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HOLONOMY GROUPOIDS OF SINGULAR FOLIATIONS

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Abstract

We give a new construction of Lie groupoids which is particularly well adapted to the generalization of holonomy groupoids to singular foliations. Given a family of local Lie groupoids on open sets of a smooth manifold M, satisfying some hypothesis, we construct a Lie groupoid which contains the whole family. This construction involves a new way of considering (local) Morita equivalences, not only as equivalence relations but also as generalized isomorphisms. In particular we prove that almost injective Lie algebroids are integrable.

Introduction

To any regular foliation is associated its holonomy groupoid. This groupoid was introduced in the topological context by C. Ehresmann [14], and later the differentiable case was done independently by J. Pradines and H.E. Winkelnkemper [25, 32]. It is the smallest Lie groupoid whose orbits are the leaves of the foliation [23], and every regular foliation is given by the action of its holonomy groupoid on its space of units.

The holonomy Lie groupoid of a foliation has been the starting point of several studies. A. Connes used it to define the Von Neumann algebra and the C^*-algebra of a foliation [8]; A. Haefliger pointed out that, considered up to a suitable equivalence, the holonomy groupoid defines the transverse structure of the foliation [16]; it plays a crucial role in the index theory for foliations which was developed by A. Connes and G. Skandalis [10] as well as for the definition of the transverse fundamental class of a foliation [9]. See [7] for a complete bibliography on this subject.

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On the other hand symplectic geometry and control theory have focused attention on foliations generated by families of vector fields. These foliations were introduced independently by P. Stefan and H.J. Sussman and are usually called Stefan foliations [28, 29]. By definition, a Stefan foliation $\mathcal{F}$ on a manifold M is the partition of M by the integral manifolds of a completely integrable differentiable distribution on M. In such foliations the dimension of the leaves can change. Our study concerns the Stefan foliations for which the union of leaves of maximal dimension is a dense open subset of the underlying manifold, these foliations are called *almost regular*.

There are plenty of examples of Stefan foliations: orbits of a Lie group action, faces of a manifold with corners, symplectic foliation of a Poisson manifold, etc. Thus it is a natural question to extend the notion of holonomy Lie groupoid to such foliations, it is the first step toward getting transverse invariants, to compute a C^*-algebra or to do index theory. This paper is devoted to the study of this question. Precisely the question we answer is: given an almost regular foliation how can one find a Lie groupoid which induces the foliation on its space of units and which is the smallest in some way?

The first idea is to define holonomy for Stefan foliations. This has already been done under some hypothesis by M. Baeur and P. Dazord [2, 12]. But this process involves loss of information. For example the holonomy group of any singular leaf reduced to a single point will always vanish and so we will miss information to compute a Lie groupoid. In [25], J. Pradines constructs the holonomy groupoid from the local transverse isomorphisms, that is the diffeomorphisms between small transversals to the foliation. This method also fails in the singular case because we lose local triviality of the foliation as well as the notion of small transversals.

We are looking at Lie groupoids which are "the smallest in some way". In other word we do not want unnecessary isotropy. This leads us to consider a special case of Lie groupoids namely the *quasi-graphoids*. A quasi-graphoid has the property that the foliation it induces on its units space is almost regular. Furthermore, restricted to the regular part, it is isomorphic to the usual holonomy groupoid of the corresponding regular foliation. So these groupoids are good candidates to be holonomy groupoids of almost regular foliations.

On the other hand, we remark that the problem of finding a Lie groupoid associated to a singular foliation $\mathcal{F}$ on a manifold M has two aspects:

The local problem which is to find for each point x of M a (local) Lie groupoid associated to the foliation induced by $\mathcal{F}$ on a neighborhood of x. This problem is obvious for the regular foliations. Since the equivalence relation *to be on the same leaf* is locally regular in this case, one can just take the graph of this regular equivalence relation over each distinguished open set. When the (singular) foliation is defined by an *almost injective* Lie algebroid over M, that is a Lie algebroid whose anchor is injective when restricted to a dense open subset of M, this problem reduced to the local integration of the algebroid. This has already been done in detail by the author in [13].

The global problem which consists in finding a Lie groupoid on M made from these (local) groupoids. Even in the regular case, this aspect is more difficult to understand because holonomy is involved.

However, the study of the regular case gives rise to the following remark. Suppose that $(M, \mathcal{F})$ is a regular foliation and take a cover of M by distinguished open sets. All the open sets of this cover are equipped with a simple foliation and these foliations fit together to give $(M, \mathcal{F})$. In the same way, to each of these open sets, the graph of the corresponding regular equivalence relation is associated and these Lie groupoids fit together to generate the holonomy groupoid of $(M, \mathcal{F})$, denoted $\mathrm{Hol}(M, \mathcal{F})$. The first step was to understand how these groupoids fit together. The conclusion of this study is the following:

If γ is an element of $\mathrm{Hol}(M, \mathcal{F})$ whose source is x and range is y, let U_0 (resp. U_1) be a distinguished neighborhood of x (resp. y), T_0 (resp. T_1) be a small transversal passing through x (resp. y) and G_0 (resp. G_1) the holonomy groupoid of the simple foliation $(U_0, \mathcal{F}|_{U_0})$ (resp. $(U_1, \mathcal{F}|_{U_1})$). Then the data of γ is equivalent to:

- The holonomy class of a path c tangent to the foliation, starting at x and ending at y (usual construction [32]).
- The germ at x of a diffeomorphism from T_0 to T_1 (J. Pradines' construction [25]).
- The *germ* of a Morita equivalence between G_0 and G_1 (defined here in Section 3).

The transverse isomorphism mentioned above will just be the holonomy isomorphism associated to the path c. Morita equivalence between two groupoids was defined by M. Hilsum and G. Skandalis [17] to be generalized isomorphisms between the orbit spaces of the groupoids. Here, by construction, the orbit space of G_i is naturally isomorphic to T_i, $i = 0, 1$.

This new approach will be very useful, because even if the notion of holonomy classes of paths tangent to the foliation and transverse isomorphisms can't be defined for almost regular foliations, the notion of Morita equivalence remains.

Finally the construction of a Lie holonomy groupoid of an almost regular foliation can be decomposed in two steps. The first step is a construction of a *generalized atlas* of the foliation. This "atlas" will be made up of local Lie groupoids. The existence of such an atlas will replace the local triviality of the foliation. The second step is the construction of a Lie groupoid associated to such an atlas using (local) Morita equivalences.

This paper is organized as follows:

In Section 1 we recall briefly the definition of Stefan foliations.

In Section 2 we study *quasi-graphoids* and we define the *general atlases of a foliation*.

In Section 3 we first extend the notion of Morita equivalence to local groupoids. Next we show that given a generalized atlas, the set of local Morita equivalence between elements of this atlas behaves almost like a pseudo-group of local diffeomorphisms. We finish by defining the germ of a Morita equivalence, and we construct the groupoids of the germs of the elements of the pseudo-group.

In the last section we apply the obtained results to almost regular foliations, and we give several examples.

I want to address special thanks to Georges Skandalis for his relevant suggestions.

1. Singular foliations

A generalized distribution $\mathcal{D} = \cup_{x \in M} \mathcal{D}_x \subset TM$ on a smooth manifold M is *smooth* when for each $(x, v) \in \mathcal{D}_x$, there exists a smooth local tangent vector field X on M such that:

$$x \in \mathrm{Dom}(X),\ X(x) = (x, v) \text{ and } \forall y \in \mathrm{Dom}(X),\ X(y) \in \mathcal{D}_y.$$

It is *integrable* when for all $x \in M$, there exists an *integral manifold* passing through x, that is, an immersed submanifold F_x of M such that x belongs to F_x and $T_yF_y = \mathcal{D}_y$ for all $y \in F_x$.

The maximal connected integral manifolds of an integrable differentiable distribution $\mathcal{D}$ on the manifold M define a partition $\mathcal{F}$ of M. Such a partition is what we call a *Stefan foliation* on the manifold M [28]. The elements of the partition are called the *leaves*. Contrary to the regular case the dimension of the leaves may change. A leaf which is not of dimension equal to $\max_{x\in M} (\dim \mathcal{D}_x)$ is *singular* and it is *regular* otherwise. Because of the rank lower semi-continuity the dimension of the leaves increases around a singular leaf.

When the union M_0 of the leaves of maximal dimension is a dense open subset of M we say that the Stefan foliation $(M, \mathcal{F})$ is *almost regular*. In this case, the foliation $\mathcal{F}$ restricted to M_0 is a regular foliation denoted by $(M_0, \mathcal{F}_0)$ and called the *maximal regular subfoliation of* $(M, \mathcal{F})$.

Examples.

1. A regular foliation is a Stefan foliation.

2. The partition of a manifold M into the orbits of a differentiable action of a Lie group on M is a Stefan foliation.

3. The integral manifolds of an involutive family of tangent vector fields on a manifold M which is locally of finite type are the leaves of a Stefan foliation [29].

4. Let N be a manifold equipped with two regular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ where $\mathcal{F}_1$ is a subfoliation of $\mathcal{F}_2$ (that is every leaf of $\mathcal{F}_1$ lies on a leaf of $\mathcal{F}_2$). Then $\mathcal{F}_1 \times \{0\}$ and $\mathcal{F}_2 \times \{t\}$ for $t \neq 0$ defines an almost regular foliation on $M \times \mathbb{R}$.

5. We recall that an *quartering* on a manifold M is the data of a finite family $\{V_i,\ i \in I\}$ of codimension 1 submanifolds of M such that for all $J \subset I$, the family of the inclusions of the V_j, $j \in J$ is transverse [19]. The faces of a quartering on M are the leaves of an almost regular foliation.

6. Recall that a *Lie algebroid* $\mathcal{A} = (p : \mathcal{A} \to TM, [\ ,\ ]_{\mathcal{A}})$ on a smooth manifold M is a vector bundle $\mathcal{A} \to M$ equipped with a bracket $[\ ,\ ]_{\mathcal{A}} : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ on the module of sections of $\mathcal{A}$ together with a morphism of fiber bundles $p : \mathcal{A} \to TM$ from $\mathcal{A}$ to the tangent bundle TM of M called the *anchor*, such that:

i) The bracket $[\ ,\ ]_{\mathcal{A}}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies to the Jacobi identity.

ii) $[X, fY]_{\mathcal{A}} = f[X,Y]_{\mathcal{A}} + p(X)(f)Y$ for all $X,\ Y \in \Gamma(\mathcal{A})$ and f a smooth function of M.

iii) $p([X,Y]_{\mathcal{A}}) = [p(X), p(Y)]$ for all $X,\ Y \in \Gamma(\mathcal{A})$.

The distribution defined by $p(\mathcal{A})$ on M is involutive and locally of finite type, so it is integrable [29]. The corresponding Stefan foliation on M is said to be *defined by the Lie algebroid* $\mathcal{A}$.

Suppose now that $G \underset{r}{\overset{s}{\rightrightarrows}} M$ is a Lie groupoid on the manifold M having s as source map and r as range map.

Recall that if x belongs to M the *orbit* of G passing through x is the set

$$\{y \in M \mid \exists \gamma \in G \text{ such that } s(\gamma) = x \text{ and } r(\gamma) = y\} = r(s^{-1}(x)).$$

We denote by $\mathcal{A}G$ the Lie algebroid of G and by $\mathcal{F}_G$ the corresponding foliation on M. The leaves of the Stefan foliation $\mathcal{F}_G$ are the connected components of the orbits of G. Thus the action of the Lie groupoid G on its space of units M defines a Stefan foliation.

2. Quasi-graphoids

These groupoids have been introduced independently by J. Renault [27] who called them essentially principal groupoids and by B. Bigonnet [3] under the name of quasi-graphoids. We will see that quasi-graphoids have the good properties to be holonomy groupoids of almost regular foliations. We recall their definition, and we complete B. Bigonnet's work by the study of their properties which leads to a geometric justification of this choice.

2.1 Definition and properties

Recall that a manifold S equipped with two submersions a and b onto a manifold B is called a *graph over* B. If $(a', b') : S' \to B \times B$ is another graph over B, a *morphism of graphs* from S to S' is a smooth map $\varphi : S \to S'$ such that $a' \circ \varphi = a$ and $b' \circ f = b$ [26].

Definition-Proposition 1 ([3])**.** *Let* $G \underset{r}{\overset{s}{\rightrightarrows}} G^{(0)}$ *be a Lie groupoid over* $G^{(0)}$*,* s *its domain map,* r *its range map and* $u : G^{(0)} \longrightarrow G$ *the units inclusion. The two following assertions are equivalent:*

1. *If* $\nu : V \longrightarrow G$ *is a local section of* s *then*

$$r \circ \nu = 1_V \text{ if and only if } \nu = u|_V.$$

2. *If* N *is a manifold,* f *and* g *are two smooth maps from* N *to* G *such that:*

   i) $s \circ f = s \circ g$ *and* $r \circ f = r \circ g$*.*

   ii) *One of the maps* $s \circ f$ *and* $r \circ f$ *is a submersion.*

   Then $f = g$*.*

A quasi-graphoid is a Lie groupoid which satisfies these equivalent properties.

In other words G is a quasi-graphoid when for any graph S over $G^{(0)}$ there exists at most one morphism of graphs from S to G.

Examples.

1. The holonomy groupoid of a regular foliation is a quasi-graphoid.

2. If $H \times M \to M$ is a differentiable action of a Lie group H on a manifold M such that there is a saturated dense open subset of M over which the action is free, the groupoid of the action is a quasi-graphoid.

One can remark that the algebraic structure of a quasi-graphoid is fixed by the source and range maps. More precisely if $G \underset{r}{\overset{s}{\rightrightarrows}} G^{(0)}$ is a quasi-graphoid then:

– The inverse map is the unique smooth map $i : G \to G$ such that $s \circ i = r$ and $r \circ i = s$.

– The product is the unique smooth map

$$p : G^{(2)} = \{(\gamma, \eta) \in G \times G \mid s(\gamma) = r(\eta)\} \to G$$

such that $s \circ p = s \circ \mathrm{pr}_2$ and $r \circ p = r \circ \mathrm{pr}_1$ where pr_1 (resp. pr_2) is the projection onto the first (resp. second) factor.

In the same way, let $G \rightrightarrows B$ and $H \rightrightarrows B$ be two Lie groupoids having the same units and $\varphi : G \to H$ a morphism of graphs. If H is a quasi-graphoid then a morphism of graphs from G to H has to be a morphism of Lie groupoids.

An other important property of quasi-graphoids is that a *local morphism of graphs* to a quasi-graphoid is always extendible.

Proposition 1. *Let $G \overset{s_G}{\underset{r_G}{\rightrightarrows}} B$ and $H \overset{s_H}{\underset{r_H}{\rightrightarrows}} B$ be two Lie groupoids having the same units space and $\varphi : V \to G$ a smooth map from a neighborhood of B in H to G which satisfies $s_H \circ \varphi = s_G$ and $r_H \circ \varphi = r_G$.*

If H is s-connected and G is a quasi-graphoid then there exists a unique morphism of Lie groupoid $\widetilde{\varphi} : H \to G$ such that the restriction of $\widetilde{\varphi}$ to V is equal to φ.

Proof. We have already noticed that if φ extends to a smooth morphism of graphs $\widetilde{\varphi}$ then $\widetilde{\varphi}$ will be a morphism of Lie groupoid. The definition of quasi-graphoids implies that such a morphism must be unique. So it remains to show that φ extends to a smooth morphism of graphs.

Recall that a local section ν of s_H is said to be *admissible* when $r_H \circ \nu$ is a local diffeomorphism of the space of units. When ν_1 and ν_2 are two local admissible sections such that $r_H(\nu_2(\mathrm{dom}(\nu_2))) \subset \mathrm{dom}(\nu_1)$ we denote by $\nu_1 \cdot \nu_2$ the local admissible section which sends $x \in \mathrm{dom}(\nu_2)$ onto the product $\nu_1(r_H(\nu_2(x))) \cdot \nu_2(x)$.

If γ is an element of H then, because H is s-connected, one can find a neighborhood O of $s_H(\gamma)$ in B and local admissible sections $\nu_1, \dots, \nu_n$ such that the image of each ν_i is a subset of V, the map $\nu_1 \cdot \nu_2 \cdots \nu_n$ is defined on O and $\nu_1 \cdot \nu_2 \cdots \nu_n(s_H(\gamma)) = \gamma$. Thus for each ν_i, the map $\varphi \circ \nu_i$ is an admissible local section of G and the product $\varphi(\nu_1) \cdot \varphi(\nu_2) \cdots \varphi(\nu_n)$ is defined.

Let U be a neighborhood of $s_H(\gamma)$ in H such that $U \subset V$ and $U \cap B = O$. Then the following map

$$\begin{array}{rccc} h: & U & \longrightarrow & H \\ & \eta & \mapsto & \nu_1 \cdot \nu_2 \cdots \nu_n(r_H(\eta)) \cdot \eta \end{array}$$

gives rise to a diffeomorphism from U onto a neighborhood W of γ in H. We define the morphism of graphs

$$\begin{array}{rccc} \widetilde{\varphi}_W : & W & \longrightarrow & G \\ & h(\eta) & \mapsto & \varphi(\nu_1) \cdot \varphi(\nu_2) \cdots \varphi(\nu_n)(r_H(\eta)) \cdot \eta \end{array} .$$

By repeating this process we obtain an open covering $\{W_i\}_{i\in I}$ of H and for each $i \in I$, a morphism of graphs $\widetilde{\varphi}_i : W_i \to G$. Because G is a quasi-graphoid, $\widetilde{\varphi}_i$ and $\widetilde{\varphi}_j$ must coincide on $W_i \cap W_j$. Thus the map $\widetilde{\varphi} : H \to G$ defined by $\widetilde{\varphi} = \widetilde{\varphi}_i$ on W_i is a morphism of graphs. By uniqueness the restriction of $\widetilde{\varphi}$ to V is equal to φ. q.e.d.

Notice that if φ is a diffeomorphism onto its image and if G and H are two s-connected quasi-graphoids then $\widetilde{\varphi}$ is an isomorphism of Lie groupoids.

Corollary 1. *Two s-connected quasi-graphoids having the same space of units are isomorphic if and only if their Lie algebroids are isomorphic.*

Proof. Suppose that G and H are two s-connected quasi-graphoids having the same space of units B such that their Lie algebroids $\mathcal{A}G$ and $\mathcal{A}H$ are isomorphic. Then using the exponential map [18, Prop. 4.12, p. 136], one can easily construct a covering $\{W_i\}_{i\in I}$ of B in G and for all $i \in I$ a morphism of graphs $\varphi_i : W_i \to H$ which integrates the isomorphism from $\mathcal{A}G$ onto $\mathcal{A}H$. Because H is a quasi-graphoid φ_i and φ_j must coincide on $W_i \cap W_j$. Thus we obtain in this way a local isomorphism of graphs from G onto H. Using Proposition 1 we conclude that G and H are isomorphic. q.e.d.

The next proposition shows that quasi-graphoids are closely related to almost regular foliations.

Proposition 2. *Let $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ be a quasi-graphoid, $(G^{(0)}, \mathcal{F}_G)$ the singular foliation induced by G on $G^{(0)}$ and $\mathcal{A}G = (p : \mathcal{A}G \to TG^{(0)}, [\ ,\ ])$ the Lie algebroid of G.*

The following assertions hold:

1. *The anchor p is almost injective, that is the set*

$$G_0^{(0)} = \{x \in G^{(0)} \mid p_x : \mathcal{A}G_x \to T_xG^{(0)} \text{ is infective } \}$$

   is a dense open subset of $G^{(0)}$.

2. *The foliation $(G^{(0)}, \mathcal{F}_G)$ is almost regular.*

3. *The restriction of the s-connected component G^c of G to the regular part $G_0^{(0)}$ is the holonomy groupoid of the maximal regular subfoliation of $\mathcal{F}_G$ denoted by $\mathrm{Hol}(G_0^{(0)}, \mathcal{F}_G|_{G_0^{(0)}})$.*

Proof. 1. The rank lower semi-continuity implies that $G_0^{(0)}$ is an open subset of $G^{(0)}$.

Let U be an open set of $G^{(0)}$, the rank lower semi-continuity implies that there is a non empty open subset V of U over which the rank of p is maximal. We can consider the fiber bundle $\ker(p|_V)$ over V. We have to check that $\ker(p|_V) = V \times \{0\}$.

Let X be a local section of $\ker(p|_V)$. According to [18] (prop. 4.1 p 126), for all $x_0 \in V$, there is an open neighborhood $V_{x_0} \subset V$ of x_0, an $\varepsilon > 0$ and a (unique) smooth family of local sections $\mathrm{Exp}(tX)$ of s defined on V_{x_0} for $|t| < \varepsilon$ such that $\{r \circ \mathrm{Exp}(tX) : V_{x_0} \to M\}$ is the one parameter group of local transformations associated to the local vector field $p(X)$ over V_{x_0} and

$$\frac{d}{dt}\mathrm{Exp}(tX)|_0 = X.$$

Since $p(X) = 0$ we get that $r \circ \mathrm{Exp}(tX) = 1_{V_{x_0}}$ for all t such that $|t| < \varepsilon$. Then, G being a quasi-graphoid, we obtain that for all such t, $\mathrm{Exp}(tX) = u|_{V_{x_0}}$, where u is the unit map of G. So $X = 0$.

2. An immediate consequence of p being almost injective is that the image by p of the restriction of $\mathcal{A}G$ to $G_0^{(0)}$ is an involutive distribution on $G_0^{(0)}$. So it induces a regular foliation. In other words the restriction of $\mathcal{F}_G$ to $G_0^{(0)}$ is regular.

3. The Lie algebroid of the restriction of G^c to $G_0^{(0)}$ is equal to the restriction of $\mathcal{A}G$ to $G_0^{(0)}$ and so it is isomorphic to the tangent space of the regular foliation $(G_0^{(0)}, \mathcal{F}_G|_{G_0^{(0)}})$. Furthermore the tangent space of the regular foliation $(G_0^{(0)}, \mathcal{F}_G|_{G_0^{(0)}})$ is the Lie algebroid of the holonomy groupoid of $(G_0^{(0)}, \mathcal{F}_G|_{G_0^{(0)}})$. Finally, the restriction of G^c to $G_0^{(0)}$ and $\mathrm{Hol}(G_0^{(0)}, \mathcal{F}_G|_{G_0^{(0)}})$ are two s-connected quasi-graphoids having isomorphic Lie algebroids, so by the previous corollary they are isomorphic as well. q.e.d.

2.2 Local quasi-graphoids

We are going to present here the local version of quasi-graphoids. First, we need to recall the notion of *local Lie groupoid* which is due to Van Est [15].

A *local Lie groupoid* is given by:

– Two smooth manifolds $\mathcal{L}$ and $\mathcal{L}^{(0)}$ and an embedding $u : \mathcal{L}^{(0)} \to \mathcal{L}$. The manifold $\mathcal{L}^{(0)}$ must be Hausdorff, it is called the *set of units*. We usually identify $\mathcal{L}^{(0)}$ with its image by u in $\mathcal{L}$.

– Two surjective submersions: $\mathcal{L} \overset{s}{\underset{r}{\rightrightarrows}} \mathcal{L}^{(0)}$ called the *range* and *source* map, they must satisfy $s \circ u = r \circ u = id$.

– A smooth involution

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{L} \\ \gamma & \mapsto & \gamma^{-1} \end{array}$$

called the *inverse* map. It satisfies $s(\gamma^{-1}) = r(\gamma)$ for $\gamma \in \mathcal{L}$.

– An open subset $\mathcal{D}^2\mathcal{L}$ of $\mathcal{L}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{L} \times \mathcal{L} \mid s(\gamma_1) = r(\gamma_2)\}$ called the *set of composable pairs* and a smooth *local product*:

$$\begin{array}{ccc} \mathcal{D}^2\mathcal{L} & \longrightarrow & \mathcal{L} \\ (\gamma_1, \gamma_2) & \mapsto & \gamma_1 \cdot \gamma_2 \end{array}.$$

The following properties must be fulfilled:

$s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$ and $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ when the product $\gamma_1 \cdot \gamma_2$ is defined.

For all $\gamma \in \mathcal{L}$ the products $r(\gamma) \cdot \gamma$, $\gamma \cdot s(\gamma)$, $\gamma \cdot \gamma^{-1}$ and $\gamma^{-1} \cdot \gamma$ are defined and respectively equal to γ, γ, $r(\gamma)$ and $s(\gamma)$.

If the product $\gamma_1 \cdot \gamma_2$ is defined then so is the product $\gamma_2^{-1} \cdot \gamma_1^{-1}$ and $(\gamma_1 \cdot \gamma_2)^{-1} = \gamma_2^{-1} \cdot \gamma_1^{-1}$.

If the products $\gamma_1 \cdot \gamma_2$, $\gamma_2 \cdot \gamma_3$ and $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$ are defined then so is the product $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ and $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$.

The only difference between groupoids and local groupoids is that in the second case the condition $s(\gamma_1) = r(\gamma_2)$ is necessary for the product $\gamma_1.\gamma_2$ to exist but not sufficient. The product is defined as soon as γ_1 and γ_2 are “small enough”, that is “close enough” from units.

Notations Let $\mathcal{L} \overset{s}{\underset{r}{\rightrightarrows}} \mathcal{L}^{(0)}$ be a (local) Lie groupoid and denote by $p : \mathcal{D}^2\mathcal{L} \to \mathcal{L}$ the product of $\mathcal{L}$.

– If O is a subset of $\mathcal{L}^{(0)}$ we denote by

$$\mathcal{L}_O = s^{-1}(O),\ \mathcal{L}^O = r^{-1}(O) \ \text{ and } \ \mathcal{L}|_O = \mathcal{L}_O^O = s^{-1}(O) \cap r^{-1}(O).$$

– If V is an open subset of $\mathcal{L}$ such that $V \cap \mathcal{L}^{(0)} = \mathcal{L}_V^{(0)} \neq \emptyset$, we denote by V^+ the set $\{\gamma \in V \mid \gamma^{-1} \in V\}$.

We define $\mathcal{L}(V) = V^+ \cap s^{-1}(\mathcal{L}_V^{(0)}) \cap r^{-1}(\mathcal{L}_V^{(0)})$ and the set of composable pairs of $\mathcal{L}(V)$ by $\mathcal{D}^2\mathcal{L}(V) = p^{-1}(\mathcal{L}(V)) \cap \mathcal{L}(V) \times \mathcal{L}(V)$. Then we define the *restriction of* $\mathcal{L}$ *to* V to be the local Lie groupoid $\mathcal{L}(V) \underset{r}{\overset{s}{\rightrightarrows}} \mathcal{L}_V^{(0)}$.

So to any open subset V of $\mathcal{L}$ which encounter $\mathcal{L}^{(0)}$ on $\mathcal{L}_V^{(0)}$ we thereby associate an open sub local Lie groupoid of $\mathcal{L}$ admitting $\mathcal{L}_V^{(0)}$ as space of units.

In particular if O is an open subset of $\mathcal{L}^{(0)}$, $\mathcal{L}|_O \underset{r}{\overset{s}{\rightrightarrows}} O$ is a local Lie groupoid also called the *restriction of* $\mathcal{L}$ *to* O when there is no risk of confusion.

As for Lie groupoid there is a natural notion of *graphs morphism* and *local graphs morphism* between two local Lie groupoids having the same space of units.

In particular two (local) Lie groupoids $\mathcal{L} \underset{r}{\overset{s}{\rightrightarrows}} \mathcal{L}^{(0)}$ and $\mathcal{L}' \underset{r'}{\overset{s'}{\rightrightarrows}} \mathcal{L}'^{(0)}$ such that $\mathcal{L}^{(0)} \cap \mathcal{L}'^{(0)} \neq \emptyset$ are *locally isomorphic as graphs* if there is a neighborhood V of $\mathcal{L}^{(0)} \cap \mathcal{L}'^{(0)}$ in $\mathcal{L}$, a neighborhood V' of $\mathcal{L}^{(0)} \cap \mathcal{L}'^{(0)}$ in $\mathcal{L}'$ and a diffeomorphism $\varphi : V \to V'$ such that $s' \circ \varphi = s$ and $r' \circ \varphi = r$. This also means that there is a restriction of $\mathcal{L}$ containing $\mathcal{L}^{(0)} \cap \mathcal{L}'^{(0)}$ and a restriction of $\mathcal{L}'$ containing $\mathcal{L}^{(0)} \cap \mathcal{L}'^{(0)}$ which are two isomorphic graphs.

Definition 1. A local quasi-graphoid is a local Lie groupoid $\mathcal{L} \rightrightarrows \mathcal{L}^{(0)}$, having the property that for all manifolds S equipped with two submersions a and b onto $\mathcal{L}^{(0)}$, there exists at most one morphism of graphs from S to $\mathcal{L}$.

Remark 1.

1. Let $\mathcal{L} \underset{r}{\overset{s}{\rightrightarrows}} \mathcal{L}^{(0)}$ be a local quasi-graphoid. If D is an open subset of $\mathcal{L}^{(2)}$ there exists at most one smooth map $p : D \to \mathcal{L}$ such that $s \circ p = s \circ \mathrm{pr}_2$ and $r \circ p = r \circ \mathrm{pr}_1$. So there exists a maximal open subset $\mathcal{D}^2_{max}\mathcal{L}$ of $\mathcal{L}^{(2)}$ which is the domain of a smooth map p which satisfies $s \circ p = s \circ \mathrm{pr}_2$ and $r \circ p = r \circ \mathrm{pr}_1$. We call $\mathcal{D}^2_{\max}\mathcal{L}$ the *maximal set of composable pairs.*

We shall always consider local quasi-graphoids equipped with the maximal set of composable pairs, and thus we shall not have to specify what is this set as soon as we know that a local product is defined on a neighborhood of $\{(\gamma, \gamma^{-1}) \mid \gamma \in \mathcal{L}\}$ in $\mathcal{L}^{(2)}$.

2. As for quasi-graphoids a (local) morphism of graphs from a local Lie groupoid to a local quasi-graphoid is a (local) morphism of groupoid.

3. Local quasi-graphoids are stable under restriction.

4. If $\mathcal{L} \overset{s}{\underset{r}{\rightrightarrows}} \mathcal{L}^{(0)}$ is a local Lie groupoid and x belongs to $\mathcal{L}^{(0)}$ we define the *orbit of* $\mathcal{L}$ *passing through* x to be the set

$$\{y \in \mathcal{L}^{(0)} \mid \exists \gamma_1, \dots, \gamma_n \in \mathcal{L}\ ;\ s(\gamma_1) = x,\ r(\gamma_1) = s(\gamma_2), \\ \dots, r(\gamma_{n-1}) = s(\gamma_n),\ r(\gamma_n) = y\}.$$

As for quasi-graphoids the Lie algebroid of a local quasi-graphoid is almost injective, and its orbits are the leaves of an almost regular foliation.

In the local context, there is a proposition similar to the Definition-Proposition 1:

Proposition 3. *Let $\mathcal{L}\rightrightarrows\mathcal{L}^{(0)}$ be a local Lie groupoid. The following assertions are equivalent:*

1. *$\mathcal{L}$ is locally isomorphic as graph to a local Lie groupoid $\mathcal{L}'\rightrightarrows\mathcal{L}^{(0)}$ having the property that the only local smooth section of both the source s' and the range r' is the inclusion of units.*

2. *$\mathcal{L}$ is locally isomorphic as graph to a local quasi-graphoid.*

Proof. A local quasi-graphoid has the property that the only local section of both the source and the range map is the unit map. So 2 implies 1.

Suppose that there is an isomorphism of graphs from V' onto V where V' (resp. V) is an open subset of $\mathcal{L}'$ (resp. $\mathcal{L}$) containing $\mathcal{L}^{(0)}$.

Because $\mathcal{D}^2\mathcal{L}$ contains $\{(\gamma, \gamma^{-1}) \mid \gamma \in \mathcal{L}\}$, one can find an open neighborhood W of $\mathcal{L}^{(0)}$ in $\mathcal{L}$ such that:

– $W = W^{-1}$ where $W^{-1} := \{\gamma^{-1} \mid \gamma \in W\}$.

– $W \times W \cap \mathcal{L}^{(2)} \subset \mathcal{D}^2\mathcal{L}$.

– The image of the local product of $\mathcal{L}$ restricted to $W \times W \cap \mathcal{L}^{(2)}$ is contained in V.

W inherits from $\mathcal{L}$ a structure of local Lie groupoid. Moreover W and $\mathcal{L}$ are obviously locally isomorphic as graphs.

Let S be a manifold equipped with two submersions a, b onto $\mathcal{L}^{(0)}$ and f_1, f_2 be two morphisms of graphs from S to W.

If $\sigma : U \to S$ is a local section of a we define

$$\begin{array}{rccc} \nu : & U & \to & W \\ & x & \mapsto & (f_1(\sigma(x)))^{-1} \cdot f_2(\sigma(x)) \end{array}$$

Then ν is a section of both the source s and the range r. Thus ν is the unit map and so $f_1 \circ \sigma = f_2 \circ \sigma$. By repeating this process for any local section of a we show that $f_1 = f_2$. Thus W is a local quasi-graphoid.

q.e.d.

2.3 Generalized Atlas

These "atlases" which are made up with local Lie groupoids are defined to replace in the case of singular foliation the atlases by distinguished charts of regular foliations. With such an atlas, we shall be able to do so without local triviality of the foliation and then compute a Lie groupoid associated to it.

Definition 2. Let M be a smooth manifold. A generalized atlas on M is a set $\mathcal{U} = \{\mathcal{L}_i \underset{r_i}{\overset{s_i}{\rightrightarrows}} O_i\}_{i \in I}$, where:

– $\{O_i\}_{i \in I}$ is a covering of M by open subsets.

– For all $i \in I$, $\mathcal{L}_i \underset{r_i}{\overset{s_i}{\rightrightarrows}} O_i$ is a local quasi-graphoid over O_i.

The following gluing condition must be fulfilled:

For all $i, j \in I$, there is a local graphs isomorphism from $\mathcal{L}_i$ onto $\mathcal{L}_j$, that is an open subset H_i^j (resp.H_j^i) of $\mathcal{L}_i$ (resp. $\mathcal{L}_j$) which contains $O_i \cap O_j$ and an isomorphism of graphs $\varphi_{ji} : H_i^j \longrightarrow H_j^i$.

Remark 2.
1. Because we are dealing with local quasi-graphoids the graphs isomorphisms φ_{ji} are unique. So there exists a maximal open subset of $\mathcal{L}_i$ containing $O_i \cap O_j$ which is the domain of a morphism of graphs to $\mathcal{L}_j$. We will always suppose that the domain H_i^j of φ_{ji} is maximal.

2. Because the $\mathcal{L}_i$ are local quasi-graphoids the following equations are fulfilled when i, j, k belong to I:

i) $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ when restricted to $H_i^j \cap H_i^k \cap \varphi_{ji}^{-1}(H_j^k)$.

ii) $\varphi_{ji}^{-1} = \varphi_{ij}$.

We will say that two generalized atlases $\mathcal{U}$ and $\mathcal{V}$ on the manifold M are *equivalent* when $\mathcal{U} \cup \mathcal{V}$ is again a generalized atlas on M. The generalized atlas $\mathcal{U}$ on M is *stable under restriction* if for any element $\mathcal{L} \rightrightarrows O$ of $\mathcal{U}$ and any open subset V of $\mathcal{L}$ which encounter O the restriction of $\mathcal{L}$ to V is an element of $\mathcal{U}$. Any generalized atlas is equivalent to a generalized atlas which is stable under restriction.

If $\mathcal{L} \rightrightarrows \mathcal{L}^{(0)}$ is a local quasi-graphoid and $\{O_i\}_{i\in I}$ an open covering of $\mathcal{L}^{(0)}$ then $\{\mathcal{L}|_{O_i}\}_{i\in I}$ defines a generalized atlas on $\mathcal{L}^{(0)}$. Two different coverings of $\mathcal{L}^{(0)}$ will give rise to two equivalent atlases by this process. Conversely, let $\mathcal{U} = \{\mathcal{L}_i \overset{s_i}{\underset{r_i}{\rightrightarrows}} O_i\}_{i\in I}$ be a generalized atlas on a manifold M. Remark 2 above implies that the relation $\sim$ defined on $\coprod_{i\in I} \mathcal{L}_i$ by:

$$(\gamma, i) \sim (\eta, j) \Leftrightarrow \begin{cases} (\gamma, i) \in H_i^j \text{ and } (\eta, j) \in H_j^i \\ \varphi_{ji}(\gamma) = \eta \ , \end{cases}$$

is a regular equivalence relation. Let us denote by $\mathcal{L}$ the quotient manifold of $\coprod_{i\in I} \mathcal{L}_i$ by this relation. There is a unique structure of local quasi-graphoid on $\mathcal{L}$ with units space M such that for all $i \in I$, $\mathcal{L}_i$ is a sub local Lie groupoid of G. Moreover, the atlases $\mathcal{U}$ and $\{\mathcal{L}\}$ are equivalent.

In conclusion, generalized atlases and local quasi-graphoids are two equivalent notions. In particular, a generalized atlas on a manifold M defines naturally an almost regular foliation on M.

Given an almost regular foliation $\mathcal{F}$ on a manifold M, we will say that a generalized atlas on M is a *a generalized atlas for* $\mathcal{F}$ when it defines $\mathcal{F}$.

Examples

1. Regular foliation: Let $\mathcal{F}$ be a regular foliation on a manifold M and $\{O_i\}_{i\in I}$ a covering of M by distinguished charts. The foliation $\mathcal{F}_i$ induced by $\mathcal{F}$ on O_i is such that its space of leaves $O_i/\mathcal{F}_i$ is a manifold. In other words the equivalence relation on O_i defined by *being on the same leaf of* $\mathcal{F}_i$ is regular. Thus the graph of this equivalence relation, $G_i = \{(x, y) \in O_i \times O_i \mid x \text{ and } y \text{ are on the same leaf of } \mathcal{F}_i\} \rightrightarrows O_i$ is a quasi-graphoid.

The family $\{G_i\}_{i\in I}$ is a generalized atlas for $\mathcal{F}$.

2. Local almost free action of a Lie group: Let $(M, \mathcal{F})$ be a singular foliation defined by a symmetric local action of a Lie group H

on M [22]. We suppose that there is a dense open subset M_0 of M on which the action is free. We define $M \rtimes_{\mathrm{loc}} H$ to be the local Lie groupoid

$$M \rtimes_{\mathrm{loc}} H := D \underset{r_\rtimes}{\overset{s_\rtimes}{\rightrightarrows}} M,$$

where:

– $D \subset H \times M$ is the domain of the local action.

– The inclusion of units is given by $x \mapsto (1_H, x)$.

– If (h,x) belongs to D then its source is $s_\rtimes(h,x) = x$ and its range is $r_\rtimes(h,x) = h \cdot x$.

– The local product is defined on

$$\mathcal{D}^2(M \rtimes_{\mathrm{loc}} H) = \{((h,x),(g,y)) \in D \times D \mid x = g \cdot y\ ,\ (hg,y) \in D\} \text{ by}$$

$$(h, g \cdot y) \cdot (g,y) = (hg, y).$$

One can easily check that $M \rtimes_{\mathrm{loc}} H$ is a local quasi-graphoid which defines $\mathcal{F}$, thus $\{M \rtimes_{\mathrm{loc}} H\}$ is a generalized atlas for $\mathcal{F}$.

3. Codimension 1 submanifold: Let M be a manifold and N be a submanifold of codimension 1 of M. We define $\mathcal{F}_N$ to be the foliation on M whose leaves are N and the connected components of $M \setminus N$.

Let O be an open subset of M such that $O \cap N \neq \emptyset$ and $f_O : O \to \mathbb{R}$ be a smooth map such that $df_{O_x} \neq 0$ for all x in $O \cap N$ and $O \cap N = f_O^{-1}(0)$. The following map is regular:

$$\begin{array}{rccc} \Phi : & O \times O \times \mathbb{R}_*^+ & \longrightarrow & \mathbb{R} \\ & (z,y,\lambda) & \mapsto & \lambda f_O(z) - f_O(y) \end{array}.$$

Thus $\Phi^{-1}(0)$ is a submanifold of $O \times O \times \mathbb{R}_*^+$ and we define:

$$GO = \Phi^{-1}(0) = \{(z,y,\lambda) \in O \times O \times \mathbb{R}_*^+ \mid \lambda f_O(z) = f_O(y)\} \underset{r_O}{\overset{s_O}{\rightrightarrows}} O$$

where:

- The inclusion of units is the map $x \mapsto (x,x,1)$.
- The source and range maps are given by $s_O(z,y,\lambda) = y$ and $r_O(z,y,\lambda) = z$.
- The product is defined on $GO^{(2)}$ by $(x,y,\lambda) \cdot (y,t,\mu) = (x,t,\lambda\mu)$.

One can find a family $\{O_i\}_{i\in I}$ of open subsets of M such that:

- For all $i \in I$, $O_i \cap N \neq \emptyset$ and O_i is equipped with a map $f_{O_i} : O_i \to \mathbb{R}$ as above.
- $N \subset \cup_{i\in I} O_i$.

Let $(M \setminus N) \boxtimes (M \setminus N)$ be the subset of $(M \setminus N) \times (M \setminus N)$ made of pairs of points which are on the same connected component of $M \setminus N$. Equipped with the usual pair groupoid structure $(M \setminus N) \boxtimes (M \setminus N)$ becomes a quasi-graphoid over $M \setminus N$.

The set $\{(M \setminus N) \boxtimes (M \setminus N)\} \cup \{G_{O_i},\ i \in I\}$ is a generalized atlas for the foliation $\mathcal{F}_N$.

4. Let N be a manifold equipped with two regular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ where $\mathcal{F}_1$ is a subfoliation of $\mathcal{F}_2$. Let $\mathcal{F}$ be the foliation on $M = N \times \mathbb{R}$ defined by $\mathcal{F}_1 \times \{0\}$ and $\mathcal{F}_2 \times \{t\}$ for $t \neq 0$. We construct a generalized atlas for $(M, \mathcal{F})$ in the following way.

Let U be an open subset of N and $\Gamma = \{X_1, \dots , X_{k+q}\}$ a family of vector fields defined on U such that:

- Γ is a basis over U of local sections of $T\mathcal{F}_2$, the tangent bundle of $\mathcal{F}_2$.
- $\Gamma' = \{X_1, \dots , X_k\}$ is a basis over U of local sections of $T\mathcal{F}_1$.
- $[X_i, X_j] = 0$ for all i, j.

Thus we have a free local action $\varphi_U : D_U \subset U \times \mathbb{R}^k \times \mathbb{R}^q \to U$ of $\mathbb{R}^k \times \mathbb{R}^q$ on U such that for (x, t, ξ) in D_U the points x and $\varphi_U(x, t, \xi)$ belong to the same leaf of $\mathcal{F}_2$ and the points x and $\varphi_U(x, t, 0)$ belong to the same leaf of $\mathcal{F}_1$. Such a local action is said to be *compatible with the foliations* $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $\varepsilon_U > 0$ be a real number such that for all $\lambda \in \mathbb{R}$, $|\lambda| < \varepsilon_U$ we have that $(y, t, \lambda\xi)$ belongs to D_U when (y, t, ξ) is in D_U.

Then we define the local quasi-graphoid

$$\mathcal{L}U = D_U \times]-\varepsilon_U, \varepsilon_U[\underset{r_U}{\overset{s_U}{\rightrightarrows}} U \times]-\varepsilon_U, \varepsilon_U[$$

as follows:

- The inclusion of units is the map $(y, \lambda) \mapsto (y, 0, 0, \lambda)$.

– The source and range maps are defined by $s_U(y,t,q,\lambda) = (y,\lambda)$ and $r_U(y,t,\xi,\lambda) = (\varphi(y,t,\lambda\xi),\lambda)$.

– The inverse is defined by $(y,t,\xi,\lambda)^{-1} = (\varphi(y,t,\lambda\xi),-t,-\xi,\lambda)$.

– The product is given by

$$(\varphi(y,t_1,\lambda\xi_1),t_2,\xi_2,\lambda)\cdot(y,t_1,\xi_1,\lambda) = (y,t_1+t_2,\xi_1+\xi_2,\lambda)$$

whenever it makes sense.

Let $\{(U_i,\varphi_i)\}_{i\in I}$ be such that $\{U_i\}_{i\in I}$ is an open covering of N and φ_i is a local action of $\mathbb{R}^k\times\mathbb{R}^q$ on U_i compatible with the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.

Let $G = H_2\times\mathbb{R}_*\rightrightarrows N\times\mathbb{R}_*$ be the product groupoid of the holonomy groupoid H_2 of $\mathcal{F}_2$ and the trivial groupoid $\mathbb{R}_* \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} \mathbb{R}_*$.

One can prove that the set $\{G\}\cup\{\mathcal{L}U_i\}_{i\in I}$ is a generalized atlas for the foliation $\mathcal{F}$.

5. Almost injective Lie algebroid: A Lie algebroid $\mathcal{A} = (p : \mathcal{A}\to TM,[\ ,\ ])$ over a manifold M is said to be *almost injective* when its anchor p is injective in restriction to a dense open subset of M. In other words the anchor p induces an injective morphism from the set of smooth local sections of $\mathcal{A}$ onto the set of smooth local tangent vector fields over M. Such a Lie algebroid defines on M an almost regular foliation $\mathcal{F}_A$. We have shown in [13] how such an algebroid integrates into a local quasi-graphoid. Here is a brief review of this construction.

Let O be an open subset of M, $\{Y_1,\dots,Y_k\}$ a local basis of sections of $\mathcal{A}$ defined on O and $X_i = p(Y_i)$ the corresponding tangent vector field on M. For simplicity, we suppose that the tangent vector fields X_i are complete, and we denote by φ_i^t the flow of X_i.

According to [13], there exist an open subset $\mathcal{L}O$ of $O\times\mathbb{R}^k$ and a local quasi-graphoid structure on $\mathcal{L}O$

$$\mathcal{L}O \overset{s}{\underset{r}{\rightrightarrows}} O$$

such that:

–The unit map is

$$\begin{array}{cccc} u: & O & \longrightarrow & \mathcal{L}O\subset O\times\mathbb{R}^k \\ & x & \mapsto & (x,0) \end{array}.$$

–The source and range maps are defined by:

$$\begin{array}{rccc} s: & \mathcal{L}O & \longrightarrow & O \\ & (x, t_1, \ldots, t_k) & \mapsto & x \end{array}$$

$$\begin{array}{rccc} r: & \mathcal{L}O & \longrightarrow & O \\ & (x, t_1, \ldots, t_k) & \mapsto & \varphi_1^{t_1} \circ \ldots \circ \varphi_k^{t_k}(x) \end{array}.$$

With the help of a local trivialisation of the bundle $\mathcal{A}$ we construct in this way a generalized atlas $\mathcal{U}_{\mathcal{A}}$ of $\mathcal{F}_{\mathcal{A}}$. Moreover up to local equivalence of generalized atlases, the generalized atlas $\mathcal{U}_{\mathcal{A}}$ only depends on $\mathcal{A}$.

Of course any of the previous examples are particular cases of this example.

3. Pseudo-group of local Morita isomorphisms and associated groupoids

In the first part of this section we extend the notion of Morita equivalence between Lie groupoids to local Lie groupoids. The following definitions are nearly the same as those encountered in [10, 16, 17]. Actions of local groupoids cannot be defined globally so the usual definitions do not make sense in this case, and we have to take some care when extending these definitions to the local context.

In the second part, we show that the set of local Morita equivalences between the elements of a generalized atlas behaves as a pseudo-group of local diffeomorphisms. We finish this section by the construction of a Lie groupoid associated to such a pseudo-group.

3.1 Local Morita isomorphisms

Definition 3. Let Z be a manifold, $a : Z \to \mathcal{L}^{(0)}$ be a smooth map and $\mathcal{L} \underset{r}{\overset{s}{\rightrightarrows}} \mathcal{L}^{(0)}$ be a local Lie groupoid:

$$\begin{array}{ccc} Z & & \mathcal{L} \\ & \overset{a}{\searrow} & r \downarrow \downarrow s \\ & & \mathcal{L}^{(0)}. \end{array}$$

Let D_Ψ be an open subset of the fiber product

$$Z \times_{a,r} \mathcal{L} = \{(z, \gamma) \in Z \times \mathcal{L} \ ; \ a(z) = r(\gamma)\}.$$

A right local action of $\mathcal{L}$ on Z with domain D_Ψ is a surjective smooth map $\Psi : D_\Psi \longrightarrow Z$ denoted by $\Psi(z,\gamma) = z * \gamma$, which satisfies the following properties:

1. For all $(z,\gamma) \in D_\Psi$ we have the equality $a(z*\gamma) = s(\gamma)$.
2. If (z,γ_1) belongs to D_Ψ, and if the product $\gamma_1.\gamma_2$ is defined, then if one of the following expression $(z*\gamma_1)*\gamma_2$ or $z*(\gamma_1 \cdot \gamma_2)$ is defined, so is the other one and they are equal.
3. For all $z \in Z$, $(z, a(z))$ belongs to D_Ψ and $z * a(z) = z$.

We will say that the local action is *free* when $z*\gamma = z$ implies that γ belongs to $\mathcal{L}^{(0)}$.

We will say that the local action is *locally proper* if all $z \in Z$ has an open neighborhood V_z such that the map $\Upsilon : D_\Psi \longrightarrow Z \times Z$ defined by $\Upsilon(z,\gamma) = (z, z*\gamma)$ is proper when restricted to $D_\Psi \cap \Upsilon^{-1}(V_z \times V_z)$.

We define *left local actions* in the same way.

Remark. A free local action of the local Lie groupoid $\mathcal{L} \rightrightarrows \mathcal{L}^{(0)}$ on the manifold Z induces a regular foliation on Z denoted by $\mathcal{F}_{\mathcal{L},Z}$. The leaf of $\mathcal{F}_{\mathcal{L},Z}$ passing trough a point z_0 of Z is the orbit of z_0 for the local action. In other words it is the set

$$\{z \in Z \mid \exists\, \gamma_1, \ldots, \gamma_n \in \mathcal{L} \text{ such that } z = (\cdots((z_0*\gamma_1)*\gamma_2)\cdots)*\gamma_n\}.$$

The tangent space of this foliation can be defined in the following way. Let $\Psi : D_\Psi \longrightarrow Z$ be the local action and $\mathrm{pr}_1 : D_\Psi \longrightarrow Z$ the projection onto the first factor. We define $F = \ker(T\mathrm{pr}_1) \subset TD_\Psi$ to be the kernel of the differential of pr_1. One can check that the image $T\Psi(F)$ of F by the differential of Ψ is an involutive subbundle of TZ. Thus $T\Psi(F)$ is the tangent bundle of a regular foliation $\mathcal{F}_{\mathcal{L},Z}$ on Z.

Let $\mathcal{L}_0 \rightrightarrows O_0$ and $\mathcal{L}_1 \rightrightarrows O_1$ be two local Lie groupoids. A *local generalized isomorphism* f from $\mathcal{L}_0$ to $\mathcal{L}_1$ is defined by its *graph*:

$$\begin{array}{ccccc} \mathcal{L}_1 & & Z_f & & \mathcal{L}_0 \\ r_1 \downarrow\downarrow s_1 & \overset{b_f}{\swarrow} & & \overset{a_f}{\searrow} & r_0 \downarrow\downarrow s_0 \\ O_1 & & & & O_0\,. \end{array}$$

The graph is a smooth manifold Z_f equipped with two surjective submersions $(b_f, a_f) : Z_f \to O_1 \times O_0$ in addition with a left local, locally proper, free action of $\mathcal{L}_1$ and a right local, locally proper, free action of $\mathcal{L}_0$. Moreover the following properties must be fulfilled:

1. For all $\gamma_0 \in \mathcal{L}_0$, $z \in Z_f$ and $\gamma_1 \in \mathcal{L}_1$ such that $\gamma_1 * z$ and $z * \gamma_0$ are defined we have that $a_f(\gamma_1 * z) = a_f(z)$, $b_f(z * \gamma_0) = b_f(z)$, and if one of the following expressions $(\gamma_1 * z) * \gamma_0$ or $\gamma_1 * (z * \gamma_0)$ is defined, so is the other one and they are equal.

2. For all $z \in Z_f$, there is an open neighborhood V_z of z in Z_f such that the action of $\mathcal{L}_0$ (resp. $\mathcal{L}_1$) is transitive on the fibers of b_f (resp. a_f) restricted to V_z. In this case $V_z/\mathcal{L}_0$ and $V_z/\mathcal{L}_1$ make sense and a_f (resp. b_f) induces a diffeomorphism from $V_z/\mathcal{L}_1$ to $a_f(V_z)$ (resp. from $V_z/\mathcal{L}_0$ to $b_f(V_z)$).

Two such graphs $(b_f, a_f) : Z_f \to O_1 \times O_0$ and $(b_g, a_g) : Z_g \to O_1 \times O_0$ are *equivalent* if there exists an isomorphism of graphs from Z_f onto Z_g which intertwines the actions of $\mathcal{L}_0$ and $\mathcal{L}_1$.

Definition 4. A local Morita isomorphism f from $\mathcal{L}_0$ onto $\mathcal{L}_1$ is an equivalence class of local generalized isomorphisms.

It will be denoted by $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ and each of its representatives will be called a graph for f.

Examples.

1. Let T and N be two manifolds and consider the trivial Lie groupoids $T \rightrightarrows T$ and $N \rightrightarrows N$ having the identity map as source and range maps. A diffeomorphism $f : T \to N$ induces a local Morita isomorphism from T onto N for which $(\mathrm{pr}_1, \mathrm{pr}_2) : Z_f = \{(f(x), x) \mid x \in T\} \to N \times T$ is a graph. In this trivial case any local Morita isomorphism is of this type.

2. Let $\mathcal{L} \rightrightarrows \mathcal{L}^{(0)}$ be a local Lie groupoid of source s and range r and T a closed embedded submanifold of $\mathcal{L}^{(0)}$ which encounters all the orbits. The manifold $\mathcal{L}_T^T = s^{-1}(T) \cap r^{-1}(T)$ inherits from $\mathcal{L}$ a structure of local Lie groupoid over T. Then $(r, s) : \mathcal{L}_T = s^{-1}(T) \to \mathcal{L}^{(0)} \times T$ is a graph of a local Morita isomorphism from $\mathcal{L}_T^T$ onto $\mathcal{L}$, the action being right and left multiplication.

3. Let $(M, \mathcal{F})$ be a regular foliation. Take two distinguished open sets $U_i \simeq P_i \times T_i$ where P_i is a plaque and T_i a transversal, $i = 0, 1$. Let $G_i \rightrightarrows U_i$ be the graph of the regular equivalence relation induced by $\mathcal{F}$ on U_i.

Combining the two previous examples one can show that there is a bijection between the set of Morita equivalences between G_0 and G_1 and the diffeomorphisms from T_0 onto T_1.

We will be especially interested in local Morita isomorphisms between local quasi-graphoids. In this situation we have the following lemma:

Lemma 1. *Let $\mathcal{L}_0 \rightrightarrows O_0$ and $\mathcal{L}_1 \rightrightarrows O_1$ be two local Lie groupoids, $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ a local Morita isomorphism and $(b_f, a_f) : Z_f \to O_1 \times O_0$ a graph for f. Let z be a point of Z_f.*

If $\mathcal{L}_0$ or $\mathcal{L}_1$ is a local quasi-graphoid there is an open neighborhood V_z of z in Z_f such that for any graph $(b_S, a_S) : S \to O_1 \times O_0$ there is at most one morphism of graphs from S onto V_z.

Proof. Let V_z be a neighborhood of z such that b_f induces a diffeomorphism between $V_z/\mathcal{L}_0$ and $b_f(V_z)$. Let $(b_S, a_S) : S \to O_1 \times O_0$ be a graph and suppose that there exist two smooth maps Φ_0, $\Phi_1 : S \to V_z$ such that $a_f \circ \Phi_0 = a_S = a_f \circ \Phi_1$ and $b_f \circ \Phi_0 = b_S = b_f \circ \Phi_1$.

Then we consider the two following differentiable maps:

$$\Psi : S \to \mathcal{L}_0 \ , \ x \mapsto \gamma \ \text{ such that } \ \Phi_0(x) = \Phi_1(x) * \gamma,$$
$$\Psi' : S \to \mathcal{L}_0 \ , \ x \mapsto a_S(x),$$

which satisfy $s_0 \circ \Psi = s_0 \circ \Psi' = a_S$ and $r_0 \circ \Psi = r_0 \circ \Psi' = a_S$.

If $\mathcal{L}_0$ is a local quasi-graphoid there is at most one morphism of graphs from $(a_s, a_s) : S \to O_0 \times O_0$ to $(r_0, s_0) : \mathcal{L}_0 \to O_0 \times O_0$. Thus $\Psi = \Psi'$ and so $\Phi_0 = \Phi_1$. q.e.d.

Let $(b_f, a_f) : Z_f \to O_1 \times O_0$ and $(b_g, a_g) : Z_g \to O_1 \times O_0$ be the graphs of two local Morita isomorphisms f and g between two local quasi-graphoids $\mathcal{L}_0$ and $\mathcal{L}_1$. The previous lemma implies that if there is an isomorphism of graphs from Z_f onto Z_g then it must intertwine the actions. So in this case, the local Morita isomorphisms f and g are equal.

This lemma will play a crucial role in the following for the definition of a smooth structure on the set of germs of local Morita isomorphisms.

3.2 The local pseudo-group structure

Local Morita isomorphisms are going to replace transverse isomorphisms in the construction of the holonomy groupoid. So we have to define identity, composition, inversion and restriction for them.

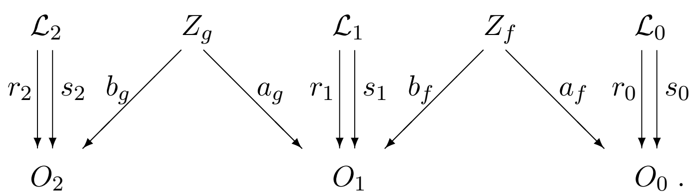
In the following $\mathcal{L}_0 \rightrightarrows O_0$ and $\mathcal{L}_1 \rightrightarrows O_1$ are two local Lie groupoids. We suppose that we have a local Morita isomorphism $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ and we take a graph $(b_f, a_f) : Z_f \to O_1 \times O_0$ for f.

Identity: Let s_0 and r_0 be respectively the source and range maps of $\mathcal{L}_0$. We denote by $Id_{\mathcal{L}_0} : \mathcal{L}_0 \curvearrowleft \mathcal{L}_0$ the local Morita isomorphism represented by the graph $Z_{Id_{\mathcal{L}_0}} = \mathcal{L}_0$, $a_{Id_{\mathcal{L}}} = s_0$ and $b_{Id_{\mathcal{L}}} = r_0$, the two actions being right and left multiplication. This local Morita isomorphism is called the *identity*.

Inversion: Let $Z_{f^{-1}} = Z_f$, $a_{f^{-1}} = b_f$ and $b_{f^{-1}} = a_f$. We consider the right local action of $\mathcal{L}_1$ on $Z_{f^{-1}}$ (resp. the left local action of $\mathcal{L}_0$ on $Z_{f^{-1}}$) which is defined by $\Psi_1(z, \gamma_1) = \gamma_1^{-1} * z$ (resp. $\Psi_0(\gamma_0, z) = z * \gamma_0^{-1}$). Equipped with these actions, $(b_{f^{-1}}, a_{f^{-1}}) : Z_{f^{-1}} \to O_0 \times O_1$ is a graph of the local Morita isomorphism $f^{-1} : \mathcal{L}_0 \curvearrowleft \mathcal{L}_1$ called the *inverse* of f.

Restriction: Let H_0 (resp. H_1) be an open sub local groupoid of $\mathcal{L}_0$ (resp. $\mathcal{L}_1$) and V be an open subset of Z_f such that $a_f(V)$ is the set of units of H_0 and $b_f(V)$ is the set of units of H_1. The *restriction of f to H_0, H_1 and V* is the local Morita isomorphism from H_0 onto H_1 which admits as graph the restriction $(b_f, a_f)| : V \to b_f(V) \times a_f(V)$ with the local action induced by $\mathcal{L}_0$ and $\mathcal{L}_1$. We denote it by $f|_{H_1,V,H_0} : H_1 \curvearrowleft H_0$.

Local composition: Let $g : \mathcal{L}_2 \curvearrowleft \mathcal{L}_1$ be another local Morita isomorphism admitting the following graph $(b_g, a_g) : Z_g \to O_2 \times O_1$, where O_2 is the set of units of $\mathcal{L}_2$:



We consider the fiber product

$$Z_g \times_{O_1} Z_f = \{(z_1, z_2) \in Z_g \times Z_f \mid a_g(z_1) = b_f(z_2)\}.$$

We define a right local free action of $\mathcal{L}_1$ on $Z_g \times_{O_1} Z_f$ in the following way: $\Psi(z_1, z_2, \gamma) = (z_1 * \gamma, \gamma^{-1} * z_2)$ when (z_1, z_2) belongs to $Z_g \times_{O_1} Z_f$ and $\gamma \in \mathcal{L}_1$ is such that $z_1 * \gamma$ and $\gamma^{-1} * z_2$ exist. This action defines a regular foliation $\mathcal{F}_{\mathcal{L}_1}$ on $Z_g \times_{O_1} Z_f$.

The local Morita isomorphisms f and g are called *composable* if the equivalence relation induced by the foliation $\mathcal{F}_{\mathcal{L}_1}$ on $Z_g \times_{O_1} Z_f$ is regular, that is the space of leaves of $\mathcal{F}_{\mathcal{L}_1}$ is a manifold. In this case we denote this space of leaves by $Z_{g\circ f} := Z_g \times_{O_1} Z_f/\mathcal{F}_{\mathcal{L}_1}$. The maps $a_f \circ \mathrm{pr}_2 : Z_g \times_{O_1} Z_f \to O_0$ and $b_g \circ \mathrm{pr}_1 : Z_g \times_{O_1} Z_f \to O_2$ induce on the quotient space the maps $a_{g\circ f} : Z_{g\circ f} \to O_0$ and $b_{g\circ f} : Z_{g\circ f} \to O_2$. Finally with the obvious local actions of $\mathcal{L}_0$ on the right and $\mathcal{L}_2$ on the left, $(b_{g\circ f}, a_{g\circ f}) : Z_{g\circ f} \to O_2 \times O_0$ is the graph of a local Morita isomorphism called the *composition of* g *and* f and denoted by $g \circ f : \mathcal{L}_2 \curvearrowleft \mathcal{L}_0$.

In the general case the foliation $\mathcal{F}_{\mathcal{L}_1}$ is regular and so it is locally trivial. Thus, by restricting f and g, one can make a composition of g and f which depends of the restrictions. Contrary to the case of local diffeomorphisms there is not a better restriction for the composition of local Morita isomorphism. Therefore we talk about *local composition.*

One must remark that for any point (z_1, z_2) of $Z_g \times_{O_1} Z_f$, one can find an open sub local groupoid H_1 of $\mathcal{L}_1$, an open neighborhood V_1 of z_1 and an open neighborhood V_2 of z_2 such that the restrictions $f|_{H_1,V_1,\mathcal{L}_0}$ and $g|_{\mathcal{L}_2,V_2,H_1}$ are composable.

The previous operations are really an inverse and a composition when you look at them locally. More precisely one can check the following proposition:

Proposition 4. *Let* $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ *be a local Morita isomorphism from* $\mathcal{L}_0 \rightrightarrows O_0$ *onto* $\mathcal{L}_1 \rightrightarrows O_1$ *for which* $(b_f, a_f) : Z_f \to O_1 \times O_0$ *is a graph. Let* z *be a point in* Z_f. *The following assertions are fulfilled:*

1. *There exists an open sub local groupoid* H_0 *of* $\mathcal{L}_0$ *with* O_0 *as space of units and an open sub local groupoid* H_1 *of* $\mathcal{L}_1$ *with* O_1 *as space of units such that:*

$$\mathrm{Id}_{H_1} \circ f|_{H_1,Z_f,H_0} = f|_{H_1,Z_f,H_0} \circ \mathrm{Id}_{H_0} = f|_{H_1,Z_f,H_0}.$$

2. *There exist an open neighborhood* V_z *of* z *in* Z_f, *an open sub local groupoid* H_0' *of* $\mathcal{L}_0$ *with* $a_f(V_z)$ *as space of units and an open sub local groupoid* H_1' *of* $\mathcal{L}_1$ *with* $b_f(V_z)$ *as space of units such that:*

$$f|_{H_1',V_z,H_0'} \circ (f|_{H_1',V_z,H_0'})^{-1} = \mathrm{Id}_{H_1'},$$
$$(f|_{H_1',V_z,H_0'})^{-1} \circ f|_{H_1',V_z,H_0'} = \mathrm{Id}_{H_0'}.$$

Definition 5. We will say that we have a local pseudo-group of local Morita isomorphisms on the smooth manifold M when we have a generalized atlas $\mathcal{U}$ on M stable under restriction together with a set $\mathcal{IU}$ of local Morita isomorphisms between elements of $\mathcal{U}$ such that:

– "Identity" is in $\mathcal{IU}$, that is for all $\mathcal{L} \rightrightarrows O$ in $\mathcal{U}$, $\mathrm{Id}_{\mathcal{L}} : \mathcal{L} \curvearrowleft \mathcal{L}$ is in $\mathcal{IU}$.

– $\mathcal{IU}$ is stable under inversion, local composition and restriction.

3.3 Associated groupoid

A generalized atlas $\mathcal{U}$ on a smooth manifold M or equivalently a local quasi-graphoid over M induces a generalized atlas $\widetilde{\mathcal{U}}$ stable under restriction on M. Then we can consider the local pseudo-group of local Morita isomorphisms between local quasi-graphoids which constitute $\widetilde{\mathcal{U}}$. We are going to associate to such a local pseudo-group the Lie groupoid of its germs as it is usually done for a pseudo-group of local diffeomorphisms. To do this we start by defining the germ of a local Morita isomorphism between two local quasi-graphoids.

Let $\mathcal{L}_0 \rightrightarrows O_0$ and $\mathcal{L}_1 \rightrightarrows O_1$ be two local quasi-graphoids. We suppose that $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ and $g : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ are two local Morita isomorphisms and we take a graph $(b_f, a_f) : Z_f \to O_1 \times O_0$ for f and a graph $(b_g, a_g) : Z_g \to O_1 \times O_0$ for g. We consider an element γ_f of Z_f and γ_g of Z_g.

We will say that the *germ* of f in γ_f is equal to the germ of g in γ_g and we will denote it by $[f]_{\gamma_f} = [g]_{\gamma_g}$ if there exists an open neighborhood V_{γ_f} of γ_f in Z_f, an open neighborhood V_{γ_g} of γ_g in Z_g and an isomorphism of graphs $\phi : V_{\gamma_f} \longrightarrow V_{\gamma_g}$ which sends γ_f onto γ_g.

One can show that the equality $[f]_{\gamma_f} = [g]_{\gamma_g}$ implies that there exists an open sub local quasi-graphoid H_0 of $\mathcal{L}_0$, an open sub local quasi-graphoid H_1 of $\mathcal{L}_1$, an open neighborhood V_{γ_f} of γ_f in Z_f and an open neighborhood V_{γ_g} of γ_g in Z_g such that $f|_{H_1, V_{\gamma_f}, H_0} = g|_{H_1, V_{\gamma_g}, H_0}$.

In the general case it may happen that $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ and $g : \mathcal{L}'_1 \curvearrowleft \mathcal{L}'_0$ are local Morita isomorphisms between different local quasi-graphoids. When $\mathcal{L}_i \rightrightarrows O_i$ and $\mathcal{L}'_i \rightrightarrows O'_i$ are locally isomorphic we can consider $\widetilde{\mathcal{L}}_i \rightrightarrows O_i \cap O'_i$ the maximal open sub quasi-graphoid of both $\mathcal{L}_i$ and $\mathcal{L}'_i$, $i = 0, 1$. Then we can look at the restrictions $f|_{\widetilde{\mathcal{L}}_1, Z_f, \widetilde{\mathcal{L}}_0}$ and $g|_{\widetilde{\mathcal{L}}_1, Z_f, \widetilde{\mathcal{L}}_0}$ and use the previous definition of germ.
The gluing condition of generalized atlases ensures that we are always in the case just mentioned when we are dealing with a local pseudo-group of local Morita isomorphisms.

From now on $\mathcal{U} = \{\mathcal{L}_i \overset{s_i}{\underset{r_i}{\rightrightarrows}} O_i\}_{i \in I}$ is a generalized atlas stable under restriction on a smooth manifold M and $\mathcal{IU}$ is the pseudo-group of local Morita isomorphisms between elements of $\mathcal{U}$.

Proposition 5. *The set of germs of elements of $\mathcal{IU}$, denoted by $\mathcal{GIU}$, is naturally endowed with a smooth manifold structure. For this structure M is an embedded submanifold of $\mathcal{GIU}$.*

Proof. Let $[f]_\gamma \in \mathcal{GIU}$, $f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0$ be one of its representatives which admits Z_f as graph and $\gamma \in Z_f$. Lemma 1 ensures the local injectivity of the map

$$\begin{array}{ccc} Z_f & \longrightarrow & \mathcal{GIU} \\ \eta & \mapsto & [f]_\eta \end{array}.$$

This allows us to provide $\mathcal{GIU}$ with a smooth structure.

The following map is an embedding of the manifold M in the manifold $\mathcal{GIU}$:

$$\begin{array}{ccc} M & \longrightarrow & \mathcal{GIU} \\ x & \mapsto & [Id_{\mathcal{L}}]_x \end{array}$$

where $\mathcal{L}$ is a *map over* x which means that $\mathcal{L}$ is an element of $\mathcal{U}$ and x a unit of $\mathcal{L}$. q.e.d.

Now we can define the Lie groupoid associated to the generalized atlas $\mathcal{U}$. We deduce the following theorem:

Theorem 1. *With the following structural maps $\mathcal{GIU}$ is a Lie groupoid over M:*

- *Source and range:*

$$\begin{array}{cccc} s: & \mathcal{GIU} & \longrightarrow & M \\ & [f:\mathcal{L}_1 \curvearrowleft \mathcal{L}_0]_z & \mapsto & a_f(z) \end{array}$$

$$\begin{array}{cccc} r: & \mathcal{GIU} & \longrightarrow & M \\ & [f:\mathcal{L}_1 \curvearrowleft \mathcal{L}_0]_z & \mapsto & b_f(z) \end{array}$$

  where $(b_f, a_f) : Z_f \to O_1 \times O_0$ is a graph for f and z belongs to Z_f.

- *Inverse:*

$$[f:\mathcal{L}_1 \curvearrowleft \mathcal{L}_0]_z^{-1} = [f^{-1}:\mathcal{L}_0 \curvearrowleft \mathcal{L}_1]_z.$$

- *Product: Let $[f:\mathcal{L}_1 \curvearrowleft \mathcal{L}_0]_z$ and $[g:\mathcal{L}_2 \curvearrowleft \mathcal{L}_1]_t$ be two elements of $\mathcal{GIU}$. Let's take a graph $(b_f, a_f) : Z_f \to O_1 \times O_0$ for f and a* *graph* $(b_g, a_g) : Z_g \to O_2 \times O_1$ *for* g *such that* z *belongs to* Z_f *and* t *belongs to* Z_g*. We suppose that* $a_g(t) = b_f(z)$*. Then*

$$[g : \mathcal{L}_2 \curvearrowleft \mathcal{L}_1]_t \cdot [f : \mathcal{L}_1 \curvearrowleft \mathcal{L}_0]_z = [g \circ f : \mathcal{L}_2 \curvearrowleft \mathcal{L}_0]_{(t,z)}.$$

– *Units:*

$$\begin{array}{ccc} M & \longrightarrow & \mathcal{GIU} \\ x & \mapsto & [\mathrm{Id}_{\mathcal{L}}]_x \end{array}$$

where $\mathcal{L}$ *is a map over* x*.*

Moreover $\mathcal{GIU} \overset{s}{\underset{r}{\rightrightarrows}} M$ *is a quasi-graphoid.*

Proof. The only point which is not obvious is that $\mathcal{GIU}$ is a quasi-graphoid. To see this let O be an open subset of M and $\nu : O \to \mathcal{GIU}$ a smooth local section of both s and r. By restricting O if necessary we can suppose that there exists an element $\mathcal{L} \overset{s_0}{\underset{r_0}{\rightrightarrows}} O$ of $\mathcal{U}$ and a local Morita isomorphism $f : \mathcal{L} \curvearrowleft \mathcal{L}$ for which $(b_f, a_f) : \mathcal{L} \to O \times O$ is a graph and such that the image of ν is a subset of $\{[f]_\gamma \mid \gamma \in Z_f\}$. Then there is a unique smooth map $\widetilde{\nu} : O \to Z_f$ such that:

i) For all $x \in O$ we have that $\nu(x) = [f]_{\widetilde{\nu}(x)}$.

ii) $a_f \circ \widetilde{\nu} = b_f \circ \widetilde{\nu} = 1_O$.

The map $\gamma \mapsto \gamma * \widetilde{\nu}(s_0(\gamma))$ induces a local isomorphism of graphs from $\mathcal{L}$ to Z_f. So there is a neighborhood W of O in $\mathcal{L}$ such that

$$[\mathrm{Id}_{\mathcal{L}}]_\gamma = [f]_{\varphi(\gamma)} \text{ for all } \gamma \text{ in } W.$$

In particular if x belongs to O we get $[\mathrm{Id}_{\mathcal{L}}]_x = [f]_{\varphi(x)} = [f]_{\widetilde{\nu}(x)} = \nu(x)$. So ν is the inclusion of units. q.e.d.

Remark.

1. If $\mathcal{L} \rightrightarrows M$ is a local quasi-graphoid let $\mathcal{U}_{\mathcal{L}}$ be the corresponding generalized atlas stable under restriction of M, that is $\mathcal{U}_{\mathcal{L}}$ contains $\mathcal{L}$ and all its restriction. The biggest quasi-graphoid over M which admits $\mathcal{L}$ as a sub local Lie groupoid is $\mathcal{GIU}_{\mathcal{L}}$. If moreover $\mathcal{L}$ is s-connected, the smallest quasi-graphoid over M which admits $\mathcal{L}$ as a sub local Lie groupoid is the s-connected component of $\mathcal{GIU}_{\mathcal{L}}$.

2. If we take $\mathcal{V}$ to be the set of all local quasi-graphoids over open subsets of M and $\mathcal{IV}$ the pseudo-group of local Morita isomorphisms

between elements of $\mathcal{V}$, an analogous construction gives rise to a Lie groupoid $\mathcal{GIV} \rightrightarrows \mathcal{GV}$ which is isomorphic to the universal convector of J. Pradines and B. Bigonnet [3, 26, 4].

Recall that an *almost injective Lie algebroid* is a Lie algebroid for which the anchor is injective when restricted to a dense open subset of the base space. The local integration of almost injective Lie algebroid [13] ensures that given an almost injective Lie algebroid on a smooth manifold M there exists a local quasi-graphoid $\mathcal{L}$ over M which integrates $\mathcal{A}$. One can consider the maximal generalized atlas corresponding to $\mathcal{L}$ and then uses the previous theorem to compute a Lie groupoid G on M. Of course the Lie algebroid of G is equal to the Lie algebroid of $\mathcal{L}$ that is $\mathcal{A}$. Finally we obtain:

Theorem 2. *Every almost injective Lie algebroid is integrable.*

In [24], J. Pradines asserted that, as for Lie algebras, every Lie algebroid integrates into a Lie groupoid. In fact this assertion, named *Lie's third theorem for Lie algebroid* is false. This was pointed out by a counter example given by P. Molino and R. Almeida in [1]. Since that time lots of work has been done around this problem. For example K. Mackenzie has found obstructions to integrability of transitive Lie algebroids [18], fundamental examples have been studied by A. Weinstein especially in relation with Poisson manifolds [30, 31] and more recently V. Nistor has studied integrability of Lie algebroids over stratified manifolds [21]. A part of this subject is discussed in the book of A. Cannas da Silva and A. Weinstein [6].

During the writing of this paper, a very interesting preprint of M. Crainic and R.L. Fernandes appeared [11]. They give a necessary and sufficient condition for the integrability of Lie algebroids. In particular, they recover the previous theorem.

4. Holonomy groupoid of an almost regular foliation

In this section, we apply the results obtained previously to almost regular foliations. We end with several examples.

The study of quasi-graphoids enables us to propose a reasonable definition of holonomy groupoid for an almost regular foliation:

Definition 6. Let $(M, \mathcal{F})$ be an almost regular foliation. A holonomy groupoid of $(M, \mathcal{F})$ is an s-connected quasi-graphoid G having M as space of units and such that the orbits of G are the leaves of $\mathcal{F}$.

Proposition 2 ensures that this definition coincides with the usual one in the case of a regular foliation.

If $\mathcal{F}$ is an almost regular foliation on a smooth manifold M such that the regular leaves are of dimension k then an obvious necessary condition for the existence of an holonomy groupoid for $\mathcal{F}$ is that $\mathcal{F}$ can be defined by an almost injective Lie algebroid $\mathcal{A}$, that is $\mathcal{A}$ is of dimension k. The previous results ensure that this condition is also sufficient. Finally we have the following theorem:

Theorem 3. *Let $(M, \mathcal{F})$ be an almost regular foliation and k denote the dimension of the regular leaves. There exists an holonomy groupoid for $(M, \mathcal{F})$ if and only if one of the following equivalent assertions is fulfilled:*

1. *$\mathcal{F}$ can be defined by a quasi-graphoid with space of units M.*
2. *There exists a generalized atlas for $\mathcal{F}$ or equivalently $\mathcal{F}$ can be defined by a local quasi-graphoid over M.*
3. *$\mathcal{F}$ can be defined by a Lie algebroid of dimension k over M.*

One must remark that for an almost regular foliation $(M, \mathcal{F})$ there exist as many holonomy groupoids as there are maximal generalized atlases or as there are almost regular Lie algebroids over M which define $\mathcal{F}$. Nevertheless some foliations admit a better holonomy groupoid. More precisely we will say that G is the *universal* holonomy groupoid of $(M, \mathcal{F})$ if for any other holonomy groupoid H of $(M, \mathcal{F})$ there is a morphism of Lie groupoids from H to G. One can check that this is equivalent to the fact that for any almost regular Lie algebroid $\mathcal{A}$ which defines $\mathcal{F}$ there is a morphism of Lie algebroids from $\mathcal{A}$ to $\mathcal{A}G$, the Lie algebroid of G. In such a situation we say that the Lie algebroid $\mathcal{A}G$ is *extremal* for $\mathcal{F}$. A particular case of extremal algebroid for a foliation $\mathcal{F}$ is when the anchor of $\mathcal{A}$ induces an isomorphism between the module of local smooth sections of $\mathcal{A}$ and the module of local vector fields tangent to the foliation $\mathcal{F}$.

In conclusion if $(M, \mathcal{F})$ is an almost regular foliation it satisfies one of the three following properties:

i) The foliation $(M, \mathcal{F})$ does not possess any holonomy groupoid, or equivalently $\mathcal{F}$ cannot be defined by an almost injective Lie algebroid.

ii) The foliation $(M, \mathcal{F})$ possesses a holonomy groupoid but not a universal one: that is $\mathcal{F}$ can be defined by an almost injective Lie algebroid, but there is no extremal Lie algebroid for $\mathcal{F}$.

iii) The foliation $(M, \mathcal{F})$ possesses an universal holonomy groupoid, or in other words $\mathcal{F}$ is defined by an extremal Lie algebroid.

We finish this paper by giving examples of each of these cases.

Examples.

1. Regular foliation: If $\mathcal{F}$ is a regular foliation on the manifold M then the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$ equipped with the inclusion as anchor is the unique almost regular Lie algebroid which defines $\mathcal{F}$. In this case the foliation admits a unique holonomy groupoid: $\mathrm{Hol}(M, \mathcal{F})$. The uniqueness of the holonomy groupoid is a characteristic property of a regular foliation.

2. Almost free action of a Lie group: Let $(M, \mathcal{F}_H)$ be a singular foliation defined by an almost free action of a connected Lie group H on M. We denote by $\Phi : H \times M \to M$ the action and by $\mathcal{H}$ the Lie algebra of H.

Let $p : M \times \mathcal{H} \to TM$ be the map defines by $p : (x, v) \mapsto T\Phi(x, v)$ where $T\Phi(x, v)$ denotes the evaluation at $(x, 0, 1_H, v) \in T_xM \times T_{1_H}H$ of the differential of Φ. The set $\mathcal{A}_H = M \times \mathcal{H}$ is naturally endowed with a structure of Lie algebroid over M whose anchor is p. This algebroid is almost injective and defines $\mathcal{F}_H$.

The crossed product groupoid $M \rtimes H \rightrightarrows M$ integrates $\mathcal{A}$ and is a holonomy groupoid for $\mathcal{F}_H$. This groupoid is not universal in general.

3. Concentric spheres: Let $\mathcal{F}$ be the foliation of $\mathbb{R}^3$ by concentric spheres and O be the singular leaf. Because the 2 sphere is not parallelizable there exists no Lie algebroid of dimension 2 which defines $\mathcal{F}$. Thus such a foliation has no holonomy groupoid.

More generally a foliation $\mathcal{F}$ on a manifold M which is locally the foliation by concentric spheres of dimension different from 1, 3 and 7 has no holonomy groupoid [5].

4. Submanifold: Let M be a manifold of dimension m and N a submanifold of M of dimension $n < m$. We define $\mathcal{F}_N$ to be the foliation on M whose leaves are N and the connected components of $M \setminus N$.

Codimension 1 submanifold: We have already seen in section 2 how to construct a generalized atlas for $\mathcal{F}_N$. The holonomy groupoid we obtain using this atlas is a particular case of groupoids related to the notion of explosion of manifolds which where constructed for the study of manifolds with corners [19, 20, 31].

In the simple case where $M = N \times \mathbb{R}$ we define

$$\begin{array}{rccc} p: & \mathcal{A}_N := TM \simeq TN \times T\mathbb{R} & \to & TM \simeq TN \times T\mathbb{R} \\ & (x, v; t, \lambda) & \mapsto & (x, v; t, t.\lambda). \end{array}$$

There is a unique structure of Lie algebroid on $\mathcal{A}_N$ whose anchor is p. This Lie algebroid is an almost injective Lie algebroid which defines $\mathcal{F}_N$, moreover it is extremal. The corresponding groupoid is

$$G = N \times N \times \mathbb{R} \times \mathbb{R}^+_* \rightrightarrows N \times \mathbb{R}$$

obtained by making the product of the pair groupoid over N with the groupoid of the action of $\mathbb{R}^+_*$ on $\mathbb{R}$ by multiplication.

The point: Suppose now that M is a manifold of dimension $n \geq 2$ and let N be a point of M. Let $f : M \to \mathbb{R}$ be a smooth function which vanishes only at N. We consider

$$\begin{array}{rccc} p_f: & \mathcal{A}_f := TM & \to & TM \\ & (x, v) & \mapsto & (x, f(x)v). \end{array}$$

There is a unique structure of Lie algebroid on $\mathcal{A}_f$ whose anchor is f. Moreover $\mathcal{A}_f$ is almost injective and it defines $\mathcal{F}_N$. Thus $\mathcal{F}_N$ admits holonomy groupoids.

Let $\mathcal{A}$ be a Lie algebroid which defines $\mathcal{F}_N$. The anchor of $\mathcal{A}$ induces a smooth map $P : U \to M_n(\mathbb{R})$, where U is a neighborhood of N in M, such that $P(N) = 0$ and $P(x)$ is invertible for $x \neq N$.

If $\mathcal{A}$ is extremal then the map P must be such that for all smooth function $f : U \to \mathbb{R}$ which vanishes only at N, the map from $U \setminus \{N\}$ to $M_n(\mathbb{R})$ definined by $x \mapsto f(x)P^{-1}(x)$ can be smoothly extended to U. Such a map P does not exist, so there are no extremal Lie algebroids and no universal holonomy groupoids for $\mathcal{F}_N$.

5. Composed foliation: Let M be a manifold equipped with two regular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ where $\mathcal{F}_1$ is a subfoliation of $\mathcal{F}_2$ and let $\mathcal{F}$ be the foliation on $M \times \mathbb{R}$ defined by $\mathcal{F}_1 \times \{0\}$ and $\mathcal{F}_2 \times \{t\}$ for $t \neq 0$.

Let G_1 (resp. G_2) be the holonomy groupoid of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) and $\varphi : G_1 \to G_2$ the immersion induced by the inclusion of $\mathcal{F}_1$ into $\mathcal{F}_2$.

We take a Euclidean metric on $T\mathcal{F}_2$ and we denote by $N = (T\mathcal{F}_1)^\perp$ the orthogonal bundle of $T\mathcal{F}_1$ in $T\mathcal{F}_2$. Then $T\mathcal{F}_2$ can be identified with $T\mathcal{F}_1 \oplus N$. We equip $\mathcal{A} = (T\mathcal{F}_1 \oplus N) \times \mathbb{R}$ with the unique structure of Lie algebroid over $M \times \mathbb{R}$ whose anchor is:

$$\begin{array}{rccc} p: & \mathcal{A} := (T\mathcal{F}_1 \oplus N) \times \mathbb{R} & \rightarrow & T(M \times \mathbb{R}) = TM \times T\mathbb{R} \\ & (v_1, v_2, t) & \mapsto & (v_1 + tv_2, (t, 0)). \end{array}$$

This Lie algebroid is almost injective, it defines $\mathcal{F}$ and it is extremal for $\mathcal{F}$. The corresponding Lie groupoid is the normal groupoid of φ defined by M. Hilsum and G. Skandalis[17]. More precisely there is an action of G_1 on $\mathcal{N} = T\mathcal{F}_2/T\mathcal{F}_1$, the normal bundle of the inclusion of $T\mathcal{F}_1$ in $T\mathcal{F}_2$.

Thus $G_1 \times_M \mathcal{N} = \{(\gamma, X) \in G_1 \times \mathcal{N} \mid X \in \mathcal{N}_{r(\gamma)}\}$ is equipped with a groupoid structure:

$$(\gamma_1, X_1) \cdot (\gamma_2, X_2) = (X_1 + \gamma_1(X_2), \gamma_1\gamma_2)$$

when $(\gamma_i, X_i) \in G_1 \times_M \mathcal{N}$ and $r(\gamma_2) = s(\gamma_1)$. At the set level, the normal groupoid of φ is the union groupoid

$$G_2 \times \mathbb{R}_* \cup (G_1 \times_M \mathcal{N}) \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

An extreme case is when $\mathcal{F}_1$ is the foliation of M by the points and $\mathcal{F}_2$ is the foliation of M with M as unique leaf. In this case we recover the tangent groupoid of M of A. Connes [7].

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