes that cluster theory ought to explain Lusztig's results. However, it does show that the conjecture is true for this important class of cluster algebras.

4 Caldero-Chapoton's formula

We describe the bijection of part a) of theorem 2. Let k be an algebraically closed field and Q a finite quiver without oriented cycles with vertex set $\{1, \ldots, n\}$. Let L be an object of the cluster category $\mathcal{C}_Q$. With L, we will associate an element X_L of the field $\mathbb{Q}(x_1, \ldots, x_n)$. According to [5], the object L decomposes into a sum of indecomposables L_i, $1 \leq i \leq s$, unique up to isomorphism and permutation. By defining

$$X_L = \prod_{i=1}^{s} X_{L_i}$$

we reduce to the case where L is indecomposable. Now again by [5], if L is indecomposable, it is either isomorphic to an object $\pi(V)$, or an object $\Sigma\pi(P_i)$, where $\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ is the canonical projection functor, Σ is the suspension functor of $\mathcal{C}_Q$, V is a representation of Q (identified with a complex of representations concentrated in degree 0) and P_i is the projective representation associated with a vertex i (P_i is characterized by the existence of a functorial isomorphism

$$\mathsf{Hom}(P_i, W) = W_i$$

for each representation W). If L is isomorphic to $\Sigma\pi(P_i)$, we put $X_L = x_i$. If L is isomorphic to $\pi(V)$, we define

$$X_L = X_V = \frac{1}{\prod_{i=1}^{n} x_i^{d_i}} \sum_{0 \leq e \leq d} \chi(\mathrm{Gr}_e(V)) \prod_{i=1}^{n} x_i^{\sum_{j \to i} e_j + \sum_{i \to j}(d_j - e_j)},$$

where $d_i = \dim V_i$, $1 \leq i \leq n$, the sum is taken over all elements $e \in \mathbb{N}^n$ such that $0 \leq e_i \leq d_i$ for all i, the *quiver Grassmannian* $\mathrm{Gr}_e(V)$ is the variety

of n-tuples of subspaces $U_i \subset V_i$ such that $\dim U_i = e_i$ and the U_i form a subrepresentation of V, the Euler characteristic χ is taken with respect to étale cohomology (or with respect to singular cohomology with coefficients in a field if $k = \mathbb{C}$) and the sums in the exponent of x_i are taken over all arrows $j \to i$ respectively all arrows $i \to j$. This formula was invented by P. Caldero and F. Chapoton in [12] for the case of a quiver whose underlying graph is a simply laced Dynkin diagram. It is still valid for arbitrary quivers without oriented cycles [15] and further generalizes to arbitrary triangulated 2-Calabi-Yau categories containing a cluster-tilting object [60].

5 Some further developments

The extension of the results presented here to quivers containing oriented cycles is the subject of ongoing research. In a series of papers [38] [34] [39] [35] [36], Geiss-Leclerc-Schröer have obtained remarkable results for a class of quivers which are important in the study of (dual semi-)canonical bases. They use an analogue [37] of the Caldero-Chapoton map due ultimately to Lusztig [55]. The class they consider has been further enlarged by Buan-Iyama-Reiten-Scott [3]. Thanks to their results, an analogue of Caldero-Chapoton's formula and a version of theorem 2 was proved in [32] for an even larger class.

Building on [57] Derksen-Weyman-Zelevinsky are developing a representation-theoretic model for mutation of general quivers in [19]. Their approach is related to Kontsevich-Soibelman's work [51], where 3-Calabi-Yau categories play an important rôle, as was already the case in [44].

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