NOTES ON MINIMAL MODELS

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Abstract. We discuss minimal models for (unbounded, augmented, differential graded) algebras, coalgebras and their strong homotopy versions. We prove the existence and uniqueness of minimal cofibrant models for a class of bigraded algebras. As a corollary, we obtain a characterization of this minimal model in module-theoretic terms. A similar characterization was proved independently by J. Chuang and A. King in [1].

1. Minimal models of A_∞-algebras and of dg coalgebras

Let k be a field. An (augmented) A_∞-algebra A over k is *minimal* if $m_1^A = 0$. If A' is an arbitrary A_∞-algebra, a *minimal model of* A' is an A_∞-quasi-isomorphism

$$A' \to A\,,$$

where A is minimal. It is known (cf. e.g. [2]) that each A_∞-algebra admits a minimal model unique up to (non unique) A_∞-isomorphism.

Let C be a fibrant dg coalgebra (=differential graded augmented coalgebra whose reduction $\overline{C}$ is cocomplete, and which is fibrant in the Quillen model category of such coalgebras, cf. [2]). The complex of primitive elements of C is

$$C_{[1]} = \mathsf{ker}(\overline{C} \to \overline{C} \otimes \overline{C}).$$

The dg coalgebra C is *minimal* if the differential induced in $C_{[1]}$ vanishes. Let C' be an arbitrary dg coalgebra. A *minimal fibrant model of* C' is a weak equivalence

$$C \to C'$$

where C is a minimal fibrant dg coalgebra.

In order to establish existence and uniqueness of minimal fibrant models of dg coalgebras, we link them to minimal models of A_∞-algebras via the bar and cobar constructions: Let C be a fibrant dg coalgebra. Then C is isomorphic to $B_\infty A$ for an A_∞-algebra A unique up to canonical isomorphism. Moreover, we have an isomorphism of complexes between the primitives $C_{[1]}$ and a shift of (A, m_1^A). Thus C is minimal fibrant iff A is minimal. Now let C' be an arbitrary dg coalgebra. Consider the category $\mathcal{C}$ of minimal A_∞-algebra models

$$\Omega C' \to A$$

and the category $\mathcal{D}$ of minimal fibrant coalgebra models

$$C' \to C.$$

We obtain a functor from $\mathcal{C}$ to $\mathcal{D}$ by sending $\Omega C' \to A$ to the composition

$$C' \to B\Omega C' = B_\infty \Omega C' \to B_\infty A.$$

This functor is essentially surjective. Indeed, if $C' \to C$ is a minimal fibrant model, then C is isomorphic to $B_\infty A$ for some minimal A_∞-algebra A and the weak equivalence $C' \to B_\infty A$ lifts to a weak equivalence $B\Omega C' \to B_\infty A$, i.e. to an A_∞-quasi-isomorphism $\Omega C' \to A$. It follows that each dg coalgebra C' admits a minimal fibrant model unique up to (non unique) isomorphism of dg coalgebras.

Date: January 3, 2003; last modified on January 3, 2003.

2. A_∞-coalgebras and cofibrant dg algebras

Let A be a cofibrant dg algebra (=augmented differential graded algebra which is cofibrant in the Quillen model category of such algebras, cf. [2]). It is *mimimal* if the differential induced in the complex of irreducibles $\overline{A}/\overline{A}^2$ vanishes. Let A' be an arbitrary dg algebra. A *minimal cofibrant model* for A' is a quasi-isomorphism

$$A \to A'$$

where A is minimal cofibrant.

Let C be an (augmented) A_∞-coalgebra. It is *minimal* if $m_1^C = 0$. Let C' be an arbitrary A_∞-coalgebra. A *minimal model for* C' is a weak equivalence of A_∞-coalgebras

$$C \to C'$$

where C is minimal. Note that in this case the complex (C, m_1^C) is isomorphic to the graded vector space H^*C' endowed with the zero differential.

An arbitrary A_∞-coalgebra need not have a minimal model: Indeed, we know that there are acyclic dg coalgebras which are not weakly equivalent to k (the bar construction of the augmentation of any non trivial unital algebra yields an example). In the following section, we will show that the bar constructions of certain bigraded algebras do admit minimal models.

Suppose that we have two minimal models $C_1 \to C$ and $C_2 \to C$. Then there is a weak equivalence $C_1 \to C_2$ which makes the obvious triangle commute in the homotopy category of A_∞-coalgebras (which is equivalent to that of cofibrant dg algebras via the functor Ω_∞). Thus a minimal model is unique up to weak equivalence. Note that the 1-component f_1 of such an equivalence f is necessarily an isomorphism (since it is a quasi-isomorphism between complexes with zero differential). Nevertheless, in general, we cannot conclude that f itself is an isomorphism. However, the k-dual f^* of f is a quasi-isomorphism between minimal A_∞-algebras and thus f^* *is* an isomorphism. We will see below that for the bar construction of certain bigraded algebras, the minimal model exists and is unique up to A_∞-isomorphism.

Let us compare minimal models for dg algebras and A_∞-coalgebras following the lines of the preceding section: Let A' be a dg algebra. Let $\mathcal{D}$ be the category of minimal cofibrant models

$$A \to A'$$

and $\mathcal{C}$ the category of minimal A_∞-coalgebra models

$$C \to BA'.$$

We obtain a functor from $\mathcal{C}$ to $\mathcal{D}$ by sending $C \to BA'$ to the composition

$$\Omega_\infty C \to \Omega_\infty BA' = \Omega BA' \to A'.$$

As in the preceding section, we see that this functor is essentially surjective. This will allow us to prove existence and uniqueness of minimal cofibrant models for certain bigraded algebras, cf. below.

3. Minimal cofibrant models for bigraded algebras

Lemma. *Let C_1 and C_2 be A_∞-coalgebras admitting additional gradings such that their reductions are graded in strictly positive degrees. Assume that $f : C_1 \to C_2$ is an A_∞-morphism respecting the additional grading and such that f_1 is a quasi-isomorphism (respectively, an isomorphism). Then f is a weak equivalence (respectively, an isomorphism).*

Proof. Let $A_1 = \Omega_\infty C_1$ and for each $p \geq 0$, let F_pA_1 be the ideal spanned by all $(S^{-1}\overline{C_1})^{\otimes n}$, $n \geq p$. Fix $p \geq 1$. Then F_pA_1 is graded in degrees $\geq p$. So the projection

$$A_1 \to A_1/F_pA_1$$

induces an isomorphism in degrees $< p$. Moreover, by assumption, the morphism

$$A_1/F_pA_1 \to A_2/F_pA_2$$

induces isomorphisms in homology (respectively, isomorphisms). Thus the morphism $A_1 \to A_2$ induced by f induces isomorphisms in homology (resp., isomorphisms) in each (second) degree $< p$. Since p is arbitrary, it follows that f induces a quasi-isomorphism (resp., an isomorphism) $A_1 \to A_2$ and thus f is a weak equivalence (resp., an isomorphism). $\surd$

We consider an augmented algebra A concentrated in (cohomological) degree 0 and endowed with an additional grading such that $\overline{A} = A/k$ is concentrated in degrees > 0. Then the bar construction $BA = T^cS\overline{A}$ has an additional grading and its reduction is graded in strictly positive degrees.

Proposition. *The A_∞-coalgebra BA admits a minimal model unique up to A_∞-isomorphism.*

Proof. Uniqueness follows from the above lemma. For the existence, we use a variant of the perturbation lemma. Choose a split short exact sequence of complexes

$$C \underset{r'}{\overset{i'}{\rightleftarrows}} \overline{BA} \underset{s'}{\overset{p'}{\rightleftarrows}} U$$

which is compatible with the additional grading and such that C has vanishing differential and U is contractile. Choose a contracting homotopy h'' of U and let $h' = s'h''p'$. Then we obtain a contraction

$$TS^{-1}C \underset{r}{\overset{i}{\rightleftarrows}} TS^{-1}\overline{BA}$$

where i and r are the algebra morphisms induced by i' and r' and both free algebras are endowed with the differentials coming from their generating complexes. So the cokernel of i is contractile and we obtain a contraction by extending h' to an $\mathbf{1}$-ir-derivation h. Now we perturb the differential d on

$$TS^{-1}\overline{BA}$$

by the contribution ∂ coming from the comultiplication on BA. We claim that the operator

$$h\partial : TS^{-1}\overline{BA} \to TS^{-1}\overline{BA}$$

is locally nilpotent. For this, we note that $TS^{-1}\overline{BA}$ is a sum of tensor powers $\overline{A}^{\otimes i}$, where we forget all parentheses and only consider the second grading. The operator ∂ takes $\overline{A}^{\otimes i}$ to a sum of copies of $\overline{A}^{\otimes i}$ for the *same* power i. Indeed this operator merely inserts parentheses (in many ways) or yields zero when no more parentheses can be inserted. The operator h is a sum of operators

$$\mathbf{1}^{\otimes k} \otimes h' \otimes (ir)^{\otimes l}.$$

Therefore it takes $\overline{A}^{\otimes i}$ to a sum of copies of $\overline{A}^{\otimes(i+1)}$ (h' increases the number of tensor factors by 1 and $\mathbf{1}^{\otimes k}$ and $(ir)^{\otimes l}$ leave it unchanged). So the operator $h\partial$ takes $\overline{A}^{\otimes i}$ to a sum of copies of $\overline{A}^{\otimes(i+1)}$. Therefore the operator $(h\partial)^N$ takes $\overline{A}^{\otimes i}$ to a sum of copies of $\overline{A}^{\otimes(i+N)}$. This is concentrated in degrees $\geq i+N$. So $h\partial$

is locally nilpotent. This is all that is needed to make the perturbation lemma work. $\surd$

4. A characterization of the minimal model

Let A be a positively graded augmented connected associative algebra. Let C be a minimal A_∞-coalgebra endowed with an additional grading. According to the above proposition, the bar construction BA admits a minimal model, so $H_*(BA)$ admits a A_∞-coalgebra structure unique up to (non canonical) isomorphism, by the above lemma.

Proposition. *The following are equivalent*

(i) *There is an isomorphism of A_∞-coalgebras $C \xrightarrow{\sim} H_*(BA)$ compatible with the additional grading.*
(ii) *There is a generalized twisting cochain $\tau : C \to A$ compatible with the additional grading such that $C \otimes_\tau A \to k$ is the minimal resolution of the A-module k.*

Proof. Let us show that (i) implies (ii). By the preceding section, the dg coalgebra BA admits a minimal A_∞-model. So there is an A_∞-coalgebra morphism

$$f : H_*(BA) \to BA$$

which is a weak equivalence and whose first component is an injection inducing the identity in homology. If we compose it with the canonical twisting cochain $\tau_0 : BA \to A$, we obtain a generalized twisting cochain

$$\tau : H_*(BA) \to A.$$

We have a canonical morphism of complexes

$$H_*(BA) \otimes_\tau A \to BA \otimes_{\tau_0} A$$

and it fits into an exact sequence

$$0 \to H_*(BA) \otimes_\tau A \to BA \otimes_{\tau_0} A \to V \otimes A \to 0$$

where V is the cokernel of $f_1 : H_*(BA) \to BA$ but the differential on the right most term is not, in general, that induced from V. However, $V \otimes A$ is a complex of free A-modules, right bounded and its image under $? \otimes_A k$ is contractile. It follows that $V \otimes A$ is contractile. So $H_*(BA) \otimes_\tau A$ is indeed the minimal resolution of k.

Let us show that (ii) implies (i). The composition of the generalized twisting cochain $\tau : C \to A$ with the canonical A_∞-morphism $B\Omega_\infty C \to C$ yields a classical twisting cochain $B\Omega_\infty C \to A$ and thus a dg coalgebra morphism $B\Omega_\infty C \to BA$ and an A_∞-coalgebra morphism $f : C \to BA$. This yields a morphism of complexes

$$C \otimes_\tau A \to BA \otimes_{\tau_0} A$$

compatible with the maps to k. Here both complexes are resolutions, so by applying $\otimes_A k$ we find that f induces an isomorphism $C \to H_*(BA)$. By the preceding section, f is a weak equivalence and C is the minimal model of BA (unique up to isomorphism by the preceding section). $\surd$

References

[1] J. Chuang, A. King, Preprint in preparation.
[2] K. Lefèvre, *Sur les A_∞-catégories*, Thèse, Université Paris 7, Janvier 2003. Available at `http://www.math.jussieu.fr/~lefevre/publ.html`