Quiver mutation and quantum dilogarithm identities

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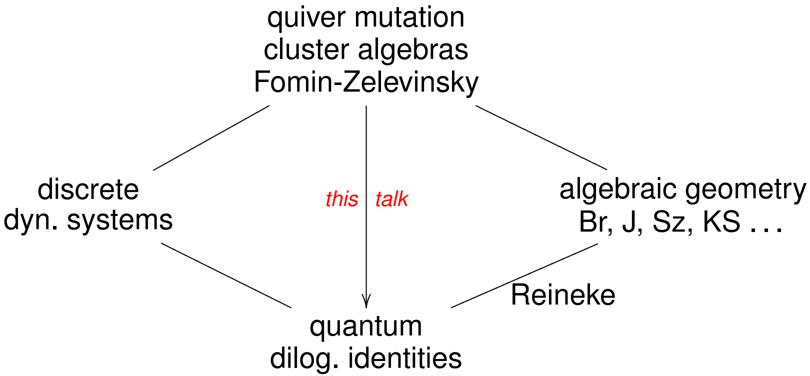
New developments in noncommutative algebra
and its applications

Isle of Skye, June 27, 2011

Context



Br=Bridgeland, J=Joyce, Sz=Szendröi, KS=Konts.-Soibelman

Plan

A quiver is an oriented graph

Definition

A *quiver* Q is an oriented graph: It is given by

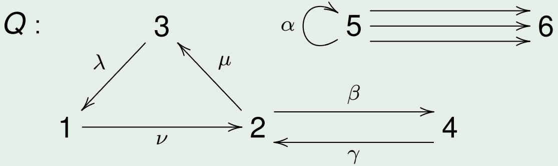
- a set Q_0 (the set of vertices)
- a set Q_1 (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (taking an arrow to its source)
  - $t : Q_1 \to Q_0$ (taking an arrow to its target).

Remark

A quiver is a ‘category without composition’.

Loops, cycles, sources and sinks of a quiver.

Example



We have $Q_0 = \{1, 2, 3, 4, 5, 6\}$, $Q_1 = \{\alpha, \beta, \dots\}$.
α is a *loop*, (β, γ) is a 2-*cycle*, (λ, μ, ν) is a 3-*cycle*.

Definition

A vertex *i* is a *source* of Q if no arrows stop at *i*.
A vertex *i* is a *sink* of Q if no arrows start at *i*.

Definition of quiver mutation

Let Q be a quiver without loops or 2-cycles.

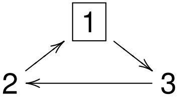
Definition (Fomin-Zelevinsky)

Let $j \in Q_0$. The *mutation* $\mu_j(Q)$ is the quiver obtained from Q as follows

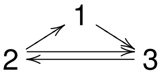
1) for each subquiver $i \xrightarrow{b} j \xrightarrow{a} k$, add a new arrow $i \xrightarrow{[ab]} k$;
2) reverse all arrows incident with j;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles (e.g. $\bullet \rightleftarrows\!\!\!\rightarrow \bullet$ yields $\bullet \longrightarrow \bullet$, '2-reduction').

Examples of quiver mutation

A simple example:



1)



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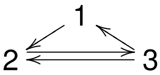
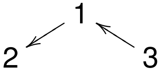
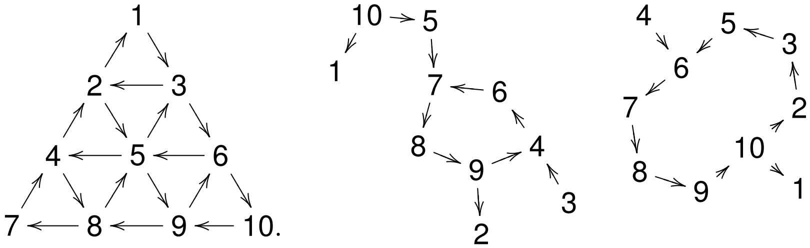


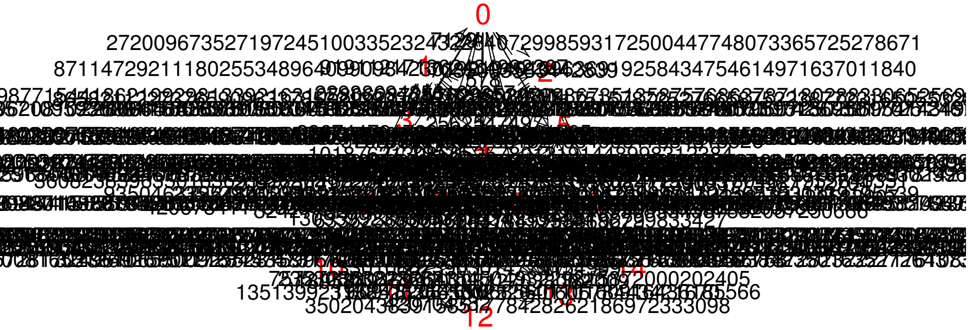
3)



More complicated examples: Google 'quiver mutation'!



Towards green quiver mutation



Next aim: Green quiver mutation!

Green quiver mutation: the framed quiver

Let Q be a quiver without loops or 2-cycles.

Definition

The framed quiver $\widetilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$.

Example

$$Q: \ 1 \longrightarrow 2 \qquad\qquad \widetilde{Q}: \ \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1' & & 2' \end{array}$$

Definition

The vertices i' are frozen vertices, i.e. we never mutate at them.

Green vertices, green sequences

Suppose that we have transformed $\widetilde{Q}$ into $\widetilde{Q}'$ by a finite sequence of mutations (at non frozen vertices).

Definition

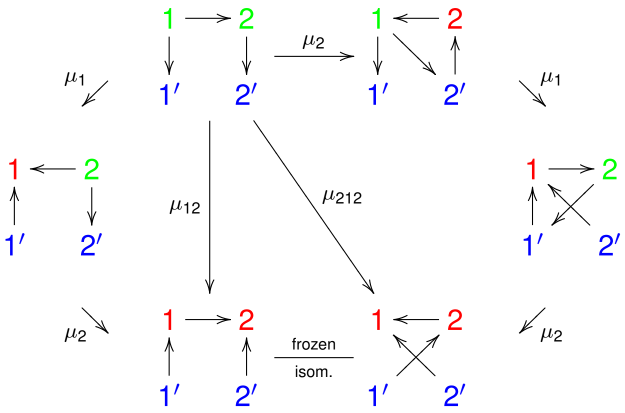
A vertex i of Q is green in $\widetilde{Q}'$ if there are no arrows $j' \to i$ in $\widetilde{Q}'$. Otherwise, it is red.

Definition

A sequence $\underline{i} = (i_1, \ldots, i_N)$ is green if for each $1 \leq t \leq N$, the vertex i_t is green in

$$\mu_{i_{t-1}} \ldots \mu_{i_2} \mu_{i_1} (\widetilde{Q}) =: \widetilde{Q}(\underline{i}, t).$$

Green quiver mutation: example



Maximal green sequences

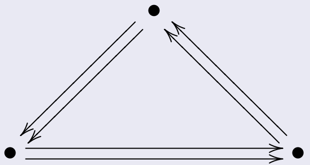
Remark 1

One can show: If $\underline{i}$ and $\underline{i}'$ are maximal green, then there is a frozen isomorphism

$$\mu_{\underline{i}}(\widetilde{Q}) \xrightarrow{\sim} \mu_{\underline{i}'}(\widetilde{Q}).$$

Remark 2

Maximal sequences do not always exist. Example:



Definition and the pentagon identity

Definition

The (exponential of the) quantum dilogarithm series is

$$\mathbb{E}(y) = 1 + \frac{q^{1/2}}{q-1} \cdot y + \cdots + \frac{q^{n^2/2} y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} + \cdots$$
$$\in \mathbb{Q}(q^{1/2})[[y]].$$

Pentagon identity (Sch. 1953, Faddeev–Kashaev–Volkov 1993)

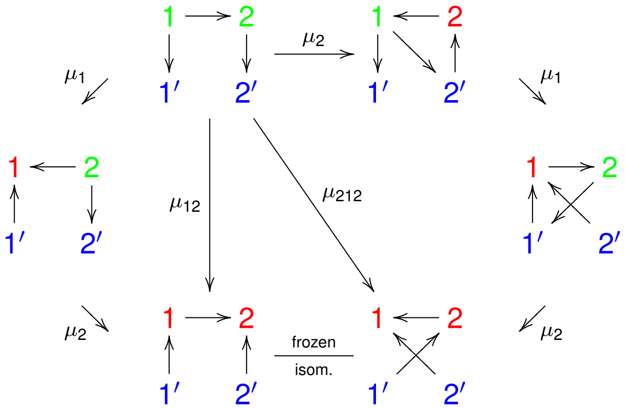
$$y_1 y_2 = q y_2 y_1 \Longrightarrow \mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1).$$

Aim

Generalize this identity using green mutations.

The pentagon corresponds to the quiver $\vec{A}_2$

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1 y_2)\mathbb{E}(y_1)$$



Key construction

Given a green sequence $\underline{i}$ we need to construct a product $\mathbb{E}(\underline{i})$ of quantum dilogarithm series. This product is taken in the algebra $\widehat{\mathbb{A}}_Q$ constructed as follows: Let $\lambda_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be bilinear antisymmetric such that

$$\lambda_Q(e_i, e_j) = \#(\text{arrows } i \to j \text{ in } Q) - \#(\text{arrows } j \to i \text{ in } Q).$$

Define

$$\widehat{\mathbb{A}}_Q = \mathbb{Q}(q^{1/2})\langle\langle y^\alpha, \alpha \in \mathbb{N}^n \mid y^\alpha y^\beta = q^{1/2\,\lambda(\alpha,\beta)} y^{\alpha+\beta} \rangle\rangle$$

$$\mathbb{E}(\underline{i}) = \mathbb{E}(y^{\alpha_1}) \ldots \mathbb{E}(y^{\alpha_N}) \,,$$

where

$$(\alpha_t)_j = \#(\text{arrows } i_t \to j' \text{ in } \widetilde{Q}(\underline{i}, t)).$$

Quantum dilogarithm identities

Theorem

Let $\underline{i}$ and $\underline{i}'$ be green sequences. If there is a frozen isomorphism

$$\mu_{\underline{i}}(\widetilde{Q}) \cong \mu_{\underline{i}'}(\widetilde{Q}'),$$

then we have

$$\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}').$$

Remark

In particular, if $\underline{i}$ and $\underline{i}'$ are *maximal* green sequences, then $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$. This power series is intrinsically associated with the quiver Q. It equals Kontsevich–Soibelman's 'refined Donaldson–Thomas–invariant' (if defined).

Example 1: Dynkin quivers

Let Q be an alternating Dynkin quiver $\vec{\Delta}$, where Δ is a simply laced Dynkin diagram, e. g.

$$Q = \vec{A}_5 : \bullet \longleftarrow \circ \longrightarrow \bullet \longleftarrow \circ \longrightarrow \bullet$$

Put

$$\begin{aligned} i_+ &= \text{ sequence of all sources } \circ \\ i_- &= \text{ sequence of all sinks } \bullet . \end{aligned}$$

Then $\underline{i} = i_+ i_-$ is maximal and so is

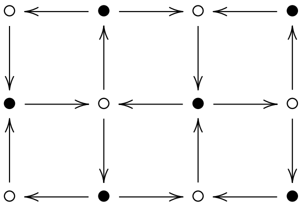
$$\underline{i}' = \underbrace{i_- i_+ i_- \ldots}_{h \text{ factors}} \quad ,$$

where h is the Coxeter number of the underlying graph of Q. Thus, we have $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$. These are Reineke's identities.

Example 2: Square products of Dynkin diagrams

$Q = \Delta \square \Delta'$, where Δ and Δ' are simply laced Dynkin diagrams, e. g.

$Q = A_4 \square A_3$:



$$
\begin{aligned}
i_+ &= \text{ sequence of all } \circ \\
i_- &= \text{ sequence of all } \bullet \\
\underline{i} &= i_+ i_- i_+ \ldots \quad \text{with } h \text{ factors, is maximal green,} \\
\underline{i}' &= i_- i_+ i_- \ldots \quad \text{with } h' \text{ factors, is maximal green.}
\end{aligned}
$$

We get a new identity $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$.

Two proofs

- ‘Proof’ based on Kontsevich-Soibelman’s theory (preprints from November 2008 and June 2010),
- Proof based on the ‘additive categorification’ of cluster algebras.

Some contributors to ‘additive categorification’ (in reverse chronological order): Nagao, Plamondon, Derksen–Weyman–Zelevinsky, . . . , Berenstein–Zelevinsky, . . . , Buan–Marsh–Reineke–Reiten–Todorov, . . . , Caldero–Chapoton, Fock–Goncharov, Fomin–Zelevinsky.

Main steps of the second proof

(1) $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$ in $\widehat{\mathbb{A}}_Q$ follows from
(2) $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$ in $\widehat{\mathbb{A}}_{\tilde{Q}}$ since $\widehat{\mathbb{A}}_Q \subset \widehat{\mathbb{A}}_{\tilde{Q}}$.
(3) The equality (2) is equivalent to

$$\mathrm{Ad}\,\mathbb{E}(\underline{i}) = \mathrm{Ad}\,\mathbb{E}(\underline{i}')$$

because the center of $\widehat{\mathbb{A}}_{\tilde{Q}}$ is $\mathbb{Q}(q^{1/2})$.
(4) The equality (3) is equivalent to its specialization at $q^{1/2} = 1$! (by a theorem on quantum cluster algebras due to Berenstein-Zelevinsky)
(5) The specialization of $\mathrm{Ad}\,\mathbb{E}(\underline{i})$ at $q^{1/2} = 1$ can be expressed in terms of Euler characteristics of quiver grassmannians of certain representations of Q (by the 'main theorem of additive categorification').
(6) One shows that these representations only depend on the class of $\mu_{\underline{i}}(\widetilde{Q})$ modulo frozen isomorphism.

Summary

- Google 'quiver mutation'!
- Quiver mutation yields quantum dilogarithm identities.
- These imply classical dilogarithm identities.
- They also imply a quantum version Zamolodchikov's periodicity conjecture for Y-systems (proved previously directly from cluster categorification).