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Grothendieck-Roos-Duality and Tilting

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Abstract (1) - We investigate the relations between the hearts of two t-structures on one triangulated category. Under suitable compatibility conditions, we obtain a common generalization of the duality theory developed by Grothendieck and Roos [10] for regular commutative rings and the tilting theory [3] [6] [4] used in the investigation of finite-dimensional algebras [5].

Dualité de Grothendieck-Roos et basculement

Résumé (1) - Nous étudions les relations entre les coeurs de deux t-structures d'une même catégorie triangulée. Des conditions de compatibilité appropriées nous permettent de généraliser à la fois la théorie de dualité développée par Grothendieck et Roos [10] et la théorie du basculement [3] [6] [4] utilisée dans l'étude des algèbres de dimension finie [5].

1. Let $\mathcal{T}$ be a triangulated category [2] [7] with suspension functor S and let $(\mathcal{V}^{<1}, \mathcal{V}^{\geq 0})$ be a t-structure on $\mathcal{T}$ in the sense of Beilinson-Bernstein-Deligne [2] [8]. The inclusion of $\mathcal{V}^{\geq n} := S^{-n}\,\mathcal{V}^{\geq 0}$ (resp. of $\mathcal{V}^{<n} := S^{-n+1}\,\mathcal{V}^{<1}$) into $\mathcal{T}$ admits a left adjoint $\tau^{\geq n} : \mathcal{T} \to \mathcal{V}^{\geq n}$ (resp. a right adjoint $\tau^{<n} : \mathcal{T} \to \mathcal{V}^{<n}$) and gives rise to triangles of the form

$$\tau^{<n}X \to X \to \tau^{\geq n}X \to S\tau^{<n}X, \quad X \in \mathcal{T}.$$

We propose ourselves to compare $(\mathcal{V}^{<1}, \mathcal{V}^{\geq 0})$ to a second t-structure $(\mathcal{U}_{\geq 0}\,,\, \mathcal{U}_{<1})$ on $\mathcal{T}$. We denote by $\tau_{\geq n} : \mathcal{T} \to \mathcal{U}_{\geq n}$ and $\tau_{<n} : \mathcal{T} \to \mathcal{U}_{<n}$ the left and right adjoints of the inclusions $\mathcal{U}_{\geq n} := S^n\,\mathcal{U}_{\geq 0}$ et $\mathcal{U}_{<n} := S^{n-1}\,\mathcal{U}_{<1}$ into $\mathcal{T}$. These give rise to triangles

$$\tau_{\geq n}X \to X \to \tau_{<n}X \to S\tau_{\geq n}X, \quad X \in \mathcal{T}.$$

The hearts [2] of the two t-structures are the abelian categories $\mathcal{A} = \mathcal{U}_{\geq 0} \cap \mathcal{U}_{<1}$ and $\mathcal{B} = \mathcal{V}^{\geq 0} \cap \mathcal{V}^{<1}$. They are related to $\mathcal{T}$ by the functors "homology"

$$H_n = \tau_{<1}\tau_{\geq 0}S^{-n} : \mathcal{T} \to \mathcal{A}$$

and "cohomology"

$$H^n = \tau^{<1}\tau^{\geq 0}S^{n} : \mathcal{T} \to \mathcal{B}$$

which transform triangles $X \to Y \to Z \to SX$ of $\mathcal{T}$ into long exact sequences [2]

$$\ldots \to H_{n+1}Z \to H_nX \to H_nY \to H_nZ \to H_{n-1}X \to \ldots$$

and

$$\ldots \to H^{n-1}Z \to H^nX \to H^nY \to H^nZ \to H^{n+1}X \to \ldots$$

2. In order to investigate the relations between the hearts $\mathcal{A}$ et $\mathcal{B}$, we set

$$\mathcal{A}^{\geq n} := \mathcal{A} \cap \mathcal{V}^{\geq n} \text{ and } \mathcal{B}_{\geq n} := \mathcal{B} \cap \mathcal{U}_{\geq n} \ ,$$

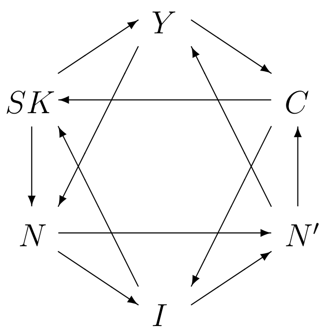
thus obtaining filtrations of $\mathcal{A}$ and $\mathcal{B}$ by subcategories which are full and stable under extensions

$$\ldots \subset \mathcal{A}^{\geq n+1} \subset \mathcal{A}^{\geq n} \subset \ldots \subset \mathcal{A} \text{ and } \mathcal{B} \supset \ldots \supset \mathcal{B}_{\geq n} \supset \mathcal{B}_{\geq n+1} \supset \ldots$$

We say that the aisle [8] $\mathcal{U}_{\geq 0}$ is *compatible* with the co-aisle $\mathcal{V}^{\geq 0}$ if $\tau^{<n}\mathcal{U}_{\geq 0} \subset \mathcal{U}_{\geq 0}$ for all $n \in \mathbf{Z}$. This implies $\tau^{\geq n}\mathcal{U}_{\geq 0} \subset \mathcal{U}_{\geq 0}$, $H^n\mathcal{U}_{\geq 0} \subset \mathcal{B}_{\geq n}$ and $H_m H^n | \mathcal{A} = 0$ for all m, n such that $m < n$.

Proposition - *If $\mathcal{U}_{\geq 0}$ is compatible with $\mathcal{V}^{\geq 0}$, the filtration $(\mathcal{B}_{\geq n})$ of $\mathcal{B}$ has the following property (*) : For each morphism $g : N \to N'$ of $\mathcal{B}$ such that $N \in \mathcal{B}_{\geq n}$ and $N' \in \mathcal{B}_{\geq n+1}$, we have* $\operatorname{Ker} g \in \mathcal{B}_{\geq n}$ *and* $\operatorname{Coker} g \in \mathcal{B}_{\geq n+1}$.

Indeed, consider the following octahedron where $K = \operatorname{Ker} g$, $I = \operatorname{Im} g$, $C = \operatorname{Coker} g$ [2].



The triangle $N \to N' \to Y \to SN$ shows that $Y \in \mathcal{U}_{\geq n+1}$. The triangle $SK \to Y \to C \to S^2K$ yields the exact sequences $0 = H^{-2}C \to K \to H^{-1}Y \to H^{-1}C = 0$ and $0 = H^1K \to H^0Y \to C \to H^2K = 0$. The compatibility condition implies that $K \xrightarrow{\sim} H^{-1}Y \in \mathcal{B}_{\geq n}$ and that $C \xrightarrow{\sim} H^0Y \in \mathcal{B}_{\geq n+1}$.

Dually, we shall say that the co-aisle $\mathcal{V}^{\geq 0}$ is *compatible* with the aisle $\mathcal{U}_{\geq 0}$ if $\tau_{<n}\mathcal{V}^{\geq 0} \subset \mathcal{V}^{\geq 0}$ for all $n \in \mathbf{Z}$. This new definition entails the dual of the foregoing proposition.

3. We say that $\mathcal{B}$ *generates* $\mathcal{T}$, if $\mathcal{T}$ coïncides with the smallest triangulated subcategory of $\mathcal{T}$ which is strictly full and contains $\mathcal{B}$. Then each object $X \in \mathcal{T}$ is obtained by successive extensions from a finite number of shifted homology groups $S^{-n}H^n X$. In particular, the t-structure $(\mathcal{V}^{<1}, \mathcal{V}^{\geq 0})$ is non-degenerated [2]. Moreover, in this case $\mathcal{U}_{\geq 0}$ is compatible with $\mathcal{V}^{\geq 0}$ iff $\mathcal{U}_{\geq 0} = \{X \in \mathcal{T} : H^n X \in \mathcal{B}_{\geq n} \text{ for all } n \in \mathbf{Z}\}$.

Proposition - *Suppose that $\mathcal{T}$ is generated by $\mathcal{A}$, as well as by $\mathcal{B}$. In order for $\mathcal{U}_{\geq 0}$ to be compatible with $\mathcal{V}^{\geq 0}$, it is then necessary and sufficient that $H_m H^n | \mathcal{A} = 0$ if $m < n$, and that the filtration $(\mathcal{B}_{\geq n})$ satisfies the condition (*) above.*

Indeed, it remains to be shown that these conditions are sufficient. Since $(\mathcal{U}_{\geq 0}\,,\ \mathcal{U}_{<1})$ is not degenerated the first condition means that $H^n\mathcal{A} \subset \mathcal{B}_{\geq n}$, $\forall n$. We conclude that $H^nX \in \mathcal{B}_{\geq n}$ for each $X \in \mathcal{U}_{\geq 0}$: if $X \neq 0$, we proceed by recursion on the greatest r such that $H_rX \neq 0$ ($\mathcal{A}$ generates $\mathcal{T}$!). The triangle $S^rH_rX \to X \to \tau_{<r}X \to S^{r+1}H_rX$ induces an exact

sequence

$$H^{n-1}\tau_{<r}X \xrightarrow{f} H^{n+r}H_r X \to H^n X \to H^n\tau_{<r}X \xrightarrow{g} H^{n+r+1}H_r X \quad ,$$

where $H^{n+r}H_r X \in \mathcal{B}_{\geq n+r}$ by what we already showed and $H^{n-1}\tau_{<r}X \in \mathcal{B}_{\geq n-1}$ by the recursion hypthesis. The condition (*) implies $\operatorname{Coker} f \in \mathcal{B}_{\geq n}$. Similarly, $\operatorname{Ker} g \in \mathcal{B}_{\geq n}$, hence $H^n X \in \mathcal{B}_{\geq n}$, since $\mathcal{B}_{\geq n}$ is stable under extensions.

Since $\mathcal{T}$ is also generated by $\mathcal{B}$, we finally obtain $\tau^{<n}\mathcal{U}_{\geq 0} \subset \mathcal{U}_{\geq 0}$ for each n.

4. *Example.* (cf. [10]) Let Λ be a *regular* ring, i.e. Λ is commutative, noetherian and of finite global dimension. We recall that [9]:

a) For each finitely generated Λ-module M , the "codimension"

$$c(M) = \inf\{\dim \Lambda_\wp : \wp \in \operatorname{Spec}(\Lambda)\ , M_\wp \neq 0\}$$

coïncides with the "grade"

$$g(M) = \inf\{i : \operatorname{Ext}^i_\Lambda(M, N) \neq 0\}.$$

b) $c(\operatorname{Ext}^n_\Lambda(M,N)) \geq n$ for all finitely generated Λ-modules M, N and each n.

The derived functor $D = R\operatorname{Hom}_\Lambda(?, \Lambda)$ induces a duality on the "bounded" derived category $\mathcal{T} := \mathcal{D}^b(\operatorname{mod}\Lambda)$ associated with the category $\operatorname{mod}\Lambda$ of finitely generated Λ-modules. We consider the natural co-aisle $\mathcal{V}^{\geq 0} = \{X \in \mathcal{T} : H^n X = 0,\ \forall n < 0\}$ defined by means of the usual cohomology functor, and the aisle $\mathcal{U}_{\geq 0} = \{Y \in \mathcal{T} : \exists X \in \mathcal{V}^{\geq 0},\ Y \xrightarrow{\sim} DX\}$. We "identify" $\operatorname{mod}\Lambda$ with $\mathcal{B} = \mathcal{V}^{\geq 0} \cap \mathcal{V}^{<1}$, $(\operatorname{mod}\Lambda)^{\mathrm{op}}$ with $\mathcal{A} = \mathcal{U}_{\geq 0} \cap \mathcal{U}_{<1}$, and the functors $H_n : \mathcal{B} \to \mathcal{A}$ and $H^n : \mathcal{A} \to \mathcal{B}$ with $\operatorname{Ext}^n_\Lambda(?, \Lambda)$.

In this case, $\mathcal{B}_{\geq n}$ is the Serre subcategory of $\mathcal{B}$ which is formed by the Λ-modules of codimension $\geq n$ and therefore satisfies (*). It is immediate from b) that $H_m H^n | \mathcal{A} = 0$ for $m < n$, hence that $\mathcal{U}_{\geq 0}$ *is compatible with* $\mathcal{V}^{\geq 0}$*, and* $\mathcal{V}^{\geq 0}$ *with* $\mathcal{U}_{\geq 0}$.

5. *Example.* (cf. [3][6][4][5]) Let k be a commutative field, Λ a finite-dimensional k-algebra, $\mathcal{B}_{\geq n} = \mathcal{B} = \operatorname{mod}\Lambda$ for $n < 0$, $\mathcal{B}_{\geq n} = 0$ for $n > 0$ and $\mathcal{B}_{\geq 0}$ a *torsion subcategory* (i. e. full and closed under extensions and quotients) of $\mathcal{B}$. Then $\mathcal{U}_{\geq 0} = \{X \in \mathcal{D}^b(\mathcal{B}) : H^n X \in \mathcal{B}_{\geq n},\ \forall n \in \mathbf{Z}\}$ is an aisle in $\mathcal{D}^b(\mathcal{B})$, which is compatible with the natural co-aisle $\mathcal{V}^{\geq 0}$ (§4). Also $\mathcal{U}_{<1}$ is compatible with $\mathcal{V}^{<1}$, $\mathcal{V}^{\geq 0}$ with $\mathcal{U}_{\geq 0}$ and $\mathcal{V}^{<1}$ with $\mathcal{U}_{<1}$.

Suppose that moreover $\mathcal{B}_{\geq 0}$ is generated by a tilting module T_Λ over Λ. If $\Gamma = \operatorname{End}(T_\Lambda)$, the derived functors $R\operatorname{Hom}_\Lambda(T, ?)$ and $L(? \otimes_\Gamma T) : \mathcal{D}^b(\operatorname{mod}\Gamma) \to \mathcal{D}^b(\operatorname{mod}\Lambda)$ are quasi-inverse [1] [5] S-equivalences [7]. They allow us to identify $\mathcal{U}_{\geq 0}$ with the natural aisle of $\mathcal{D}^b(\operatorname{mod}\Gamma)$, $\mathcal{A}$ with $\operatorname{mod}\Gamma$, $H_n|\,\mathcal{A}$ with $\operatorname{Tor}_n^\Gamma(?, T)$ and $H^n|\,\mathcal{B}$ with $\operatorname{Ext}_\Lambda^n(T, ?)$.

6. *Example.* (cf. [2] [8]) Let k be a commutative field, Q a finite quiver without oriented cycle, I an admissible ideal in the path category kQ, Λ the quotient kQ/I and $\operatorname{mod}\Lambda$ the category of Λ-"modules" $M : \Lambda^{\mathrm{op}} \to \operatorname{mod} k$. We consider the natural co-aisle $\mathcal{V}^{\geq 0}$ of $\mathcal{T} = \mathcal{D}^b(\operatorname{mod}\Lambda)$.

In order to construct an (artificial) aisle in $\mathcal{T}$, we start from a function $p : \{\text{ points of } \Lambda\} \to \mathbf{Z}$ such that $p(x) \geq p(y)$ if $\operatorname{Hom}(x, y) \neq 0$. We denote by $\Lambda_{\geq n}$ (resp. $\Lambda_{<n}$, resp. Λ_n) the full subcategory of Λ formed by the $x \in \Lambda$ such that $p(x) \geq n$ (resp. $p(x) < n$, resp. $p(x) = n$) and we set

$$\mathcal{U}' = \{X \in \mathcal{T} : \operatorname{supp} H^n X \subset \Lambda_{\geq n},\ \forall n\}$$

$$\mathcal{U}'' = \{X \in \mathcal{T} : \operatorname{supp} H^n X \subset \Lambda_{<n},\ \forall n\}$$

It is clear that $\mathcal{U}'$ is stable under S and $\mathcal{U}''$ under S^{-1}. Moreover, $\operatorname{Hom}(X, Y) = 0$ if $X \in \mathcal{U}'$ and $Y \in \mathcal{U}''$: indeed, $X \in \mathcal{U}'$ is equivalent to the existence of a quasi-isomorphism $P \to X$, where the P^n are projective and such that $\operatorname{supp} P^n \subset \Lambda_{\geq n}$, $\forall n$. And $Y \in \mathcal{U}''$ means that there is a quasi-isomorphism $Y \to J$, where the J^n are injective and such that $\operatorname{supp} J^n \subset \Lambda_{<n}$, $\forall n$.

For each $X \in \mathcal{T}$ we denote by X' the subcomplex such that $X'^n = (X^n|\,\Lambda_{\geq n+1})_0 + (Z^n X|\,\Lambda_{\geq n})_0$, where the index 0 stands for the extension by zero. We obtain a triangle $X' \to X \to X'' = X/X' \to SX'$ of $\mathcal{T}$ where $X' \in \mathcal{U}'$ and $X'' \in \mathcal{U}''$, as can be seen from the restrictions $X'|\,\Lambda_n$ et $X''|\,\Lambda_n$. We finally establish that:

- $\mathcal{U}_{\geq 0} = \mathcal{U}'$ is an aisle in $\mathcal{T}$, and we recover the situation described in §1 with $\mathcal{U}_{<0} = \mathcal{U}''$.

- $\tau_{\geq 0}$ and $\tau_{<0}$ can be chosen such that $\tau_{\geq 0}X = X'$ et $\tau_{<0}X = X''$ with the above notations.

- The functor $X \mapsto (H^n\, X)_{n \in \mathbf{Z}}$ induces an equivalence between the heart

$$\mathcal{A} = \{X \in \mathcal{T} : \operatorname{supp} H^n\, X \subset \Lambda_n,\ \forall n\}$$

  and the direct sum of the categories $\operatorname{mod}\Lambda_n$; in particular, $\mathcal{A}$ generates $\mathcal{T}$.

- If $X \in \mathcal{T}$, the complex $H_0\, X$ of $\operatorname{mod}\Lambda$ has homology groups $H^n\, H_0\, X \xrightarrow{\sim} (H^n\, X|\, \Lambda_n)_0$.

By construction, $\mathcal{U}_{\geq 0}$ is compatible with $\mathcal{V}^{\geq 0}$ and $\mathcal{U}_{<1}$ and $\mathcal{V}^{<1}$. The description of $\tau_{\geq 0}$ and $\tau_{<1}$ shows that $\mathcal{V}^{\geq 0}$ is also compatible with $\mathcal{U}_{\geq 0}$ and $\mathcal{V}^{<1}$ with $\mathcal{U}_{<1}$. In particular, for each $N \in \operatorname{mod}\Lambda$, we have $H^n\, H_n\, N \xrightarrow{\sim} (N|\, \Lambda_n)_0$ for each n and $H^m\, H_n\, N = 0$ if $m \neq n$.

7. We return to the situation of §1. In the following paragraphs, *we always suppose that $\mathcal{U}_{\geq 0}$ is compatible with $\mathcal{V}^{\geq 0}$ and $\mathcal{V}^{\geq 0}$ with $\mathcal{U}_{\geq 0}$.*

Each $N \in \mathcal{B}$ gives rise to a triangle $\tau_{\geq n}N \to N \to \tau_{<n}N \to S\tau_{\geq n}N$ of $\mathcal{T}$. Since $\tau_{\geq n}N \in \mathcal{V}^{\geq 0}$ and $\tau_{<n}N \in \mathcal{V}^{\geq 0}$, the associated cohomology sequence reduces to

$$0 \to H^0\tau_{\geq n}N \to N \to H^0\tau_{<n}N \to H^1\tau_{\geq n}N \to 0$$

and to the isomorphisms $H^i\tau_{<n}N \xrightarrow{\sim} H^{i+1}\tau_{\geq n}N$ $(i > 0)$. Since $\tau_{\geq n}N \in \mathcal{U}_{\geq n}$, it follows that $H^0\tau_{\geq n}N \in \mathcal{B}_{\geq n}$ and $H^1\tau_{\geq n}N \in \mathcal{B}_{\geq n+1}$.

8. Proposition -

a) The subcategory $\mathcal{B}_{\geq n}$ of $\mathcal{B}$ contains with N all the quotients of N.

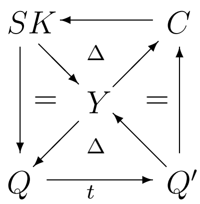
b) For each $N \in \mathcal{B}$, $N_{\geq n} := H^0\tau_{\geq n}N$ is the largest subobject of N belonging to $\mathcal{B}_{\geq n}$.

Indeed, each morphism $N' \to N$ of $\mathcal{B}$ such that $N' \in \mathcal{B}_{\geq n}$ factors through $N_{\geq n} \to N$, which proves b) and a).

9. We say that a morphism $t : N \to N'$ of $\mathcal{B}$ is a *n-quasi-isomorphism* if $\operatorname{Ker} t \in \mathcal{B}_{\geq n}$ and $\operatorname{Coker} t \in \mathcal{B}_{\geq n+1}$. An objet $B \in \mathcal{B}$ is called *n-closed* if for each n-quasi-isomorphism $t : N \to N'$ the map $\operatorname{Hom}(t, B) : \operatorname{Hom}(N', B) \to \operatorname{Hom}(N, B)$ is bijective.

Proposition - *The inclusion of the full subcategory of $\mathcal{B}$ formed by the n-closed objects has the functor $N \mapsto H^0\tau_{<n}N$ ($=$ n-closure of N) as a left adjoint.*

Indeed, the canonical morphism $N \to H^0\tau_{<n}N$ is an n-quasi-isomorphism (§7). Thus it suffices to show that $H^0\tau_{<n}N$ is n-closed. Now let $t : Q \to Q'$ be an n-quasi-isomorphism. Since $\tau_{<n}N \in \mathcal{V}^{\geq 0}$, $H^0\tau_{<n}N$ coïncides with $\tau^{<1}\tau_{<n}N$, and $\operatorname{Hom}(t, H^0\tau_{<n}N)$ identifies with $\operatorname{Hom}(t, \tau_{<n}N)$. The lower triangle of the diagram



where $K = \operatorname{Ker} t$, $C = \operatorname{Coker} t$ and $Y \in \mathcal{U}_{\geq n+1}$ (cf. §2), yields an exact sequence

$$0 = \operatorname{Hom}(Y, \tau_{<n}N) \to \operatorname{Hom}(Q', \tau_{<n}N) \to$$

$$\operatorname{Hom}(Q, \tau_{<n}N) \to \operatorname{Hom}(S^{-1}Y, \tau_{<n}N) = 0 \ ,$$

which proves our assertion.

Dually, one defines "n-coquasi-isomorphisms" and n-co-closed objects in $\mathcal{A}$.

10. Consider the following strictly full subcategories of $\mathcal{A}$ and $\mathcal{B}$:

$$\underline{\mathcal{A}}^n = \{M \in \mathcal{A}^{\geq n} : M \text{ is } (n+1)\text{-co-closed } \}$$

$$\overline{\mathcal{B}}_n = \{N \in \mathcal{B}_{\geq n} : N \text{ is } (n+1)\text{-closed } \}$$

Proposition - *The functors H_n et H^n induce a pair of adjoint functors*

$$H_n : \mathcal{B}_{\geq n} \to \mathcal{A}^{\geq n} \;, \quad H^n : \mathcal{A}^{\geq n} \to \mathcal{B}_{\geq n}$$

and of quasi-inverse equivalences

$$H_n : \overline{\mathcal{B}}_n \xrightarrow{\sim} \underline{\mathcal{A}}^n \;, \quad H^n : \underline{\mathcal{A}}^n \xrightarrow{\sim} \overline{\mathcal{B}}_n.$$

Indeed, we know that $H_n \mathcal{B}_{\geq n} \subset \mathcal{A}^{\geq n}$ et $H^n \mathcal{A}^{\geq n} \subset \mathcal{B}_{\geq n}$. For each $N \in \mathcal{B}_{\geq n}$, we have $H_n N = \tau_{<1} S^{-n} N$; also, $H^n M = \tau^{<1} S^n M$ for each $M \in \mathcal{A}^{\geq n}$. Thus we obtain a functorial isomorphism

$$\mathrm{Hom}\,(H_n N, M) \xrightarrow{\sim} \mathrm{Hom}\,(S^{-n} N, M) \xrightarrow{\sim} \mathrm{Hom}\,(N, S^n M)$$
$$\xrightarrow{\sim} \mathrm{Hom}\,(N, H^n M),$$

which the first assertion.

If $n \in \mathcal{B}_{\geq n}$, we also have $S^n H_n N = \tau_{<n+1} N$, and the adjunction morphism $N \to H^n H_n N$ coïncides with the canonical morphism $N \to H^0 \tau_{<n+1} N$ to the $(n+1)$-closure of N. It follows that $N \to H^n H_n N$ is invertible iff N if $(n+1)$-closed . By duality, one sees that $H_n H^n M \to M$ is invertible iff M is $(n+1)$-co-closed.

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