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Derived equivalences from mutations of quivers with potential

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Abstract

We show that Derksen–Weyman–Zelevinsky's mutations of quivers with potential yield equivalences of suitable 3-Calabi–Yau triangulated categories. Our approach is related to that of Iyama–Reiten and 'Koszul dual' to that of Kontsevich–Soibelman. It improves on previous work by Vitória. In Appendix A, the first-named author studies pseudo-compact derived categories of certain pseudo-compact dg algebras.
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Contents

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1. Introduction

1.1. Previous work and context

A quiver is an oriented graph. Quiver mutation is an elementary operation on quivers. In mathematics, it first appeared in the definition of cluster algebras invented by Fomin and Zelevinsky at the beginning of this decade [16] (though, with hindsight, Gabrielov transformations [19,13] may be seen as predecessors). In physics, it appeared a few years before in Seiberg duality [47]. In this article, we 'categorify' quiver mutation by interpreting it in terms of equivalences between derived categories. Our approach is based on Ginzburg's Calabi–Yau algebras [21] and on mutation of quivers with potential. Let us point out that mutation of quivers with potential goes back to the insight of mathematical physicists, cf. for example Section 5.2 of [14], Section 2.3 of [6], Section 6 of [15] and Section 3 of [35]. Their methods were made rigorous by Derksen, Weyman and Zelevinsky [12]. The idea of interpreting quiver mutation via derived categories goes back at least to Berenstein and Douglas [7] and was further developed by Mukhopadhyay and Ray [39] in physics and by Bridgeland [10], Iyama and Reiten [25], Vitória [53], Kontsevich and Soibelman [36], etc. in mathematics. The main new ingredient in our approach is Ginzburg's algebra [21] associated to a quiver with potential. This differential graded (= dg) algebra is concentrated in negative degrees. Several previous approaches have only used its homology in degree 0, namely the Jacobian algebra of [12] or vacualgebra of the physics literature. By taking into account the full dg algebra, we are able to get rid of the restrictive hypotheses that one had to impose and to prove the homological interpretation in full generality.

Our second source of inspiration and motivation is the theory which links cluster algebras [16] to representations of quivers and finite-dimensional algebras, cf. [28] and [27] for recent surveys. This theory is based on the discovery [38] of the striking similarity between the combinatorics of cluster algebras and those of tilting theory in the representation theory of finite-dimensional algebras. The cluster category, invented in [4], and the module category over the preprojective algebra [20] provided a conceptual framework for this remarkable phenomenon. These categories are triangulated respectively exact and 2-Calabi–Yau. Recall that a (triangulated or exact) category is d-Calabi–Yau if there are non-degenerate bifunctorial pairings

$$\mathsf{Ext}^i(X, Y) \times \mathsf{Ext}^{d-i}(Y, X) \to k,$$

where k is the ground field. Iyama and Reiten [25] used both 2- and 3-Calabi–Yau categories to categorify mutations of different classes of quivers. We first learned about the possibility of categorifying arbitrary quiver mutations using 3-Calabi–Yau categories from Kontsevich, who constructed the corresponding categories using A_∞-structures, cf. Section 5 of [32] for a sketch. This idea is fully developed in Section 8 of Kontsevich and Soibelman's work [36]. The Ginzburg dg algebras which we use are Koszul dual to Kontsevich–Soibelman's A_∞-algebras. Since a Ginzburg dg algebra is concentrated in negative degrees, its derived category has a canonical t-structure given by the standard truncation functors. The heart is equivalent to the module category over the zeroth homology of the Ginzburg algebra, i.e. over the Jacobian algebra. This allows us

to establish the link to previous approaches, especially to the one based on the cluster category and the one based on decorated representations [38,12,11].

1.2. The classical paradigm and the main result

Like Derksen, Weyman and Zelevinsky [12], we model our approach on the reflection functors introduced by Bernstein, Gelfand and Ponomarev [8]. Let us recall their result in a modern form similar to the one we will obtain: Let Q be a finite quiver and i a *source* of Q, i.e. a vertex without incoming arrows. Let Q' be the quiver obtained from Q by reversing all the arrows going out from i. Let k be a field, kQ the path algebra of Q (we multiply paths in the same way as we compose morphisms so that for paths $p : u \to v$ and $q : v \to w$, the product, denoted by qp, goes from u to w) and $\mathcal{D}(kQ)$ the derived category of the category of all right kQ-modules. For a vertex j of Q' respectively Q, let P'_j respectively P_j be the projective indecomposable associated with the vertex j (i.e. the right ideal generated by the idempotent e_j of the respective path algebra). Then the main result of [8] reformulated in terms of derived categories following Happel [22] says that there is a canonical triangle equivalence

$$F : \mathcal{D}(kQ') \to \mathcal{D}(kQ)$$

which takes P'_j to P_j for $j \neq i$ and P'_i to the cone over the morphism

$$P_i \to \bigoplus P_j$$

whose components are the left multiplications by all arrows going out from i. In Theorem 3.2, we will obtain an analogous result for the mutation of a quiver with potential (Q, W) at an *arbitrary* vertex i, where the rôle of the quiver with reversed arrows is played by the quiver with potential (Q', W') obtained from (Q, W) by mutation at i in the sense of Derksen, Weyman and Zelevinsky [12]. The rôle of the derived category $\mathcal{D}(kQ)$ will be played by the derived category $\mathcal{D}(\Gamma)$ of the complete differential graded algebra $\Gamma = \Gamma(Q, W)$ associated with (Q, W) by Ginzburg [21]. Let us point out that our result is analogous to but not a generalization of Bernstein, Gelfand and Ponomarev's since even if the potential W vanishes, the derived category $\mathcal{D}(\Gamma)$ is *not* equivalent to $\mathcal{D}(kQ)$. Notice that we work with completed path algebras because this is the setup in which Derksen–Weyman–Zelevinsky's results are proved. Nevertheless, with suitable precautions, one can prove similar derived equivalences for non-complete Ginzburg algebras, cf. Section 7.6 of [29].

1.3. Plan of the paper and further results

In Section 2, we first recall Derksen–Weyman–Zelevinsky's results on mutations of quivers with potential. In Section 2.6, we recall the definition of the Ginzburg dg algebra associated to a quiver with potential and show that reductions of quivers with potential yield quasi-isomorphisms between Ginzburg dg algebras. Then we recall the basics on derived categories of differential graded algebras in Sections 2.12 and 2.14. In Section 2.18, we prove the useful Theorem 2.19, which could be viewed as an 'Artin–Rees lemma' for derived categories of suitable complete dg algebras. In Section 3, we state and prove the main result described above. In Section 4, we establish a link with cluster categories in the sense of Amiot [1], who considered the Jacobi-finite case, and Plamondon [43,42], who is considering the general case. We use this to give another

proof of the fact that neighboring Jacobian algebras are nearly Morita equivalent in the sense of Ringel [46]. This was previously proved in [3] and [11]. We include another proof to illustrate the links between our approach and those of these two papers. In Section 5, we compare the canonical t-structures in the derived categories associated with a quiver with potential and its mutation at a vertex. In general, the hearts of these t-structures are not related by a tilt in the sense of Happel, Reiten and Smalø [23]. However, this is the case if the homology of the Ginzburg dg algebra is concentrated in degree 0. In Section 6, we show that this last condition is preserved under mutation. In a certain sense, this explains the beautiful results of Iyama and Reiten [25] on mutation of 3-Calabi–Yau algebras (concentrated in degree 0): In fact, these algebras are quasi-isomorphic to the Ginzburg algebras associated to the corresponding quivers with potential. Strictly speaking, the classical derived category, which we use in the main body of the paper, is not well-adapted to topological dg algebras like the completed Ginzburg dg algebra. In Appendix A, the first-named author defines the pseudo-compact derived category, which is the 'sophisticated' derived category associated with a topological dg algebra (satisfying suitable boundedness and regularity hypotheses). He shows that the main result also holds for pseudocompact derived categories. In more detail, in Appendix A, he first recalls the basics of the theory of pseudocompact algebras from [18]. As shown in Corollary A.8, the theory immediately yields a positive answer to question 12.1 of [12], where the authors asked whether a Jacobian algebra is determined by its category of finite-dimensional modules. In Section A.11, the theory of pseudocompact algebras and modules is then adapted, under certain boundedness and regularity assumptions, to pseudocompact differential graded algebras. The analogue of the main theorem also holds in this setting (Section A.22). Moreover, the results of Amiot [1] on Jacobi-finite quivers with potentials can be extended to potentials in complete path algebras as we see in Section A.20.

2. Preliminaries

2.1. Quivers with potential

We follow [12]. Let k be a field. Let Q be a finite quiver (possibly with loops and 2-cycles). We denote its set of vertices by Q_0 and its set of arrows by Q_1. For an arrow a of Q, let $s(a)$ denote its source and $t(a)$ denote its target. The lazy path corresponding to a vertex i will be denoted by e_i.

The *complete path algebra* $\widehat{kQ}$ is the completion of the path algebra kQ with respect to the ideal generated by the arrows of Q. Let $\mathfrak{m}$ be the ideal of $\widehat{kQ}$ generated by the arrows of Q. A *potential* on Q is an element of the closure $Pot(kQ)$ of the space generated by all non-trivial cyclic paths of Q. We say two potentials are *cyclically equivalent* if their difference is in the closure of the space generated by all differences $a_1 \ldots a_s - a_2 \ldots a_s a_1$, where $a_1 \ldots a_s$ is a cycle.

For a path p of Q, we define $\partial_p : Pot(kQ) \to \widehat{kQ}$ as the unique continuous linear map which takes a cycle c to the sum $\sum_{c=upv} vu$ taken over all decompositions of the cycle c (where u and v are possibly lazy paths). Obviously two cyclically equivalent potentials have the same image under ∂_p. If $p = a$ is an arrow of Q, we call ∂_a the *cyclic derivative with respect to a*.

Remark 2.2. The space of cyclic equivalence classes of elements in $Pot(kQ)$ identifies with the *reduced continuous Hochschild homology* $HH_0^{\mathrm{red}}(\widehat{kQ})$, the quotient of $\widehat{kQ}$ by the closure of the subspace generated by the lazy paths and the commutators of elements of kQ.

Let W be a potential on Q such that W is in $\mathfrak{m}^2$ and no two cyclically equivalent cyclic paths appear in the decomposition of W [12, (4.2)]. Then the pair (Q, W) is called a *quiver with potential*. Two quivers with potential (Q, W) and (Q', W') are *right-equivalent* if Q and Q' have the same set of vertices and there exists an algebra isomorphism $\varphi : \widehat{kQ} \to \widehat{kQ'}$ whose restriction on vertices is the identity map and $\varphi(W)$ and W' are cyclically equivalent. Such an isomorphism φ is called a *right-equivalence*.

The Jacobian algebra of a quiver with potential (Q, W), denoted by $J(Q, W)$, is the quotient of the complete path algebra $\widehat{kQ}$ by the closure of the ideal generated by $\partial_a W$, where a runs over all arrows of Q. It is clear that two right-equivalent quivers with potential have isomorphic Jacobian algebras. A quiver with potential is called *trivial* if its potential is a linear combination of cycles of length 2 and its Jacobian algebra is the product of copies of the base field k. A quiver with potential is called *reduced* if $\partial_a W$ is contained in $\mathfrak{m}^2$ for all arrows a of Q.

Let (Q', W') and (Q'', W'') be two quivers with potential such that Q' and Q'' have the same set of vertices. Their *direct sum*, denoted by $(Q', W') \oplus (Q'', W'')$, is the new quiver with potential (Q, W), where Q is the quiver whose vertex set is the same as the vertex set of Q' (and Q'') and whose arrow set the union of the arrow set of Q' and the arrow set of Q'', and $W = W' + W''$.

Theorem 2.3. *In the following statements, all quivers are quivers without loops.*

(a) *(See [12, Proposition 4.4].) A quiver with potential is trivial if and only if it is right-equivalent, via a right-equivalence which takes arrows to linear combinations of arrows, to a quiver with potential (Q, W) such that Q_1 consists of $2n$ distinct arrows*

$$Q_1 = \{a_1, b_1, \ldots, a_n, b_n\},$$

where each $a_i b_i$ is a 2-cycle, and

$$W = \sum_{i=1}^{n} a_i b_i.$$

(b) *(See [12, Proposition 4.5].) Let (Q', W') be a quiver with potential and (Q'', W'') be a trivial quiver with potential. Let (Q, W) be their direct sum. Then the canonical embedding $\widehat{kQ'} \to \widehat{kQ}$ induces an isomorphism of Jacobian algebras $J(Q', W') \to J(Q, W)$.*

(c) *(See [12, Theorem 4.6].) For any quiver with potential (Q, W), there exist a trivial quiver with potential $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and a reduced quiver with potential $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ such that (Q, W) is right-equivalent to the direct sum $(Q_{\mathrm{tri}}, W_{\mathrm{tri}}) \oplus (Q_{\mathrm{red}}, W_{\mathrm{red}})$. Furthermore, the right-equivalence class of each of $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ is determined by the right-equivalence class of (Q, W). We call $(Q_{\mathrm{tri}}, W_{\mathrm{tri}})$ and $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ respectively the trivial part and the reduced part of (Q, W).*

2.4. Mutations of quivers with potential

We follow [12]. Let (Q, W) be a quiver with potential. Let i be a vertex of Q. Assume the following conditions:

(c1) the quiver Q has no loops;
(c2) the quiver Q does not have 2-cycles at i;
(c3) no cyclic path occurring in the expansion of W starts and ends at i [12, (5.2)].

Note that under the condition (c1), any potential is cyclically equivalent to a potential satisfying (c3). We define a new quiver with potential $\tilde{\mu}_i(Q, W) = (Q', W')$ as follows. The new quiver Q' is obtained from Q by

Step 1 For each arrow β with target i and each arrow α with source i, add a new arrow $[\alpha\beta]$ from the source of β to the target of α.
Step 2 Replace each arrow α with source or target i with an arrow $\alpha^\star$ in the opposite direction.

The new potential W' is the sum of two potentials W_1' and W_2'. The potential W_1' is obtained from W by replacing each composition $\alpha\beta$ by $[\alpha\beta]$, where β is an arrow with target i. The potential W_2' is given by

$$W_2' = \sum_{\alpha, \beta}[\alpha\beta]\beta^\star\alpha^\star,$$

where the sum ranges over all pairs of arrows α and β such that β ends at i and α starts at i. It is easy to see that $\tilde{\mu}_i(Q, W)$ satisfies (c1)–(c3). We define $\mu_i(Q, W)$ as the reduced part of $\tilde{\mu}_i(Q, W)$, and call μ_i the *mutation at the vertex i*.

Theorem 2.5.

(a) *(See [12, Theorem 5.2].) The right-equivalence class of $\tilde{\mu}_i(Q, W)$ is determined by the right-equivalence class of (Q, W).*
(b) *(See [12].) The quiver with potential $\tilde{\mu}_i^2(Q, W)$ is right-equivalent to the direct sum of (Q, W) with a trivial quiver with potential.*
(c) *(See [12, Theorem 5.7].) The correspondence μ_i acts as an involution on the right-equivalence classes of reduced quivers with potential satisfying* (c1) *and* (c2).

2.6. The complete Ginzburg dg algebra

Let (Q, W) be a quiver with potential. The *complete Ginzburg dg algebra* $\hat{\Gamma}(Q, W)$ is constructed as follows [21]: Let $\tilde{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (they all have degree 0),
- an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q.

The underlying graded algebra of $\hat{\Gamma}(Q, W)$ is the completion of the graded path algebra $k\tilde{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $\tilde{Q}$. Thus, the n-th component of $\hat{\Gamma}(Q, W)$ consists of elements of the form $\sum_p \lambda_p p$, where p runs over all

paths of degree n. The differential of $\hat{\Gamma}(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = (du)v + (-1)^p u\, dv,$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\tilde{Q}$:

- $da = 0$ for each arrow a of Q,
- $d(a^*) = \partial_a W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q, where e_i is the lazy path at i and the sum runs over the set of arrows of Q.

Remark 2.7. The authors thank N. Broomhead for pointing out that Ginzburg's original definition in Section 5.3 of [21] concerns the case where the potential W lies in the non-completed path algebra kQ and that he completes with respect to the ideal generated by the cyclic derivatives of W rather than the ideal generated by the arrows of Q.

The following lemma is an easy consequence of the definition.

Lemma 2.8. *Let (Q, W) be a quiver with potential. Then the Jacobian algebra $J(Q, W)$ is the 0-th cohomology of the complete Ginzburg dg algebra $\hat{\Gamma}(Q, W)$, i.e.*

$$J(Q, W) = H^0 \hat{\Gamma}(Q, W).$$

Let (Q, W) and (Q', W') be two quivers with potential. Let $\Gamma = \hat{\Gamma}(Q, W)$ and $\Gamma' = \hat{\Gamma}(Q', W')$ be the associated complete Ginzburg dg algebras.

Lemma 2.9. *Assume (Q, W) and (Q', W') are right-equivalent. Then the complete Ginzburg dg algebras Γ and Γ' are isomorphic to each other.*

Proof. Let φ be a right-equivalence from (Q, W) to (Q', W'). Let $\varphi_* : \Gamma \to \Gamma'$ be the unique continuous algebra homomorphism such that its restriction to Q_0 is the identity and for $\rho \in Q_1$ and $i \in Q_0$

$$\varphi_*(\rho) = \varphi(\rho),$$

$$\varphi_*(\rho^*) = \sum_{\rho' \in Q'_1} \sum_p \sum_{1 \leqslant j \leqslant l(p)} b_{p,\rho'} \delta_{\rho, \rho_j^p} \varphi(\rho_{j+1}^p \cdots \rho_{l(p)}^p) \rho'^* \varphi(\rho_1^p \cdots \rho_{j-1}^p),$$

$$\varphi_*(t_i) = t_i',$$

where the middle summation is over all paths $p = \rho_1^p \cdots \rho_{l(p)}^p$ ($\rho_j^p \in Q_1$) of Q, δ is the Kronecker symbol, and for $\rho' \in Q'_1$

$$\varphi^{-1}(\rho') = \sum_{p: \text{ path of } Q} b_{p,\rho'} p,$$

and t_i' is the loop of $\tilde{Q}'$ of degree -2 at the vertex i.

It is straightforward to prove that φ_* commutes with the differentials. Thus φ_* is a homomorphism of dg algebras. Moreover, φ_* induces a linear map from $k\tilde{Q}_1$ (the k-vector space with basis the arrows of $\tilde{Q}$) to $k\tilde{Q}'_1$, whose matrix is invertible. Indeed, this matrix is quasi-diagonal with diagonal blocks the matrix of the linear map from kQ_1 to kQ'_1 induced from φ, the matrix $(b_{\rho,\rho'})_{\rho \in Q_1, \rho' \in Q'_1}$ (which we read from the definition of $\varphi_*(\rho^*)$), and the identity matrix of size the cardinality of Q_0 (which we read from the definition of $\varphi(t_i)$), all the three of which are invertible. Therefore φ_* defined as above is an isomorphism of dg algebras from Γ to Γ'. $\quad\square$

Lemma 2.10. *Suppose that (Q, W) is the direct sum of (Q', W') and (Q'', W'') and that (Q'', W'') is trivial. Then the canonical projection $\Gamma \to \Gamma'$ induces quasi-isomorphisms*

$$\Gamma/\mathfrak{m}^n \to \Gamma'/\mathfrak{m}'^n$$

for all $n \geqslant 1$ and a quasi-isomorphism $\Gamma \to \Gamma'$.

Proof. Let K be the kernel of the projection from Γ onto Γ'. For $i \geqslant 1$, let $K_i \subset K$ be $\mathfrak{m}^i \cap K$. Thus, K_i is the topological span of the paths of length $\geqslant i$ with at least one arrow in Q'' or Q''^*. Then the K_i form a decreasing filtration of K and the complex K_i/K_{i+1} is isomorphic to

$$(U \oplus V)^{\otimes_R i}/U^{\otimes_R i},$$

where $R = \prod_{i \in Q_0} k$, $V = kQ''_1 \oplus kQ''^*_1$ as a complex endowed with the differential given by W'' (thus, V is contractible) and $U = kQ'_1 \oplus kQ'^*_1 \oplus k\{t_i \mid i \in Q_0\}$ as a complex with the differential which is zero on $k\{t_i \mid i \in Q_0\}$ and, on $kQ'_1 \oplus kQ'^*_1$, is given by the projection of W onto the subspace of cycles of length 2. Thus K_i/K_{i+1} is contractible. It follows that K/K_n is contractible for each $n \geqslant 1$, which implies the first assertion. The second one now follows from the Mittag–Leffler Lemma A.4. $\quad\square$

Remark 2.11. As a consequence of Lemma 2.10, the canonical projection $\widehat{kQ} \to \widehat{kQ'}$ induces an isomorphism of Jacobian algebras $J(Q, W) \to J(Q', W')$, which is the inverse of the isomorphism in Theorem 2.3(b).

2.12. Derived categories

We follow [30]. Let A be a dg k-algebra with differential d_A (all differentials are of degree 1). A *(right) dg module* M over A is a graded A-module equipped with a differential d_M such that

$$d_M(ma) = d_M(m)a + (-1)^{|m|}m\, d_A(a)$$

where m in M is homogeneous of degree $|m|$, and $a \in A$.

Given two dg A-modules M and N, we define the *morphism complex* to be the graded k-vector space $\mathcal{H}om_A(M, N)$ whose i-th component $\mathcal{H}om_A^i(M, N)$ is the subspace of the product $\prod_{j \in \mathbb{Z}} \mathsf{Hom}_k(M^j, N^{j+i})$ consisting of morphisms f such that

$$f(ma) = f(m)a,$$

for all m in M and all a in A, together with the differential d given by

$$d(f) = f \circ d_M - (-1)^{|f|} d_N \circ f$$

for a homogeneous morphism f of degree $|f|$.

The category $\mathcal{C}(A)$ of dg A-modules is the category whose objects are the dg A-modules, and whose morphisms are the 0-cycles of the morphism complexes. This is an abelian category and a Frobenius category for the conflations which are split exact as sequences of graded A-modules. Its stable category $\mathcal{H}(A)$ is called the *homotopy category* of dg A-modules, which is equivalently defined as the category whose objects are the dg A-modules and whose morphism spaces are the 0-th homology groups of the morphism complexes. The homotopy category $\mathcal{H}(A)$ is a triangulated category whose suspension functor Σ is the shift of dg modules $M \mapsto M[1]$. The *derived category* $\mathcal{D}(A)$ of dg A-modules is the localization of $\mathcal{H}(A)$ at the full subcategory of acyclic dg A-modules. A short exact sequence

$$0 \to M \to N \to L \to 0$$

in $\mathcal{C}(A)$ yields a triangle

$$M \to N \to L \to \Sigma M$$

in $\mathcal{D}(A)$. A dg A-module P is *cofibrant* if

$$\mathsf{Hom}_{\mathcal{C}(A)}(P, L) \xrightarrow{s_*} \mathsf{Hom}_{\mathcal{C}(A)}(P, M)$$

is surjective for each quasi-isomorphism $s : L \to M$ which is surjective in each component. We use the term "cofibrant" since these are actually the objects which are cofibrant for a certain structure of Quillen model category on the category $\mathcal{C}(A)$, cf. [31, Theorem 3.2].

Proposition 2.13. *A dg A-module is cofibrant if and only if it is a direct summand of a dg module P which admits a filtration*

$$\cdots \subset F_{p-1} \subset F_p \subset \cdots \subset P, \quad p \in \mathbb{Z}$$

in $\mathcal{C}(A)$ such that

(F1) *we have $F_p = 0$ for all $p \ll 0$, and P is the union of the F_p, $p \in \mathbb{Z}$;*
(F2) *the inclusion morphism $F_{p-1} \subset F_p$ splits in the category of graded A-modules, for all $p \in \mathbb{Z}$;*
(F3) *the subquotient F_p/F_{p-1} is isomorphic in $\mathcal{C}(A)$ to a direct summand of a direct sum of shifted copies of A, for all $p \in \mathbb{Z}$.*

Let P be a cofibrant dg A-module. Then the canonical map

$$\mathsf{Hom}_{\mathcal{H}(A)}(P, N) \to \mathsf{Hom}_{\mathcal{D}(A)}(P, N)$$

is bijective for all dg A-modules N. The canonical projection from $\mathcal{H}(A)$ to $\mathcal{D}(A)$ admits a left adjoint functor $\mathbf{p}$ which sends a dg A-module M to a cofibrant dg A-module $\mathbf{p}M$ quasi-isomorphic to M. We call $\mathbf{p}M$ the *cofibrant resolution* of M. In other words, we have

$$\operatorname{Hom}_{\mathcal{D}(A)}(M, N) = \operatorname{Hom}_{\mathcal{H}(A)}(\mathbf{p}M, N) = H^0\big(\mathcal{H}om_A(\mathbf{p}M, N)\big).$$

The *perfect derived category* $\operatorname{per}(A)$ is the smallest full subcategory of $\mathcal{D}(A)$ containing A which is stable under taking shifts, extensions and direct summands. An object M of $\mathcal{D}(A)$ belongs to $\operatorname{per}(A)$ if and only if it is *compact*, i.e. the functor $\operatorname{Hom}_{\mathcal{D}(A)}(M, ?)$ commutes with arbitrary (set-indexed) direct sums, cf. Section 5 of [30] and [41]. The *finite-dimensional derived category* $\mathcal{D}_{fd}(A)$ is the full subcategory of $\mathcal{D}(A)$ consisting of those dg A-modules whose homology is of finite total dimension, or equivalently those dg A-modules M such that the sum of the dimensions of the spaces

$$\operatorname{Hom}_{\mathcal{D}(A)}\big(P, \Sigma^p M\big), \quad p \in \mathbb{Z},$$

is finite for any P in $\operatorname{per}(A)$.

2.14. Cofibrant resolutions of simples over a tensor algebra

Let Q be a finite graded quiver, and $\widehat{kQ}$ the complete path algebra. Thus, the ith component of the graded algebra $\widehat{kQ}$ is the completion of the space of paths of total degree i with respect to the descending filtration by path length. Let $\mathfrak{m}$ be the two-sided ideal of $\widehat{kQ}$ generated by arrows of Q. Let $A = (\widehat{kQ}, d)$ be a topological dg algebra whose differential takes each arrow of Q to an element of $\mathfrak{m}$.

For a vertex i of Q, let $P_i = e_i A$, and let S_i be the simple module corresponding to i. Then in $\mathcal{C}(A)$ we have a short exact sequence

$$0 \to \ker(\pi) \to P_i \xrightarrow{\pi} S_i \to 0,$$

where π is the canonical projection from P_i to S_i. Explicitly, we have

$$\ker(\pi) = \sum_{\rho \in Q_1 : \, t(\rho) = i} \rho P_{s(\rho)} = \bigoplus_{\rho \in Q_1 : \, t(\rho) = i} \rho P_{s(\rho)}$$

with the induced differential. Notice that the direct sum decomposition only holds in the category of graded A-modules and that, if $d(\rho)$ does not vanish, the summand $\rho P_{s(\rho)}$ is not stable under the differential of $\ker(\pi)$. The simple module S_i is quasi-isomorphic to

$$P = \operatorname{Cone}\big(\ker(\pi) \to P_i\big),$$

whose underlying graded space is

$$\bigoplus_{\rho \in Q_1 : \, t(\rho) = i} \Sigma \rho P_{s(\rho)} \oplus P_i.$$

Let n be the maximum of the degrees of the arrows ending in i. Let

$$F_p = 0 \quad \text{for } p < -n, \qquad F_{-n} = P_i,$$

and for $p > -n$, let

$$F_p = \bigoplus_\rho \Sigma \rho \, P_{s(\rho)} \oplus P_i,$$

where the sum is taken over the arrows ρ of Q ending in i and of degree at least $1 - p$. Then each F_p, $p \in \mathbb{Z}$, is a dg A-submodule of P. It is easy to check that (F1), (F2) and (F3) hold. Therefore, P is a cofibrant dg A-module, and hence it is a cofibrant resolution of S_i. In particular, S_i belongs to the perfect derived category $\mathsf{per}(A)$.

Lemma 2.15. *Let i and j be two vertices of Q, and let n be an integer. Let a_{ij}^n denote the number of arrows of Q starting from j and ending at i and of degree $-n + 1$. Then the dimension over k of $\mathsf{Hom}_{\mathcal{D}(A)}(S_i, \Sigma^n S_j)$ equals $a_{ij}^n + 1$ if $i = j$ and $n = 0$, and equals a_{ij}^n otherwise.*

Proof. First one notices that the morphism complex $\mathcal{H}om_A(P_i, S_j)$ is 1-dimensional and concentrated in degree 0 if $i = j$ and vanishes otherwise. Now let P be the cofibrant resolution of S_i constructed as above. By the assumption on the differential of A, the differential of $\mathcal{H}om_A(P, S_j)$ vanishes. Therefore, we have

$$\mathsf{Hom}_{\mathcal{D}(A)}(S_i, \Sigma^n S_j) = \mathsf{Hom}_{\mathcal{H}(A)}(P, \Sigma^n S_j) = H^n \mathcal{H}om_A(P, S_j) = \mathcal{H}om_A^n(P, S_j),$$

which has a basis

$$\{\pi_\rho \mid \rho \in Q_1, \ s(\rho) = j, \ t(\rho) = i, \ |\rho| = -n + 1\} \cup \{\pi_i\}$$

if $i = j$ and $n = 0$, and

$$\{\pi_\rho \mid \rho \in Q_1, \ s(\rho) = j, \ t(\rho) = i, \ |\rho| = -n + 1\}$$

otherwise, where π_i is the projection from P_i to S_i, and for an arrow ρ of Q starting from j, π_ρ is the projection from $\Sigma \rho \, P_j \cong \Sigma^n P_j$ to S_j. The assertion follows immediately. □

2.16. The perfect derived category is Krull–Schmidt

As in Section 2.14, let Q be a finite graded quiver, $\widehat{kQ}$ the complete path algebra, $\mathfrak{m}$ the two-sided ideal of $\widehat{kQ}$ generated by the arrows of Q and $A = (\widehat{kQ}, d)$ a topological dg algebra whose differential takes each arrow of Q to an element of $\mathfrak{m}$.

Lemma 2.17. *The perfect derived category $\mathsf{per}(A)$ is Krull–Schmidt, i.e. any object in $\mathsf{per}(\Gamma)$ is isomorphic to a finite direct sum of objects whose endomorphism rings are local.*

Proof. Clearly the category of finitely generated projective graded A-modules is Krull–Schmidt and its radical is the ideal generated by the morphisms $A[i] \to A[i]$, $i \in \mathbb{Z}$, given by the left multiplication by an element of the ideal $\mathfrak{m}$. Moreover, in this category, each endomorphism f of an object P admits a 'Fitting–Jordan decomposition', which can be constructed as follows: Let $i > 0$. Then the morphism f_i induced by f in $P/P\mathfrak{m}^i$ admits a Fitting–Jordan decomposition into the direct sum of $I_i = \mathsf{Im}\, f_i^{m_i}$ and $K_i = \mathsf{Ker}\, f_i^{m_i}$ for some integer m_i. Clearly, by induction, we can even construct an increasing sequence of integers m_i, $i > 0$, with this property. Then we

have natural projection morphisms $K_{i+1} \to K_i$ and $I_{i+1} \to I_i$ and clearly P decomposes into the direct sum of the inverse limits K_∞ and I_∞ and f acts nilpotently in each finite-dimensional quotient of K_∞ and invertibly in I_∞. Thus, the morphism induced by f in K_∞ belongs to the radical and the morphism induced in I_∞ is invertible. Now we consider the category of strictly perfect dg A-modules, i.e. the full subcategory of the category of dg A-modules obtained as the closure of A under shifts, direct factors and extensions which split in the category of graded A-modules (equivalently: cones). If P belongs to this category, its underlying graded A-module is finitely generated projective. It follows that P decomposes into a finite sum of indecomposable dg A-modules. Now assume that P is indecomposable as a dg A-module and that f an endomorphism of P in the category of dg A-modules. If we apply the Fitting–Jordan decomposition to the morphism of graded modules underlying f, we find a decomposition of P as the direct sum of K_∞ and I_∞ and the construction shows that these are dg submodules of P, that f acts by a radical morphism in K_∞ and by an invertible morphism in I_∞. Since P is indecomposable, f is radical or invertible. We conclude that the category of strictly perfect dg A-modules is a Krull–Schmidt category. It follows that the canonical functor from the category of strictly perfect dg A-modules to the perfect derived category is essentially surjective and that the perfect derived category is a Krull–Schmidt category since it is the quotient of the category of strictly perfect dg A-modules by the ideal of morphisms factoring through contractible strictly perfect dg A-modules. □

2.18. On finite-dimensional dg modules

As in Section 2.14, let Q be a finite graded quiver, $\widehat{kQ}$ the complete path algebra, $\mathfrak{m}$ the two-sided ideal of $\widehat{kQ}$ generated by the arrows of Q and $A = (\widehat{kQ}, d)$ a topological dg algebra whose differential takes each arrow of Q to an element of $\mathfrak{m}$. Let $\mathcal{H}_{fd}(A)$ be the full subcategory of $\mathcal{H}(A)$ formed by the dg modules of finite total dimension and let $\mathcal{A}c_{fd}(A)$ be its full subcategory formed by the acyclic dg modules of finite total dimension. Let $\mathcal{D}_0(A)$ be the localizing subcategory of $\mathcal{D}(A)$ generated by $\mathcal{D}_{fd}(A)$. Thus, the subcategory $\mathcal{D}_0(A)$ is the closure of $\mathcal{D}_{fd}(A)$ under arbitrary coproducts and finite extensions.

Theorem 2.19. *Assume that all arrows of Q are of degree $\leqslant 0$.*

(a) *The objects of $\mathcal{D}_{fd}(A)$ are compact in $\mathcal{D}(A)$ and the canonical functor*

$$\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A) \to \mathcal{D}_{fd}(A)$$

is an equivalence.

(b) *For each perfect dg module P and each dg module M belonging to $\mathcal{D}_0(A)$, the canonical map*

$$\mathrm{colim}(\mathcal{D}A)\big(P \overset{L}{\otimes}_A A/\mathfrak{m}^n, M\big) \to (\mathcal{D}A)(P, M)$$

is bijective.

Proof. (a) Let us first show that both categories, $\mathcal{D}_{fd}(A)$ and $\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A)$ are generated by the simple modules S_i. Indeed, if M is a dg module, then, since A is concentrated in negative degrees, we have an exact sequence of dg modules

$$0 \to \tau_{\leqslant 0}M \to M \to \tau_{>0}(M) \to 0.$$

If M belongs to $\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A)$ respectively $\mathcal{D}_{fd}(A)$, this sequence yields a triangle in the respective triangulated category. Thus, it is enough to check that an object M whose only non-zero component is in degree 0 belongs to the triangulated category generated by the S_i. This is easy by induction on the length of the zeroth component as a module over the semilocal algebra $H^0(A)$. Since the S_i are compact, it follows that $\mathcal{D}_{fd}(A)$ consists of compact objects. Moreover, in order to show the equivalence claimed in (a), it is enough to show that the canonical functor is fully faithful. For this, it suffices to show that the canonical map

$$\big(\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A)\big)(S_i, M) \to (\mathcal{D}_{fd}A)(S_i, M)$$

is bijective for each vertex i of Q and for each dg module M of finite total dimension. Let us first show that it is surjective. Let $P_i \to S_i$ be the canonical projection and R its kernel. Let j be the inclusion of R in P_i. By Section 2.14, the cone C over j is a cofibrant resolution of S_i. Since M is of finite total dimension, it is annihilated by some power $\mathfrak{m}^n$ of the ideal $\mathfrak{m}$. Let $f : S_i \to M$ be a morphism of $\mathcal{D}_{fd}(A)$. It is represented by a morphism of dg modules $g : C \to M$. Since C is the cone over j, the datum of g is equivalent to that of a morphism of dg modules $g_0 : P_i \to M$ and a morphism of graded modules $g_1 : R \to M$ homogeneous of degree -1 and such that $g_0 \circ j = d(g_1)$. Since M is annihilated by $\mathfrak{m}^n$, the morphism g_0 factors canonically through the projection $P_i \to P_i/P_i\mathfrak{m}^{n+1}$ and the morphism g_1 through the projection $R \to R/R\mathfrak{m}^n$. We choose these exponents because we have

$$P_i\mathfrak{m}^{n+1} \cap R = R\mathfrak{m}^n.$$

Indeed, the graded module $P_i\mathfrak{m}^{n+1}$ has a topological basis consisting of the paths of length $\geqslant n+1$ ending in i, the graded module R has a topological basis consisting of the paths of length $\geqslant 1$ ending in i and so $R\mathfrak{m}^n$ also has as topological basis the set of paths of length $\geqslant n+1$ ending in i. It follows that we have an exact sequence of dg modules

$$0 \to R/R\mathfrak{m}^n \to P_i/P_i\mathfrak{m}^{n+1} \to S_i \to 0$$

and the cone C' over the map

$$R/R\mathfrak{m}^n \to P_i/P_i\mathfrak{m}^{n+1}$$

is still quasi-isomorphic to S_i. By construction, the map $g : C \to M$ factors through the canonical projection $C \to C'$ and C' is of finite total dimension. So the given morphism f is represented by the fraction

$$S_i \leftarrow C' \to M$$

and hence is the image of a morphism of $\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A)$. To show injectivity of the canonical map, we first observe that the functor

$$\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A) \to \mathcal{D}_{fd}(A)$$

detects isomorphisms. Indeed, if a morphism $f : L \to M$ becomes invertible after applying this functor, then its cone is acyclic and so f was already invertible. Now we show that the map

$$\big(\mathcal{H}_{fd}(A)/\mathcal{A}c_{fd}(A)\big)(S_i, M) \to (\mathcal{D}_{fd}A)(S_i, M)$$

is injective. Indeed, if f is a morphism in its kernel, then in the triangle

$$\Sigma^{-1}Y \xrightarrow{\varepsilon} S_i \xrightarrow{f} M \to Y,$$

the morphism ε admits a section s in $\mathcal{D}_{fd}(A)$. This section lifts to a morphism $\tilde{s}$ by the surjectivity we have already shown. Then $\varepsilon\tilde{s}$ is invertible since it becomes invertible in $\mathcal{D}_{fd}(A)$ and so ε is a retraction and f vanishes.

(b) Both sides are homological functors in M which commute with arbitrary coproducts. So it is enough to show the claim for M in $\mathcal{D}_{fd}(A)$. By (a), we may assume that M belongs to $\mathcal{H}_{fd}(A)$. We may also assume that P is strictly perfect. Let us show that the map is surjective. A morphism from P to M in $\mathcal{D}(A)$ is represented by a morphism of dg modules $f : P \to M$. Since M is of finite total dimension, it is annihilated by some power $\mathfrak{m}^n$ and so f factors through the projection $P \to P/P\mathfrak{m}^n$, which shows that f is in the image of the map. Let us now show injectivity. Let $f : P/P\mathfrak{m}^n \to M$ be a morphism of $\mathcal{D}A$ such that the composition

$$P \to P/P\mathfrak{m}^n \to M$$

vanishes in $\mathcal{D}A$. By the calculus of left fractions, there is a quasi-isomorphism $M \to M'$ such that the morphism f is represented by a fraction

$$P/P\mathfrak{m}^n \xrightarrow{f'} M' \leftarrow M,$$

where f' is a morphism in $\mathcal{H}_{fd}(A)$. By forming the shifted cone over f' we obtain an exact sequence of dg modules

$$0 \to \Sigma^{-1}M' \to E \to P/P\mathfrak{m}^n \to 0$$

which splits as a sequence of graded A-modules and whose terms all have finite total dimension. Since the composition $P \to P/P\mathfrak{m}^n \to M'$ vanishes in $\mathcal{D}A$, the canonical surjection $P \to P/P\mathfrak{m}^n$ factors through E in the category of dg modules:

$$P \to E \to P/P\mathfrak{m}^n.$$

Since E is of finite total dimension, the map $P \to E$ factors through the canonical projection $P \to P/P\mathfrak{m}^N$ for some $N \gg 0$. The composition

$$P/P\mathfrak{m}^N \to E \to P/P\mathfrak{m}^n$$

is the canonical projection since its composition with $P \to P/P\mathfrak{m}^N$ is the canonical projection $P \to P/P\mathfrak{m}^n$ and $P \to P/P\mathfrak{m}^n$ is surjective. The composition

$$P/P\mathfrak{m}^N \to E \to P/P\mathfrak{m}^n \to M'$$

vanishes in $\mathcal{D}_{fd}(A)$ since it contains two successive morphisms of a triangle. Thus the composition of the given map $f : P/P\mathfrak{m}^n \to M$ with the projection $P/P\mathfrak{m}^N \to P/P\mathfrak{m}^n$ vanishes in $\mathcal{D}_{fd}(A)$, which implies that the given map vanishes in the passage to the colimit. □

2.20. The Nakayama functor

We keep the notations and assumptions of Section 2.18. We will construct a fully faithful functor from $\mathsf{per}(A)$ to $\mathcal{D}_0(A)$. Let P be in $\mathsf{per}(A)$. Then the functor

$$D\,\mathsf{Hom}_{\mathcal{D}(A)}(P, ?) : \mathcal{D}_0(A)^{op} \to \mathsf{Mod}\,k$$

is cohomological and takes coproducts of $\mathcal{D}_0(A)$ to products because P is compact. Now the triangulated category $\mathcal{D}_0(A)$ is compactly generated by $\mathcal{D}_{fd}(A)$. So by the Brown representability theorem, we have an object νP in $\mathcal{D}_0(A)$ and a canonical isomorphism

$$D\,\mathsf{Hom}_{\mathcal{D}(A)}(P, L) = \mathsf{Hom}_{\mathcal{D}(A)}(L, \nu P)$$

functorial in the object L of $\mathcal{D}_0(A)$. As in the case of the Serre functor (cf. [9,52]), one shows that the assignment $P \mapsto \nu P$ underlies a canonical triangle functor from $\mathsf{per}(A)$ to $\mathcal{D}_0(A)$. We call it the *Nakayama functor*. It takes a strictly perfect object P to the colimit $\mathsf{colim}\,D(P/P\mathfrak{m}^n)$.

Proposition 2.21.

(a) *The Nakayama functor $\nu : \mathsf{per}(A) \to \mathcal{D}_0(A)$ is fully faithful. Its image is formed by the objects M such that for each object F in $\mathcal{D}_{fd}(A)$, the sum*

$$\sum_{i \in \mathbb{Z}} \dim \mathsf{Hom}\big(F, \Sigma^i M\big)$$

is finite.
(b) *The functor*

$$\Phi : \mathsf{per}(A) \to \mathsf{Mod}\big(\mathcal{D}_{fd}(A)^{op}\big)$$

taking an object P to the restriction of $\mathsf{Hom}(P, ?)$ to $\mathcal{D}_{fd}(A)$ is fully faithful.

Proof. (a) If P and Q are perfect objects, we have

$$\mathsf{Hom}(\nu P, \nu Q) = D\,\mathsf{Hom}(Q, \nu P) = D\,\mathsf{colim}\,\mathsf{Hom}\big(Q \overset{L}{\otimes}_A A/\mathfrak{m}^n, \nu P\big),$$

where we have used part (b) of Theorem 2.19. Now by definition of νP, we have

$$D\,\mathsf{colim}\,\mathsf{Hom}\big(Q \overset{L}{\otimes}_A A/\mathfrak{m}^n, \nu P\big) = D\,\mathsf{colim}\,D\,\mathsf{Hom}\big(P, Q \overset{L}{\otimes}_A A/\mathfrak{m}^n\big)$$

and the last space identifies with

$$\lim D D \operatorname{Hom}\big(P, Q \overset{L}{\otimes}_A A/\mathfrak{m}^n\big) = \lim \operatorname{Hom}\big(P, Q \overset{L}{\otimes}_A A/\mathfrak{m}^n\big) = \operatorname{Hom}(P, Q).$$

Thus, the functor ν is fully faithful.

(b) The given functor takes its values in the full subcategory $\mathcal{U}$ of left modules on $\mathcal{D}_{fd}(A)$ which take values in the category of finite-dimensional vector spaces. The duality functor $D: \mathcal{U} \to \operatorname{Mod}(\mathcal{D}_{fd}(A))$ is fully faithful and the composition $D \circ \Phi$ is isomorphic to the composition of the Yoneda embedding with ν. By part (a), the composition $D \circ \Phi$ and therefore Φ are fully faithful. $\quad\square$

3. The derived equivalence

3.1. The main theorem

Let (Q, W) be a quiver with potential and i a fixed vertex of Q. We assume (c1), (c2) and (c3) as in Section 2.4. Write $\tilde{\mu}_i(Q, W) = (Q', W')$. Let $\Gamma = \hat{\Gamma}(Q, W)$ and $\Gamma' = \hat{\Gamma}(Q', W')$ be the complete Ginzburg dg algebras associated to (Q, W) and (Q', W') respectively, cf. Section 2.6.

For a vertex j of Q, let $P_j = e_j \Gamma$ and $P'_j = e_j \Gamma'$.

Theorem 3.2.

(a) *There is a triangle equivalence*

$$F: \mathcal{D}\big(\Gamma'\big) \to \mathcal{D}(\Gamma),$$

which sends the P'_j to P_j for $j \neq i$ and to the cone T_i over the morphism

$$P_i \to \bigoplus_\alpha P_{t(\alpha)} \tag{1}$$

for $i = j$, where we have a summand $P_{t(\alpha)}$ for each arrow α of Q with source i and the corresponding component of the map is the left multiplication by α. The functor F restricts to triangle equivalences from $\operatorname{per}(\Gamma')$ to $\operatorname{per}(\Gamma)$ and from $\mathcal{D}_{fd}(\Gamma')$ to $\mathcal{D}_{fd}(\Gamma)$.

(b) *Let Γ_{red} respectively Γ'_{red} be the complete Ginzburg dg algebra associated with the reduction of (Q, W) respectively the reduction $\mu_i(Q, W)$ of $\tilde{\mu}_i(Q, W)$. The functor F yields a triangle equivalence*

$$F_{\mathrm{red}}: \mathcal{D}\big(\Gamma'_{\mathrm{red}}\big) \to \mathcal{D}(\Gamma_{\mathrm{red}}),$$

which restricts to triangle equivalences from $\operatorname{per}(\Gamma'_{\mathrm{red}})$ to $\operatorname{per}(\Gamma_{\mathrm{red}})$ and from $\mathcal{D}_{fd}(\Gamma'_{\mathrm{red}})$ to $\mathcal{D}_{fd}(\Gamma_{\mathrm{red}})$.

Proof. (a) We will prove this in the following two subsections: we will first construct a Γ'–Γ-bimodule T, and then we will show that the left derived tensor functor and the right derived Hom functor associated to T are quasi-inverse equivalences.

(b) This follows from (a) and Lemma 2.10, since quasi-isomorphisms of dg algebras induce triangle equivalences in their derived categories. $\quad\square$

Remark 3.3. In the situation of the theorem, there is also a triangle equivalence $F' : \mathcal{D}(\Gamma') \to \mathcal{D}(\Gamma)$ which, for $j \neq i$, sends the P'_j to P_j and, for $i = j$, to the shifted cone

$$T'_i = \Sigma^{-1}\left(\bigoplus_{\beta} P_{s(\beta)} \to P_i\right), \tag{2}$$

where we have a summand $P_{s(\beta)}$ for each arrow β of Q with target i and the corresponding component of the morphism is left multiplication by β. To construct F', one can either adapt the proof below or put

$$F' = t_{S_i} \circ F$$

where t_{S_i} is the twist functor [48] with respect to the 3-spherical object S_i, which gives rise to a triangle

$$\mathsf{RHom}(S_i, X) \otimes S_i \to X \to t_{S_i}(X) \to \Sigma\,\mathsf{RHom}(S_i, X) \otimes S_i$$

for each object X of $\mathcal{D}(\Gamma)$.

3.4. A dg Γ'–Γ-bimodule

First let us analyze the definition of T_i. For an arrow α of Q starting at i, let P_α be a copy of $P_{t(\alpha)}$. We denote the element $e_{t(\alpha)}$ of this copy by e_α. Let $P_{\Sigma i}$ be a copy of ΣP_i. We denote the element e_i of this copy by $e_{\Sigma i}$. Let T_i be the mapping cone of the canonical inclusion

$$P_i \xrightarrow{\iota} \bigoplus_{\alpha \in Q_1 : s(\alpha)=i} P_\alpha,$$

$$a \mapsto \sum_{\alpha \in Q_1 : s(\alpha)=i} e_\alpha \alpha a.$$

Therefore, as a graded space, T_i has the decomposition

$$T_i = P_{\Sigma i} \oplus \bigoplus_{\alpha \in Q_1 : s(\alpha)=i} P_\alpha,$$

and the differential is given by

$$d_{T_i}\left(e_{\Sigma i} a_i + \sum_{\alpha \in Q_1 : s(\alpha)=i} e_\alpha a_\alpha\right) = -e_{\Sigma i} d_\Gamma(a_i) + \sum_{\alpha \in Q_1 : s(\alpha)=i} e_\alpha\big(d_\Gamma(a_\alpha) + \alpha a_i\big).$$

Note that each P_α is a dg submodule of T_i.

In this and the next subsection, T_j will be P_j for a vertex j of Q different from i. Let T be the direct sum of the T_j, where j runs over all vertices of Q, i.e.

$$T = \bigoplus_{j \in Q_0} T_j.$$

In the following, for each arrow ρ' of $\tilde{Q}'$, we will construct a morphism $f_{\rho'}$ in $\Lambda = \mathcal{H}om_\Gamma(T, T)$ and show that the assignment $\rho' \to f_{\rho'}$ extends to a morphism of dg algebras from Γ' to Λ.

For a vertex j of Q, let $f_j : T_j \to T_j$ be the identity map.

For an arrow α of Q starting at i, we define the morphism $f_{\alpha^\star} : T_{t(\alpha)} \to T_i$ of degree 0 as the composition of the isomorphism $T_{t(\alpha)} \cong P_\alpha$ with the canonical embedding $P_\alpha \hookrightarrow T_i$, i.e.

$$f_{\alpha^\star} : T_{t(\alpha)} \to T_i, \quad a \mapsto e_\alpha a,$$

and define the morphism $f_{\alpha^{\star\star}} : T_i \to T_{t(\alpha)}$ of degree -1 by

$$f_{\alpha^{\star\star}}\left(e_{\Sigma i} a_i + \sum_{\rho \in Q_1 : s(\rho) = i} e_\rho a_\rho \right) = -\alpha t_i a_i - \sum_{\rho \in Q_1 : s(\rho) = i} \alpha \rho^* a_\rho.$$

For an arrow β of Q ending at i, we define the morphism $f_{\beta^\star} : T_i \to T_{s(\beta)}$ of degree 0 by

$$f_{\beta^\star}\left(e_{\Sigma i} a_i + \sum_{\rho \in Q_1 : s(\rho) = i} e_\rho a_\rho \right) = -\beta^* a_i - \sum_{\rho \in Q_1 : s(\rho) = i} (\partial_{\rho\beta} W) a_\rho,$$

and define the morphism $f_{\beta^{\star\star}} : T_{s(\beta)} \to T_i$ of degree -1 as the composition of the morphism $T_{s(\beta)} \to P_{\Sigma i}, a \mapsto e_{\Sigma i} \beta a$ and the canonical embedding $P_{\Sigma i} \hookrightarrow T_i$, i.e.

$$f_{\beta^{\star\star}} : T_{s(\beta)} \to T_i, \quad a \mapsto e_{\Sigma i} \beta a.$$

For a pair of arrows α, β of Q such that α starts at i and β ends at i, we define

$$f_{[\alpha\beta]} : T_{s(\beta)} \to T_{t(\alpha)}, \quad a \mapsto \alpha\beta a,$$

and

$$f_{[\alpha\beta]^*} = 0 : T_{t(\alpha)} \to T_{s(\beta)}.$$

For an arrow γ of Q not incident to i, we denote by f_γ and f_{γ^*} respectively the left-multiplication by γ from $T_{s(\gamma)}$ to $T_{t(\gamma)}$ and the left-multiplication by γ^* from $T_{t(\gamma)}$ to $T_{s(\gamma)}$, i.e.

$$f_\gamma : T_{s(\gamma)} \to T_{t(\gamma)}, \quad a \mapsto \gamma a,$$

$$f_{\gamma^*} : T_{t(\gamma)} \to T_{s(\gamma)}, \quad a \mapsto \gamma^* a.$$

For a vertex j of Q different from i, we define $f_{t'_j}$ as the left-multiplication by t_j from T_j to itself, i.e.

$$f_{t'_j} : T_j \to T_j, \quad a \mapsto t_j a.$$

It is a morphism of degree -2. We define $f_{t'_i}$ as the Γ-linear morphism of degree -2 from T_i to itself given by

$$f_{t'_i}\left(e_{\Sigma i} a_i + \sum_{\rho \in Q_1 : s(\rho)=i} e_\rho a_\rho\right) = -e_{\Sigma i}\left(t_i a_i + \sum_{\rho \in Q_1 : s(\rho)=i} \rho^* a_\rho\right).$$

We extend all the morphisms defined above trivially to morphisms from T to itself.

Proposition 3.5. *The correspondence taking e_j to f_j for all vertices j of Q and taking an arrow ρ' of $\tilde{Q}'$ to $f_{\rho'}$ extends to a homomorphism of dg algebras from Γ' to $\Lambda = \mathcal{H}om_\Gamma(T, T)$. In this way, T becomes a left dg Γ'-module.*

Proof. Recall that $\Lambda = \mathcal{H}om_\Gamma(T, T)$ is the dg endomorphism algebra of the dg Γ-module T. The $\mathfrak{m}$-adic topology of Γ induces an $\mathfrak{m}$-adic topology of Λ: If we let

$$\mathfrak{n} = \left\{ f \in \Lambda \mid \mathrm{im}(f) \in T\mathfrak{m} \right\},$$

then the powers of $\mathfrak{n}$ form a basis of open neighborhoods of 0. The topological algebra Λ is complete with respect to this topology. Note that the image $f_{a^\star}$ of the arrow $a^\star$ does not lie in $\mathfrak{n}$ but that the composition of the images of any two arrows of $\tilde{Q}'$ does lie in $\mathfrak{n}$. Thus the correspondence defined in the proposition uniquely extends to a continuous algebra homomorphism. We denote by f_p the image of an element p in Γ'. It remains to check that this map commutes with the differentials on generators ρ' of Γ'. Namely, we need to show

$$d_\Lambda(f_{\rho'}) = 0 \quad \text{for each arrow } \rho' \text{ of } Q',$$

$$d_\Lambda(f_{\rho'^*}) = f_{\partial_{\rho'} W'} \quad \text{for each arrow } \rho' \text{ of } Q',$$

$$d_\Lambda(f_{t'_j}) = f_j \left(\sum_{\rho' \in Q'_1} [f_{\rho'}, f_{\rho'^*}] \right) f_j \quad \text{for each vertex } j \text{ of } Q'.$$

We will prove the first two equalities in the following Lemmas 3.6–3.9 and the last equality in Lemma 3.10. $\square$

Lemma 3.6. *Let α be an arrow of Q starting at i. Then*

$$d_\Lambda(f_{\alpha^\star}) = 0, \qquad d_\Lambda(f_{\alpha^{\star*}}) = f_{\partial_{\alpha^\star} W'}.$$

Proof. It suffices to check the equalities on the generator $e_{t(\alpha)}$ of $T_{t(\alpha)}$ and on the generators $e_{\Sigma i}$ and e_a, $a : i \to t(a)$ in Q, of T_i. We have

$$d_\Lambda(f_{\alpha^\star})(e_{t(\alpha)}) = d_{T_i}\big(f_{\alpha^\star}(e_{t(\alpha)})\big) - (-1)^0 f_{\alpha^\star}\big(d_{T_{t(\alpha)}}(e_{t(\alpha)})\big)$$
$$= d_{T_i}(e_\alpha)$$
$$= 0.$$

Therefore $d_\Lambda(f_{\alpha^\star}) = 0$.

Let a be any arrow of Q starting from i. We have

$$d_\Lambda(f_{\alpha^{\star *}})(e_a) = d_{T_{t(\alpha)}}\big(f_{\alpha^{\star *}}(e_a)\big) - (-1)^{-1} f_{\alpha^{\star *}}\big(d_{T_i}(e_a)\big)$$
$$= d_{T_{t(\alpha)}}\big(-\alpha a^*\big)$$
$$= -\alpha \partial_a W,$$

$$d_\Lambda(f_{\alpha^{\star *}})(e_{\Sigma i}) = d_{T_{t(\alpha)}}\big(f_{\alpha^{\star *}}(e_{\Sigma i})\big) - (-1)^{-1} f_{\alpha^{\star *}}\big(d_{T_i}(e_{\Sigma i})\big)$$
$$= d_{T_{t(\alpha)}}(-\alpha t_i) + f_{\alpha^{\star *}}\Bigg(\sum_{a\in Q_1:\, s(a)=i} e_a a\Bigg)$$
$$= -\alpha\Bigg(\sum_{\beta\in Q_1:\, t(\beta)=i} \beta\beta^* - \sum_{a\in Q_1:\, s(a)=i} a^*a\Bigg) + \sum_{a\in Q_1:\, s(a)=i}(-\alpha a^*)a$$
$$= -\alpha\Bigg(\sum_{\beta\in Q_1:\, t(\beta)=i} \beta\beta^*\Bigg).$$

It follows from the definition of W' that

$$\partial_{\alpha^\star} W' = \partial_{\alpha^\star}\big(W_1' + W_2'\big) = \partial_{\alpha^\star} W_2' = \sum_{\beta\in Q_1:\, t(\beta)=i} [\alpha\beta]\beta^\star.$$

Thus

$$f_{\partial_{\alpha^\star} W'}(e_a) = \sum_{\beta\in Q_1:\, t(\beta)=i} f_{[\alpha\beta]} f_{\beta^\star}(e_a)$$
$$= \sum_{\beta\in Q_1:\, t(\beta)=i} f_{[\alpha\beta]}(-\partial_{a\beta} W)$$
$$= \sum_{\beta\in Q_1:\, t(\beta)=i} (-\alpha\beta \partial_{a\beta} W)$$
$$= -\alpha \sum_{\beta\in Q_1:\, t(\beta)=i} (\beta \partial_{a\beta} W)$$
$$= -\alpha \partial_a W,$$

$$f_{\partial_{\alpha^\star} W'}(e_{\Sigma i}) = \sum_{\beta\in Q_1:\, t(\beta)=i} f_{[\alpha\beta]} f_{\beta^\star}(e_{\Sigma i})$$
$$= \sum_{\beta\in Q_1:\, t(\beta)=i} f_{[\alpha\beta]}(-\beta^*)$$

$$= -\alpha \left(\sum_{\beta \in Q_1 : \, t(\beta)=i} \beta \beta^* \right).$$

Therefore $d_\Lambda(f_{\alpha^{\star\star}}) = f_{\partial_{\alpha^\star} W'}$. $\quad \square$

Lemma 3.7. *Let β be an arrow of Q ending at i. Then*

$$d_\Lambda(f_{\beta^\star}) = 0, \qquad d_\Lambda(f_{\beta^{\star\star}}) = f_{\partial_{\beta^\star} W'}.$$

Proof. For an arrow α of Q starting at i, we have

$$
\begin{aligned}
d_\Lambda(f_{\beta^\star})(e_\alpha) &= d_{T_{s(\beta)}}\big(f_{\beta^\star}(e_\alpha)\big) - (-1)^0 f_{\beta^\star}\big(d_{T_k}(e_\alpha)\big) \\
&= d_{T_{s(\beta)}}(-\partial_{\alpha\beta} W) \\
&= 0, \\
d_\Lambda(f_{\beta^\star})(e_{\Sigma i}) &= d_{T_{s(\beta)}}\big(f_{\beta^\star}(e_{\Sigma i})\big) - (-1)^0 f_{\beta^\star}\big(d_{T_i}(e_{\Sigma i})\big) \\
&= d_{T_{s(\beta)}}(-\beta^*) - f_{\beta^\star}\left(\sum_{\alpha \in Q_1 : \, s(\alpha)=i} e_\alpha \alpha \right) \\
&= d_{T_{s(\beta)}}(-\beta^*) - \sum_{\alpha \in Q_1 : \, s(\alpha)=i} (-(\partial_{\alpha\beta} W)\alpha) \\
&= -\partial_\beta W + \sum_{\alpha \in Q_1 : \, s(\alpha)=i} (\partial_{\alpha\beta} W)\alpha \\
&= 0.
\end{aligned}
$$

This shows the first equality. The second equality holds because

$$
\begin{aligned}
d_\Lambda(f_{\beta^{\star\star}})(e_{s(\beta)}) &= d_{T_i}\big(f_{\beta^{\star\star}}(e_{s(\beta)})\big) - (-1)^{-1} f_{\beta^{\star\star}}\big(d_{T_{s(\beta)}}(e_{s(\beta)})\big) \\
&= d_{T_i}(e_{\Sigma i}\beta) \\
&= \sum_{\alpha \in Q_1 : \, s(\alpha)=i} e_\alpha \alpha \beta, \\
f_{\partial_{\beta^\star} W'}(e_{s(\beta)}) &= f_{\partial_{\beta^\star} W_2'}(e_{s(\beta)}) \\
&= \sum_{\alpha \in Q_1 : \, s(\alpha)=i} f_{\alpha^\star} f_{[\alpha\beta]}(e_{s(\beta)}) \\
&= \sum_{\alpha \in Q_1 : \, s(\alpha)=i} f_{\alpha^\star}(\alpha\beta) \\
&= \sum_{\alpha \in Q_1 : \, s(\alpha)=i} e_\alpha \alpha \beta. \quad \square
\end{aligned}
$$

Lemma 3.8. *Let α, β be arrows of Q starting and ending at i respectively. Then*

$$d_\Lambda(f_{[\alpha\beta]}) = 0, \qquad d_\Lambda(f_{[\alpha\beta]^*}) = f_{\partial_{[\alpha\beta]}W'}.$$

Proof. The first equality follows from

$$d_\Lambda(f_{[\alpha\beta]})(e_{s(\beta)}) = d_{T_{t(\alpha)}}\big(f_{[\alpha\beta]}(e_{s(\beta)})\big) - (-1)^0 f_{[\alpha\beta]}\big(d_{T_{s(\beta)}}(e_{s(\beta)})\big)$$
$$= d_{T_{t(\alpha)}}(\alpha\beta)$$
$$= 0.$$

Since $f_{[\alpha\beta]^*}$ is the zero morphism, its differential $d_\Lambda(f_{[\alpha\beta]^*})$ is zero as well. On the other hand,

$$f_{\partial_{[\alpha\beta]}W'}(e_{t(\alpha)}) = f_{\partial_{[\alpha\beta]}W'_1 + \beta^\star\alpha^\star}(e_{t(\alpha)})$$
$$= f_{\partial_{[\alpha\beta]}W'_1}(e_{t(\alpha)}) + f_{\beta^\star\alpha^\star}(e_{t(\alpha)})$$
$$= f_{[\partial_{\alpha\beta}W]}(e_{t(\alpha)}) + (-\partial_{\alpha\beta}W)$$
$$= 0.$$

Here we applied the following observation: Let p be a path of Q neither starting or ending at i and let $[p]$ be the path of Q' obtained from p by replacing all ab by $[ab]$, where a, b are a pair of arrows of Q such that a starts at i and b ends at i. Then the map $f_{[p]}$ is the left multiplication by p. Therefore

$$f_{\partial_{[\alpha\beta]}W'} = 0 = d_\Lambda(f_{[\alpha\beta]^*}). \qquad \square$$

Lemma 3.9. *Let γ be an arrow of Q not starting or ending at i. Then*

$$d_\Lambda(f_\gamma) = 0, \qquad d_\Lambda(f_{\gamma^*}) = f_{\partial_\gamma W'}.$$

Proof. The first equality follows from

$$d_\Lambda(f_\gamma)(e_{s(\gamma)}) = d_{T_{t(\gamma)}}\big(f_\gamma(e_{s(\gamma)})\big) - (-1)^0 f_\gamma\big(d_{T_{s(\gamma)}}(e_{s(\gamma)})\big)$$
$$= d_{T_{t(\gamma)}}(\gamma)$$
$$= 0.$$

For the second, we have

$$d_\Lambda(f_{\gamma^*})(e_{t(\gamma)}) = d_{T_{s(\gamma)}}\big(f_{\gamma^*}(e_{t(\gamma)})\big) - (-1)^{-1} f_{\gamma^*}\big(d_{T_{t(\gamma)}}(e_{t(\gamma)})\big)$$
$$= d_{T_{s(\gamma)}}(\gamma^*)$$
$$= \partial_\gamma W,$$

$$f_{\partial_\gamma W'}(e_{t(\gamma)}) = f_{\partial_\gamma W'_1}(e_{t(\gamma)})$$

$$= f_{[\partial_\gamma W]}(e_{t(\gamma)})$$

$$= \partial_\gamma W.$$

Therefore $d_\Lambda(f_{\gamma^*}) = f_{\partial_\gamma W'}$. $\quad\square$

Lemma 3.10. *Let* $j \in Q_0$. *Then*

$$d_\Lambda(f_{t'_j}) = \sum_{\rho' \in Q'_1: \, t(\rho')=j} f_{\rho'} f_{\rho'^*} - \sum_{\rho' \in Q'_1: \, s(\rho')=j} f_{\rho'^*} f_{\rho'}.$$

Proof. We divide the proof into two cases.

Case $j \neq i$. We have

$$d_\Lambda(f_{t'_j})(e_j) = d_{T_j}\big(f_{t'_j}(e_j)\big) - (-1)^{-2} f_{t'_j}\big(d_{T_j}(e_j)\big)$$

$$= d_{T_j}(t_j),$$

$$\sum_{\rho' \in Q'_1: \, t(\rho')=j} f_{\rho'} f_{\rho'^*}(e_j) = \left(\sum_{\beta: \, j \to i \text{ in } Q_1} f_{\beta^\star} f_{\beta^{\star\star}} + \sum_{\gamma: \, s(\gamma) \to j \text{ in } Q_1, s(\gamma) \neq i} f_\gamma f_{\gamma^*} \right)(e_j)$$

$$= \sum_{\beta: \, j \to i \text{ in } Q_1} (-\beta^* \beta) + \sum_{\gamma: \, s(\gamma) \to j \text{ in } Q_1, s(\gamma) \neq i} \gamma \gamma^*,$$

$$\left(- \sum_{\rho' \in Q'_1: \, s(\rho')=j} f_{\rho'^*} f_{\rho'} \right)(e_j) = \left(- \sum_{\alpha: \, i \to j \text{ in } Q_1} f_{\alpha^{\star\star}} f_{\alpha^\star} - \sum_{\gamma: \, j \to t(\gamma) \neq i \text{ in } Q_1} f_{\gamma^*} f_\gamma \right)(e_j)$$

$$= - \sum_{\alpha: \, i \to j \text{ in } Q_1} (-\alpha \alpha^*) - \sum_{\gamma: \, j \to t(\gamma) \neq i \text{ in } Q_1} \gamma^* \gamma,$$

$$\text{the sum of the last two} = \sum_{\rho: \, s(\rho) \to i \text{ in } Q_1} \rho \rho^* - \sum_{\rho: \, j \to t(\rho) \text{ in } Q_1} \rho^* \rho$$

$$= d_{T_j}(t_j).$$

Therefore

$$d_\Lambda(f_{t'_j}) = \sum_{\rho' \in Q'_1: \, t(\rho')=j} f_{\rho'} f_{\rho'^*} - \sum_{\rho' \in Q'_1: \, s(\rho')=j} f_{\rho'^*} f_{\rho'}.$$

Case $j = i$. For an arrow a of Q starting at i, we have

$$d_\Lambda(f_{t'_i})(e_a) = d_{T_i}\big(f_{t'_i}(e_a)\big) - (-1)^{-2} f_{t'_i}\big(d_{T_i}(e_a)\big)$$

$$= d_{T_i}\big(-e_{\Sigma i} a^*\big)$$

$$= - \sum_{\alpha \in Q_1: \, s(\alpha)=i} e_\alpha \alpha a^* + e_{\Sigma i} \partial_a W,$$

$$d_\Lambda(f_{t_i'})(e_{\Sigma i}) = d_{T_i}\left(f_{t_i'}(e_{\Sigma i})\right) - (-1)^{-2} f_{t_i'}\left(d_{T_i}(e_{\Sigma i})\right)$$

$$= d_{T_i}(-e_{\Sigma i} t_i) - f_{t_i'}\left(\sum_{\alpha \in Q_1:\, s(\alpha)=i} e_\alpha \alpha\right)$$

$$= -\sum_{\alpha \in Q_1:\, s(\alpha)=i} e_\alpha \alpha t_i + e_{\Sigma i} d_\Gamma(t_k) - \sum_{\alpha \in Q_1:\, s(\alpha)=i} (-e_{\Sigma i}\alpha^*)\alpha$$

$$= -\sum_{\alpha \in Q_1:\, s(\alpha)=i} e_\alpha \alpha t_i + \sum_{\beta \in Q_1:\, t(\beta)=i} e_{\Sigma i} \beta \beta^*.$$

On the other hand,

$$\sum_{\alpha \in Q_1:\, s(\alpha)=i} f_{\alpha^\star} f_{\alpha^{\star\star}}(e_a) = \sum_{\alpha \in Q_1:\, s(\alpha)=i} (-e_\alpha \alpha a^*)$$

$$= -\sum_{\alpha \in Q_1:\, s(\alpha)=i} e_\alpha \alpha a^*,$$

$$\left(-\sum_{\beta \in Q_1:\, t(\beta)=i} f_{\beta^{\star\star}} f_{\beta^\star}\right)(e_a) = -\sum_{\beta \in Q_1:\, t(\beta)=i} (-e_{\Sigma i}\beta \partial_{a\beta} W)$$

$$= e_{\Sigma i} \partial_a W,$$

$$\sum_{\alpha \in Q_1:\, s(\alpha)=i} f_{\alpha^\star} f_{\alpha^{\star\star}}(e_{\Sigma i}) = \sum_{\alpha \in Q_1:\, s(\alpha)=i} (-e_\alpha \alpha t_i)$$

$$= -\sum_{\alpha \in Q_1:\, s(\alpha)=i} e_\alpha \alpha t_i,$$

$$\left(-\sum_{\beta \in Q_1:\, t(\beta)=i} f_{\beta^{\star\star}} f_{\beta^\star}\right)(e_{\Sigma i}) = -\sum_{\beta \in Q_1:\, t(\beta)=i} (-e_{\Sigma i}\beta^*)$$

$$= \sum_{\beta \in Q_1:\, t(\beta)=i} e_{\Sigma i} \beta \beta^*.$$

Therefore

$$d_\Lambda(f_{t_i'}) = \sum_{\alpha \in Q_1:\, s(\alpha)=i} f_{\alpha^\star} f_{\alpha^{\star\star}} - \sum_{\beta \in Q_1:\, t(\beta)=i} f_{\beta^{\star\star}} f_{\beta^\star}$$

$$= \sum_{\rho' \in Q_1':\, t(\rho')=i} f_{\rho'} f_{\rho'^*} - \sum_{\rho' \in Q_1':\, s(\rho')=i} f_{\rho'^*} f_{\rho'}. \qquad \square$$

3.11. *Proof of the equivalence*

In the preceding subsection we constructed a dg $\Gamma'-\Gamma$-bimodule T. Clearly T is cofibrant as a right dg Γ-module. Consequently we obtain a pair of adjoint triangle functors $F = ? \overset{L}{\otimes}_{\Gamma'} T$ and $G = \mathcal{H}om_\Gamma(T, ?)$ between the derived categories $\mathcal{D}(\Gamma')$ and $\mathcal{D}(\Gamma)$. In this subsection we will prove that they are quasi-inverse equivalences.

We will denote by S_j and S'_j respectively the simple modules over Γ and Γ' attached to the vertex j of Q. As shown in the proof of Theorem 2.19(a), they respectively generate the triangulated categories $\mathcal{D}_{fd}(\Gamma)$ and $\mathcal{D}_{fd}(\Gamma')$.

Lemma 3.12.

(a) *Let j be a vertex of Q. If $j = i$, we have an isomorphism in $\mathcal{D}(\Gamma)$*

$$F(S'_i) \cong \Sigma S_i.$$

If $j \neq i$, then $F S'_j$ is isomorphic in $\mathcal{D}(\Gamma)$ to the cone of the canonical map

$$\Sigma^{-1} S_j \rightarrow \mathsf{Hom}_{\mathcal{D}(\Gamma)}(\Sigma^{-1} S_j, S_i) \otimes_k S_i.$$

(b) *For any vertex j of Q, we have an isomorphism in $\mathcal{D}(\Gamma')$*

$$GF(S'_j) \cong S'_j.$$

Proof. As shown in Section 2.14, a cofibrant resolution of S'_j is given by the graded vector space

$$\mathbf{p}S'_j = \Sigma^3 P'_j \oplus \bigoplus_{\rho \in Q'_1: \, s(\rho)=j} \Sigma^2 P'_{t(\rho)} \oplus \bigoplus_{\tau \in Q'_1: \, t(\tau)=j} \Sigma P'_{s(\tau)} \oplus P'_j,$$

and the differential

$$d_{\mathbf{p}S'_j} = \begin{pmatrix} d_{\Sigma^3 P'_j} & 0 & 0 & 0 \\ \rho & d_{\Sigma^2 P'_{t(\rho)}} & 0 & 0 \\ -\tau^* & -\partial_{\rho\tau} W' & d_{\Sigma P'_{s(\tau)}} & 0 \\ t'_j & \rho^* & \tau & d_{P'_j} \end{pmatrix}$$

where the obvious summation symbols over ρ, τ and j are omitted. Therefore

$$F(S'_j) = \mathbf{p}S'_j \otimes_{\Gamma'} T$$

is the dg Γ-module whose underlying graded vector space is

$$\Sigma^3 T_j \oplus \bigoplus_{\rho \in Q'_1: \, s(\rho)=j} \Sigma^2 T^{\rho}_{t(\rho)} \oplus \bigoplus_{\tau \in Q'_1: \, t(\tau)=j} \Sigma T^{\tau}_{s(\tau)} \oplus T_j,$$

where $T^{\rho}_{t(\rho)}$ (respectively, $T^{\tau}_{s(\tau)}$) is the direct summands corresponding to ρ (respectively, τ), and whose differential is

$$d_{F(S'_j)} = \begin{pmatrix} d_{\Sigma^3 T_j} & 0 & 0 & 0 \\ f_\rho & d_{\Sigma^2 T^{\rho}_{t(\rho)}} & 0 & 0 \\ -f_{\tau^*} & -f_{\partial_{\rho\tau} W'} & d_{\Sigma T^{\tau}_{s(\tau)}} & 0 \\ f_{t'_j} & f_{\rho^*} & f_\tau & d_{T_j} \end{pmatrix}.$$

Then $F(S'_j)$ is the mapping cone of the morphism

$$M \xrightarrow{\ f=(f_{t'_i},f_{\rho*},f_\tau)\ } T_j,$$

where M is the dg Γ-module whose underlying graded vector space is

$$\Sigma^2 T_j \oplus \bigoplus_{\rho \in Q'_1:\, s(\rho)=j} \Sigma T^\rho_{t(\rho)} \oplus \bigoplus_{\tau \in Q'_1:\, t(\tau)=j} T^\tau_{s(\tau)}$$

and whose differential is

$$d_M = \begin{pmatrix} d_{\Sigma^2 T_j} & 0 & 0 \\ -f_\rho & d_{\Sigma T^\rho_{t(\rho)}} & 0 \\ f_{\tau*} & f_{\partial_{\rho\tau} W'} & d_{T^\tau_{s(\tau)}} \end{pmatrix}.$$

We will simplify the description of FS'_j by showing that the kernel $\ker(f)$ of $f : M \to T_j$ is contractible and computing its cokernel. Then we will apply the functor G to the cokernel. We distinguish two cases: $j = i$ and $j \neq i$.

Case $j = i$. Let us compute the effect of f on the generators of the three families of summands of M. Recall that T_i is generated by $e_{\Sigma i}$ and e_α, $\alpha : i \to t(\alpha)$ in Q, and T_j, $j \neq i$, is generated by e_j.

(1) The morphism $f_{t'_i} : \Sigma^2 T_i \to T_i$ sends $e_{\Sigma i}$ to $-e_{\Sigma i}t_i$, and e_α ($\alpha : i \to t(\alpha)$ in Q) to $-e_{\Sigma i}\alpha^*$.

(2) For $\rho : i \to t(\rho)$: in this case $\rho = b^\star$ for some $b : s(b) \to i$ in Q. The morphism $f_{b^{\star*}} : \Sigma T^{b^\star}_{s(b)} \to T_i$ sends $e^{b^\star}_{s(b)}$ to $e_{\Sigma i}b$.

(3) For $\tau : s(\tau) \to i$: in this case $\tau = a^\star$ for some $a : i \to t(a)$ in Q. The morphism $f_{a^\star} : T^{a^\star}_{t(a)} \to T_i$ sends $e^{a^\star}_{t(a)}$ to e_a.

This description shows that the morphism f is injective. Moreover, the image $\mathrm{im}(f)$ is the dg Γ-module generated by $e_{\Sigma i}t_i$, $e_{\Sigma i}\alpha^*$, $e_{\Sigma i}b$, and e_α (where $\alpha : i \to t(\alpha)$ and $b : s(b) \to i$ are arrows in Q). Thus the cokernel of f is ΣS_i. So

$$F(S'_i) \cong \Sigma S_i,$$

and

$$GF(S'_i) \cong G(\Sigma S_i) = \mathcal{H}om_\Gamma(T, \Sigma S_i) = \mathcal{H}om_\Gamma(P_{\Sigma i}, \Sigma S_i) = S'_i.$$

Case $j \neq i$.

(1) The morphism $f_{t'_j} : \Sigma^2 T_j \to T_j$ sends e_j to t_j.

(2) For $\rho : j \to t(\rho)$, we have $\rho = \gamma$ for some $\gamma : j \to t(\gamma) \neq i$ in Q, or $\rho = [ab]$ for some $a : i \to t(a)$ and $b : j \to i$ in Q, or $\rho = a^\star$ for some $a : i \to j$ in Q. In the first case, the morphism $f_{\gamma*} : \Sigma T^\gamma_{t(\gamma)} \to T_j$ sends $e^\gamma_{t(\gamma)}$ to γ^*. In the second case, the morphism $f_{[ab]*} : \Sigma T^{[ab]}_{t(ab)} \to T_i$ is zero. In the third case, the morphism $f_{a^{\star*}} : \Sigma T^{a^\star}_i \to T_i$ sends $e^{a^\star}_\alpha$ to $-a\alpha^*$ and sends $e^{a^\star}_{\Sigma i}$ to $-at_i$.

(3) For $\tau : s(\tau) \to j$, we have $\tau = \gamma$ for some $\gamma : i \neq s(\gamma) \to j$ in Q, or $\tau = [ab]$ for some $a : i \to j$ and $b : s(b) \to i$ in Q, or $\tau = b^\star$ for some $b : j \to i$ in Q. In the first case, the morphism $f_\gamma : T^\gamma_{s(\gamma)} \to T_j$ sends $e^\gamma_{s(\gamma)}$ to γ. In the second case, the morphism $f_{[ab]} : T^{[ab]}_{s(ab)} \to T_j$ sends

$e^{[ab]}_{s(ab)}$ to ab. In the third case, the morphism $f_{b^\star} : T_i^{b^\star} \to T_j$ sends $e_\alpha^{b^\star}$ to $-\partial_{\alpha b}W$ and sends $e_{\Sigma i}^{b^\star}$ to $-b^*$.

By (2), the kernel $\ker(f)$ of f contains all $\Sigma T^{[ab]}_{t(ab)}$. For a pair of arrows $a : i \to t(a)$ and $b : j \to i$ in Q, let $R_{a,b}$ denote the graded Γ-submodule of M generated by

$$r_{a,b} = \left(\left(e^\gamma_{s(\gamma)} \partial_{ab\gamma} W \right)_\gamma, \left(e^{[a'b']}_{s(a'b')} \partial_{aba'b'} W \right)_{a',b'}, e_a^{b^\star} \right)$$

where γ runs over all $\gamma : i \neq s(\gamma) \to j$ in Q and a' and b' run over all $a' : i \to j$ and $b' : s(b') \to i$ in Q. The degree of $r_{a,b}$ is zero, so $R_{a,b}$ is indeed a dg Γ-submodule of M. It is contained in $\ker(f)$ since

$$
\begin{aligned}
f(r_{a,b}) &= \sum_\gamma f_\gamma \left(e^\gamma_{s(\gamma)} \partial_{ab\gamma} W \right) + \sum_{a',b'} f_{[a'b']} \left(e^{[a'b']}_{s(a'b')} \partial_{aba'b'} W \right) + f_{b^\star}\left(e_a^{b^\star} \right) \\
&= \sum_\gamma \gamma \partial_{ab\gamma} W + \sum_{a',b'} a'b' \partial_{aba'b'} W - \partial_{ab} W \\
&= 0.
\end{aligned}
$$

Moreover, it is straightforward to show that the sum of all $R_{a,b}$ is a direct sum and that the underlying graded space of $\ker(f)$ is

$$\bigoplus_{a,b} \Sigma T^{[ab]}_{t(ab)} \oplus \bigoplus_{a,b} R_{a,b},$$

and its differential is

$$d_{\ker(f)} = \begin{pmatrix} d_{\Sigma T^{[ab]}_{t(ab)}} & 0 \\ (f_{\partial_{[ab]\tau} W'})_\tau & d_{R_{a,b}} \end{pmatrix},$$

where τ runs over all arrows $\tau : s(\tau) \to j$ of Q'. If $\tau = \gamma$ for some $\gamma : i \neq s(\gamma) \to j$ in Q, then

$$f_{\partial_{[ab]\gamma} W'}\left(e^{[ab]}_{t(ab)} \right) = \left(e^\gamma_{s(\gamma)} \partial_{ab\gamma} W \right)_\gamma.$$

If $\tau = [a'b']$ for some $a' : i \to j$ and $b' : s(b') \to i$ in Q, then

$$f_{\partial_{[ab][a'b']} W'}\left(e^{[ab]}_{t(ab)} \right) = e^{[a'b']}_{s(a'b')} \partial_{aba'b'} W.$$

If $\tau = b'^\star$ for some $b' : j \to i$ in Q, then

$$f_{\partial_{[ab]b'^\star} W'}\left(e^{[ab]}_{t(ab)} \right) = \delta_{b,b'} e_a^{b^\star}.$$

Summing them up, we see that the differential of $\ker(f)$ sends $e^{[ab]}_{t(ab)}$ to $r_{a,b}$, and this induces an isomorphism of degree 1 from $\Sigma T^{[ab]}_{t(ab)}$ to $R_{a,b}$. In particular, $\ker(f)$ is contractible.

Now $\mathrm{im}(f)$ is generated by t_j, γ^* ($\gamma : j \to t(\gamma) \neq i$), $a\alpha^*$ ($a : i \to j$ and $\alpha : i \to t(\alpha)$ in Q), at_i ($a : i \to j$ in Q), γ ($\gamma : i \neq s(\gamma) \to j$ in Q), and ab ($a : i \to j$ and $b : s(b) \to i$ in Q),

b^* ($b : j \to i$ in Q). Therefore $\mathsf{cok}(f)$ is the vector space $k\{\bar{e}_j, \bar{a} \mid a : i \to j$ in $Q\}$ concentrated in degree 0 with the obvious Γ-action. Thus we have that

$$FS'_j \cong \mathsf{cok}(f)$$

is the universal extension of S_j by S_i, or in other words, it is isomorphic to the cone of the canonical map

$$\Sigma^{-1}S_j \to \mathsf{Hom}_{\mathcal{D}(\Gamma)}\left(\Sigma^{-1}S_j, S_i\right) \otimes_k S_i.$$

Further,

$$GF\left(S'_j\right) \cong G\left(\mathsf{cok}(f)\right)$$
$$= \mathcal{H}om_\Gamma\left(T, \mathsf{cok}(f)\right),$$

and so the complex underlying GFS'_j is

$$0 \to k\{g_j, g_a \mid a : i \to j \text{ in } Q\} \xrightarrow{d} k\{h_a \mid a : i \to j \text{ in } Q\} \to 0,$$

where $g_j : P_j \to \mathsf{cok}(f)$ is the canonical projection, $g_a : P_a \to \mathsf{cok}(f)$ is a copy of g_j, and $h_a : \Sigma P_i \to \mathsf{cok}(f)$ is of degree 1 sending $e_{\Sigma i}$ to $\bar{a}$. The differential d sends g_a to h_a. Thus the above dg Γ'-module is quasi-isomorphic to $k\{g_j\}$ concentrated in degree 0. From the left Γ'-action on T we deduce that $g_j e_j = g_j$. Therefore $GF(S'_j)$ is isomorphic to S'_j. $\quad\square$

Proposition 3.13. *The functors F and G induce a pair of quasi-inverse triangle equivalences*

$$\mathcal{D}_{fd}(\Gamma) \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}_{fd}\left(\Gamma'\right).$$

Proof. Let $\eta : Id \to GF$ be the unit of the adjoint pair (F, G). Let j be a vertex of Q. We would like to show that

$$\eta_{S'_j} : S'_j \to GFS'_j$$

is invertible. Under the adjunction

$$\mathsf{Hom}_{\mathcal{D}_{fd}(\Gamma)}\left(FS'_j, FS'_j\right) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}_{fd}(\Gamma')}\left(S'_j, GFS'_j\right),$$

the morphism $\eta_{S'_j}$ corresponds to the identity of FS'_j. So $\eta_{S'_j}$ is non-zero. Since S'_j is isomorphic to GFS'_j (Lemma 3.12), and $\mathsf{Hom}_{\mathcal{D}_{fd}(\Gamma')}(S'_j, S'_j)$ is one-dimensional, it follows that $\eta_{S'_j}$ is invertible. Therefore, so are its shifts $\eta_{\Sigma^p S'_j} = \Sigma^p \eta_{S'_j}$ ($p \in \mathbb{Z}$). Let

$$X_1 \xrightarrow{f} X \xrightarrow{g} X_2 \xrightarrow{h} \Sigma X_1$$

be a triangle in $\mathcal{D}_{fd}(\Gamma')$. Then we obtain a commutative diagram

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{\ f\ } & X & \xrightarrow{\ g\ } & X_2 & \xrightarrow{\ h\ } & \Sigma X_1 \\
\downarrow{\scriptstyle \eta_{X_1}} & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{X_2}} & & \downarrow{\scriptstyle \Sigma\eta_{X_1}} \\
GFX_1 & \xrightarrow{GFf} & GFX & \xrightarrow{GFg} & GFX_2 & \xrightarrow{GFh} & \Sigma GFX_1.
\end{array}
$$

Thus if η_{X_1} and η_{X_2} are isomorphisms, then so is η_X. Since $\mathcal{D}_{fd}(\Gamma')$ is generated by all the S_j''s, we deduce that for all objects X of $\mathcal{D}_{fd}(\Gamma')$, the morphism η_X is an isomorphism. It follows that F is fully faithful. Moreover, by Lemma 3.12(a), the objects FS_j', $j \in Q_0$, generate the triangulated category $\mathcal{D}_{fd}(\Gamma)$, since they generate all the simples S_j. Therefore F is an equivalence, and it follows that G is an equivalence as well. $\quad\square$

Proof of Theorem 3.2. As in Section 2.18, let $\mathcal{D}_0(\Gamma)$ denote the localizing subcategory of $\mathcal{D}(\Gamma)$ generated by $\mathcal{D}_{fd}(\Gamma)$. Now observe that the functors F and G commute with arbitrary coproducts. Since they induce equivalences between $\mathcal{D}_{fd}(\Gamma)$ and $\mathcal{D}_{fd}(\Gamma')$ and these subcategories are formed by compact objects, the functors F and G also induce equivalences between $\mathcal{D}_0(\Gamma)$ and $\mathcal{D}_0(\Gamma')$. Now let us check that F is compatible with the Nakayama functor ν defined in Section 2.20. For P' in $\mathsf{per}(\Gamma')$ and M in $\mathcal{D}_0(\Gamma)$, the object FP' is perfect in $\mathcal{D}(\Gamma)$ and we have

$$\mathsf{Hom}\big(M, \nu FP'\big) = D\,\mathsf{Hom}\big(FP', M\big) = D\,\mathsf{Hom}\big(P', GM\big) = \mathsf{Hom}\big(GM, \nu P'\big).$$

Since F and G are quasi-inverse to each other on the subcategories $\mathcal{D}_0$, we also have

$$\mathsf{Hom}\big(GM, \nu P'\big) = \mathsf{Hom}\big(M, F\nu P'\big).$$

We obtain that when restricted to $\mathsf{per}(\Gamma')$, the functors $F\nu$ and νF are isomorphic. Since $F\nu$ is fully faithful on $\mathsf{per}(\Gamma')$ and ν is fully faithful on $\mathsf{per}(\Gamma)$, it follows that F is fully faithful on $\mathsf{per}(\Gamma')$. Since $F\Gamma' = T$ generates $\mathsf{per}(\Gamma)$ it follows that F induces an equivalence from $\mathsf{per}(\Gamma')$ to $\mathsf{per}(\Gamma')$. Since F commutes with arbitrary coproducts, F itself is an equivalence. $\quad\square$

4. Nearly Morita equivalence for neighboring Jacobian algebras

Let (Q, W) be a quiver with potential, and Γ the associated complete Ginzburg dg algebra. For a vertex j of Q, let $P_j = e_j\Gamma$ and let S_j be the corresponding simple dg Γ-module concentrated in degree 0.

Recall that the finite-dimensional derived category $\mathcal{D}_{fd}(\Gamma)$ is generated by the S_j's and all of them belong to the perfect derived category $\mathsf{per}(\Gamma)$ (cf. Sections 2.14 and 2.18). It follows that $\mathcal{D}_{fd}(\Gamma)$ is a triangulated subcategory of $\mathsf{per}(\Gamma)$. We call the idempotent completion $\mathcal{C}_{Q,W}$ of the triangle quotient category

$$\mathsf{per}(\Gamma)/\mathcal{D}_{fd}(\Gamma)$$

the generalized cluster category associated with (Q, W). This category was introduced by C. Amiot in the case when the Jacobian algebra $J(Q, W) = H^0\Gamma$ is finite-dimensional, cf. [1]. Let π denote the projection functor from $\mathsf{per}(\Gamma)$ to $\mathcal{C}_{Q,W}$.

Remark 4.1. For a compactly generated triangulated category $\mathcal{T}$, let us denote by $\mathcal{T}^c$ the sub-category of compact objects, i.e. the objects C such that the functor $\mathsf{Hom}(C, ?)$ commutes with arbitrary coproducts, cf. [41]. Recall that $\mathcal{D}_0(\Gamma)$ denotes the localizing subcategory of $\mathcal{D}(\Gamma)$ generated by $\mathcal{D}_{fd}(\Gamma)$. We have $(\mathcal{D}\Gamma)^c = \mathsf{per}(\Gamma)$ and $(\mathcal{D}_0\Gamma)^c = \mathcal{D}_{fd}(\Gamma)$. Thus, by a theorem of Neeman [40], we have an equivalence

$$\mathcal{C}_{Q,W} \xrightarrow{\sim} \big(\mathcal{D}(\Gamma)/\mathcal{D}_0(\Gamma)\big)^c.$$

This shows that the quotient $\mathcal{D}(\Gamma)/\mathcal{D}_0(\Gamma)$ is the 'unbounded version' of the cluster category.

Let $\mathsf{fpr}(\Gamma)$ be the full subcategory of $\mathcal{C}_{Q,W}$ consisting of cones of morphisms of $\mathsf{add}(\pi(\Gamma))$. If the Jacobian algebra is finite-dimensional, this subcategory equals $\mathcal{C}_{Q,W}$, cf. Proposition 2.9 and Lemma 2.10 of [1]. The following proposition generalizes [1, Proposition 2.9].

Proposition 4.2. *(See [43].) Let $\mathcal{F}$ be the full subcategory of $\mathsf{per}(\Gamma)$ consisting of cones of morphisms of $\mathsf{add}(\Gamma)$. Then the projection functor $\pi : \mathsf{per}(\Gamma) \to \mathcal{C}_{Q,W}$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathsf{fpr}(\Gamma)$.*

As immediate consequences of Proposition 4.2, we have

$$\mathsf{Hom}_{\mathcal{C}_{Q,W}}(\Gamma, \Gamma) = \mathsf{Hom}_{\mathsf{per}(\Gamma)}(\Gamma, \Gamma) = J(Q, W),$$

and

$$\mathsf{Hom}_{\mathcal{C}_{Q,W}}(\Gamma, \Sigma\Gamma) = \mathsf{Hom}_{\mathsf{per}(\Gamma)}(\Gamma, \Sigma\Gamma) = 0.$$

Proposition 4.3. *The functor $\mathsf{Hom}_{\mathcal{C}_{Q,W}}(\Gamma, ?)$ induces an equivalence from the additive quotient category $\mathsf{fpr}(\Gamma)/(\Sigma\Gamma)$ to the category $\mathsf{mod}\, J(Q, W)$ of finitely presented modules over the Jacobian algebra $J(Q, W)$.*

Proof. As in [33, Proposition 2.1(c)], cf. also [5, Theorem 2.2]. □

We denote the functor $\mathsf{Hom}_{\mathcal{C}_{Q,W}}(\Gamma, ?)$ by Ψ and also use this symbol for the induced equivalence as in the preceding proposition.

Lemma 4.4. *(See [43].) Short exact sequences in $\mathsf{mod}\, J(Q, W)$ lift to triangles in $\mathsf{fpr}(\Gamma)$. More precisely, given a short exact sequence in $\mathsf{mod}\, J(Q, W)$, we can find a triangle in $\mathcal{C}_{Q,W}$ whose terms are in $\mathsf{fpr}(\Gamma)$ and whose image under the functor Ψ is isomorphic to the given short exact sequence.*

Assume that Q does not have loops. Let j be any vertex of Q. In this section (in contrast to Section 3), we denote by T_j the mapping cone of the morphism of dg modules

$$P_j \xrightarrow{(\rho)} \bigoplus_\rho P_{t(\rho)},$$

where we have a summand $P_{t(\rho)}$ for each arrow ρ of Q with source j, and the corresponding component of the morphism is the left multiplication by ρ. We denote by R_j the mapping cone of the morphism of dg modules

$$\bigoplus_{\tau} P_{s(\tau)} \xrightarrow{(\tau)} P_j,$$

where we have a summand $P_{s(\tau)}$ for each arrow τ of Q with target j, and the corresponding component of the morphism is the left multiplication by τ. From ΣT_j to R_j there is a morphism of dg modules, which we will denote by φ_j, given in matrix form as

$$\varphi_j = \begin{pmatrix} -\tau^* & -\partial_{\rho\tau} W \\ t_j & \rho^* \end{pmatrix} : \Sigma T_j \to R_j.$$

The mapping cone of φ_j is isomorphic as a dg Γ-module to the standard cofibrant resolution of S_j given in Section 2.14. Thus ΣT_j and R_j are isomorphic in $\mathcal{C}_{Q,W}$. In particular, ΣT_j belongs to $\mathrm{fpr}(\Gamma)$. Direct calculation shows that the $J(Q, W)$-module $\Psi(R_j)$ and hence $\Psi(\Sigma T_j)$ are isomorphic to the simple module S_j.

From now on, we fix a vertex i which does not lie on a 2-cycle. Let $(Q', W') = \tilde{\mu}_i(Q, W)$, let Γ' be the associated complete Ginzburg dg algebra, and let $P'_j = e_j \Gamma'$ for each vertex j and let S'_j be the corresponding simple dg Γ'-module concentrated in degree 0. It follows from Theorem 3.2 that we have a triangle equivalence

$$\bar{F} : \mathcal{C}_{(Q', W')} \to \mathcal{C}_{Q,W},$$

which sends P'_j to P_j for $j \neq i$, and P'_i to T_i.

Proposition 4.5. *(See [43].) The equivalence $\bar{F} : \mathcal{C}_{(Q', W')} \to \mathcal{C}_{Q,W}$ restricts to an equivalence from $\mathrm{fpr}(\Gamma')$ to $\mathrm{fpr}(\Gamma)$.*

Corollary 4.6. *There is a canonical equivalence of categories of finitely presented modules*

$$\mathrm{mod}\, J(Q', W')/(S'_i) \to \mathrm{mod}\, J(Q, W)/(S_i),$$

which restricts to an equivalence between the subcategories of finite-dimensional modules.

Remark 4.7. The restriction of the equivalence in Corollary 4.6 to the subcategories of finite-dimensional modules is a nearly Morita equivalence in the sense of Ringel [46]. It was first shown in [3, Theorem 7.1] that the two Jacobian algebras $J(Q, W)$ and $J(Q', W')$ are nearly Morita equivalent in the sense of Ringel. Another proof was given in [11, Proposition 6.1].

Proof. For each vertex j of Q', we define dg Γ'-modules T'_j and R'_j analogously to the dg Γ-modules T_j and R_j. Explicitly, R'_i is the mapping cone of the morphism of dg modules

$$\bigoplus_{\alpha} P'_{t(\alpha)} \xrightarrow{(\alpha^\star)} P'_i,$$

where α runs over all arrows of Q with source i. The dg functor underlying F takes this morphism of dg modules to the morphism

$$\bigoplus_{\alpha} P_{t(\alpha)} \xrightarrow{(f_{\alpha^\star})} T_i,$$

where $f_{\alpha^\star}$ was defined in Section 3.4. In fact, it is exactly the inclusion of the summand $P_{t(\alpha)}$ corresponding to the arrow α into T_i. We immediately deduce that the mapping cone of this latter morphism is quasi-isomorphic to ΣP_i. So the functor F takes R_i' to ΣP_i. Thus we have isomorphisms in $\mathcal{C}_{Q,W}$

$$\Sigma \bar{F}(T_i') \cong \bar{F}(R_i') \cong \Sigma P_i.$$

By Proposition 4.5, the equivalence $\bar{F}$ induces an equivalence

$$\mathsf{fpr}(\Gamma')/(\Sigma\Gamma' \oplus \Sigma T_i') \to \mathsf{fpr}(\Gamma)/(\Sigma\Gamma \oplus \Sigma T_i).$$

Thus, we have the following square

$$
\begin{array}{ccc}
\mathsf{fpr}(\Gamma')/(\Sigma\Gamma' \oplus \Sigma T_i') & \xrightarrow{\;\simeq\;} & \mathsf{mod}(J(Q',W'))/(S_i') \\
\Big\downarrow{\simeq} & & \Big\downarrow{\Phi} \\
\mathsf{fpr}(\Gamma)/(\Sigma\Gamma \oplus \Sigma T_i) & \xrightarrow{\;\simeq\;} & \mathsf{mod}(J(Q,W))/(S_i).
\end{array}
$$

The induced functor Φ from $\mathsf{mod}(J(Q',W'))/(S_i')$ to $\mathsf{mod}(J(Q,W))/(S_i)$, represented by the dashed arrow, is an equivalence as well. Explicitly, for a finitely presented module M over $J(Q,W')$, the module $\Phi(M)$ is defined as $\Psi \circ \bar{F}(X)$, where X is an object of $\mathsf{fpr}(\Gamma')$ such that $\Psi'(X) \cong M$.

It remains to prove that the functor Φ and its quasi-inverse Φ^{-1} take finite-dimensional modules to finite-dimensional modules. We will prove the assertion for Φ by induction on the dimension of the module: it is similar for Φ^{-1}.

Let j be a vertex of Q. We have seen that the simple $J(Q',W')$-module S_j' is the image of $\Sigma T_j'$ under the functor Ψ'. First of all, we have

$$\Phi(S_i') = \Psi \circ \bar{F}(\Sigma T_i') = \Psi(\Sigma P_i) = 0.$$

In the following we assume that j is different from i. We have a triangle in $\mathcal{C}_{(Q',W')}$:

$$\bigoplus_{\rho} P_{s(\rho)}' \xrightarrow{(\rho)} P_j' \to \Sigma T_j' \to \Sigma \bigoplus_{\rho} P_{s(\rho)}',$$

where ρ ranges over all arrows of Q' with target j. We apply the functor $\bar{F}$ and obtain a triangle in $\mathcal{C}_{Q,W}$:

$$\bigoplus_\rho P_{s(\rho)} \oplus \bigoplus_{\alpha,b} P_{s(b)} \oplus \bigoplus_\beta T_i \xrightarrow{(f_\rho, f_{[\alpha b]}, f_{\beta \star})} P_j$$

$$\to \Sigma F(T'_j) \to \bigoplus_\rho \Sigma P_{s(\rho)} \oplus \bigoplus_{\alpha,b} \Sigma P_{s(b)} \oplus \bigoplus_\beta \Sigma T_i,$$

where ρ ranges over arrows of Q with target j and with source different from i, α ranges over all arrows of Q with source i and target j, b ranges all arrows of Q with target i and β ranges over all arrows of Q with source j and target i. Note that the first two terms from the left are in the fundamental domain $\mathcal{F}$. Thus, applying the functor Ψ, we obtain an exact sequence

$$H^0 \left(\bigoplus_\rho P_{s(\rho)} \oplus \bigoplus_{\alpha,b} P_{s(b)} \oplus \bigoplus_\beta T_i \right) \xrightarrow{H^0(f_\rho, f_{[\alpha b]}, f_{\beta \star})} H^0 P_j \to \Phi(S'_j) \to \bigoplus_\beta S_i.$$

The image of the first map is the $J(Q, W)$-submodule of $H^0 P_j$ generated by ρ, αb, and $\partial_{a\beta} W$, where ρ, α, β, a and b are arrows of Q such that $s(\rho) \neq i$, $t(\rho) = t(\alpha) = s(\beta) = j$, $s(\alpha) = t(b) = s(a) = t(\beta) = i$. It has finite codimension in $H^0 P_j$. Therefore, the module $\Phi(S'_j)$ is finite-dimensional.

Now let M be a finite-dimensional $J(Q', W')$-module. Suppose

$$0 \to L \to M \to N \to 0$$

is a short exact sequence in $\mathsf{mod}\, J(Q', W')$ such that L and N are non-trivial. Following Lemma 4.4, we have a triangle in $\mathsf{fpr}(\Gamma')$:

$$X \to Y \to Z \to \Sigma X,$$

whose image under the functor Ψ' is isomorphic to the above short exact sequence. We apply the functor $\bar{F}$ to this triangle and obtain a triangle in $\mathsf{fpr}(\Gamma)$:

$$\bar{F}X \to \bar{F}Y \to \bar{F}Z \to \Sigma \bar{F}X.$$

Applying Ψ to this triangle, we obtain an exact sequence in $\mathsf{mod}\, J(Q, W)$:

$$\Phi(L) \to \Phi(M) \to \Phi(N).$$

By induction hypothesis, both $\Phi(L)$ and $\Phi(N)$ are finite-dimensional, and hence so is $\Phi(M)$. This completes the proof. $\square$

5. Tilting between Jacobian algebras

5.1. The canonical t-structure

Let A be a dg k-algebra such that the homology $H^p(A)$ vanishes for all $p > 0$. Let $\mathcal{D}_{\leqslant 0}$ be the full subcategory of the derived category $\mathcal{D}(A)$ whose objects are the dg modules M such that the homology $H^p(M)$ vanishes for all $p > 0$. The following lemma already appears in Section 2.1 of [1]. For the convenience of the reader, we include a detailed proof (slightly different from the one in [1]).

Lemma 5.2.

(a) *The subcategory $\mathcal{D}_{\leqslant 0}$ is a left aisle* [34] *in the derived category $\mathcal{D}(A)$.*

(b) *The functor $M \mapsto H^0(M)$ induces an equivalence from the heart of the corresponding t-structure to the category $\mathsf{Mod}\, H^0(A)$ of all right $H^0(A)$-modules.*

Proof. (a) Clearly, the subcategory $\mathcal{D}_{\leqslant 0}$ is stable under the suspension functor Σ and under extensions. We have to show that for each object M of $\mathcal{D}(A)$, there is a triangle

$$M' \to M \to M'' \to \Sigma M'$$

such that M' belongs to $\mathcal{D}_{\leqslant 0}$ and M'' is right orthogonal to $\mathcal{D}_{\leqslant 0}$. The map of complexes $\tau_{\leqslant 0}A \to A$ is a quasi-isomorphism of dg algebras. Thus we may assume that the components A^p vanish for all $p > 0$. If M is a dg module, the subcomplex $\tau_{\leqslant 0}M$ of M is then a dg submodule and thus the sequence

$$0 \to \tau_{\leqslant 0}M \to M \to \tau_{>0}M \to 0$$

is an exact sequence of dg modules. Of course, the two truncation functors preserve quasi-isomorphisms and thus induce functors from $\mathcal{D}(A)$ to itself so that the above sequence yields a triangle functorial in the object M of $\mathcal{D}(A)$. In this triangle, the object $\tau_{\leqslant 0}M$ lies in of course in $\mathcal{D}_{\leqslant 0}$. Let us show that $\tau_{>0}M$ lies in the right orthogonal subcategory of $\mathcal{D}_{\leqslant 0}$. Indeed, if L lies in $\mathcal{D}_{\leqslant 0}$ and we have a morphism $f : L \to \tau_{>0}M$, then, by the functoriality of the triangle, f factors through $\tau_{>0}L$, which vanishes.

(b) As in the proof of (a), we may assume that the components A^p vanish for $p > 0$. We then have the morphism of dg algebras $A \to H^0(A)$. The restriction along this morphism yields a functor from $\mathsf{Mod}\, H^0(A)$ to the heart $\mathcal{H}$ of the t-structure. Using the truncation functors defined in (a), we see that this functor is essentially surjective. Let us show that it is fully faithful. Let L and M in be in $\mathsf{Mod}\, H^0(A)$. We compute morphisms between their images in $\mathcal{D}(A)$. We have

$$\big(\mathcal{D}(A)\big)(L, M) = \mathsf{colim}\, \mathcal{H}(A)\big(L, M'\big)$$

where M' ranges through the category of quasi-isomorphisms with source M in $\mathcal{H}(A)$. Now for each object $M \to M'$ in this category, we have the object $M \to M' \to \tau_{\geqslant 0}M'$. This shows that the objects M' with $M'^n = 0$ for $n < 0$ form a cofinal subcategory. We restrict the colimit to this cofinal subcategory. We find

$$\mathsf{colim}\, \mathcal{H}(A)\big(L, M'\big) = \mathsf{colim}\, \mathcal{H}(A)\big(L, \tau_{\leqslant 0}M'\big) \xrightarrow{\sim} \mathsf{colim}\, \mathcal{H}(A)\big(L, H^0(M')\big) = \mathcal{H}(A)(L, M).$$

Thus the functor $\mathsf{Mod}\, H^0(A) \to \mathcal{D}(A)$ is fully faithful. $\quad\square$

5.3. Comparison of t-structures

Assume that (Q, W) and (Q', W') are two quivers with potential related by the mutation at a vertex i. Let Γ and Γ' be the corresponding complete Ginzburg algebras and

$$F : \mathcal{D}\big(\Gamma'\big) \to \mathcal{D}(\Gamma)$$

the associated triangle equivalence taking $P'_j = e_j \Gamma'$ to P_j for $j \neq i$ and P'_i to the cone over the morphism

$$P_i \to \bigoplus_{\alpha:\, s(\alpha)=i} P_{t(\alpha)}$$

whose components are the left multiplications by the corresponding arrows α. The image under F of the canonical t-structure on $\mathcal{D}(\Gamma')$ (cf. Section 5.1) is a new t-structure on $\mathcal{D}(\Gamma)$. Let us denote its left aisle by $\mathcal{D}'_{\leq 0}$ and its right aisle by $\mathcal{D}'_{> 0}$. Let us denote the left aisle of the canonical t-structure on $\mathcal{D}(\Gamma)$ by $\mathcal{D}_{\leq 0}$ and let $\mathcal{A}$ be its heart. By Lemma 5.2(b), the category $\mathcal{A}$ is equivalent to the category $\mathsf{Mod}(J(Q, W))$ of all right modules over the Jacobian algebra of (Q, W).

The following lemma shows how the new t-structure is obtained from the old one and that the two hearts are 'piecewise equivalent'. However, in general, the new heart is not tilted from the old one in the sense of [23], cf. also [10], because these hearts are not faithful in general (i.e. the higher extension groups computed in the hearts are different from the higher extension groups computed in the ambient triangulated categories). They are, however, faithful if the homology of Γ is concentrated in degree 0 and then the homology of Γ' is also concentrated in degree 0, as we show below in Theorem 6.2.

Lemma 5.4.

(a) *We have $\Sigma \mathcal{D}_{\leq 0} \subset \mathcal{D}'_{\leq 0} \subset \mathcal{D}_{\leq 0}$.*
(b) *Let $\mathcal{F} \subset \mathcal{A}$ be the subcategory of modules M supported at i and $\mathcal{T} \subset \mathcal{A}$ the left orthogonal subcategory of $\mathcal{F}$. Then an object X of $\mathcal{D}(\Gamma)$ belongs to $\mathcal{D}'_{\leq 0}$ (respectively $\mathcal{D}'_{> 0}$) iff $H^n(X)$ vanishes in all degrees $n > 0$ and $H^0(X)$ lies in $\mathcal{T}$ (respectively $H^n(X)$ vanishes in all degrees $n < 0$ and $H^0(X)$ lies in $\mathcal{F}$).*
(c) *The pair $(\mathcal{F}, \mathcal{T})$ is a torsion pair, i.e. $\mathcal{F}$ is the right orthogonal of $\mathcal{T}$ and $\mathcal{T}$ the left orthogonal of $\mathcal{F}$.*

Proof. An object X of $\mathcal{D}(\Gamma)$ belongs to $\mathcal{D}'_{\leq 0}$ iff it satisfies $\mathsf{Hom}(FP'_j, \Sigma^n X) = 0$ for all $n > 0$ and all $1 \leq j \leq n$. Using this characterization, parts (a) and (b) are easy to check. Also part (c) is easy to check using that $\mathcal{D}'_{\leq 0}$ and $\mathcal{D}'_{> 0}$ are the two aisles of a t-structure. $\square$

Corollary 5.5. *Let $\mathcal{A}'$ be the heart of the t-structure $(\mathcal{D}'_{\leq 0}, \mathcal{D}'_{\geq 0})$. Then an object X of $\mathcal{D}(\Gamma)$ belongs to $\mathcal{A}'$ if and only if $H^n(X) = 0$ for $n \neq 0, -1$, $H^0(X)$ belongs to $\mathcal{T}$ and $H^{-1}(X)$ belongs to $\mathcal{F}$.*

6. Stability under mutation of Ginzburg algebras concentrated in degree 0

6.1. The statement

Let k be a field. Let Q be a non-empty finite quiver and W a potential in $\widehat{kQ}$. The quiver Q may contain loops and 2-cycles but we assume that W is reduced (i.e. no paths of length ≤ 1 occur in the relations deduced from W). Let Γ be the complete Ginzburg algebra associated with (Q, W) and put $A = H^0(\Gamma)$.

For each vertex i of Q, we denote by S_i the associated simple module and by $P_i = e_i A$ its projective cover. Fix a vertex i of Q and consider the complex T' which is the sum of the P_j, $j \neq i$, concentrated in degree 0 and of the complex

$$0 \to P_i \xrightarrow{f} B \to 0$$

where P_i is in degree -1, B is the sum over the arrows α starting in i of the $P_{t(\alpha)}$ and the components of f are the left multiplications by the corresponding arrows.

Theorem 6.2. *Suppose that the complete Ginzburg algebra $\Gamma = \hat{\Gamma}(Q, W)$ has its homology concentrated in degree 0. If the simple module S_i corresponding to the vertex i is spherical (i.e. there are no loops at i in Q), then T' is a tilting object in the perfect derived category $\mathsf{per}(A)$. Thus, under the assumptions (c1), (c2) and (c3), the complete Ginzburg algebra Γ' associated with the mutated quiver with potential $\mu_i(Q, W)$ still has its homology concentrated in degree 0 and the Jacobian algebras A and A' associated with (Q, W) and $\mu_i(Q, W)$ are derived equivalent.*

Using the theorem and Corollary 5.5 we obtain the following corollary.

Corollary 6.3. *Under the hypothesis of the theorem, the category $\mathsf{Mod}\, J(\mu_i(Q, W))$ is obtained by tilting, in the sense of [23,10], from the category $\mathsf{Mod}\, J(Q, W)$ at the simple module S_i.*

Let us prove the theorem.

Proof of Theorem 6.2. We would like to use Proposition 6.5 below and have to check conditions (1), (2) and (3). Condition (1) holds since B belongs to $\mathcal{B}$, the additive closure of the P_j, $j \neq i$. Condition (2) holds since $f : P_i \to B$ is a left $\mathcal{B}$-approximation. Finally, in order to show condition (3), it suffices to show that f is injective. This can be deduced from Ginzburg's results [21] but we can also show it as follows: Since the homology of Γ is concentrated in degree 0, the functor $? \overset{L}{\otimes}_\Gamma A$ is an equivalence from $\mathsf{per}(\Gamma)$ to $\mathsf{per}(A)$ whose inverse is given by the restriction along the projection morphism $\Gamma \to H^0\Gamma = A$. If we apply the equivalence $? \overset{L}{\otimes}_\Gamma A$ to the cofibrant resolution $\mathbf{p}S_i$ constructed in Section 2.14, we obtain the complex of projective A-modules

$$0 \to P_i \xrightarrow{f} B \to B' \to P_i \to 0,$$

as we see by using the explicit description of $\mathbf{p}S_i$ given at the beginning of the proof of Lemma 3.12. The image of this complex under the restriction along $\Gamma \to A$ is again quasi-isomorphic to S_i. So the complex itself is quasi-isomorphic to S_i. In particular, the map f is injective. $\square$

6.4. Tilts of tilting objects in triangulated categories

The following lemma is well-known to the experts in tilting theory although it is not easy to point to a specific reference. One could cite Auslander, Platzeck and Reiten [2], Riedtmann and

Schofield [45], Happel and Unger [24], Buan, Marsh, Reineke, Reiten and Todorov [4], Geiss, Leclerc and Schröer [20] and many others.

Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. Let T be a *tilting object in $\mathcal{T}$*, i.e.

(a) $\mathcal{T}$ coincides with the closure of T under taking suspensions, desuspensions, extensions and direct factors and
(b) we have $\mathcal{T}(T, \Sigma^n T) = 0$ for all integers $n \neq 0$.

Assume that we are given a decomposition

$$T = T_0 \oplus T_1.$$

Let $\mathsf{add}(T_1)$ denote the closure of T_1 under taking finite direct sums and direct summands. Assume that there exists a map $f : T_0 \to B$ such that

(1) B belongs to $\mathsf{add}(T_1)$;
(2) the map $f^* : \mathcal{T}(B, T_1) \to \mathcal{T}(T_0, T_1)$ is surjective and
(3) the map $f_* : \mathcal{T}(T_1, T_0) \to \mathcal{T}(T_1, B)$ is injective.

Choose a triangle

$$T_0 \xrightarrow{f} B \xrightarrow{g} T_0^* \xrightarrow{h} \Sigma T_0. \tag{3}$$

Proposition 6.5. *The object $T' = T_0^* \oplus T_1$ is a tilting object in $\mathcal{T}$.*

Proof. It is clear from the triangle (3) that T' still generates $\mathcal{T}$ in the sense of condition (a). It remains to be checked that there are no morphisms between non-trivial shifts of the two given summands of T'. We distinguish several cases:

Case 1: We have $\mathcal{T}(T_0^*, \Sigma T_0^*) = 0$. Consider the triangles

$$B \xrightarrow{g} T_0^* \xrightarrow{h} \Sigma T_0 \xrightarrow{-\Sigma f} \Sigma B,$$

$$\Sigma B \xrightarrow{\Sigma g} \Sigma T_0^* \xrightarrow{\Sigma h} \Sigma^2 T_0 \to \Sigma^2 B.$$

Using condition (b) and the long exact sequences associated with these triangles we obtain the following diagram whose first two rows and last column are exact (we write (X, Y) instead of $\mathcal{T}(X, Y)$):

$$(\Sigma B, \Sigma T_0) \longrightarrow (\Sigma B, \Sigma B) \longrightarrow (\Sigma B, \Sigma T_0^*) \longrightarrow 0$$

$$(\Sigma T_0, \Sigma T_0) \longrightarrow (\Sigma T_0, \Sigma B) \longrightarrow (\Sigma T_0, \Sigma T_0^*) \longrightarrow 0$$
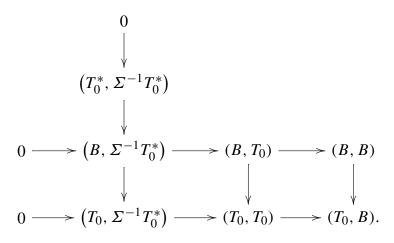
$$(T_0^*, \Sigma T_0^*)$$

$$0.$$

This implies that the space of morphisms $(T_0^*, \Sigma T_0^*)$ is isomorphic to the space of maps up to homotopy from the complex $T_0 \to B$, with T_0 in degree 0, to its shift by one degree to the left. Clearly conditions (1) and (2) suffice for this space to vanish.

Case 2: *We have* $\mathcal{T}(T_0^*, \Sigma^{-1} T_0^*) = 0$. Consider the triangles

$$B \xrightarrow{g} T_0^* \xrightarrow{h} \Sigma T_0 \xrightarrow{-\Sigma f} \Sigma B,$$

$$\Sigma^{-1} B \xrightarrow[\Sigma^{-1} g]{} \Sigma^{-1} T_0^* \xrightarrow[\Sigma^{-1} h]{} T_0 \xrightarrow{f} B.$$

By using condition (b) and the long exact sequences associated with these triangles we obtain the following commutative diagram whose first column and two last rows are exact

$$0$$

$$(T_0^*, \Sigma^{-1} T_0^*)$$

$$0 \longrightarrow (B, \Sigma^{-1} T_0^*) \longrightarrow (B, T_0) \longrightarrow (B, B)$$

$$0 \longrightarrow (T_0, \Sigma^{-1} T_0^*) \longrightarrow (T_0, T_0) \longrightarrow (T_0, B).$$

This shows that the space of morphisms $(T_0^*, \Sigma^{-1} T_0^*)$ is isomorphic to the space of maps up to homotopy from the complex $T_0 \to B$, with T_0 in degree 0, to its shift by one degree to the right. Clearly conditions (2) and (3) suffice to imply the vanishing of this space.

Case 3: *We have* $\mathcal{T}(T_0^*, \Sigma^n T_0^*) = 0$ *for all integers n different from* -1, *0 and* 1. This follows easily by considering the long exact sequences associated with the triangle (3).

Case 4: *We have* $\mathcal{T}(T_0^*, \Sigma^n T_1) = 0$ *for all integers n different from* 0 *and* 1. Again this is easy.

Case 5: *We have* $\mathcal{T}(T_0^*, \Sigma T_1) = 0$. This follows if we apply $\mathcal{T}(?, \Sigma T_1)$ to the triangle

$$B \xrightarrow{g} T_0^* \xrightarrow{h} \Sigma T_0 \xrightarrow{-\Sigma f} \Sigma B$$

and use condition (2).

Case 6: *We have* $\mathcal{T}(T_1, \Sigma^n T_0^*) = 0$ *for* $n \neq 0$. This follows if we apply $\mathcal{T}(T_1, ?)$ to the triangle

$$\Sigma^n B \xrightarrow{\Sigma^n f} \Sigma^n T_0^* \xrightarrow{\Sigma^n h} \Sigma^{n+1} T_0 \to \Sigma^{n+1} B$$

and use condition (3) in the case where $n = -1$.

Case 7: *We have* $\mathcal{T}(T_1, \Sigma^n T_1) = 0$. This is clear since T is tilting. $\quad\square$

Acknowledgments

Appendix A. Pseudocompact dg algebras and derived categories by Bernhard Keller

A.1. Motivation

As pointed out by D. Simson [49], the Jacobian algebra of a quiver with potential is a topological algebra in a natural way and it is natural and useful to take this structure into account and to consider (suitable) topological modules (or dually, to consider coalgebras and comodules, cf. for example [50]). It is then necessary to consider derived categories of topological modules as well. This allows one, for example, to extend the results of Amiot [1] from potentials belonging to kQ to potentials belonging to the completion $\widehat{kQ}$. We briefly recall some classical facts on the relevant class of topological algebras and modules: the pseudocompact ones. We show their usefulness in reconstructing Jacobian algebras from their categories of finite-dimensional modules. Then we adapt the notion of pseudocompact module to the dg setting and show that mutation also yields equivalences between pseudocompact derived categories. Notice that the definition of the pseudocompact derived category which we give after Lemma A.13 may seem 'too simple' because it uses the notion of ordinary quasi-isomorphism instead of a more refined notion of weak equivalence as in [37,44,51]. The category we obtain is nevertheless equivalent to the 'sophisticated one' because of the boundedness and smoothness hypothesis which we impose.

A.2. Reminder on pseudocompact algebras and modules

Let k be a field and R a finite-dimensional separable k-algebra (i.e. R is projective as a bimodule over itself). By an *R-algebra*, we mean an algebra in the monoidal category of R-bimodules. Following Chapter IV, Section 3 of [18], we call an R-algebra A *pseudocompact* if it is endowed with a linear topology for which it is complete and separated and admits a basis of neighborhoods of zero formed by left ideals I such that A/I is finite-dimensional over k. Let us fix a pseudocompact R-algebra A. A right A-module M is *pseudocompact* if it is endowed with a linear topology admitting a basis of neighborhoods of zero formed by submodules N of M such

that M/N is finite-dimensional over k. A *morphism* between pseudocompact modules is a continuous A-linear map. We denote by $\mathsf{Pcm}(A)$ the *category of pseudocompact right A-modules* thus defined. For example, if A equals R, then we have the equivalence

$$M \mapsto \mathsf{Hom}_R(M, R)$$

from the opposite of the category $\mathsf{Mod}\,R$ to $\mathsf{Pcm}(A)$. In particular, since $\mathsf{Mod}\,R$ is semisimple, so is $\mathsf{Pcm}(R)$.

Recall that a *Grothendieck category* is an abelian category where all colimits exist, where the passage to a filtered colimit is an exact functor and which admits a generator; a *locally finite category* is a Grothendieck category where each object is the filtered colimit of its subobjects of finite length.

Theorem A.3. *(See [18].)*

(a) *The category $\mathsf{Pcm}(A)$ is an abelian category and the forgetful functor $\mathsf{Pcm}(A) \to \mathsf{Mod}\,R$ is exact.*
(b) *The opposite category $\mathsf{Pcm}(A)^{op}$ is a locally finite category.*

The main ingredient of the proof of the theorem is the following well-known 'Mittag–Leffler lemma'.

Lemma A.4. *If*

$$0 \to L_i \to M_i \to N_i \to 0, \quad i \in I,$$

is a filtered inverse system of exact sequences of vector spaces and all the L_i are finite-dimensional, then the inverse limit of the sequence is still exact.

Using classical facts on locally finite Grothendieck categories we deduce the following corollary from the theorem.

Corollary A.5.

(a) *The dual of the Krull–Schmidt theorem holds in $\mathsf{Pcm}(A)$ (decompositions into possibly infinite products of objects with local endomorphism rings are unique up to permutation and isomorphism).*
(b) *Each object of the category $\mathsf{Pcm}(A)$ admits a projective cover and a minimal projective resolution.*
(c) *The category $\mathsf{Pcm}(A)$ is determined, up to equivalence, by its full subcategory of finite-length objects $\mathsf{mod}\,A$. More precisely, we have the equivalence*

$$\mathsf{Pcm}(A) \to \mathsf{Lex}(\mathsf{mod}\,A, \mathsf{Mod}\,k), \qquad M \mapsto \mathsf{Hom}_A(M, ?) \,|\, \mathsf{mod}\,A,$$

where $\mathsf{Lex}(\mathsf{mod}\,A, \mathsf{Mod}\,k)$ is the category of left exact functors from $\mathsf{mod}\,A$ to the category $\mathsf{Mod}\,k$ of vector spaces.
(d) *The category $\mathsf{Pcm}(A)$ determines the pseudocompact algebra A up to isomorphism. Namely, the algebra A is the endomorphism algebra of any minimal projective generator of $\mathsf{Pcm}(A)$.*

Proof. (a) Indeed, the Krull–Schmidt theorem holds in Grothendieck abelian categories by Theorem 1, Section 6, Chapter I of [18].

(b) Indeed, injective hulls exist in an arbitrary Grothendieck category by Theorem 2, Section 6, Chapter III of [18].

(c) This equivalence is a special case of the general description of the smallest locally finite (or, more generally, locally noetherian) abelian category containing a given finite length category, cf. Theorem 1, Section 4, Chapter II of [18].

(d) This follows from the description of finite length categories via pseudocompact algebras in Section 4, Chapter IV of [18]. $\quad\square$

A.6. Pseudocompact modules over Jacobian algebras

Let k be a field of characteristic 0. Let Q be a non-empty finite quiver and W a potential in $\widehat{kQ}$. The quiver Q may contain loops and 2-cycles but we assume that W is reduced, i.e. no cycles of length $\leqslant 2$ appear in W (and thus no paths of length $\leqslant 1$ occur in the relations deduced from W). Let Γ be the complete Ginzburg algebra associated with (Q, W) and put $A = H^0(\Gamma)$.

Lemma A.7. *The algebra A is pseudocompact in the sense of Section* A.2*, i.e. endowed with its natural topology, it is complete and separated and there is a basis of neighborhoods of the origin formed by right ideals I such that A/I is finite-dimensional over k.*

Proof. This follows from the fact that Γ is pseudocompact as a dg algebra and Lemma A.12 below. $\quad\square$

We consider the category $\mathsf{Pcm}(A)$ of *pseudocompact* modules over A, i.e. topological right modules which are complete and separated and have a basis of neighborhoods of the origin formed by submodules of finite codimension over k, cf. Section A.2. As recalled there, the category $\mathsf{Pcm}(A)$ is abelian and its opposite category $\mathsf{Pcm}(A)^{op}$ is a locally finite Grothendieck category and therefore has all the good properties enumerated in Corollary A.5. If we combine parts (c) and (d) of that corollary, we obtain the following corollary.

Corollary A.8. *The category $\mathsf{mod}\, A$ of finite-dimensional A-modules determines the algebra A up to isomorphism.*

This yields a positive answer to question 12.1 of [12], where it was asked whether the category of finite-dimensional modules over the Jacobian algebra of a quiver with potential determined the Jacobian algebra.

A.9. The pseudocompact derived category of a pseudocompact algebra

Let A be a pseudocompact R-algebra as in Section A.2. We define the *pseudocompact derived category of A* by $\mathcal{D}_{pc}(A) = \mathcal{D}(\mathsf{Pcm}(A))$. Notice that the opposite category $\mathcal{D}_{pc}(A)^{op}$ is isomorphic to $\mathcal{D}(\mathsf{Pcm}(A)^{op})$ and that $\mathsf{Pcm}(A)^{op}$ is a locally finite Grothendieck category. As shown in [17], cf. also [26], the category of complexes over a Grothendieck category admits a structure of Quillen model category whose weak equivalences are the quasi-isomorphisms and whose cofibrations are the monomorphisms. This applies in particular to the category $\mathsf{Pcm}(A)^{op}$. Notice that the free A-module $E = {}_A A$ is an injective cogenerator of this category and that the

algebra A is canonically isomorphic to the endomorphism algebra of E both in $\mathsf{Pcm}(A)$ and in $\mathcal{D}_{pc}(A)$.

Lemma A.10. *If a locally finite category $\mathcal{A}$ is of finite homological dimension, then the derived category $\mathcal{D}\mathcal{A}$ is compactly generated by the simple objects of $\mathcal{A}$. In particular, if the category $\mathcal{A} = \mathsf{Pcm}(A)$ is of finite homological dimension, then the opposite $\mathcal{D}_{pc}(A)^{op}$ of the pseudocompact derived category is compactly generated by the simple A-modules.*

Proof. If $\mathcal{A}$ is of finite homological dimension, then a complex over $\mathcal{A}$ is fibrant-cofibrant for the above model structure iff it has injective components. Moreover, since $\mathcal{A}$ is in particular locally noetherian, arbitrary coproducts of injective objects are injective. Thus, to check that a simple object S is compact, it suffices to check that any morphism from S to a coproduct of a family of complexes with injective components factors through a finite sub-coproduct. This is clear. Let $\mathcal{T}$ be the closure in $\mathcal{D}\mathcal{A}$ of the simple objects under suspensions, desuspensions, extensions and arbitrary coproducts. Since $\mathcal{T}$ is a compactly generated subcategory of $\mathcal{D}\mathcal{A}$, to conclude that $\mathcal{T}$ coincides with $\mathcal{D}\mathcal{A}$, it suffices to check that the right orthogonal of $\mathcal{T}$ in $\mathcal{D}\mathcal{A}$ vanishes. Clearly $\mathcal{T}$ contains all objects of finite length. These form a generating family for $\mathcal{A}$. Now it is easy to check that a complex I of injectives is right orthogonal to $\mathcal{T}$ in the category of complexes modulo homotopy iff I is acyclic. Thus the right orthogonal subcategory of $\mathcal{T}$ in $\mathcal{D}\mathcal{A}$ vanishes and $\mathcal{T}$ equals $\mathcal{D}\mathcal{A}$. $\square$

A.11. Extension to the dg setting

Let k and R be as in Section A.2. Let A be a differential algebra in the category of graded R-bimodules. We assume that A is *bilaterally pseudocompact*, i.e. it is endowed with a complete separated topology admitting a basis of neighborhoods of 0 formed by bilateral differential graded ideals I of finite total codimension in A.

For example, if Q is a finite graded quiver, we can take R to be the product of copies of k indexed by the vertices of Q and A to be the completed path algebra, i.e. for each integer n, the component A^n is the product of the spaces kc, where c ranges over the paths in Q of total degree n. We endow A with a continuous differential sending each arrow to a possibly infinite linear combination of paths of length $\geqslant 2$. For each n, we define I_n to be the ideal generated by the paths of length $\geqslant n$ and we define the topology on A to have the I_n as a basis of neighborhoods of 0. Then A is bilaterally pseudocompact.

A right dg A-module is *pseudocompact* if it is endowed with a topology for which it is complete and separated (in the category of graded A-modules) and which admits a basis of neighborhoods of 0 formed by dg submodules of finite total codimension. Clearly A is a pseudocompact dg module over itself.

Lemma A.12.

(a) *The homology $H^*(A)$ is a bilaterally pseudocompact graded R-algebra. In particular, $H^0(A)$ is a pseudocompact R-algebra.*

(b) *For each pseudocompact dg module M, the homology $H^*(M)$ is a pseudocompact graded module over $H^*(A)$.*

Proof. (a) By the Mittag–Leffler Lemma A.4, the homology $H^*(A)$ identifies with the limit of the $H^*(A/I)$, where I runs through a basis of neighborhoods of zero formed by bilateral dg ideals of finite codimension. This inverse limit is also the inverse limit of the images of the maps $H^*(A) \to H^*(A/I)$. Now clearly, the kernels of these maps are of finite codimension and form a basis of neighborhoods of zero for a complete separated topology on $H^*(A)$ (in the category of graded R-modules). (b) In analogy with (a), the homology $H^*(M)$ is endowed with the topology where a basis of neighborhoods of zero is formed by the kernels of the maps $H^*(M) \to H^*(M/M')$, where M' runs through a basis of neighborhoods of zero formed by dg submodules of finite codimension. $\square$

A morphism $L \to M$ between pseudocompact dg modules is a *quasi-isomorphism* if it induces an isomorphism $H^*(L) \to H^*(M)$. Let $\mathcal{C}_{pc}(A)$ be the category of pseudocompact dg right A-modules. It becomes a dg category in the natural way. Let $\mathcal{H}_{pc}(A)$ be the associated zeroth homology category, i.e. the category up to homotopy of pseudocompact dg A-modules. The *strictly perfect derived category of A* is the thick subcategory of $\mathcal{H}_{pc}(A)$ generated by the A-module A. A pseudocompact dg A-module is *strictly perfect* if it lies in this subcategory.

The opposite algebra A^{op} is still a bilaterally pseudocompact dg algebra. If A and A' are two bilaterally pseudocompact dg algebras, then so is their completed tensor product $A \hat{\otimes}_k A'$. The dg algebra A is a pseudocompact dg module over the enveloping algebra $A^{op} \hat{\otimes} A$ (this is the reason why we use bilaterally pseudocompact algebras).

The bilaterally pseudocompact dg algebra A is *topologically homologically smooth* if the module A considered as a pseudocompact dg module over $A^{op} \hat{\otimes}_k A$ is quasi-isomorphic to a strictly perfect dg module.

Lemma A.13. *(See [29].) If A is the completed path algebra of a finite graded quiver endowed with a continuous differential sending each arrow to a possibly infinite linear combination of paths of length $\geqslant 2$, then A is topologically homologically smooth.*

We define the *pseudocompact derived category* $\mathcal{D}_{pc}(A)$ to be the localization of $\mathcal{H}_{pc}(A)$ at the class of quasi-isomorphisms. As we will see below, this definition does yield a reasonable category if A is topologically homologically smooth and concentrated in non-positive degrees. For arbitrary pseudocompact dg algebras, the category $\mathcal{D}_{pc}(A)$ should be defined with more care, cf. [37,44,51]. We define the *perfect derived category* $\mathsf{per}(A)$ to be the thick subcategory of $\mathcal{D}_{pc}(A)$ generated by the free A-module of rank 1. We define the *finite-dimensional derived category* $\mathcal{D}_{fd}(A)$ to be the full subcategory whose objects are the pseudocompact dg modules M such that $(\mathcal{D}_{pc}(A))(P, M)$ is finite-dimensional for each perfect P.

Proposition A.14. *Assume that A is topologically homologically smooth and that $H^p(A)$ vanishes for all $p > 0$.*

(a) *The canonical functor $\mathcal{H}_{pc}(A) \to \mathcal{D}_{pc}(A)$ has a left adjoint $M \mapsto \mathbf{p}M$.*
(b) *The triangulated category $\mathcal{D}_{fd}(A)$ is generated by the dg modules of finite dimension concentrated in degree 0.*
(c) *The full subcategory $\mathcal{D}_{fd}(A)$ of $\mathcal{D}_{pc}(A)$ is contained in the perfect derived category $\mathsf{per}(A)$. In the setup of Section 2.18, it coincides with the subcategory of the same name defined there.*
(d) *The opposite category $\mathcal{D}_{pc}(A)^{op}$ is compactly generated by $\mathcal{D}_{fd}(A)$.*

(e) *Let $A \to A'$ be a quasi-isomorphism of bilaterally pseudocompact, topologically homologically smooth dg algebras whose homology is concentrated in non-positive degrees. Then the restriction functor $\mathcal{D}_{pc}(A') \to \mathcal{D}_{pc}(A)$ is an equivalence. In particular, if the homology of A is concentrated in degree 0, there is an equivalence $\mathcal{D}_{pc}(A) \to \mathcal{D}_{pc}(H^0 A)$. Moreover, in this case $\mathcal{D}_{pc}(H^0 A)$ is equivalent to the derived category of the abelian category $\mathsf{Pcm}(H^0 A)$.*

(f) *Assume that A is a complete path algebra as in Section 2.18. There is an equivalence between $\mathcal{D}_{pc}(A)^{op}$ and the localizing subcategory $\mathcal{D}_0(A)$ of the ordinary derived category $\mathcal{D}(A)$ generated by the finite-dimensional dg A-modules.*

Proof. (a) Let $\phi \colon P \to A$ be a quasi-isomorphism of $A^{op} \hat\otimes A$-modules where P is strictly perfect. Then the underlying morphism of pseudocompact left A-modules is an homotopy equivalence. Indeed, its cone is contractible since it is acyclic and lies in the thick subcategory of $\mathcal{H}_{pc}(A^{op})$ generated by the $A \hat\otimes_k V$, where V is a pseudocompact dg k-module. Thus, for each pseudocompact dg A-module M, the induced morphism

$$M \hat\otimes_A P \to M \hat\otimes_A A = M$$

is a quasi-isomorphism. We claim that the canonical map

$$\mathsf{Hom}_{\mathcal{H}_{pc}(A)}(M \hat\otimes_A P, ?) \to \mathsf{Hom}_{\mathcal{D}_{pc}(A)}(M \hat\otimes_A P, ?)$$

is bijective (which shows that $M \mapsto M \hat\otimes_A P$ is left adjoint to the canonical functor). Indeed, since P is strictly perfect, it suffices to show that if we replace P by $A \hat\otimes_k A$, then the corresponding map is bijective. But we have

$$M \hat\otimes_A (A \hat\otimes_k A) = M \hat\otimes_k A$$

and the functor

$$\mathsf{Hom}_{\mathcal{H}_{pc}(A)}(M \hat\otimes_k A, ?) = \mathsf{Hom}_{\mathcal{H}_{pc}(R)}(M, ?)$$

makes quasi-isomorphisms invertible, which implies the claim.

(b) After replacing A with $\tau_{\leqslant 0} A$, cf. Section A.18 below, we may and will assume that the component A^p vanishes for all $p > 0$. Then, for each dg module M, in the exact sequence of complexes

$$0 \to \tau_{<0} M \to M \to \tau_{\geqslant 0} M \to 0,$$

the complex $\tau_{<0} M$ is a dg A-submodule of M. Thus, the sequence is a sequence of dg A-modules. It is clear that the truncation functors $\tau_{<0}$ and $\tau_{\geqslant 0}$ preserve quasi-isomorphisms and easy to see that they define t-structures on $\mathcal{D}(A)$, $\mathcal{D}_{pc}(A)$ and $\mathcal{D}_{fd}(A)$, cf. Section A.18 below. Clearly, the t-structure obtained on $\mathcal{D}_{fd}(A)$ is non-degenerate and each object M of its heart is quasi-isomorphic to the dg module $H^0(M)$, which is concentrated in degree 0 and finite-dimensional. This clearly implies the claim.

(c) Let M be in $\mathcal{D}_{fd}(A)$. By (b), we may assume that M is of finite total dimension. Let P be as in the proof of (a). Then $M \hat\otimes_A P$ is perfect and quasi-isomorphic to M. The second claim follows because the two categories identify with the same full subcategory of $\mathsf{per}(A)$.

(d) We have to check that $\mathcal{D}_{pc}(A)$ has arbitrary products. Indeed, they are given by products of pseudocompact dg modules, which in turn are given by the products of the underlying dg modules. We have to check that k-finite-dimensional pseudocompact dg modules are co-compact in $\mathcal{D}_{pc}(A)$. For this, we first observe that they are co-compact in $\mathcal{H}_{pc}(A)$. Now let P be as in the proof of (a) and let X_i, $i \in I$, be a family of objects of $\mathcal{D}_{pc}(A)$. We consider the space of morphisms

$$(\mathcal{H}A)\left(\left(\prod_i X_i\right) \hat{\otimes}_A P, M\right).$$

Since the module M is finite-dimensional, it is annihilated by some bilateral dg ideal I of finite codimension in A. Thus the above space of morphisms is isomorphic to

$$(\mathcal{H}A)\left(\left(\prod_i X_i\right) \hat{\otimes}_A (P/PI), M\right).$$

Now the bimodule P/PI is perfect as a left dg A-module (since $A^{op} \hat{\otimes}_A (A/I) = A^{op} \otimes_A (A/I)$ is left perfect). Thus, we have the isomorphisms

$$\left(\prod_i X_i\right) \hat{\otimes}_A (P/PI) = \left(\prod_i X_i\right) \otimes_A (P/PI) = \prod_i \left(X_i \otimes_A (P/PI)\right).$$

Since M is co-compact in $\mathcal{H}A$, we find the isomorphism

$$(\mathcal{H}A)\left(\prod_i \left(X_i \otimes_A (P/PI)\right), M\right) = \coprod_i (\mathcal{H}A)\left(X_i \otimes_A (P/PI), M\right).$$

Now the last space is isomorphic to

$$\coprod_i (\mathcal{H}A)(X_i \hat{\otimes}_A P, M) = \coprod_i (\mathcal{D}A)(X_i, M).$$

Finally, we have to show that the left orthogonal of $\mathcal{D}_{fd}(A)$ vanishes in $\mathcal{D}_{pc}(A)$. Indeed, if M belongs to the left orthogonal of $\mathcal{D}_{fd}(A)$, then all the maps $M \to M/M'$, where M' is a dg submodule of finite codimension, vanish in $\mathcal{D}_{pc}(A)$. Thus the maps $H^*(M) \to H^*(M/M')$ vanish. But $H^*(M)$ is the inverse limit of the system of the $H^*(M/M')$, where M' ranges over a basis of neighborhoods formed by dg submodules of finite codimension. So $H^*(M)$ vanishes and M vanishes in $\mathcal{D}_{pc}(A)$.

(e) Let $R : \mathcal{D}_{pc}(A') \to \mathcal{D}_{pc}(A)$ be the restriction functor. Let P be as in the proof of (a). Then R admits the left adjoint L given by $M \mapsto (M \hat{\otimes}_A P) \hat{\otimes}_A A'$. It is easy to check that R and L induce quasi-inverse equivalences between $\mathsf{per}(A)$ and $\mathsf{per}(A')$. An object M of $\mathsf{per}(A)$ belongs to the subcategory $\mathcal{D}_{fd}(A)$ if and only if the space $\mathsf{Hom}_{\mathsf{per}(A)}(P, M)$ is finite-dimensional for each P in $\mathsf{per}(A)$. Therefore, the equivalences L and R must induce quasi-inverse equivalences between $\mathcal{D}_{fd}(A)$ and $\mathcal{D}_{fd}(A')$. Clearly the functor R commutes with arbitrary products. Now the

claim follows from (c). We obtain the equivalence between $\mathcal{D}_{pc}(A)$ and $\mathcal{D}_{pc}(H^0 A)$ by using the quasi-isomorphism

$$A \leftarrow \tau_{\leqslant 0} A \rightarrow H^0 A.$$

The canonical functor $\mathcal{D}(\mathsf{Pcm}(H^0 A)) \rightarrow \mathcal{D}_{pc}(H^0 A)$ is an equivalence because its restriction to the subcategory of finite-dimensional dg modules concentrated in degree 0 is an equivalence and this subcategory compactly generates the opposites of both categories.

(f) Let $\mathcal{H}_0(A)$ be the full subcategory of $\mathcal{H}(A)$ formed by the dg modules which are filtered unions of their finite-dimensional submodules and let $\mathcal{Ac}_0(A)$ be its full subcategory formed by the acyclic dg modules N such that each finite-dimensional dg submodule of N is contained in an acyclic finite-dimensional dg submodule. Notice that the quotient $\mathcal{H}_0(A)/\mathcal{Ac}_0(A)$ inherits arbitrary coproducts from $\mathcal{H}_0(A)$. We have a natural functor

$$\mathcal{H}_0(A)/\mathcal{Ac}_0(A) \rightarrow \mathcal{D}_0(A)$$

and a natural duality functor

$$\mathcal{H}_0(A)/\mathcal{Ac}_0(A) \rightarrow \left(\mathcal{D}_{pc}(A)\right)^{op}$$

taking M to $\mathsf{Hom}_R(M, R)$. We will show that $\mathcal{H}_0(A)/\mathcal{Ac}_0(A)$ is compactly generated by its full subcategory $\mathcal{F}$ of finite-dimensional dg modules and that the above functors induce equivalences

$$\mathcal{F} \xrightarrow{\sim} \mathcal{D}_{fd}(A) \quad \text{and} \quad \mathcal{F} \xrightarrow{\sim} \mathcal{D}_{fd}(A)^{op}.$$

Since the functors

$$\mathcal{H}_0(A)/\mathcal{Ac}_0(A) \rightarrow \mathcal{D}_0(A) \quad \text{and} \quad \mathcal{H}_0(A)/\mathcal{Ac}_0(A) \rightarrow \left(\mathcal{D}_{pc}(A)\right)^{op}$$

both commute with arbitrary coproducts, they must be equivalences because $\mathcal{D}_{fd}(A)$ compactly generates $\mathcal{D}_0(A)$ (by Theorem 2.19) and its opposite compactly generates $\mathcal{D}_{pc}(A)^{op}$ (by part (d)). So let us show that $\mathcal{F}$ consists of compact objects. Indeed, it follows from the definition of $\mathcal{Ac}_0(A)$ that for an object F of $\mathcal{F}$, the category of $\mathcal{Ac}_0(A)$-quasi-isomorphisms $F' \rightarrow F$ with finite-dimensional F' is cofinal in the category of all $\mathcal{Ac}_0(A)$-quasi-isomorphisms with target F. Now it follows from the calculus of fractions that F is compact in $\mathcal{H}_0(A)/\mathcal{Ac}_0(A)$. Moreover, we see that $\mathcal{F}$ is naturally equivalent to the category $\mathcal{H}_{fd}(A)/\mathcal{Ac}_{fd}(A)$ of Section 2.18. Now the fact that both functors

$$\mathcal{F} \xrightarrow{\sim} \mathcal{D}_{fd}(A) \quad \text{and} \quad \mathcal{F} \xrightarrow{\sim} \mathcal{D}_{fd}(A)^{op}$$

are equivalences follows from part (a) of Theorem 2.19. □

A.15. The Calabi–Yau property

Keep the assumptions on k, R, A from Section A.11. In particular, A is assumed to be topologically homologically smooth. For two objects L and M of $\mathcal{D}_{pc}(A)$, define

$$\mathsf{RHom}_A(L, M) = \mathsf{Hom}_A(\mathbf{p}L, M).$$

Put $A^e = A^{op} \hat{\otimes}_k A$ and define

$$\Omega = \mathsf{RHom}_{A^e}(A, A^e).$$

Lemma A.16. *(See [32].) We have a canonical isomorphism*

$$D\,\mathsf{Hom}(L, M) = \mathsf{Hom}(M \hat{\otimes}_A \Omega, L)$$

bifunctorial in $L \in \mathcal{D}_{fd}(A)$ and $M \in \mathcal{D}_{pc}(A)$.

Let n be an integer. For an object L of $\mathcal{D}_{pc}(A^e)$, define

$$L^{\#} = \mathsf{RHom}_{A^e}(L, A^e)[n].$$

The dg algebra A is *n-Calabi–Yau as a bimodule* if there is an isomorphism

$$f : A \xrightarrow{\sim} A^{\#}$$

in $\mathcal{D}_{cd}(A^e)$ such that $f^{\#} = f$. The preceding lemma implies that in this case, the category $\mathcal{D}_{fd}(A)$ is n-Calabi–Yau as a triangulated category.

Theorem A.17. *(See [29].) Completed Ginzburg algebras are topologically homologically smooth and 3-Calabi–Yau.*

A.18. t-Structure

Keep the assumptions on k, R, A from the preceding section. Let us assume that $H^n(A)$ vanishes for all $n > 0$. We have the truncated complex $\tau_{\leqslant 0}A$, which becomes a dg algebra in a natural way. The dg algebra $\tau_{\leqslant 0}A$ is still a bilaterally pseudo-compact and topologically homologically smooth. Indeed, we have $\tau_{\leqslant 0}(A/I) = \tau_{\leqslant 0}(A)/\tau_{\leqslant 0}I$ for each dg ideal I of A, which yields that $\tau_{\leqslant 0}A$ is bilaterally pseudo-compact. Moreover, the morphism $\tau_{\leqslant 0}A \to A$ is a quasi-isomorphism, which yields the homological smoothness. Let us therefore assume that the components A^n vanish in all degrees $n > 0$.

Lemma A.19.

(a) *For each pseudocompact dg A-module M, the truncations $\tau_{\leqslant 0}M$ and $\tau_{>0}M$ are pseudocompact dg A-modules.*
(b) *The functors $\tau_{\leqslant 0}$ and $\tau_{>0}$ define a t-structure on $\mathcal{D}_{pc}(A)$.*
(c) *The heart of the t-structure is canonically equivalent to the category $\mathsf{Pcm}(H^0(A))$ of pseudocompact modules over the pseudocompact algebra $H^0(A)$.*

Proof. (a) Let M' be a dg submodule of M. The canonical map

$$Z^0(M)/Z^0(M') \to Z^0(M/M')$$

is an isomorphism because the map $B^1(M') \to B^1(M)$ is injective. Thus, we have an isomorphism

$$\tau_{\leqslant 0} M = \lim\left(\tau_{\leqslant 0}(M)/\tau_{\leqslant 0}(M')\right),$$

where M' ranges through a basis of neighborhoods of 0 formed by dg submodules of finite codimension. It follows that $\tau_{\leqslant 0} M$ is endowed with a complete separated topology and is pseudocompact. Similarly for $\tau_{>0} M = M/\tau_{\leqslant 0} M$.

(b) They define functors in the homotopy categories which preserve quasi-isomorphisms and thus descend to $\mathcal{D}_{pc}(A)$. As in the proof of Lemma 5.2, it is not hard to check that the induced functors define a t-structure.

(c) We omit the proof, which is completely analogous to that of part (b) of Lemma 5.2. $\quad\square$

A.20. Jacobi-finite quivers with potentials

Let k be a field and R a finite-dimensional separable k-algebra. Let A be a dg R-algebra which is bilaterally pseudocompact (Section A.11). Assume moreover that

(1) A is topologically homologically smooth (Section A.11),
(2) for each $p > 0$, the space $H^p A$ vanishes,
(3) the algebra $H^0(A)$ is finite-dimensional,
(4) A is 3-Calabi–Yau as a bimodule (Section A.15).

Thanks to assumption (1), the finite-dimensional derived category $\mathcal{D}_{fd}(A)$ is contained in the perfect derived category. The *generalized cluster category* [1] is the triangle quotient $\mathcal{C} = \mathrm{per}(A)/\mathcal{D}_{fd}(A)$. Let π be the canonical projection functor.

Theorem A.21. *The generalized cluster category $\mathcal{C}$ is* Hom-*finite and 2-Calabi–Yau. The object $\pi(A)$ is cluster-tilting in $\mathcal{C}$.*

This theorem was proved for non-topological dg algebras A by C. Amiot, cf. Theorem 2.1 of [1]. Using the material of the preceding sections, one can easily imitate Amiot's proof to obtain the above topological version. We leave the details to the reader. In particular, by Theorem A.17, the statement holds for complete Ginzburg dg algebras associated with quivers with potential (Q, W), where W belongs to the completed path algebra of Q and the Jacobian algebra is finite-dimensional.

A.22. Mutation

Let (Q, W) be a quiver with potential and let Γ be the associated completed Ginzburg dg algebra. By definition, Γ is concentrated in non-positive degrees and by Theorem A.17, it is topologically homologically smooth and 3-Calabi–Yau. Let i be a vertex of Q such that conditions (c1), (c2) and (c3) of Section 2.4 hold. Let (Q', W') be the mutation $\tilde{\mu}_i(Q, W)$ and Γ' the associated Ginzburg dg algebra.

Theorem A.23.

(a) *There is a triangle equivalence*

$$F_{pc} : \mathcal{D}_{pc}(\Gamma') \to \mathcal{D}_{pc}(\Gamma)$$

which sends the P'_j to P_j for $j \neq i$ and to the cone T_i over the morphism

$$P_i \to \bigoplus_{\alpha} P_{t(\alpha)}$$

for $i = j$, where we have a summand $P_{t(\alpha)}$ for each arrow α of Q with source i and the corresponding component of the map is the left multiplication by α. The functor F_{pc} restricts to triangle equivalences from $\mathrm{per}(\Gamma')$ to $\mathrm{per}(\Gamma)$ and from $\mathcal{D}_{fd}(\Gamma')$ to $\mathcal{D}_{fd}(\Gamma)$.

(b) *Let Γ_{red} respectively Γ'_{red} be the complete Ginzburg dg algebra associated with the reduction of (Q, W) respectively the reduction $\mu_i(Q, W)$ of $\tilde{\mu}_i(Q, W)$. The functor F_{pc} yields a triangle equivalence*

$$F_{\mathrm{red}} : \mathcal{D}_{pc}(\Gamma'_{\mathrm{red}}) \to \mathcal{D}_{pc}(\Gamma_{\mathrm{red}}),$$

which restricts to triangle equivalences from $\mathrm{per}(\Gamma'_{\mathrm{red}})$ to $\mathrm{per}(\Gamma_{\mathrm{red}})$ and from $\mathcal{D}_{fd}(\Gamma'_{\mathrm{red}})$ to $\mathcal{D}_{fd}(\Gamma_{\mathrm{red}})$.

Remark A.24. By combining the theorem with the facts on t-structures from Section A.18 we obtain a comparison of the categories of pseudocompact modules

$$\mathrm{Pcm}(H^0(\Gamma)) \quad \text{and} \quad \mathrm{Pcm}(H^0(\Gamma'))$$

completely analogous to the one in Section 5.3.

Proof of Theorem A.23. (a) Let X be the Γ'–Γ-bimodule constructed in Section 3.4. It gives rise to a pair of adjoint functors: the left adjoint $F_{pc} : \mathcal{D}_{pc}(\Gamma') \to \mathcal{D}_{pc}(\Gamma)$ is the left derived functor of the completed tensor product functor $M \mapsto M \hat{\otimes}_{\Gamma'} X$. The right adjoint G_{pc} is induced by the functor $\mathrm{Hom}_\Gamma(X, ?)$. These functors restrict to a pair of adjoint functors between $\mathcal{D}_{fd}(\Gamma)$ and $\mathcal{D}_{fd}(\Gamma')$. The induced functors are isomorphic to those of Proposition 3.13. Indeed, by Proposition A.14, the categories $\mathcal{D}_{fd}$ are contained in the perfect derived categories and the canonical morphism from the left derived functor of the (non-completed) tensor product $M \mapsto M \otimes_{\Gamma'} X$ to F_{pc} restricts to an isomorphism on the perfect derived category and similarly for G_{pc}. Thus, the restriction of G_{pc} is an equivalence from $\mathcal{D}_{fd}(\Gamma)$ to $\mathcal{D}_{fd}(\Gamma')$. Moreover, the functor G_{pc} commutes with arbitrary products since it admits a left adjoint. It follows that G_{pc} is an equivalence since the opposites of the categories $\mathcal{D}_{pc}$ are compactly generated by the subcategories $\mathcal{D}_{fd}$, by Proposition A.14.

(b) This follows from (a) and part (e) of Proposition A.14. □

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