On differential graded categories

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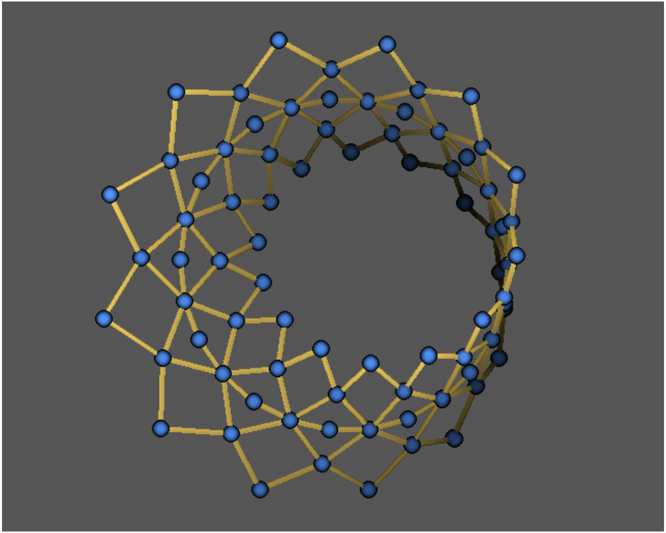


Philosophy

Question: What is a non commutative (=NC) scheme?

- Grothendieck, Manin, ... :

$$\begin{array}{ccc} \text{NC scheme} & = & \text{abelian category} \\ \text{classical scheme } X & \leftrightarrow & \mathrm{Qcoh}(X) \end{array}$$

- Drinfeld, Kontsevich, ... :

$$\begin{array}{ccc} \text{NC scheme} & = & \text{triangulated category} \\ \text{class. scheme } X & \leftrightarrow & \text{derived category } \mathcal{D}(\mathrm{Qcoh}(X)) \end{array}$$

From philosophy to mathematics

$$\text{NC scheme} \quad = \quad \text{triangulated category}$$

Problems:

1. Tensor product is not triangulated
2. Functor category with triangulated target is not triangulated
3. Hochschild homology, cyclic homology ... not defined

Partial solutions: ..., Bondal-Kapranov (1990), ...
Complete solution: Toën (2004), based on Drinfeld ...

Main aim: Construct the closed tensor category of

$$\text{NC schemes} = \text{enhanced triangulated cat.}$$

following Toën.

Outline

Complexes form a closed monoidal category

- k is a commutative ring.
- $\mathcal{C}(k)$ is the category of complexes of k-modules

$$\cdots \xrightarrow{d} C^p \xrightarrow{d} C^{p+1} \xrightarrow{d} \cdots \; , \; p \in \mathbb{Z}.$$

- $\otimes$ is the tensor product of complexes over k.

Lemma

- *$(\mathcal{C}(k), \otimes)$ is a tensor category (i.e. monoidal and symmetric).*
- *Moreover, it is closed, i.e. it has an* internal Hom-functor HOM *defined by*

$$\mathrm{Hom}(C \otimes D, E) = \mathrm{Hom}(C, \mathrm{HOM}(D, E)).$$

DG category = cat. enriched in complexes

Definition

A differential graded (=dg) category is given by

- a class of objects obj($\mathcal{A}$),
- complexes of morphisms $\mathcal{A}(X, Y)$, $X, Y \in$ obj($\mathcal{A}$),
- composition maps

$$\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)\,,\ X, Z \in \text{obj}(\mathcal{A})\,,$$

  which are associative and unital.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is given by ...

Example

(dg cat. $\mathcal{A}$ with one object $*$) = (dg algebra $A = \mathcal{A}(*,*)$)
In particular, each ordinary (non com.) algebra $A = A^0$ is a dg category.

Example: DG categories of complexes

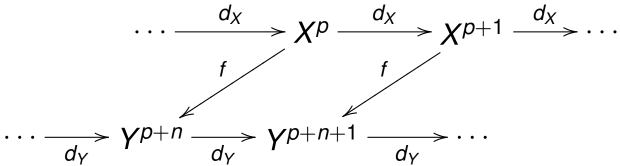
Example

- $\mathcal{M}$ a k-linear category (*e.g.* a category of modules)
- $\mathcal{A} = \mathcal{C}_{dg}(\mathcal{M}) =$ dg category of complexes over $\mathcal{M}$

objects: complexes of objects of $\mathcal{M}$

morphisms: $\mathcal{C}_{dg}(\mathcal{M})(X, Y) = \mathrm{HOM}_{\mathcal{M}}(X, Y)$

$\mathrm{HOM}_{\mathcal{M}}(X, Y)^n$:



Differential:

$$d(f) = d_Y \circ f - (-1)^n f \circ d_X.$$

DG categories form a closed tensor category

Definition

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two dg categories is the dg category with

objects: (U, V), $U \in \mathcal{A}$, $V \in \mathcal{B}$,

morphisms: $\mathcal{A}(U, U') \otimes \mathcal{B}(V, V')$

Lemma

The category dgcat *of dg categories endowed with* $\otimes$ *is a closed tensor category with*

$$\mathrm{HOM}(\mathcal{A}, \mathcal{B}) = \{\textit{dg functors } \mathcal{A} \to \mathcal{B}\}$$

Note:

$\mathrm{HOM}(\mathcal{A}, \mathcal{B})$ has many objects even if $\mathcal{A}$ and $\mathcal{B}$ only have one. This is the main reason for working with categories rather than algebras.

Zero-homology

Definition
The *zero-homology category* $H^0(\mathcal{A})$ has

objects: same as $\mathcal{A}$,
morphisms: $(H^0\mathcal{A})(X, Y) = H^0(\mathcal{A}(X, Y))$.

Example

$$H^0(\mathcal{C}_{dg}(\mathcal{M})) = \mathcal{C}(\mathcal{M})/(\text{homotopy}) = \mathcal{H}(\mathcal{M})$$

Definition
An object N of $\mathcal{A}$ is *contractible* if N becomes a zero object in $H^0(\mathcal{A})$.

Remark
N is contractible iff 1_N is a coboundary in $\mathcal{A}(N, N)$.

DG modules

$\mathcal{A}$ a dg category
$\mathcal{A}^{op}$ its opposite: $f \circ_{op} g = (-1)^{|f||g|} g \circ f$.

Definition

- A *dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k)$.
- A *morphism of dg modules* $f : L \to M$ is a morphism of dg functors such that $fX : LX \to MX$ is a morphism of complexes for all X in $\mathcal{A}$.

Example

A an ordinary algebra:

$$\text{dg } A\text{-module} = \text{complex of right } A\text{-modules.}$$

$\mathcal{A}$ a dg category: For each $X \in \mathcal{A}$, we have the *free $\mathcal{A}$-module*

$$X^{\wedge} : Y \mapsto \mathcal{A}(Y, X).$$

The derived category

Definition

- $f : L \to M$ is a *quasi-isomorphism* if fX is a quasi-isomorphism for all X in $\mathcal{A}$.
- The *derived category* $\mathcal{DA}$ is the localization of the category of dg modules with respect to the class of quasi-isomorphisms.

Remark

$\mathcal{DA}$ is a triangulated category with shift induced by the shift of complexes and triangles coming from short exact sequences of complexes. The Yoneda functor $X \mapsto X^{\wedge}$ induces an embedding $H^0(\mathcal{A}) \to \mathcal{DA}$.

Saturated dg categories

Definition

- The *perfect derived category* $\mathrm{per}\,\mathcal{A} \subset \mathcal{D}\mathcal{A}$ is the triangulated subcategory closed under retracts generated by the free modules.
- The dg category $\mathcal{A}$ is *saturated* if the Yoneda functor induces an equivalence $H^0(\mathcal{A}) \to \mathrm{per}(\mathcal{A})$.

Examples

The dg category of complexes $\mathcal{C}_{dg}(\mathcal{M})$ is saturated.
A non zero dg algebra A is not saturated.

Definition (Bondal-Kapranov)

An enhanced triangulated category is a saturated dg category $\mathcal{A}$. The associated triang. cat. is $\mathrm{per}\,\mathcal{A}$.

Definition

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a *Morita morphism* if it induces an equivalence $F^* : \mathcal{DB} \to \mathcal{DA}$.

Example

Let A be an ordinary algebra and proj A the category of finitely generated projective A-modules. The inclusions

$$A \to \operatorname{proj} A \to \mathcal{C}^b_{dg}(\operatorname{proj} A)$$

are Morita morphisms.

Definition

The category NCS of non commutative schemes is the localization of the category dgcat of dg categories with respect to the Morita morphisms. Two dg categories $\mathcal{A}$, $\mathcal{B}$ are derived equivalent if they become isomorphic in NCS.

Model structure and tensor product

Theorem (Tabuada)

The category dgcat *carries a Quillen model structure whose*

- *weak equivalences are the Morita morphisms,*
- *fibrant objects are (certain) saturated dg categories.*

Consequences

- NCS is well-defined,
- each object of NCS is isomorphic to an enhanced triangulated category (=saturated dg category),
- NCS becomes a tensor category via the derived tensor product.

Functor categories

Problem

The HOM-functor of dgcat cannot be derived to yield a HOM-functor for NCS (ultimate reason: tensor products of free NC algebras are not free NC!).

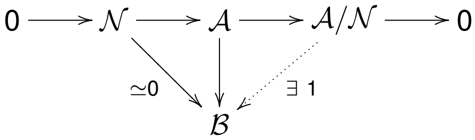
Main Theorem 1 (Toën)

- The tensor category NCS does have an internal HOM-functor.
- If k is a field and $\mathcal{B}$ saturated, then

$$\mathrm{HOM}(\mathcal{A}, \mathcal{B}) = A_\infty\!-\!\mathrm{Fun}(\mathcal{A}, \mathcal{B})$$

where $A_\infty-\mathrm{Fun}$ is the A_∞-functor category (Fukaya, Kontsevich, Lyubashenko, . . .).

Quotients



Main Theorem 2 (K, Drinfeld)

- If $\mathcal{N} \subset \mathcal{A}$ is a full dg subcategory, there is a universal morphism

$$Q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$$

  in NCS such that $Q(N) = 0$ in $H^0(\mathcal{A}/\mathcal{N})$, $N \in \mathcal{N}$.
- If k is a field, then $\mathcal{A}/\mathcal{N}$ is obtained by adjoining to $\mathcal{A}$ a contracting homotopy for each object of $\mathcal{N}$ (Drinfeld).
- We have $\mathrm{per}(\mathcal{A}/\mathcal{N}) \xleftarrow{\sim} (\mathrm{per}(\mathcal{A})/\mathrm{per}(\mathcal{N}))^{\sim}$.

From schemes to NC schemes

Definition

The *dg derived category* of an abelian category $\mathcal{E}$ is

$$\mathcal{D}_{dg}(\mathcal{E}) = \frac{\text{dg category of complexes over } \mathcal{E}}{\text{dg subcategory of acyclic complexes}}$$

Remark

- The theorem yields $H^0(\mathcal{D}_{dg}(\mathcal{E})) \xrightarrow{\sim} \mathcal{D}(\mathcal{E})$.
- Via $X \mapsto \mathrm{Qcoh}(X) \mapsto \mathcal{D}_{dg}(\mathrm{Qcoh}(X))$, we obtain

$$\begin{array}{ccc} \{\text{classical schemes}\} & \longrightarrow & \{\text{abelian categories}\} \\ & \searrow & \downarrow \\ & & \{\text{NC schemes}\} \end{array}$$

Mukai transforms

Question

What are the morphisms between the NC schemes associated with classical schemes ?

Let X, Y be quasi-compact, separated, flat schemes. Put $\mathcal{D}_{dg}(X) = \mathcal{D}_{dg}(\mathrm{Qcoh}(X))$.

Main Theorem 3 (Toën)

We have a canonical bijection

$$\begin{array}{c} \{\text{isoclasses of } \mathcal{D}(X \times Y)\} \\ \Big\downarrow \text{bijective} \\ \{\text{continuous morphisms } \mathcal{D}_{dg}(X) \to \mathcal{D}_{dg}(Y)\}, \end{array}$$

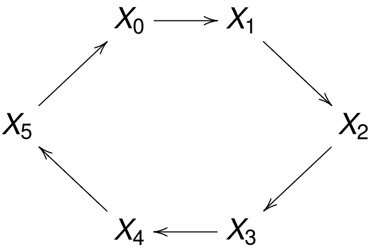
where a morphism of NCS is *continuous* if the associated triangle functor $\mathcal{D}(X) \to \mathcal{D}(Y)$ commutes with infinite sums. Compare with Orlov's theorem.

Cyclic homology I

Suppose for simplicity that k is a field. Let $\mathcal{A}$ be a dg k-category. Let

$$
\begin{aligned}
C_0(\mathcal{A}) &= \coprod_{X_0 \in \mathcal{A}} \mathcal{A}(X_0, X_0)\,, \\
C_1(\mathcal{A}) &= \coprod_{X_0, X_1 \in \mathcal{A}} \mathcal{A}(X_1, X_0) \otimes \mathcal{A}(X_0, X_1)\,, \\
C_p(\mathcal{A}) &= \coprod_{X_0, \ldots, X_p} \mathcal{A}(X_{p-1}, X_p) \otimes \ldots \otimes \mathcal{A}(X_0, X_1).
\end{aligned}
$$

For example, for $p = 5$, $C_p(\mathcal{A})$ is formed by sums of diagrams



Cyclic homology II

Definition

The *Hochschild (-Mitchell) complex* $C(\mathcal{A})$ is the sum-total complex of the bicomplex

$$\ldots \to C_p(\mathcal{A}) \to \ldots \to C_1(\mathcal{A}) \to C_0(\mathcal{A}) \to 0$$

with the horizontal differential given by

$$b(f_0, \ldots, f_p) = \sum_{i=0}^{p-1} \pm(f_0, \ldots, f_i f_{i+1}, \ldots f_p) \pm (f_p f_0, \ldots, f_{p-1}).$$

The *Hochschild homology* $HH_*(\mathcal{A})$ is the homology of $(C(\mathcal{A}), b_{tot})$. The cyclic actions on the $C_p(\mathcal{A})$ yield a mixed complex $M(\mathcal{A})$ and the *cyclic homology* $HC_*(\mathcal{A})$.

Invariance and localization

Theorem (K)

a) *Cyclic homology extends to a functor* NCS $\to$ Vec(k).

b) *Each exact sequence*

$$0 \to \mathcal{N} \to \mathcal{A} \to \mathcal{A}/\mathcal{N} \to 0$$

of NCS *yields a long exact sequence*

$$\dots HC_p(\mathcal{N}) \to HC_p(\mathcal{A}) \to HC_p(\mathcal{A}/\mathcal{N}) \to HC_{p-1}(\mathcal{N}) \dots$$

Remarks

- An analogous theorem holds for K-theory (Thomason-Trobaugh).
- Sequences as in b) arise when $X = U \coprod Z$, X a commutative (quasi-compact quasi-separated) scheme, U open.

The Hodge-to-de Rham degeneration conj.

Definition (Kontsevich)

- A dg category $\mathcal{A}$ is *smooth* if the bimodule $(X, Y) \mapsto \mathcal{A}(X, Y)$ is in $\mathrm{per}(\mathcal{A}^{op} \otimes \mathcal{A})$.
- It is *proper* if it is isomorphic in NCS to a dg algebra whose homology is of finite total dimension.

Endow $M(\mathcal{A}) \otimes k[[u]]$, $\deg(u) = 2$, with the differential

$$d = d \otimes \mathbf{1} + d' \otimes u.$$

Hodge-to-de Rham degeneration Conjecture (Kontsevich)

If $\mathcal{A}$ is a smooth proper dg category over a field k of characteristic 0, then the homology of $M(\mathcal{A}) \otimes k[[u]]/(u^n)$ is a flat $k[u]/(u^n)$-module for all $n \geq 1$.

Further topics

Topics not covered:

- Dugger, Schwede, Shipley:
  derived equivalence $\not\Longleftarrow\!\!\!\Longrightarrow$ topological der. eq.
- Toën-Vaquié: higher moduli spaces
- Toën: Hall algebras
- …

NCS and homological geometry

NCS = framework for 'homological geometry' as used in

- algebraic geometry: *cf.* T. Bridgeland's ICM talk
- symplectic geometry: Fukaya, Kontsevich, ...
- Lie representation theory: *cf.* R. Bezrukavnikov's ICM talk
- modular group rep. theory: *cf.* R. Rouquier's ICM talk
- representation theory of quivers and its links to cluster algebras (Fomin-Zelevinsky, ...)
- ...