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Blow-up constructions for Lie groupoids and a Boutet de Monvel type calculus

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Abstract. We present natural and general ways of building Lie groupoids, by using the classical procedures of blow-ups and of deformations to the normal cone. Our constructions are seen to recover many known ones involved in index theory. The deformation and blow-up groupoids obtained give rise to several extensions of C^*-algebras and to full index problems. We compute the corresponding K-theory maps. Finally, as an application, we use the blow-up of a manifold sitting in a transverse way in the space of objects of a Lie groupoid to construct a calculus which is quite similar to the Boutet de Monvel calculus for manifolds with boundary.

Contents

1. Introduction

Let $G \rightrightarrows M$ be a Lie groupoid. The Lie groupoid G comes with its natural pseudodifferential calculus. For example:

- if the groupoid G is just the pair groupoid $M \times M$, the associated calculus is the ordinary (pseudo)differential calculus on M;

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- if the groupoid G is a family groupoid $M \times_B M$ associated with a fibration $p\colon M \to B$, the associated (pseudo)differential operators are families of operators acting on the fibers of p (those of [3]);
- if the groupoid G is the holonomy groupoid of a foliation, the associated (pseudo)differential operators are longitudinal operators as defined by Connes in [7];
- if the groupoid G is the monodromy groupoid, *i.e.,* the groupoid of homotopy classes (with fixed endpoints) of paths in a (compact) manifold M, the associated (pseudo)differential operators are the $\pi_1(M)$-invariant operators on the universal cover of M...

The groupoid G defines therefore a class of partial differential equations.

Our study will focus here on the corresponding index problems on M. The index takes values naturally in the K-theory of the C^*-algebra of G.

Let then V be a submanifold of M. We will consider V as bringing a singularity into the problem: it forces operators of G to "slow down" near V, at least in the normal directions. Inside V, they should only propagate along a sub-Lie-groupoid $\Gamma \rightrightarrows V$ of G. One can just take $\Gamma = V$ (no action) in order to encode that the propagation slows down in all directions near V.

In these cases, this behavior is nicely encoded by a groupoid $\mathrm{SBlup}_{r,s}(G, \Gamma)$ obtained by using a blow-up construction of the inclusion $\Gamma \to G$.

The blow-up construction (Blup) is a well known construction in algebraic geometry as well as in differential geometry. It is closely related to the *deformation to the normal cone* which has been used in quite a few cases in connection with non-commutative geometry.

Let X be a submanifold of a manifold Y. Denote by N_X^Y the normal bundle.

- The *deformation to the normal cone* of X in Y is a smooth manifold $\mathrm{DNC}(Y, X)$ obtained by naturally gluing $N_X^Y \times \{0\}$ with $Y \times \mathbb{R}^*$.
- The *blow-up* of X in Y is a smooth manifold $\mathrm{Blup}(Y, X)$, where X is inflated to the projective space $\mathbb{P}N_X^Y$. It is obtained by gluing $Y \setminus X$ with $\mathbb{P}N_X^Y$ in a natural way. We will mainly consider its variant the *spherical blow-up* $\mathrm{SBlup}(Y, X)$ (which is a manifold with boundary), in which the sphere bundle $\mathbb{S}N_X^Y$ replaces the projective bundle $\mathbb{P}N_X^Y$.

The *functoriality* of the DNC and Blup constructions allows to naturally endow $\mathrm{DNC}(G, \Gamma)$ and a large open subset $\mathrm{SBlup}_{r,s}(G, \Gamma)$ of $\mathrm{SBlup}(G, \Gamma)$ with a Lie groupoid structure for *any* Lie subgroupoid Γ of a Lie groupoid G. This turns out to be very useful in order to analyze index type problems in many geometric situations.

The first use of deformation groupoids in connection with index theory appeared in [9]. Connes showed there that the analytic index on a compact manifold M can be described using a groupoid, called the "tangent groupoid". This groupoid was obtained as a deformation to the normal cone of the diagonal inclusion of M into the pair groupoid $M \times M$.

Since Connes' construction, deformation groupoids were used by many authors in various contexts.

- This idea of Connes was extended in [21] by considering the same construction of a deformation to the normal cone for smooth immersions which are groupoid morphisms. The groupoid obtained was used in order to define the wrong way functoriality for immersions of foliations [21, Section 3]. An analogous construction for submersions of foliations was also given in [21, Remark 3.19].
- In [32, 35] Monthubert and Pierrot, and Nistor, Weinstein and Xu considered the deformation to the normal cone of the inclusion $G^{(0)} \to G$ of the space of units of a smooth groupoid G. This generalization of Connes' tangent groupoid was called the adiabatic groupoid of G and denoted by G_{ad}. It was shown that this adiabatic groupoid still encodes the analytic index associated with G.
- Many other important articles use this idea of deformation groupoids. We will briefly discuss some of them in the sequel of the paper.

Let us come back to our discussion above. By construction, the propagation in the blow-up groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma))$ is tangent to V: its orbits are either contained in the open subset $\mathring{M} = M \setminus V$ or in its complement—the manifold $\mathbb{S}N_V^M$. In other words, the groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma)$ is the union of two Lie subgroupoids:

- an open subgroupoid, which is the restriction $G_{\mathring{M}}^{\mathring{M}} \rightrightarrows \mathring{M}$ of G to $\mathring{M} = M\setminus V$;
- a closed subgroupoid, its restriction to the boundary which is a Lie groupoid $\mathcal{S}N_\Gamma^G \rightrightarrows \mathbb{S}N_V^M$.

This gives rise to a C^*-algebraic exact sequence

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^G) \longrightarrow 0. \qquad (E^\partial_{\mathrm{SBlup}})$$

In order to study the index theory of $C^*(\mathrm{SBlup}_{r,s}(G,\Gamma))$, we will study this exact sequence and, in particular, its connecting map—more precisely the class of this exact sequence in the Kasparov group $KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}}))$.

There is a very natural parallel of this exact sequence with a corresponding exact sequence for the deformation to the normal cone groupoid, which reads

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(\mathrm{DNC}_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0. \qquad (E^\partial_{\mathrm{DNC}_+})$$

The sequence $(E^\partial_{\mathrm{DNC}_+})$ is somewhat easier to compute than $(E^\partial_{\mathrm{SBlup}})$ using [14]. In particular, if $\Gamma = V$ is just a space, the class of the exact sequence $(E^\partial_{\mathrm{DNC}_+})$ is the composition of a wrong way functoriality map with the (analytic) index map of the groupoid G (*cp.* Proposition 4.14 (iv)).

It turns out that there is a natural Connes–Thom map (in the sense of [8]) comparing these exact sequences. Even better, if the original propagation along G is nowhere tangent to V (we say that V is $\mathfrak{A}G$-*small*), these Connes–Thom maps are isomorphisms (KK^1-equivalences, *cp.* Theorem 4.9). We therefore naturally deduce the computation of the class of $(E^\partial_{\mathrm{SBlup}})$ when $\Gamma = V$ and V is $\mathfrak{A}G$-small (*cp.* Proposition 4.15)

We will, in addition, address another index question on the new groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma)$. This one concerns *full ellipticity, i.e.,* (pseudo)differential operators on the groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma)$ which are invertible modulo $C^*(G_{\mathring{M}}^{\mathring{M}})$. This question naturally leads us to the study of the exact sequence:

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow \Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \xrightarrow{\sigma_{\mathrm{full}}} \Sigma_{\mathrm{SBlup}}(G,\Gamma) \longrightarrow 0, \quad (\widetilde{E^{\mathrm{ind}}_{\mathrm{SBlup}}})$$

where $\Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma))$ is the C^*-algebra of pseudodifferential operators of order ≤ 0 on the groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma)$ and $\Sigma_{\mathrm{SBlup}}(G,\Gamma)$ is the quotient $\Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}})$.

The *full symbol algebra* $\Sigma_{\mathrm{SBlup}}(G,\Gamma)$ is naturally a fibered product (see [14, Section 4]):

$$\Sigma_{\mathrm{SBlup}}(G,\Gamma) = C(\mathbb{S}\mathfrak{A}^*\mathrm{SBlup}_{r,s}(G,\Gamma)) \times_{C(\mathbb{S}\mathfrak{A}^*\mathcal{S}N_\Gamma^G)} \Psi^*(\mathcal{S}N_\Gamma^G).$$

The first component corresponds to the principal symbol of an order 0 pseudodifferential operator; the second one is the restriction to the boundary.

We wish to compute the connecting map of the exact sequence $(\widetilde{E^{\mathrm{ind}}_{\mathrm{SBlup}}})$, and use again its parallel (DNC) exact sequence:

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow \Psi^*(\mathrm{DNC}_+(G,\Gamma)) \xrightarrow{\sigma_{\mathrm{full}}} \Sigma_{\mathrm{DNC}_+}(G,\Gamma) \longrightarrow 0, \ (\widetilde{E^{\mathrm{ind}}_{\mathrm{DNC}_+}})$$

which we directly relate (using [14]) to the analytic index map of the groupoid G when $\Gamma = V$.

As for the case discussed above, the sequences $(\widetilde{E^{\mathrm{ind}}_{\mathrm{SBlup}}})$ and $(\widetilde{E^{\mathrm{ind}}_{\mathrm{DNC}_+}})$ are related through natural Connes–Thom maps. When V is $\mathfrak{A}G$-small, in fact the maps relating sequences $(\widetilde{E^{\mathrm{ind}}_{\mathrm{SBlup}}})$ and $(\widetilde{E^{\mathrm{ind}}_{\mathrm{DNC}_+}})$ are KK^1-equivalences.

Finally, as an application, we consider the particular case when $\Gamma = V$ is a submanifold of M *transverse* to the action of G. We construct a calculus which resembles the Boutet de Monvel calculus for manifolds with boundary. We are planning to investigate the relations between these two calculi.

The paper is organized as follows:

- In Section 2 we review two geometric constructions: deformation to the normal cone and blow-up, and their functorial properties.
- In Section 3, using this functoriality, we study deformation to the normal cone and blow-up in the Lie groupoid context. We outline examples which recover groupoids constructed previously by several authors.
- In Section 4, applying the results obtained in [14], we compute the connecting maps and index maps of the groupoids constructed in Section 4.
- In Section 5 we describe the above mentioned Boutet de Monvel type calculus.
- In Appendix A we recall a few facts on the notion of $\mathcal{VB}$ groupoids, and study the particular case of a $\mathcal{VB}$ groupoid over a manifold.
- The present paper is the second part of the article that appeared on the arXiv (arXiv:1705.09588). Since this paper was quite long and addressed a large variety of situations, we decided to split it into two pieces hoping to make it easier to read. The first part is [14].

Our constructions involved a large amount of notation, that we tried to choose as coherent as possible. We found it, however, helpful to list several items of the notation introduced in [14] and the one introduced here in an index at the end of the paper.

2. Classical geometric constructions: normal bundle, deformation to the normal cone, blow-up and functoriality

As mentioned in the introduction, index problems in a large number of geometrical situations lead to consider two geometric constructions of groupoids: deformation to the normal cone and blow-up. These two constructions are classical in algebraic geometry. In this section we recall these constructions and we emphasize their properties that are relevant for our purposes.

Throughout this section, Y will be a smooth manifold and X a locally closed submanifold (the same constructions hold if we are given an injective immersion $X \to Y$). Let us call such a pair (Y, X) a *manifold pair*.

Given two manifold pairs (Y, X) and (Y', X'), a morphism f from (Y, X) to (Y', X') is a smooth map $f\colon Y \to Y'$ which restricts to a smooth map $f_X\colon X \to X'$. Thus we have a commutative diagram of smooth maps

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f"] \\ X' \arrow[r, hook] & Y', \end{tikzcd}$$

where the horizontal arrows are inclusions of submanifolds.

2.1. Normal bundle.

We begin by a few remarks on the normal bundle construction.

We denote by $N_X^Y = \bigcup_{x\in X} T_xY/T_xX$ the (total space) of the normal bundle of X in Y.

Functoriality. The differential $df\colon TY \to TY'$ of a morphism f of manifold pairs

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f"] \\ X' \arrow[r, hook] & Y' \end{tikzcd}$$

maps TX to TX'. Thus it induces a natural smooth map $N(f)\colon N_X^Y \to N_{X'}^{Y'}$ given by $N(f)_x \circ p_x = p'_{f(x)} \circ df_x$ for all $x \in X$ (where $p_x\colon T_xY \to (N_X^Y)_x$ and $p'_x\colon T_xY' \to (N_{X'}^{Y'})_x$).

Exponential map. An *exponential map* (or tubular neighborhood construction) for the manifold pair (Y, X) is a diffeomorphism $\theta\colon U' \to U$ from an open neighborhood U' of the 0-section in N_X^Y to an open neighborhood U of X which satisfies:

- the restriction of θ to X viewed as the 0-section in N_X^Y is the identity map:
$$\theta(x,0) = x \quad \text{for all } x \in X;$$
- the differential of θ is equal to the "identity on the normal direction to X":
$$p_x \circ d\theta_x = q_x \quad \text{for all } x \in X,$$
where $p_x\colon T_xY \to (N_X^Y)_x = (T_xY)/(T_xX)$ and $q_x\colon T_xN_X^Y \simeq (N_X^Y)_x \oplus (T_xX) \to (N_X^Y)_x$ are the projections.

Note that one can just use such a construction locally, *i.e.,* taking small open subsets of X and diffeomorphisms of N_V^Y with small open subsets in Y. One actually needs to use this local construction when X is just an immersed submanifold of Y.

2.2. **Deformation to the normal cone.** The deformation to the normal cone $\mathrm{DNC}(Y,X)$ is obtained by gluing $N_X^Y \times \{0\}$ with $Y \times \mathbb{R}^*$. The smooth structure of $\mathrm{DNC}(Y,X)$ is described by use of any exponential map $\theta\colon U' \to U$ from an open neighborhood U' of the 0-section in N_X^Y to an open neighborhood U of X. The manifold structure of $\mathrm{DNC}(Y,X)$ is then given by the requirement that:

(i) the inclusion $Y \times \mathbb{R}^* \to \mathrm{DNC}(Y,X)$ and

(ii) the map $\Theta\colon \Omega' = \{((x,\xi),\lambda) \in N_X^Y \times \mathbb{R};\, (x,\lambda\xi) \in U'\} \to \mathrm{DNC}(Y,X)$ defined by $\Theta((x,\xi),0) = ((x,\xi),0)$ and $\Theta((x,\xi),\lambda) = (\theta(x,\lambda\xi),\lambda) \in Y \times \mathbb{R}^*$ if $\lambda \neq 0$

are diffeomorphisms onto open subsets of $\mathrm{DNC}(Y,X)$.

It is easily shown that $\mathrm{DNC}(Y,X)$ has indeed a smooth structure satisfying these requirements and that this smooth structure does not depend on the choice of θ. (See, for example, [6] for a detailed description of this structure).

In other words, $\mathrm{DNC}(Y,X)$ is obtained by gluing $Y \times \mathbb{R}^*$ with Ω' by means of the diffeomorphism $\Theta\colon \Omega' \cap (N_X^Y \times \mathbb{R}^*) \to U \times \mathbb{R}^*$.

Let us recall the following facts which are essential in our construction.

Definition 2.3 (The zooming action of $\mathbb{R}^*$)**.** The group $\mathbb{R}^*$ acts on $\mathrm{DNC}(Y,X)$ by $\lambda.(w,t) = (w,\lambda t)$ and $\lambda.((x,\xi),0) = ((x,\lambda^{-1}\xi),0)$ (with $\lambda, t \in \mathbb{R}^*$, $w \in Y$, $x \in X$ and $\xi \in (N_X^Y)_x$).

Remarks 2.4. (i) The zooming action is easily seen to be free and proper on the open subset $\mathrm{DNC}(Y,X) \setminus X \times \mathbb{R}$. Indeed, for $(x,\xi,t) \in \Omega' \subset N_X^Y \times \mathbb{R}$, the zooming action is given by $\lambda.(x,\xi,t) = (x,\lambda^{-1}\xi,\lambda t)$ under the map Θ^{-1}.

(ii) In the following sections we will apply this construction to Lie groupoids, and many natural Lie groupoids are non Hausdorff manifolds. If the manifolds X and Y are not assumed to be Hausdorff (but of course *locally* Hausdorff) and $X \subset Y$ is *locally* closed, then $\mathrm{DNC}(Y,X)$ is also locally Hausdorff. The subset $\mathrm{DNC}(Y,X) \setminus X \times \mathbb{R}$ is a submanifold if $X \subset Y$ is closed. In that case, the zooming action of $\mathbb{R}_+^*$ restricted to $\mathrm{DNC}(Y,X) \setminus X \times \mathbb{R}$ is *locally* proper. By this, we mean that every point has a neighborhood invariant under the action, on which the action is proper (*cp.* [14, Remark 2.5]).

Definition 2.5 (Functoriality)**.** Given a morphism f of manifold pairs:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f"] \\ X' \arrow[r, hook] & Y', \end{tikzcd}$$

we naturally obtain a smooth map $\mathrm{DNC}(f)\colon \mathrm{DNC}(Y,X) \to \mathrm{DNC}(Y',X')$. This map is defined by $\mathrm{DNC}(f)(y,\lambda) = (f(y),\lambda)$ for $y \in Y$ and $\lambda \in \mathbb{R}_*$ and $\mathrm{DNC}(f)(x,\xi,0) = (N(f)(x,\xi),0)$ for $x \in X$ and $\xi \in (N_X^Y)_x$. This map is of course equivariant with respect to the zooming action of $\mathbb{R}^*$.

Remarks 2.6. Let us make a few remarks concerning the DNC construction.

(i) The map equal to identity on $X \times \mathbb{R}^*$ and sending $X \times \{0\}$ to the zero section of N_X^Y leads to an embedding of $X \times \mathbb{R}$ into $\mathrm{DNC}(Y,X)$, we may often identify $X \times \mathbb{R}$ with its image in $\mathrm{DNC}(Y,X)$. As $\mathrm{DNC}(X,X) = X \times \mathbb{R}$, this corresponds to the functoriality of DNC for the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X \arrow[d] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

(ii) We have a natural smooth map $\pi\colon \mathrm{DNC}(Y,X) \to Y \times \mathbb{R}$ defined by $\pi(y,\lambda) = (y,\lambda)$ (for $y \in Y$ and $\lambda \in \mathbb{R}^*$) and $\pi((x,\xi),0) = (x,0)$ (for $x \in X \subset Y$ and $\xi \in (N_X^Y)_x$ a normal vector). This corresponds to the functoriality of DNC for the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ Y \arrow[r, hook] & Y. \end{tikzcd}$$

(iii) To see that the smooth structure on $\mathrm{DNC}(Y,X)$ is well defined and establish functoriality, one may also note that the following maps are smooth:

- the map $\pi\colon \mathrm{DNC}(Y,X) \to Y \times \mathbb{R}$ defined above;
- given a smooth function $f\colon Y \to \mathbb{R}$ whose restriction to X is 0, the map $F_f\colon \mathrm{DNC}(Y,X) \to \mathbb{R}$ defined by $F_f(y,\lambda) = \frac{f(y)}{\lambda}$ (for $y \in Y$ and $\lambda \in \mathbb{R}^*$) and $F_f(x,p_x(\xi),0) = df_x(\xi)$ for $x \in X$ and $\xi \in T_xY$, where $p_x\colon T_xY \to (N_X^Y)_x = T_xY/T_xX$ is the quotient map (note that df_x vanishes on T_xX).

These maps describe the smooth structure of $\mathrm{DNC}(Y,X)$. Indeed, given a manifold Z, a map $g\colon Z \to \mathrm{DNC}(Y,X)$ is smooth if and only if $\pi\circ g$ and the maps $F_f \circ g$ are smooth. Actually, a finite number of those give rise to an immersion $\mathrm{DNC}(Y,X) \to Y \times \mathbb{R} \times \mathbb{R}^k$ (at least locally, if we do not assume

X to be compact). This offers an alternative proof of the independence of the smooth structure relative to the choice of the exponential map.

(iv) If Y_1 is an open subset of Y_2 such that $X \subset Y_1$, then $\mathrm{DNC}(Y_1, X)$ is an open subset of $\mathrm{DNC}(Y_2, X)$, and $\mathrm{DNC}(Y_2, X)$ is the union of the open subsets $\mathrm{DNC}(Y_1, X)$ and $Y_2 \times \mathbb{R}^*$. This reduces to the case when Y_1 is a tubular neighborhood, and therefore to the case where Y is (diffeomorphic to) the total space of a real vector bundle over X. In that case, one gets $\mathrm{DNC}(Y, X) = Y \times \mathbb{R}$ and the zooming action of $\mathbb{R}^*$ on $\mathrm{DNC}(Y, X) = Y \times \mathbb{R}$ is given by $\lambda.((x, \xi), t) = ((x, \lambda^{-1}\xi), \lambda t)$ (with $\lambda \in \mathbb{R}^*$, $t \in \mathbb{R}$, $x \in X$ and $\xi \in Y_x$).

(v) More generally, let X be a submanifold of Y and let E be (the total space of) a real vector bundle over Y. Then $\mathrm{DNC}(E, X)$ identifies with the total space of the pull back vector bundle $\hat{\pi}^*(E)$ over $\mathrm{DNC}(Y, X)$, where $\hat{\pi}$ is the composition of $\pi \colon \mathrm{DNC}(Y, X) \to Y \times \mathbb{R}$ (remark (ii)) with the projection $Y \times \mathbb{R} \to Y$. The zooming action of $\mathbb{R}^*$ is $\lambda.(w, \xi) = (\lambda.w, \lambda^{-1}\xi)$ for $w \in \mathrm{DNC}(Y, X)$ and $\xi \in E_{\hat{\pi}(w)}$.

(vi) Let X_1 be a (locally closed) smooth submanifold of a smooth manifold Y_1 and let $f \colon Y_2 \to Y_1$ be a smooth map transverse to X_1. Put $X_2 = f^{-1}(X_1)$. Then the normal bundle $N_{X_2}^{Y_2}$ identifies with the pull back of $N_{X_1}^{Y_1}$ by the restriction $X_2 \to X_1$ of f. It follows that $\mathrm{DNC}(Y_2, X_2)$ identifies with the fibered product $\mathrm{DNC}(Y_1, X_1) \times_{Y_1} Y_2$.

(vii) More generally, let Y, Y_1, Y_2 be smooth manifolds and $f_i \colon Y_i \to Y$ be smooth maps. Assume that f_1 is transverse to f_2. Let $X \subset Y$ and $X_i \subset Y_i$ be (locally closed) smooth submanifolds. Assume that $f_i(X_i) \subset X$ and that the restrictions $g_i \colon X_i \to X$ of f_i are also transverse. We thus have a diagram

$$\begin{tikzcd} X_1 \arrow[r, "g_1"] \arrow[d, hook] & X \arrow[d, hook] & X_2 \arrow[l, "g_2"'] \arrow[d, hook] \\ Y_1 \arrow[r, "f_1"] & Y & Y_2. \arrow[l, "f_2"'] \end{tikzcd}$$

Then the maps $\mathrm{DNC}(f_i) \colon \mathrm{DNC}(Y_i, X_i) \to \mathrm{DNC}(Y, X)$ are transverse and the deformation to the normal cone of fibered products $\mathrm{DNC}(Y_1 \times_Y Y_2, X_1 \times_X X_2)$ identifies with the fibered product $\mathrm{DNC}(Y_1, X_1) \times_{\mathrm{DNC}(Y,X)} \mathrm{DNC}(Y_2, X_2)$.

Note that construction (vi) is the particular case $X = Y = Y_1$ of our construction here.

Notation 2.7. We denote by $\mathrm{DNC}_+(Y, X)$ the closed subset $\mathrm{DNC}_+(Y, X) = Y \times \mathbb{R}_+^* \cup N_X^Y \times \{0\} = \pi^{-1}(Y \times \mathbb{R}_+)$ of $\mathrm{DNC}(Y, X)$.

2.8. **Blow-up constructions.** The blow-up $\mathrm{Blup}(Y, X)$ is a smooth manifold which is a union of $Y \setminus X$ with the (total space) $\mathbb{P}(N_X^Y)$ of the projective space of the normal bundle N_X^Y of X in Y. We will also use the "spherical version" $\mathrm{SBlup}(Y, X)$ of $\mathrm{Blup}(Y, X)$ which is a manifold with boundary obtained by gluing $Y \setminus X$ with the (total space of the) sphere bundle $\mathbb{S}(N_X^Y)$. We have an obvious smooth onto map $\mathrm{SBlup}(Y,X) \to \mathrm{Blup}(Y,X)$ with fibers 1 or 2 points. These spaces are of course similar and we will often give details in our constructions to the one of them which is the most convenient for our purposes.

We may view $\mathrm{Blup}(Y,X)$ as the quotient space of a submanifold of the deformation to the normal cone $\mathrm{DNC}(Y,X)$ under the zooming action of $\mathbb{R}^*$.

Recall that the group $\mathbb{R}^*$ acts on $\mathrm{DNC}(Y,X)$ by $\lambda.(w,t) = (w,\lambda t)$ and $\lambda.((x,\xi),0) = ((x,\lambda^{-1}\xi),0)$ (with $\lambda, t \in \mathbb{R}^*$, $w \in Y$, $x \in X$ and $\xi \in (N_X^Y)_x$). According to Remark 2.4 (i), this action is free and (locally) proper on the open subset $\mathrm{DNC}(Y,X) \setminus X \times \mathbb{R}$.

Definition 2.9. We put

$$\mathrm{Blup}(Y,X) = (\mathrm{DNC}(Y,X) \setminus X \times \mathbb{R})/\mathbb{R}^*$$

and

$$\mathrm{SBlup}(Y,X) = (\mathrm{DNC}_+(Y,X) \setminus X \times \mathbb{R}_+)/\mathbb{R}_+^*.$$

Remark 2.10. With the notation of Section 2.2, $\mathrm{Blup}(Y,X)$ is thus obtained by gluing $Y \setminus X = ((Y \setminus X) \times \mathbb{R}^*)/\mathbb{R}^*$ with $(\Omega' \setminus (X \times \mathbb{R}))/\mathbb{R}^*$, using the map Θ which is equivariant with respect to the zooming action of $\mathbb{R}^*$.

Choose a Euclidean metric on N_X^Y. Let $\mathbb{S} = \{((x,\xi),\lambda) \in \Omega';\ \|\xi\| = 1\}$ and let τ be the involution of $\mathbb{S}$ given by $((x,\xi),\lambda) \mapsto ((x,-\xi),-\lambda)$. The map Θ induces a diffeomorphism of $\mathbb{S}/\tau$ with an open neighborhood $\widetilde{\Omega}$ of $\mathbb{P}(N_X^Y)$ in $\mathrm{Blup}(Y,X)$.

Since $\hat{\pi}\colon \mathrm{DNC}(Y,X) \to Y$ is invariant by the zooming action of $\mathbb{R}^*$, we obtain a natural smooth map $\tilde{\pi}\colon \mathrm{Blup}(Y,X) \to Y$ whose restriction to $Y \setminus X$ is the identity and whose restriction to $\mathbb{P}(N_X^Y)$ is the canonical projection $\mathbb{P}(N_X^Y) \to X \subset Y$. This map is easily seen to be proper.

Remark 2.11 (*cp.* Remark 2.4 (ii))**.** If X and Y are not assumed to be Hausdorff, we may still form the manifold $\mathrm{Blup}(Y,X)$ since the action of $\mathbb{R}_+^*$ on $\mathrm{DNC}(Y,X) \setminus (X \times \mathbb{R})$ is *locally* proper. Also, the map $\mathrm{Blup}(Y,X) \to Y \times \mathbb{R}$ is locally proper.

Remark 2.12. Note that, according to Remark 2.6 (v), $\mathrm{DNC}(Y,X)$ canonically identifies with the open subset $\mathrm{Blup}(Y \times \mathbb{R}, X \times \{0\}) \setminus \mathrm{Blup}(Y \times \{0\}, X \times \{0\})$ of $\mathrm{Blup}(Y \times \mathbb{R}, X \times \{0\})$. Thus, since the map $\mathrm{Blup}(Y \times \mathbb{R}, X \times \{0\}) \to Y \times \mathbb{R}$ is proper, one may think at $\mathrm{Blup}(Y \times \mathbb{R}, X \times \{0\})$ as a "local compactification" of $\mathrm{DNC}(Y,X)$.

Example 2.13. In the case where Y is a real vector bundle over X, $\mathrm{Blup}(Y,X)$ identifies noncanonically with an open submanifold of the bundle of projective spaces $\mathbb{P}(Y \times \mathbb{R})$ over X. Indeed, in that case, $\mathrm{DNC}(Y,X) = Y \times \mathbb{R}$; choose a Euclidean structure on the bundle Y. Consider the smooth involution Φ from $(Y \setminus X) \times \mathbb{R}$ onto itself, which to (x,ξ,t) associates $(x, \frac{\xi}{\|\xi\|^2}, t)$ (for $x \in X$, $\xi \in Y_x$, $t \in \mathbb{R}$). This map transforms the zooming action of $\mathbb{R}^*$ on $\mathrm{DNC}(Y,X)$ into the action of $\mathbb{R}^*$ by dilations on the vector bundle $Y \times \mathbb{R}$ over X and thus defines a diffeomorphism of $\mathrm{Blup}(Y,X)$ into its image, which is the open set

$\mathbb{P}(Y\times\mathbb{R})\setminus X$, where X embeds into $\mathbb{P}(Y\times\mathbb{R})$ by mapping $x\in X$ to the line $\{(x,0,t),\, t\in\mathbb{R}\}$.

Functoriality.

Definition 2.14 (Functoriality)**.** Let f be a morphism of manifold pairs:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f"] \\ X' \arrow[r, hook] & Y'. \end{tikzcd}$$

Let $U_f=\mathrm{DNC}(Y,X)\setminus\mathrm{DNC}(f)^{-1}(X'\times\mathbb{R})$ be the inverse image by $\mathrm{DNC}(f)$ of the complement in $\mathrm{DNC}(Y',X')$ of the subset $X'\times\mathbb{R}$. We thus obtain a smooth map $\mathrm{Blup}(f)\colon \mathrm{Blup}_f(Y,X)\to\mathrm{Blup}(Y',X')$, where $\mathrm{Blup}_f(Y,X)\subset\mathrm{Blup}(Y,X)$ is the quotient of U_f by the zooming action of $\mathbb{R}^*$.

In particular:

(i) If $X\subset Y_1$ are (locally) closed submanifolds of a manifold Y_2, then $\mathrm{Blup}(Y_1,X)$ is a submanifold of $\mathrm{Blup}(Y_2,X)$.

(ii) Also, if Y_1 is an open subset of Y_2 such that $X\subset Y_1$, then $\mathrm{Blup}(Y_1,X)$ is an open subset of $\mathrm{Blup}(Y_2,X)$ and $\mathrm{Blup}(Y_2,X)$ is the union of the open subsets $\mathrm{Blup}(Y_1,X)$ and $Y_2\setminus X$. This allows to reduce to the case when Y_1 is a tubular neighborhood.

Fibered products. Let X_1 be a (locally closed) smooth submanifold of a smooth manifold Y_1 and let $f\colon Y_2\to Y_1$ be a smooth map transverse to X_1. Put $X_2=f^{-1}(X_1)$. Recall from Remark 2.6 (vi) that in this situation $\mathrm{DNC}(Y_2,X_2)$ identifies with the fibered product $\mathrm{DNC}(Y_1,X_1)\times_{Y_1}Y_2$. Thus $\mathrm{Blup}(Y_2,X_2)$ identifies with the fibered product $\mathrm{Blup}(Y_1,X_1)\times_{Y_1}Y_2$.

Vector bundles over blow-ups.

Fact 2.15. Let $p\colon E\to Y$ be a (real) vector bundle and let $F\to X$ be a subbundle of the restriction of E to X. Then $\mathrm{Blup}(p)\colon\mathrm{Blup}_p(E,F)\to\mathrm{Blup}(Y,X)$ is a vector bundle. Indeed, $N(p)\colon N_F^E\to N_X^Y$ carries a natural vector bundle structure; therefore $\mathrm{DNC}(p)\colon\mathrm{DNC}(E,F)\to\mathrm{DNC}(Y,X)$ is also a vector bundle, as well as its restriction to $\mathrm{DNC}(Y,X)\setminus X\times\mathbb{R}$. Since this structure is invariant by the action of $\mathbb{R}^*$, it passes to the quotient.

The tangent bundle of $\mathrm{Blup}(Y,X)$ is naturally seen to be $\mathrm{Blup}_p(TY,TX)$.

Note also that, given vector bundles $p_i\colon E_i\to Y$ $(i=1,2)$ and subbundles $F_i\to X$ of the restrictions of E_i to X, the blow-up construction of a linear bundle map $f\colon E_1\to E_2$ such that $f(F_1)\subset F_2$ induces a linear bundle map $\mathrm{Blup}(f)\colon\mathrm{Blup}_{p_1}(E_1,F_1)\to\mathrm{Blup}_{p_2}(E_2,F_2)$.

3. Constructions of groupoids

We start this section with a quick reminder of some generalities on Lie groupoids which will be useful for the sequel of this paper. Then we use the functoriality of the normal space, deformation to the normal cone and blow-up constructions to apply these constructions in the groupoid setting and look at various examples.

3.1. **Generalities around transversality and Morita equivalence of groupoids.**

3.1.1. *Some notation.* Let $G \overset{r,s}{\rightrightarrows} G^{(0)}$ be a groupoid with source s, range r and space of units $G^{(0)}$. For any maps $f\colon A \to G^{(0)}$ and $g\colon B \to G^{(0)}$, define

$$G^f = \{(a,x) \in A \times G;\, r(x) = f(a)\}, \quad G_g = \{(x,b) \in G \times B;\, s(x) = g(b)\}$$

and

$$G_g^f = \{(a,x,b) \in A \times G \times B;\, r(x) = f(a),\, s(x) = g(b)\}.$$

In particular, for $A, B \subset G^{(0)}$, we put $G^A = \{x \in G;\, r(x) \in A\}$ and $G_A = \{x \in G;\, s(x) \in A\}$; we also put $G_A^B = G_A \cap G^B$.

3.1.2. *Transversality.* Let us recall the following definition (see, *e.g.,* [42] for details):

Definition 3.2. Let $G \overset{r,s}{\rightrightarrows} M$ be a Lie groupoid with set of objects $G^{(0)} = M$ and Lie algebroid $\mathfrak{A}G$ with anchor map $\natural$. Let V be a manifold. A smooth map $f\colon V \to M$ is said to be *transverse* to (the action of the groupoid) G if for every $x \in V$, $df_x(T_xV) + \natural_{f(x)}\mathfrak{A}_{f(x)}G = T_{f(x)}M$.

An equivalent condition is that the map $(\gamma, y) \mapsto r(\gamma)$ defined on the fibered product $G_f = G \underset{s,f}{\times} V$ is a submersion from G_f to M.

A submanifold V of M is *transverse* to G if the inclusion $V \to M$ is transverse to G—equivalently, if for every $x \in V$, the composition $q_x = p_x \circ \natural_x \colon \mathfrak{A}_xG \to (N_V^M)_x = T_xM/T_xV$ is onto.

Remark 3.3. Let V be a (locally) closed submanifold of M transverse to a groupoid $G \overset{r,s}{\rightrightarrows} M$. Denote by N_V^M the (total space) of the normal bundle of V in M. Upon arguing locally, we can assume that V is compact.

By the transversality assumption, the anchor $\natural\colon \mathfrak{A}G_{|V} \to TM_{|V}$ induces a surjective bundle morphism $\mathfrak{A}G_{|V} \to N_V^M$. Choose then

(i) an exponential map $\theta\colon U' \to U$ which is a diffeomorphism from a neighborhood U' in $\mathfrak{A}G \to G$ of M onto a neighborhood U of $M = G^{(0)}$ in G such that $s \circ \theta(x,\xi) = x$ for all $x \in M$ and $\xi \in (\mathfrak{A}G)_x$,
(ii) a subbundle $F \subset \mathfrak{A}G_{|V}$ of the restriction $\mathfrak{A}G_{|V}$ such that $F \to N_V^M$ is an isomorphism.

We thus obtain a submanifold $W = \theta(F \cap U') \subset G$ such that $r\colon W \to M$ is étale at every point of V and s is a submersion from W onto V. Replacing U' by a an open subset, we may assume that $r\colon W \to M$ is a diffeomorphism onto a tubular neighborhood of V in M, diffeomorphic to N_V^M. The map $W \times_V G_V^V \times_V W \to G$ defined by $(\gamma_1, \gamma_2, \gamma_3) \mapsto \gamma_1 \circ \gamma_2 \circ \gamma_3^{-1}$ is a diffeomorphism and a groupoid isomorphism from the pull back groupoid (see next section) $(G_V^V)_s^s = W \times_V G_V^V \times_V W$ onto the open subgroupoid $G_{r(W)}^{r(W)}$ of G.

3.3.1. *Pull back.* If $f\colon V \to M$ is transverse to a Lie groupoid $G \overset{r,s}{\rightrightarrows} M$, then G_f^f is a submanifold of $V \times G \times V$ naturally equipped with a structure of Lie groupoid $G_f^f \rightrightarrows V$. It is called the *pull back groupoid*.

If $f_i\colon V_i \to M$ are transverse to G (for $i = 1, 2$), then we obtain a Lie groupoid $G_{f_1 \sqcup f_2}^{f_1 \sqcup f_2} \rightrightarrows V_1 \sqcup V_2$. The *linking manifold* $G_{f_2}^{f_1}$ is a clopen submanifold. We denote by $C^*(G_{f_2}^{f_1})$ the closure in $C^*(G_{f_1 \sqcup f_2}^{f_1 \sqcup f_2})$ of the space of functions (half densities) with support in $G_{f_2}^{f_1}$; it is a $C^*(G_{f_1}^{f_1}) - C^*(G_{f_2}^{f_2})$ bimodule.

Fact 3.4. The bimodule $C^*(G_{f_2}^{f_1})$ is full if all the G-orbits meeting $f_2(V_2)$ meet also $f_1(V_1)$.

3.4.1. *Morita equivalence.* Two Lie groupoids $G_1 \overset{r,s}{\rightrightarrows} M_1$ and $G_2 \overset{r,s}{\rightrightarrows} M_2$ are *Morita equivalent* if there exists a groupoid $G \overset{r,s}{\rightrightarrows} M$ and smooth maps $f_i\colon M_i \to M$ transverse to G such that the pull back groupoids $G_{f_i}^{f_i}$ identify to G_i, and $f_i(M_i)$ meets all the orbits of G.

Equivalently, a Morita equivalence is given by a linking manifold X with extra data: surjective smooth submersions $r\colon X \to G_1^{(0)}$ and $s\colon X \to G_2^{(0)}$ and compositions $G_1 \times_{s,r} X \to X$, $X \times_{s,r} G_2 \to X$, $X \times_{r,r} X \to G_2$ and $X \times_{s,s} X \to G_1$ with natural associativity conditions (see [33] for details). In the above situation, X is the manifold $G_{f_2}^{f_1}$ and the extra data are the range and source maps and the composition rules of the groupoid $G_{f_1 \sqcup f_2}^{f_1 \sqcup f_2} \rightrightarrows M_1 \sqcup M_2$ (see [33]).

If the map $r\colon X \to G_1^{(0)}$ is surjective but $s\colon X \to G_2^{(0)}$ is not necessarily surjective, then G_1 is Morita equivalent to the restriction of G_2 to the open saturated subspace $s(X)$. We say that G_1 is *sub-Morita* equivalent to G_2.

3.4.2. *Remarks on possible singularities.*

About corners. We wish to emphasize a remark already made in [14]:

Many manifolds and groupoids that occur in our constructions have boundaries or corners. In fact, all the groupoids we consider sit naturally inside Lie groupoids *without boundaries* as restrictions to closed *saturated* subsets. This means that we consider subgroupoids $G_V^V = G_V$ of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$, where V is a closed subset of $G^{(0)}$. Such groupoids have a natural algebroid, adiabatic deformation, pseudodifferential calculus, *etc.* that are restrictions to V and G_V of the corresponding objects on $G^{(0)}$ and G. We chose to give our definitions and constructions for Lie groupoids for the clarity of the exposition. The case of a longitudinally smooth groupoid over a manifold with corners is a straight-forward generalization using a convenient restriction.

About non-Haudorffness. Our groupoids need not be Hausdorff. Precisely, for $G \rightrightarrows G^{(0)}$, the manifold G may be a non-Haudorff manifold, but $G^{(0)}$ will always be assumed to be Hausdorff. Of course a non-Hausdorff manifold is locally Hausdorff.

3.5. **Normal groupoids, deformation groupoids and blow-up groupoids.**

3.5.1. *Definitions.* Let Γ be a closed Lie subgroupoid of a Lie groupoid G. Using functoriality (*cp.* Definitions 2.5, 2.14) of the normal bundle, the DNC and the Blup constructions, we may construct a normal, a deformation and a blow-up groupoid.

(i) The normal bundle N_Γ^G carries a Lie groupoid structure with objects $N_{\Gamma^{(0)}}^{G^{(0)}}$: its source and range maps are $N(s)$ and $N(r)$; the space of composable arrows identifies with $N(G^{(2)},\Gamma^{(2)})$ and its product with $N(m)$, where m denotes both products $G^{(2)}\to G$ and $\Gamma^{(2)}\to\Gamma$. We denote by $\mathcal{N}_\Gamma^G \rightrightarrows N_{\Gamma^{(0)}}^{G^{(0)}}$ this *normal groupoid*. Note that the source and range maps of $\mathcal{N}_\Gamma^G$ are not equal as soon as the source and range maps of G restricted to Γ are different (it is not the vector bundle viewed as a groupoid). This is a typical example of a $\mathcal{VB}$ groupoid in the sense of Pradines ([37, 23] see also the appendix, Definition A.2).

(ii) The manifold $\mathrm{DNC}(G,\Gamma)$ is naturally a Lie groupoid (unlike what was asserted in [21, Remark 3.19]). Its unit space is $\mathrm{DNC}(G^{(0)},\Gamma^{(0)})$; its source and range maps are $\mathrm{DNC}(s)$ and $\mathrm{DNC}(r)$; the space of composable arrows identifies with $\mathrm{DNC}(G^{(2)},\Gamma^{(2)})$ and its product with $\mathrm{DNC}(m)$.

(iii) The subset $\widetilde{\mathrm{DNC}}(G,\Gamma)=U_r\cap U_s$ of $\mathrm{DNC}(G,\Gamma)$ consisting of elements whose image by $\mathrm{DNC}(r)$ and $\mathrm{DNC}(s)$ is not in $G_1^{(0)}\times\mathbb{R}$ is an open subgroupoid of $\mathrm{DNC}(G,\Gamma)$: it is the restriction of $\mathrm{DNC}(G,\Gamma)$ to the open subspace $\mathrm{DNC}(G^{(0)},G_1^{(0)})\setminus G_1^{(0)}\times\mathbb{R}$.

(iv) The group $\mathbb{R}^*$ acts on $\mathrm{DNC}(G,\Gamma)$ via the zooming action by groupoid morphisms. Its action on $\widetilde{\mathrm{DNC}}(G,\Gamma)$ is (locally) proper. Therefore the open subset $\mathrm{Blup}_{r,s}(G,\Gamma)=\widetilde{\mathrm{DNC}}(G,\Gamma)/\mathbb{R}^*$ of $\mathrm{Blup}(G,\Gamma)$ inherits a groupoid structure as well: its space of units is $\mathrm{Blup}(G_2^{(0)},G_1^{(0)})$; its source and range maps are $\mathrm{Blup}(s)$ and $\mathrm{Blup}(r)$ and the product is $\mathrm{Blup}(m)$.

(v) In the same way, we define the groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma)$. It is the quotient of the restriction $\widetilde{\mathrm{DNC}}_+(G,\Gamma)$ of $\widetilde{\mathrm{DNC}}(G,\Gamma)$ to $\mathbb{R}_+$ by the action of $\mathbb{R}_+^*$.

(vi) The singular part of $\mathrm{SBlup}_{r,s}(G,\Gamma)$, *i.e.,* its restriction to the boundary $\mathbb{S}N_V^M$ is the *spherical normal groupoid* $\mathcal{S}N_\Gamma^G$. It is the quotient by the action of $\mathbb{R}_+^*$ of the restriction of $\mathcal{N}_\Gamma^G\rightrightarrows N_V^M$ to the open subset $N_V^M\setminus V$ of its objects.

An analogous result about the groupoid structure on $\mathrm{Blup}_{r,s}(G,\Gamma)$ in the case of $\Gamma^{(0)}$ being a hypersurface of $G^{(0)}$ can be found in [16, Thm. 2.8] (*cp.* also [17]).

3.5.2. *Algebroids and anchors.* The (total space of the) Lie algebroid $\mathfrak{A}\Gamma$ is a closed submanifold (and a subbundle) of $\mathfrak{A}G$. The functoriality enables to get the Lie algebroids of the previous construction. Indeed, we have the following:

(i) The Lie algebroid of $\mathcal{N}_\Gamma^G$ is $N_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}$. Its anchor map is $N(\natural_G)\colon N_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}\to N_{T\Gamma^{(0)}}^{TG^{(0)}}\simeq TN_{\Gamma^{(0)}}^{G^{(0)}}$.

(ii) The groupoid $\mathrm{DNC}(G,\Gamma)$ is the union of its open subgroupoid $G\times\mathbb{R}^*$ with its closed Lie sub-groupoid $\mathcal{N}_\Gamma^G$. The Lie algebroid of $G\times\mathbb{R}^*$ is $\mathfrak{A}G\times\mathbb{R}^*$ and the anchor is just the map $\natural_G\times\mathrm{id}\colon \mathfrak{A}G\times\mathbb{R}^*\to T(G^{(0)}\times\mathbb{R}^*_+)$. It follows that the Lie algebroid of $\mathrm{DNC}(G,\Gamma)$ is $\mathrm{DNC}(\mathfrak{A}G,\mathfrak{A}\Gamma)$. Its anchor map is $\mathrm{DNC}(\natural_G)\colon \mathrm{DNC}(\mathfrak{A}G,\mathfrak{A}\Gamma)\to \mathrm{DNC}(TG^{(0)},T\Gamma^{(0)})\subset T\mathrm{DNC}(G^{(0)},\Gamma^{(0)})$.

(iii) Similarly, the Lie algebroid of $\mathrm{Blup}_{r,s}(G,\Gamma)$ is $\mathrm{Blup}_p(\mathfrak{A}G,\mathfrak{A}\Gamma)$. Its anchor map is $\mathrm{Blup}(\natural_G)\colon \mathrm{Blup}_p(\mathfrak{A}G,\mathfrak{A}\Gamma)\to\mathrm{Blup}_q(TG^{(0)},T\Gamma^{(0)})$. Here, $p\colon\mathfrak{A}G\to G^{(0)}$ and $q\colon TG^{(0)}\to G^{(0)}$ denote the bundle projections, see Fact 2.15.

3.5.3. *Stability under Morita equivalence.* Let $G_1\rightrightarrows G_1^{(0)}$ and $G_2\rightrightarrows G_2^{(0)}$ be Lie groupoids, $\Gamma_1\subset G_1$ and $\Gamma_2\subset G_2$ Lie subgroupoids. A *Morita equivalence of the pair* $(\Gamma_1\subset G_1)$ with the pair $(\Gamma_2\subset G_2)$ is given by a pair $(X\subset Y)$, where Y is a linking manifold which is a Morita equivalence between G_1 and G_2 and $X\subset Y$ is a submanifold of Y such that the maps r,s and products of Y (see page 12) restrict to a Morita equivalence X between Γ_1 and Γ_2.

Then, by functoriality,

- $\mathrm{DNC}(Y,X)$ is a Morita equivalence between $\mathrm{DNC}(G_1,\Gamma_1)$ and $\mathrm{DNC}(G_2,\Gamma_2)$,
- $\mathrm{DNC}_+(Y,X)$ is a Morita equivalence between $\mathrm{DNC}_+(G_1,\Gamma_1)$ and $\mathrm{DNC}_+(G_2,\Gamma_2)$,
- $\mathrm{Blup}_{r,s}(Y,X)$ is a Morita equivalence between $\mathrm{Blup}_{r,s}(G_1,\Gamma_1)$ and $\mathrm{Blup}_{r,s}(G_2,\Gamma_2)$,
- $\mathrm{SBlup}_{r,s}(Y,X)$ is a Morita equivalence between $\mathrm{SBlup}_{r,s}(G_1,\Gamma_1)$ and $\mathrm{SBlup}_{r,s}(G_2,\Gamma_2)$, ...

Note that if Y and X are sub-Morita equivalences, the above linking spaces are also sub-Morita equivalences.

3.5.4. *Groupoids on manifolds with boundary.* Let M be a manifold and V a hypersurface in M, and suppose that V cuts M into two manifolds with boundary $M=M_-\cup M_+$ with $V=M_-\cap M_+$. Then by considering a tubular neighborhood of V in M, $\mathrm{DNC}(M,V)=M\times\mathbb{R}^*\cup\mathcal{N}_V^M\times\{0\}$ identifies with $M\times\mathbb{R}$, the quotient $\widetilde{\mathrm{DNC}}(M,V)/\mathbb{R}^*_+$ identifies with two copies of M, and $\mathrm{SBlup}(M,V)$ identifies with the disjoint union $M_-\sqcup M_+$. Under this last identification, the class under the zooming action of a normal vector in $\mathcal{N}_V^M\setminus V\times\{0\}$ pointing in the direction of M_+ is an element of $V\subset M_+$.

Let M_b be manifold with boundary V. A *piece of Lie groupoid* is the restriction $G=\widetilde{G}_{M_b}^{M_b}$ to M_b of a Lie groupoid $\widetilde{G}\rightrightarrows M$, where M is a neighborhood of M_b and $\widetilde{G}$ is a groupoid without boundary.

With the above notation, since V is of codimension 1 in M, $\mathrm{SBlup}(M,V)=M_b\sqcup M_-$, where $M_-=M\setminus\mathring{M}$ is the complement in M of the interior $\mathring{M}=M_b\setminus V$ of M_b in M.

Let then $\Gamma\rightrightarrows V$ be a Lie subgroupoid of $\widetilde{G}$.

We may construct $\mathrm{SBlup}_{r,s}(\widetilde{G},\Gamma)$ and consider its restriction to the open subset M_b of $\mathrm{SBlup}(M,V)$. We thus obtain a longitudinally smooth groupoid that will be denoted $\mathrm{SBlup}_{r,s}(G,\Gamma)$.

Note that the groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma) \rightrightarrows M_b$ is the restriction to M_b of a Lie groupoid $\mathcal{G} \rightrightarrows M$ for which M_b is saturated. Indeed, $\mathrm{SBlup}_{r,s}(G,\Gamma)$ is an open subgroupoid of $\mathrm{SBlup}_{r,s}(\widetilde{G},\Gamma) \rightrightarrows M_b \sqcup M_-$, which is a piece of the Lie groupoid $\widetilde{\mathrm{DNC}}(\widetilde{G},\Gamma)/\mathbb{R}_+^* \rightrightarrows \widetilde{\mathrm{DNC}}(M,V)/\mathbb{R}_+^* \simeq M \sqcup M$. We may then let $\mathcal{G}$ be the restriction of $\widetilde{\mathrm{DNC}}(M,V)/\mathbb{R}_+^*$ to one of the copies of M.

In this way, we may treat by induction a finite number of mutually transverse hypersurfaces and, in particular, groupoids on manifolds with (embeded) corners.

Remarks 3.6. (i) Let us highlight that we do *not* assume V to be saturated for G. In particular, the boundary V can happen to be transverse to the groupoid $\widetilde{G}$. In that case G is in fact a manifold with corners. The blow-up groupoid $\mathrm{SBlup}_{r,s}(G,\Gamma)$ coincides with G outside V, and V becomes a saturated subset in this new groupoid.

(ii) If M is a manifold with boundary V and $G = M \times M$ is the pair groupoid, then $\mathrm{SBlup}_{r,s}(G,V)$ is in fact the groupoid associated with the 0 calculus in the sense of Mazzeo (*cp.* [24, 27, 26]), *i.e.,* the canonical pseudodifferential calculus associated with $\mathrm{SBlup}_{r,s}(G,V)$ is the Mazzeo–Melrose's 0-calculus. Indeed, the sections of the algebroid of $\mathrm{SBlup}_{r,s}(G,V)$ are exactly the vector fields of M vanishing at the boundary V, *i.e.,* those generating the 0-calculus.

(iii) In a recent paper [34], an alternative description of $\mathrm{SBlup}_{r,s}(G,V)$ is given under the name of *edge modification* for G along the "$\mathfrak{A}G$-tame manifold" V, thus, in particular, V is transverse to G. This is essentially the gluing construction described in 3.7.4 below.

3.7. **Examples of deformation groupoids and blow-up groupoids.** We examine some particular cases of inclusions of groupoids $G_1 \subset G_2$. The various constructions of deformation to the normal cone and blow-up allow us to recover many well-known groupoids.

As already noted in Section 3.4.2, our constructions immediately extend to the case where we restrict to a closed saturated subset of a smooth groupoid, in particular, for manifolds with corners.

3.7.1. *Inclusion $F \subset E$ of vector bundles—seen as groupoids.* Let E be a real vector space (considered as a group) and F a vector subspace of E. The inclusion of groups $F \to E$ gives rise to a groupoid $\mathrm{DNC}(E,F)$. Using any complementary subspace of F in E, the space E can be seen as a vector bundle over F; we thus identify the groupoid $\mathrm{DNC}(E,F)$ with $E \times \mathbb{R} \rightrightarrows \mathbb{R}$. Its C^*-algebra identifies then with $C_0(E^* \times \mathbb{R})$.

More generally, if F is a vector-subbundle of a vector bundle E over a manifold M (considered as a family of groups indexed by M), then the groupoid $\mathrm{DNC}(E,F) \rightrightarrows M \times \mathbb{R}$ identifies with $E \times \mathbb{R}$ and its C^*-algebra is $C_0(E^* \times \mathbb{R})$.

Let $p_E \colon E \to M$ be a vector bundle over a manifold M and let V be a submanifold of M. Let $p_F \colon F \to V$ be a subbundle of the restriction of E to V. We use a tubular construction and find an open subset U of M which is a vector bundle $\pi\colon Q \to V$. Using π, we may extend F to a subbundle F_U of the restriction to F on U. Using that, we may identify $\mathrm{DNC}(E,F)$ with the open subset $E\times\mathbb{R}^*\cup p_E^{-1}(U)\times\mathbb{R}$ of $E\times\mathbb{R}$. Its C^*-algebra identifies then with $C_0(E^*\times\mathbb{R}^*\cup p_{E^*}^{-1}(U)\times\mathbb{R})$.

3.7.2. *Inclusion $G^{(0)}\subset G$: adiabatic groupoid.* The deformation to the normal cone $\mathrm{DNC}(G,G^{(0)})$ is the adiabatic groupoid G_{ad} ([32, 35]), which is obtained by using the deformation to the normal cone construction for the inclusion of $G^{(0)}$ as a Lie subgroupoid of G. The normal bundle $N_{G^{(0)}}^G$ is the total space of the Lie algebroid $\mathfrak{A}(G)$ of G. Note that in this situation its groupoid structure coincides with its vector bundle structure. Thus,

$$\mathrm{DNC}(G,G^{(0)})=G\times\mathbb{R}^*\cup\mathfrak{A}(G)\times\{0\}\rightrightarrows G^{(0)}\times\mathbb{R}.$$

We often denote $\mathrm{DNC}(G,G^{(0)})$ by G_{ad} and G_{ad}^+, $G_{ad}^{[0,1]}$, $G_{ad}^{[0,1)}$ its restriction respectively to the saturated subset $G^{(0)}\times\mathbb{R}_+$, to $G^{(0)}\times[0,1]$ and to $G^{(0)}\times[0,1)$ of $G^{(0)}\times\mathbb{R}=G_{ad}^{(0)}$.

Note that $\mathrm{Blup}(G^{(0)},G^{(0)})=\varnothing=\mathrm{Blup}_{r,s}(G,G^{(0)})$.

The particular case where G is the pair groupoid $M\times M$ is the original construction of the "tangent groupoid" of Alain Connes [9].

3.7.3. *Gauge adiabatic groupoid.* Start with a Lie groupoid $G\rightrightarrows V$.

Let $G\times(\mathbb{R}\times\mathbb{R})\overset{\tilde r,\tilde s}{\rightrightarrows} V\times\mathbb{R}$ be the product groupoid of G with the pair groupoid over $\mathbb{R}$. First notice that since $V\times\{0\}$ is a codimension 1 submanifold in $V\times\mathbb{R}$, $\mathrm{SBlup}(V\times\mathbb{R},V\times\{0\})$ is canonically isomorphic to $V\times(\mathbb{R}_-\sqcup\mathbb{R}_+)$. Then $\mathrm{SBlup}_{\tilde r,\tilde s}(G\times(\mathbb{R}\times\mathbb{R}),V\times\{(0,0)\})_{V\times\mathbb{R}_+}^{V\times\mathbb{R}_+}$ is the semi-direct product groupoid $G_{ad}(V\times\mathbb{R}_+)\rtimes\mathbb{R}_+^*$:

$$\mathrm{SBlup}_{\tilde r,\tilde s}(G\times(\mathbb{R}\times\mathbb{R}),V\times\{(0,0)\})_{V\times\mathbb{R}_+}^{V\times\mathbb{R}_+}=G_{ad}^+\rtimes\mathbb{R}^*\rightrightarrows V\times\mathbb{R}_+.$$

In other words, $\mathrm{SBlup}_{\tilde r,\tilde s}(G\times(\mathbb{R}\times\mathbb{R}),V\times\{(0,0)\})_{V\times\mathbb{R}_+}^{V\times\mathbb{R}_+}$ is the gauge adiabatic groupoid used in [12]; we often denote it by G_{ga} .

Indeed, as $G\times(\mathbb{R}\times\mathbb{R})$ is a vector bundle over G, $\mathrm{DNC}(G\times(\mathbb{R}\times\mathbb{R}),V\times\{(0,0)\})\simeq\mathrm{DNC}(G,V)\times\mathbb{R}^2$ (Remark 2.6 (v)). Under this identification, the zooming action of $\mathbb{R}^*$ is given by $\lambda.(w,t,t')=(\lambda.w,\lambda^{-1}t,\lambda^{-1}t')$. The maps $\mathrm{DNC}(\tilde s)$ and $\mathrm{DNC}(\tilde r)$ are, respectively, $(w,t,t')\mapsto(\mathrm{DNC}(s)(w),t')$ and $(w,t,t')\mapsto(\mathrm{DNC}(r)(w),t)$. It follows that $\mathrm{SBlup}_{\tilde r,\tilde s}(G\times(\mathbb{R}\times\mathbb{R}),V\times\{(0,0)\})$ is the quotient by the diagonal action of $\mathbb{R}_+^*$ of the open subset $\mathrm{DNC}(G,V)\times(\mathbb{R}^*)^2$ of $\mathrm{DNC}_+(G,V)\times\mathbb{R}^2$.

According to the description of the groupoid of a group action on a groupoid given in [14, Section 2.3], it is isomorphic to $\mathrm{DNC}(G,V)_+\rtimes\mathbb{R}_+^*\times\{-1,+1\}^2$, where $\{-1,+1\}^2$ is the pair groupoid over $\{-1,+1\}$.

3.7.4. *Inclusion of a transverse submanifold of the unit space.* Let G be a Lie groupoid with set of objects $M=G^{(0)}$ and let V be a transverse submanifold of M. Put $\mathring{G} = G_{M\setminus V}^{M\setminus V}$. Upon arguing locally, we can assume that V is compact.

By Remark 3.3, V admits a tubular neighborhood $W \simeq N_V^M$ such that G_W^W is the pull back of G_V^V by the retraction $q\colon W \to V$.

The DNC groupoid $\mathrm{DNC}(G_W^W, V)$ identifies with the pull back groupoid $(\mathrm{DNC}(G_V^V, V))_q^q$ of the adiabatic deformation $\mathrm{DNC}(G_V^V, V) = (G_V^V)_{ad}$ by the map $q\colon N_V^M \to V$.

The (spherical) blow-up groupoid $\mathrm{SBlup}_{r,s}(G_W^W, V)$ identifies with the pull back groupoid $(\mathrm{DNC}_+(G_V^V, V) \rtimes \mathbb{R}_+^*)_p^p$ of the gauge adiabatic deformation $\mathrm{DNC}_+(G_V^V, V) \rtimes \mathbb{R}_+^* = (G_V^V)_{ga}$ by the map $p\colon \mathbb{S}N_V^M \to V$.

In order to get $\mathrm{SBlup}_{r,s}(G, V)$, we then may glue $(\mathrm{DNC}_+(G_V^V, V) \rtimes \mathbb{R}_+^*)_p^p$ with $\mathring{G}$ in their common open subset $((G_V^V)_q^q)_{W\setminus V}^{W\setminus V} \simeq G_{W\setminus V}^{W\setminus V}$.

3.7.5. *Inclusion $G_V^V \subset G$ for a transverse hypersurface V of G: b-groupoid.* If V is a hypersurface of M, the blow-up $\mathrm{Blup}(M \times M, V \times V)$ is just the construction of Melrose of the b-space. Its open subspace $\mathrm{Blup}_{r,s}(M \times M, V \times V)$ is the associated groupoid of Monthubert [30, 31]. Moreover, if G is a groupoid on M and V is transverse to G, we can form the restriction groupoid $G_V^V \subset G$, which is a submanifold of G. The corresponding blow-up construction $\mathrm{Blup}_{r,s}(G, G_V^V)$ identifies with the fibered product $\mathrm{Blup}_{r,s}(M\times M, V\times V) \times_{M\times M} G$ (*cp.* Remark 2.6 (vi)).

Iterating (at least locally) this construction, we obtain the b-groupoid of Monthubert for manifolds with corners (*cp.* [30, 31]).

Remark 3.8. The groupoid $\mathrm{Blup}_{r,s}(G, V)$ corresponds to inflating all the distances when getting close to V.

The groupoid $\mathrm{Blup}_{r,s}(G, G_V^V)$ is a kind of *cylindric deformation groupoid* which is obtained by pushing the boundary V at infinity but keeping the distances along V constant.

Remark 3.9. Intermediate examples between these two are given by a subgroupoid $\Gamma \rightrightarrows V$ of G_V^V.

In the case where $G = M \times M$, such a groupoid Γ is nothing else than the holonomy groupoid $\mathrm{Hol}(V, \mathcal{F})$ of a regular foliation $\mathcal{F}$ of V (with trivial holonomy groups). The groupoid $\mathrm{SBlup}_{r,s}(M \times M, \mathrm{Hol}(V, \mathcal{F}))$ is a holonomy groupoid of a singular foliation of M: the sections of its algebroid. Its leaves are $M \setminus V$ and the leaves of $(V, \mathcal{F})$. The corresponding calculus, when M is a manifold with a boundary V, is Rochon's generalization [38] of the ϕ calculus of Mazzeo and Melrose [25].

Iterating (at least locally) this construction, we obtain the holonomy groupoid associated to a stratified space in [11].

3.9.1. *Inclusion $G_V^V \subset G$ for a saturated submanifold V of G: shriek map for immersions.* Suppose now that V is saturated, thus $G_V^V = G_V = G^V$.

In such case the groupoid G_V^V acts on the normal bundle $N_{G_V^V}^G = r^*(N_V^{G^{(0)}})$ and $\mathrm{DNC}(G, G_V^V) \rightrightarrows \mathrm{DNC}(G^{(0)}, V)$ coincides with the normal groupoid of the immersion $\varphi\colon G_V^V \to G$. This construction was defined in the case of foliation groupoids in [21, Section 3] and was used in order to define $\varphi_!$ as its associated KK-element.

3.9.2. *Inclusion $G_1 \subset G_2$ with $G_1^{(0)} = G_2^{(0)}$.* This is the case for the tangent and adiabatic groupoid discussed above. Let us mention two other kinds of this situation[1] that can be encountered in the literature:

(i) The case of a subfoliation $\mathcal{F}_1$ of a foliation $\mathcal{F}_2$ on a manifold M: shriek map for submersion. As pointed out in [21, Remark 3.19], the corresponding deformation groupoid $\mathrm{DNC}(G_2, G_1)$ gives an alternative construction of the element $\varphi_!$, where $\varphi\colon M/\mathcal{F}_1 \to M/\mathcal{F}_2$ is a submersion of leaf spaces.

(ii) The case of a subgroup of a Lie group.
   - If K is a maximal compact subgroup of a reductive Lie group G, the connecting map associated to the exact sequence of $\mathrm{DNC}(G,K)$ is the Dirac extension mapping the twisted K-theory of K to the K-theory of $C_r^*(G)$ (see [18]).
   - In the case where Γ is a dense (nonamenable) countable subgroup of a compact Lie group K, the groupoid $\mathrm{DNC}(K,\Gamma)$ was used in [19] in order to produce a Hausdorff groupoid for which the Baum–Connes map is not injective.

3.9.3. *Wrong way functoriality.* Let $f\colon G_1 \to G_2$ be a morphism of Lie groupoids. If f is an (injective) immersion, the construction of $\mathrm{DNC}_+(G_2, G_1)$ gives rise to a short exact sequence

$$0 \longrightarrow C^*(G_2 \times \mathbb{R}_+^*) \longrightarrow C^*(\mathrm{DNC}_+(G_2, G_1)) \longrightarrow C^*(\mathcal{N}_{G_1}^{G_2}) \longrightarrow 0,$$

and consequently to a connecting map from the K-theory of the C^*-algebra of the groupoid $\mathcal{N}_{G_1}^{G_2}$, which is a $\mathcal{VB}$ groupoid over G_1, to the K-theory of $C^*(G_2)$ (*cp.* Definition A.2 for a discussion on $\mathcal{VB}$ groupoids). This wrong way functoriality map will be discussed in the next section.

More generally, let $\mathcal{G} = G_1^{(0)} \times G_2 \times G_1^{(0)}$ be the product of G_2 by the pair groupoid of $G_1^{(0)}$. Assume that the map $x \mapsto (r(x), f(x), s(x))$ is an immersion from $G_1 \to \mathcal{G}$.

The above construction gives a map from $K_*(C^*(\mathcal{N}_{G_1}^{\mathcal{G}}))$ to $K_*(C^*(\mathcal{G}))$ which is isomorphic to $K_*(C^*(G_2))$, since the groupoids G_2 and $\mathcal{G}$ are canonically Morita equivalent.

3.9.4. *Some more recent examples.* Since the present paper appeared as a preprint, several papers have used it and applied our DNC and Blup constructions in order to build interesting groupoids illustrating important geometric phenomena. See, *e.g.,* [1, 28, 29, 36, 40, 42, 41].

[1] Note that in this case $\mathrm{Blup}(G_2^{(0)}, G_1^{(0)}) = \varnothing$, whence $\mathrm{Blup}_{r,s}(G_2, G_1) = \varnothing$.

4. The C^*-algebra of a deformation and of a blow-up groupoid, full symbol and index map

Let $G \rightrightarrows M$ be a Lie groupoid and $\Gamma \rightrightarrows V$ a Lie subgroupoid of G. The groupoids $\mathrm{DNC}_+(G,\Gamma)$ and $\mathrm{SBlup}_{r,s}(G,\Gamma)$ that we constructed admit the closed saturated subsets $N_V^M \times \{0\}$ and $\mathbb{S}N_V^M$, respectively. We apply results of [14] in order to compute various KK-elements involved in index theory for such situation.

In order to shorten the notation we put $\mathring{M} = M \setminus V$.

The *full symbol algebras* are the quotient C^*-algebras:

- $\Sigma_{\mathrm{DNC}_+}(G,\Gamma) = \Psi^*(\mathrm{DNC}_+(G,\Gamma))/C^*(G \times \mathbb{R}+^*)$,
- $\Sigma_{\mathrm{SBlup}}(G,\Gamma) = \Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}})$.

They give rise to the exact sequences

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^G) \longrightarrow 0 \qquad (E^\partial_{\mathrm{SBlup}})$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(\mathrm{DNC}_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \qquad (E^\partial_{\mathrm{DNC}_+})$$

of groupoid C^*-algebras as well as index type ones

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow \Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \longrightarrow \Sigma_{\mathrm{SBlup}}(G,\Gamma) \longrightarrow 0 \qquad (E^{\widetilde{\mathrm{ind}}}_{\mathrm{SBlup}})$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow \Psi^*(\mathrm{DNC}_+(G,\Gamma)) \longrightarrow \Sigma_{\mathrm{DNC}_+}(G,\Gamma) \longrightarrow 0. \quad (E^{\widetilde{\mathrm{ind}}}_{\mathrm{DNC}_+})$$

We will compare the K-theory exact sequences given by DNC and by SBlup.

If V is $\mathfrak{A}G$*-small* (see Notation 4.6 below), we will show that, in a sense, DNC and SBlup give rise to equivalent exact sequences—both for the "connecting" ones and for the "index" ones.

We will then compare these KK-elements with a coboundary construction.

We will compute these exact sequences when $\Gamma = V \subset M$. Finally, we will study a refinement of these constructions using relative K-theory.

4.1. "DNC" versus "Blup".

Let $\Gamma \rightrightarrows V$ be a submanifold and a subgroupoid of a Lie-groupoid $G \rightrightarrows M$. We will further assume that the groupoid Γ is amenable. We still put $\mathring{M} = M \setminus V$ and let $\mathring{\mathcal{N}}_\Gamma^G$ be the restriction of the groupoid $\mathcal{N}_\Gamma^G$ to the open subset $\mathring{N}_V^M = N_V^M \setminus V$ of its unit space N_V^M.

4.1.1. *The connecting KK-element.* As the groupoid Γ is amenable, we have exact sequences both for the reduced and for the maximal C^*-algebras:

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^G) \longrightarrow 0 \qquad (E^\partial_{\mathrm{SBlup}})$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(\mathrm{DNC}_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0. \qquad (E^\partial_{\mathrm{DNC}_+})$$

By amenability, these exact sequences admit completely positive cross sections and therefore define elements $\partial^{G,\Gamma}_{\mathrm{SBlup}} \in KK^1(C^*(\mathring{\mathcal{N}}^G_\Gamma/\mathbb{R}_+^*), C^*(G^{\mathring{M}}_{\mathring{M}}))$ and $\partial^{G,\Gamma}_{\mathrm{DNC}_+} \in KK^1(C^*(\mathcal{N}^G_\Gamma), C^*(G \times \mathbb{R}_+^*))$.

With the notation of Section 3.5, write DNC_+ for DNC restricted to $\mathbb{R}_+$ and $\widetilde{\mathrm{DNC}}_+$ for $\widetilde{\mathrm{DNC}}$ restricted to $\mathbb{R}_+$.

By [14, Section 5.3], we have a diagram, where the vertical arrows are KK^1-equivalences and the squares commute in KK-theory:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G^{\mathring{M}}_{\mathring{M}}) \arrow[r] \arrow[d, dash, "\beta'"] & C^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \arrow[r] \arrow[d, dash, "\beta"] & C^*(\mathring{\mathcal{N}}^G_\Gamma/\mathbb{R}_+^*) \arrow[r] \arrow[d, dash, "\beta''"] & 0 & (E^\partial_{\mathrm{SBlup}}) \\
0 \arrow[r] & C^*(G^{\mathring{M}}_{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] & C^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \arrow[r] & C^*(\mathring{\mathcal{N}}^G_\Gamma) \arrow[r] & 0. & (E^\partial_{\widetilde{\mathrm{DNC}}_+})
\end{tikzcd}$$

Denote by $\partial^{G,\Gamma}_{\widetilde{\mathrm{DNC}}_+}$ the connecting KK-element associated to $(E^\partial_{\widetilde{\mathrm{DNC}}_+})$. We thus have, according to [14, Prop. 5.3]:

Fact 4.2. $\partial^{G,\Gamma}_{\mathrm{SBlup}} \otimes \beta' = -\beta'' \otimes \partial^{G,\Gamma}_{\widetilde{\mathrm{DNC}}_+} \in KK^1(C^*(\mathcal{S}N^G_\Gamma), C^*(G^{\mathring{M}}_{\mathring{M}} \times \mathbb{R}_+^*))$.

We also have a commutative diagram, where the vertical maps are inclusions:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G^{\mathring{M}}_{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] \arrow[d, "j'"] & C^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \arrow[r] \arrow[d, "j"] & C^*(\mathring{\mathcal{N}}^G_\Gamma) \arrow[r] \arrow[d, "j''"] & 0 \\
0 \arrow[r] & C^*(G \times \mathbb{R}_+^*) \arrow[r] & C^*(\mathrm{DNC}_+(G,\Gamma)) \arrow[r] & C^*(\mathcal{N}^G_\Gamma) \arrow[r] & 0.
\end{tikzcd} \tag{1}$$

We thus find:

Fact 4.3. $(j'')^*(\partial^{G,\Gamma}_{\mathrm{DNC}_+}) = j'_*(\partial^{G,\Gamma}_{\widetilde{\mathrm{DNC}}_+}) \in KK^1(C^*(\mathring{\mathcal{N}}^G_\Gamma), C^*(G \times \mathbb{R}_+^*))$.

4.3.1. *The full symbol index.* We now compare the elements

$$\widetilde{\mathrm{ind}}^{G,\Gamma}_{\mathrm{SBlup}} \in KK^1(\Sigma_{\mathrm{SBlup}}(G,\Gamma), C^*(G^{\mathring{M}}_{\mathring{M}}))$$

and

$$\widetilde{\mathrm{ind}}^{G,\Gamma}_{\mathrm{DNC}_+} \in KK^1(\Sigma_{\mathrm{DNC}_+}(G,\Gamma), C^*(G \times \mathbb{R}_+^*)),$$

defined by the semi-split exact sequences

$$0 \longrightarrow C^*(G^{\mathring{M}}_{\mathring{M}}) \longrightarrow \Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \longrightarrow \Sigma_{\mathrm{SBlup}}(G,\Gamma) \longrightarrow 0 \qquad (E^{\widetilde{\mathrm{ind}}}_{\mathrm{SBlup}})$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow \Psi^*(\mathrm{DNC}_+(G,\Gamma)) \longrightarrow \Sigma_{\mathrm{DNC}_+}(G,\Gamma) \longrightarrow 0. \quad (E^{\widetilde{\mathrm{ind}}}_{\mathrm{DNC}_+})$$

Put $\Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma) = \Psi^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*)$. By [14, Prop. 5.4], we have a diagram where the vertical arrows are KK^1-equivalences and the squares commute in KK-theory:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[d, dash, "\beta'"] & \Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma)) \arrow[r] \arrow[d, dash, "\beta_\Psi"] & \Sigma_{\mathrm{SBlup}}(G,\Gamma) \arrow[r] \arrow[d, dash, "\beta_\Sigma"] & 0 \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] & \Psi^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \arrow[r] & \Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma) \arrow[r] & 0.
\end{tikzcd}$$

We let $\widetilde{\mathrm{ind}}_{\widetilde{\mathrm{DNC}}_+}^{G,\Gamma} \in KK^1(\Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma), C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*))$ be the connecting map induced by the second exact sequence.

Fact 4.4. We have

$$\widetilde{\mathrm{ind}}_{\mathrm{SBlup}}^{G,\Gamma} \otimes \beta' = -\beta_\Sigma \otimes \widetilde{\mathrm{ind}}_{\widetilde{\mathrm{DNC}}_+}^{G,\Gamma} \in KK^1(\Sigma_{\mathrm{SBlup}}(G,\Gamma), C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*)).$$

We also have a commutative diagram where the vertical maps are inclusions:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] \arrow[d, "j'"] & \Psi^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \arrow[r] \arrow[d, "j_\Psi"] & \Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma) \arrow[r] \arrow[d, "j_\Sigma"] & 0 \\
0 \arrow[r] & C^*(G \times \mathbb{R}_+^*) \arrow[r] & \Psi^*(\mathrm{DNC}_+(G,\Gamma)) \arrow[r] & \Sigma_{\mathrm{DNC}_+}(G,\Gamma) \arrow[r] & 0.
\end{tikzcd} \tag{2}$$

We thus find:

Fact 4.5. $j_\Sigma^*(\widetilde{\mathrm{ind}}_{\mathrm{DNC}_+}^{G,\Gamma}) = j'_*(\widetilde{\mathrm{ind}}_{\widetilde{\mathrm{DNC}}_+}^{G,\Gamma}) \in KK^1(\Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma), C^*(G \times \mathbb{R}_+^*))$.

4.5.1. *When V is $\mathfrak{A}G$-small.* If V is small in each G orbit, *i.e.,* if the Lebesgue measure (in the manifold G^x) of G_V^x is 0 for every x, it follows from Proposition 4.7 below that the inclusion $i\colon C^*(G_{\mathring{M}}^{\mathring{M}}) \hookrightarrow C^*(G)$ is an isomorphism. Also, if $\mathring{M}$ meets all the orbits of G, the inclusion i is a Morita equivalence. In these cases $\partial_{\mathrm{DNC}_+}^{G,\Gamma}$ determines $\partial_{\mathrm{SBlup}}^{G,\Gamma}$.

Definition 4.6. We will say that V is *$\mathfrak{A}G$-small* if for every $x \in V$, the composition $\mathfrak{A}G_x \xrightarrow{\natural_x} T_xM \longrightarrow (N_V^M)_x$ is not the zero map.

If V is $\mathfrak{A}G$-small, then the orbits of the groupoid $\mathcal{N}_\Gamma^G$ are never contained in the 0 section, *i.e.,* they meet the open subset $\mathring{N}_V^M$, and in fact the set $V \times \{0\}$ is small in every orbit of the groupoid $\mathrm{DNC}(G,\Gamma)$. It follows that the map j is an isomorphism—as well of course as j' and j'' of diagram (1). In that case, $\partial_{\mathrm{DNC}_+}^{G,\Gamma}$ and $\partial_{\mathrm{SBlup}}^{G,\Gamma}$ correspond to each other under these isomorphisms.

Proposition 4.7 (*cp.* [20, 13])**.** *Let $\mathcal{G} \rightrightarrows Y$ be a Lie groupoid and let $Z \subset Y$ be a (locally closed) submanifold. Assume that, for every $x \in Z$, the composition $\mathfrak{A}\mathcal{G}_x \xrightarrow{\natural_x} T_xY \longrightarrow (N_Z^Y)_x$ is not the zero map. Then the inclusion $C^*(\mathcal{G}_{Y\setminus Z}^{Y\setminus Z}) \to C^*(\mathcal{G})$ is an isomorphism.*

Proof. For every $x \in Z$, we can find a neighborhood U of $x \in Y$, a section X of $\mathfrak{A}\mathcal{G}$ such that, for every $y \in U$, $\natural_y(X(y)) \neq 0$, and if $y \in U \cap Z$, then $\natural_y(X(y)) \notin T_y(Z)$. Denote by $\mathcal{F}$ the foliation of U associated with the vector field X. It follows from [20, Lemme 4] that $C_0(U \setminus Z)C^*(U, \mathcal{F}) = C^*(U, \mathcal{F})$. As $C^*(U, \mathcal{F})$ acts in a nondegenerate way on the Hilbert-$C^*(G)$ module $C^*(G^U)$, we deduce that $C_0(U \setminus Z)C^*(G^U) = C^*(G_U)$. We conclude, using a partition of the identity argument, that $C_c(Y \setminus Z)C^*(G) = C_c(Y)C^*(G)$, whence we have $C_0(Y \setminus Z)C^*(G) = C_0(Y)C^*(G) = C^*(G)$. ☐

Proposition 4.8. *We assume that Γ is amenable and that V is $\mathfrak{A}G$-small. Then the inclusions $j_\Sigma : \Sigma_{\widetilde{\mathrm{DNC}}_+}(G, \Gamma) \to \Sigma_{\mathrm{DNC}_+}(G, \Gamma)$, $j_\Psi : \Psi^*(\widetilde{\mathrm{DNC}}_+(G, \Gamma)) \to \Psi^*(\mathrm{DNC}_+(G, \Gamma))$ and $j_{\mathrm{symb}} : C_0(\mathbb{S}\mathfrak{A}^*(\widetilde{\mathrm{DNC}}_+(G, \Gamma))) \to C_0(\mathbb{S}\mathfrak{A}^*(\mathrm{DNC}_+(G, \Gamma)))$ are KK-equivalences.*

Proof. We have a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & C^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \arrow[r] \arrow[d, "j"] & \Psi^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \arrow[r] \arrow[d, "j_\Psi"] & C_0(\mathbb{S}\mathfrak{A}^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma))) \arrow[r] \arrow[d, "j_{\mathrm{symb}}"] & 0 \\
0 \arrow[r] & C^*(\mathrm{DNC}_+(G,\Gamma)) \arrow[r] & \Psi^*(\mathrm{DNC}_+(G,\Gamma)) \arrow[r] & C_0(\mathbb{S}\mathfrak{A}^*(\mathrm{DNC}_+(G,\Gamma)) \arrow[r] \arrow[d] & 0 \\
 & & & C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+) \arrow[d] & \\
 & & & 0. &
\end{tikzcd}$$

As j is an equality, we find an exact sequence

$$0 \longrightarrow \Psi^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma)) \xrightarrow{j_\Psi} \Psi^*(\mathrm{DNC}_+(G,\Gamma)) \longrightarrow C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+) \longrightarrow 0.$$

As $j' : C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \to C^*(G \times \mathbb{R}_+^*)$ is also an equality, we find (using diagram (2)) an exact sequence

$$0 \longrightarrow \Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma)) \xrightarrow{j_\Sigma} \Sigma_{\mathrm{DNC}_+}(G,V) \longrightarrow C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+) \longrightarrow 0.$$

As the algebra $C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+)$ is contractible, we deduce that j_{symb} and then j_Ψ and j_Σ are KK-equivalences. ☐

As a summary of these considerations, we find:

Theorem 4.9. *Let $G \rightrightarrows M$ be a Lie groupoid and $\Gamma \rightrightarrows V$ a Lie subgroupoid of G. Assume that Γ is amenable and put $\mathring{M} = M \setminus V$. Let $i : C^*(G_{\mathring{M}}^{\mathring{M}}) \to$*

$C^*(G)$ be the inclusion. Put $\hat\beta'' = j''_*(\beta'') \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(\mathcal{N}_\Gamma^G))$ and $\hat\beta_\Sigma = (j_\Sigma)_*(\beta_\Sigma) \in KK^1(\Sigma_{\mathrm{SBlup}}(G,\Gamma), \Sigma_{\mathrm{DNC}_+}(G,V))$.

(i) *We have equalities*
- $\partial^{G,\Gamma}_{\mathrm{SBlup}} \otimes [i] = \hat\beta'' \otimes \partial^{G,\Gamma}_{\mathrm{DNC}_+} \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(G))$ *and*
- $\widetilde{\mathrm{ind}}^{G,\Gamma}_{\mathrm{SBlup}} \otimes [i] = \hat\beta_\Sigma \otimes \widetilde{\mathrm{ind}}^{G,\Gamma}_{\mathrm{DNC}_+} \in KK^1(C^*(\Sigma_{\mathrm{SBlup}}(G,\Gamma), C^*(G)))$.

(ii) *If V is $\mathfrak{A}G$-small, then i is an isomorphism and the elements $\hat\beta''$ and $\hat\beta_\Sigma$ are invertible.*

4.10. **The KK-element associated with DNC.** The connecting element $\partial^{G,\Gamma}_{\mathrm{DNC}_+}$ can be expressed in the following way: let $\mathcal{G}$ be the restriction of $\mathrm{DNC}(G,\Gamma)$ to $[0,1]$, *i.e.,* $\mathcal{G} = \mathcal{N}_\Gamma^G \times \{0\} \cup G \times (0,1]$. We have a semi-split exact sequence:

$$0 \to C^*(G \times (0,1]) \to C^*(\mathcal{G}) \xrightarrow{\mathrm{ev}_0} C^*(\mathcal{N}_\Gamma^G) \to 0.$$

As $C^*(G \times (0,1])$ is contractible, ev_0 is a KK-equivalence. Let $ev_1 \colon C^*(\mathcal{G}) \to C^*(G)$ be evaluation at 1 and let $\delta_\Gamma^G = [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(\mathcal{N}_\Gamma^G), C^*(G))$. Let $[\mathrm{Bott}] \in KK^1(\mathbb{C}, C_0(\mathbb{R}_+^*))$ be the Bott element.

We then have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G(0,1)) \arrow[r] \arrow[d, equal] & C^*(\mathrm{DNC}(G,\Gamma)) \arrow[r, "ev_0"] \arrow[d] & C^*(\mathcal{N}_\Gamma^G) \arrow[r] \arrow[d] & 0 & (E^{\partial}_{\mathrm{DNC}_+}) \\
0 \arrow[r] & C^*(G(0,1)) \arrow[r] & C^*(\mathcal{G}) \arrow[r, "ev_0 \oplus ev_1"] & C^*(\mathcal{N}_\Gamma^G) \oplus C(G) \arrow[r] & 0 & \\
0 \arrow[r] & C^*(G(0,1)) \arrow[r] \arrow[u, equal] & C^*(G(0,1]) \arrow[r, "ev_1"] \arrow[u] & C^*(G) \arrow[r] \arrow[u] & 0. & (E_{-\mathrm{Bott}})
\end{tikzcd}$$

It follows that, $[ev_0] \otimes \partial^{G,\Gamma}_{\mathrm{DNC}_+} + [ev_1] \otimes \partial_{-\mathrm{Bott}} = 0$. As $\partial_{-\mathrm{Bott}}$ define the opposite of the Bott element in $KK^1(C^*(G), C^*(G \times (0,1))$, we find:

Fact 4.11. $\partial^{G,\Gamma}_{\mathrm{DNC}_+} = \delta_\Gamma^G \underset{\mathbb{C}}{\otimes} [\mathrm{Bott}]$.

Consider now the groupoid $\mathbb{G} = \mathcal{G}_{ad}^{[0,1]}$. It is a family of groupoids indexed by $[0,1] \times [0,1]$:
- its restriction to $\{s\} \times [0,1]$ for $s \neq 0$ is $G_{ad}^{[0,1]}$;
- its restriction to $\{0\} \times [0,1]$ is $(\mathcal{N}_\Gamma^G)_{ad}^{[0,1]}$;
- its restriction to $[0,1] \times \{s\}$ for $s \neq 0$ is $\mathcal{G}$;
- its restriction to $[0,1] \times \{0\}$ is the algebroid $\mathfrak{A}\mathcal{G}$ which is the restriction of $\mathrm{DNC}(\mathfrak{A}G, \mathfrak{A}\Gamma)$ to $[0,1]$.

For every locally closed subset $X \subset [0,1] \times [0,1]$, denote by $\mathbb{G}^X$ the restriction of $\mathbb{G}$ to X.

For every closed subset $X \subset [0,1] \times [0,1]$, denote by $q_X \colon C^*(\mathbb{G}) \to C^*(\mathbb{G}^X)$ the restriction map.

We thus have the following commutative diagram:

$$\begin{tikzcd}
C^*(\mathcal{N}_\Gamma^G) \arrow[rr, dotted, bend left, "\delta_\Gamma^G"] & \mathcal{G} \arrow[l] \arrow[r] & C^*(G) \\
C^*((\mathcal{N}_\Gamma^G)_{ad}^{[0,1]}) \arrow[u] \arrow[d] & C^*(\mathbb{G}) \arrow[l, "q_{\{0\}\times[0,1]}"'] \arrow[r, "q_{\{1\}\times[0,1]}"] \arrow[u, "q_{[0,1]\times\{1\}}"] \arrow[d, "q_{[0,1]\times\{0\}}"] \arrow[ul, "q_{(0,1)}"'] \arrow[ur, "q_{(1,1)}"] \arrow[dl, "q_{(0,0)}"] \arrow[dr, "q_{(1,0)}"'] & C^*(G_{ad}^{[0,1]}) \arrow[u] \arrow[d] \\
C^*(N_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}) \arrow[u, dotted, bend left=60, "\mathrm{ind}_{\mathcal{N}_\Gamma^G}"] \arrow[rr, dotted, bend right, "\delta_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}"'] & C^*(\mathfrak{A}\mathcal{G},) \arrow[l] \arrow[r] & C_0(\mathfrak{A}^*G). \arrow[u, dotted, bend right=60, "\mathrm{ind}_G"']
\end{tikzcd}$$

For every locally closed subset $T \subset [0,1]$, the C^*-algebras $C^*(\mathbb{G}^{(0,1]\times T})$ and $C^*(\mathbb{G}^{T\times(0,1]})$ are contractible as well as $C^*(\mathbb{G}^{[0,1]^2\setminus\{0,0)\}})$. It follows that $q_{\{0\}\times[0,1]}$, $q_{[0,1]\times\{0\}}$ and $q_{\{(0,0)\}}$ are KK-equivalences.

Now $[q_{(0,0)}]^{-1}\otimes[q_{(0,1)}] = \mathrm{ind}_{\mathcal{N}_\Gamma^G}$ and it follows that $[q_{(0,0)}]^{-1}\otimes[q_{(1,1)}] = \mathrm{ind}_{\mathcal{N}_\Gamma^G}\otimes\delta_\Gamma^G$.

In the same way, $[q_{(0,0)}]^{-1}\otimes[q_{(1,0)}] = \delta_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}$ and it follows that $[q_{(0,0)}]^{-1}\otimes [q_{(1,1)}] = \delta_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}\otimes\mathrm{ind}_G$.

Finally, it follows from Example 3.7.1 that $\delta_{\mathfrak{A}\Gamma}^{\mathfrak{A}G}$ is associated with a morphism $\varphi\colon C_0(\mathfrak{A}^*(\mathcal{N}_V^G)) \hookrightarrow C_0(\mathfrak{A}^*G)$ corresponding to an inclusion of $\mathfrak{A}^*(\mathcal{N}_\Gamma^G)$ in $\mathfrak{A}^*G$ as a tubular neighborhood.

We thus have established:

Fact 4.12. $\mathrm{ind}_{\mathcal{N}_\Gamma^G}\otimes\delta_\Gamma^G = [\varphi]\otimes\mathrm{ind}_G$.

Similar groupoids and commutative diagrams for the special case of $\mathcal{V}$ being the normal bundle of the inclusion of a manifold M into some $\mathbb{R}^n$, $G = \mathcal{V}\times\mathcal{V}$ and $\Gamma = \mathcal{V}\underset{M}{\times}\mathcal{V}$ appeared in [10, Section 6.1] in order to give a K-theoretical proof using groupoids of the Atiyah–Singer index theorem.

4.13. **The case of a submanifold of the space of units.** Let G be a Lie groupoid with objects M and let $\Gamma = V \subset M$ be a closed submanifold of M. In this section, we push further the computations of the connecting maps and indices, *i.e.,* the connecting maps of the exact sequences $(E^\partial_{\mathrm{SBlup}})$, $(E^\partial_{\mathrm{DNC}_+})$, $(E^{\widetilde{\mathrm{ind}}}_{\mathrm{SBlup}})$ and $(E^{\widetilde{\mathrm{ind}}}_{\mathrm{DNC}_+})$.

Let $N = N_V^G$ and $N' = N_V^M$ be the normal bundles. We identify N' with a subbundle of N by means of the inclusion $M \subset G$. The submersions $r,s\colon G\to M$ give rise to bundle morphisms $r^N, s^N\colon N\to N'$ that are sections of the inclusion $N'\to N$. By construction, using Remark (iii) (a), the groupoid $\mathrm{DNC}(G,V)$ is the union of $G\times\mathbb{R}^*$ with the family of linear groupoids $\mathcal{N}_{r^N,s^N}(N)$. It follows that $\mathrm{SBlup}_{r,s}(G,V)$ is the union of $G_{M\setminus V}^{M\setminus V}$ with the family $(\mathcal{S}N, r^N, s^N)$ of spherical groupoids.

If V is transverse to G, the bundle map $r^N - s^N \colon N = N_V^G \to N' = N_V^M$ is surjective; it follows that

- $\mathcal{N}_{r^N,s^N}(N)$ identifies with the pullback groupoid $(\mathfrak{A}(G_V^V))_q^q$, where $q : N' \to V$ is the projection,
- $(\mathcal{S}N, r^N, s^N)$ with the pullback groupoid $(\mathfrak{A}(G_V^V)\rtimes\mathbb{R}_+^*)_p^p$, where $p : \mathbb{S}(N') \to V$ is the projection.

4.13.1. *Connecting map and index map.* From Proposition A.9, [14, Propositions 4.1, 4.6, 4.7] and Fact 4.12, we find:

Proposition 4.14. (i) *The index element* $\mathrm{ind}_{\mathcal{N}_V^G} \in KK(C_0(\mathfrak{A}^*N_V^G), C^*(\mathcal{N}_V^G))$ *is invertible.*

(ii) *The inclusion* $j\colon \Sigma_{N_V^M\times\{0\}}(\mathrm{DNC}_+(G,V)) \hookrightarrow \Sigma_{\mathrm{DNC}_+}(G,V)$ *is invertible in* KK*-theory.*

(iii) *The* C^**-algebra* $\Sigma_{\mathrm{DNC}_+}(G,V)$ *is naturally* KK^1*-equivalent with the mapping cone* C_χ *of the map* $\chi\colon C_0(\mathfrak{A}^*G\times\mathbb{R}_+^*) \to C_0(\mathrm{DNC}_+(M,V))$ *defined by*

$$\chi(f)(x) = \begin{cases} f(x,0) & \text{if } x\in M\times\mathbb{R}_+^*,\\ 0 & \text{if } x\in N_V^M. \end{cases}$$

(iv) *The connecting element* $\partial_{\mathrm{DNC}_+}^{G,V} \in KK^1(C^*(\mathcal{N}_V^G), C^*(G\times\mathbb{R}_+^*)) = KK(C^*(\mathcal{N}_V^G), C^*(G))$ *is* $\delta_V^G = \mathrm{ind}_{\mathcal{N}_V^G}^{-1}\otimes[\varphi]\otimes\mathrm{ind}_G$*, where* $\varphi\colon C_0(\mathfrak{A}^*N_V^G)\to C_0(\mathfrak{A}^*G)$ *is the inclusion using the tubular neighborhood construction.*

(v) *Under the* KK^1 *equivalence of* (c)*, the full index element*

$$\widetilde{\mathrm{ind}}_{\mathrm{DNC}_+}^{G,V} \in KK^1(\Sigma_{\mathrm{DNC}_+}(G,V), C^*(G\times\mathbb{R}_+^*)) = KK^1(\mathsf{C}_\chi, C^*(G))$$

is $q^*([\mathrm{Bott}]\underset{\mathbb{C}}{\otimes}\mathrm{ind}_G)$*, where* $q\colon \mathsf{C}_\chi \to C_0(\mathfrak{A}^*G\times\mathbb{R}_+^*)$ *is evaluation at* 0.

The element $[\chi]\in KK(C_0(\mathfrak{A}^*G\times\mathbb{R}_+^*), C_0(\mathrm{DNC}_+(M,V)))$ is the Kasparov product of the "Euler element" of the bundle $\mathfrak{A}^*G$, which is the class in $KK(C_0(\mathfrak{A}^*G), C_0(M)) = KK(C_0(\mathfrak{A}^*G\times\mathbb{R}_+^*), C_0(M\times\mathbb{R}_+^*))$ of the map $x\mapsto (x,0)$ with the inclusion $C_0(M\times\mathbb{R}_+^*)\to C_0(\mathrm{DNC}_+(M,V))$. It follows that $[\chi]$ is often the zero element of $KK(C_0(\mathfrak{A}^*G\times\mathbb{R}_+^*), C_0(\mathrm{DNC}_+(M,V)))$. In particular, this is the case when the Euler class of the bundle $\mathfrak{A}^*G$ vanishes. In that case, the algebra $\Sigma_{\mathrm{DNC}_+}(G,V)$ is KK-equivalent to $C_0(\mathfrak{A}^*G)\oplus C_0(\mathrm{DNC}_+(M,V))$.

If V is $\mathfrak{A}G$ small, then, by Theorem 4.9, $\partial_{\mathrm{SBlup}}^{G,V}$ and $\widetilde{\mathrm{ind}}_{\mathrm{SBlup}}^{G,V}$ are immediately deduced from Proposition 4.14.

Proposition 4.15. *Let* $G\rightrightarrows M$ *be a Lie groupoid and let* $V\subset M$ *be a* $\mathfrak{A}G$ *small submanifold. Then the algebra* $C^*(\mathring{\mathcal{N}}_V^G)$ *is naturally* KK^1*-equivalent to* $C_0(U)$*, where* U *is a tubular neighborhood of* V *in* $\mathfrak{A}^*G$*. Under this* KK*-equivalence, the connecting element of the exact sequence* $(E_{\mathrm{SBlup}}^\partial)$ *is the composition of the index element* $[\mathrm{ind}_G]\in KK(C_0(\mathfrak{A}^*G), C^*(G))$ *with the inclusion* $C_0(U)\to C_0(\mathfrak{A}^*G)$.

Remark 4.16. Let M_b be a manifold with boundary and $V = \partial M_b$. Put $\mathring{M} = M_b \setminus V$. Let G be a piece of Lie groupoid on M_b in the sense of Section 3.5.4. Thus G is the restriction of a Lie groupoid $\widetilde{G} \rightrightarrows M$, where M is a neighborhood of M_b. Recall that in this situation, $\mathrm{SBlup}(M,V) = M_b \sqcup M_-$, where $M = M_b \cup M_-$ and $M \cap M_- = V$, and we let $\mathrm{SBlup}_{r,s}(G,V) \rightrightarrows M_b$ be the restriction of $\mathrm{SBlup}_{r,s}(\widetilde{G},V)$ to M_b.

Let us denote by $\mathring{\mathcal{N}}_V^G$ the open subset of $N_V^{\widetilde{G}}$ made of (normal) tangent vectors whose image under the differential of the source and range maps of $\widetilde{G}$ are nonvanishing elements of N_V^M pointing in the direction of M_b. The groupoid $\mathrm{SBlup}_{r,s}(G,V)$ is the union $\mathring{\mathcal{N}}_V^G/\mathbb{R}_+^* \cup G_{\mathring{M}}^{\mathring{M}}$.

We have exact sequences

$$0 \to C^*(G_{\mathring{M}}^{\mathring{M}}) \to C^*(\mathrm{SBlup}_{r,s}(G,V)) \to C^*(\mathring{\mathcal{N}}_V^G/\mathbb{R}_+^*) \to 0,$$
$$0 \to C^*(G_{\mathring{M}}^{\mathring{M}}) \to \Psi^*(\mathrm{SBlup}_{r,s}(G,V)) \to \Sigma_{\mathrm{SBlup}}(G,V) \to 0.$$

As V is of codimension 1, we find that V is $\mathfrak{A}\widetilde{G}$-small if and only if it is transverse to $\widetilde{G}$. In that case, Proposition 4.14 computes the KK-theory of $C^*(\mathring{\mathcal{N}}_V^G/\mathbb{R}_+^*)$ and of $\Sigma_{\mathrm{SBlup}}(G,V)$ and the KK-class of the connecting maps of these exact sequences.

In particular, we obtain a six term exact sequence

$$\begin{tikzcd}
K_0(C(M_b)) \arrow[r] & K_0(\Sigma_{\mathrm{SBlup}}(G,V)) \arrow[r] & K_1(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}})) \arrow[d, "\chi"] \\
K_0(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}})) \arrow[u, "\chi"] & K_1(\Sigma_{\mathrm{SBlup}}(G,V)) \arrow[l] & K_0(C(M_b)) \arrow[l]
\end{tikzcd}$$

and the index map $K_*(\Sigma_{\mathrm{SBlup}}(G,V)) \to K_{*+1}(G_{\mathring{M}}^{\mathring{M}})$ is the composition of $K_*(\Sigma_{\mathrm{SBlup}}(G,V)) \to K_{*+1}(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}}))$ with the index map of the groupoid $G_{\mathring{M}}^{\mathring{M}}$.

This holds, in particular, if $G = M_b \times M_b$ since the boundary $V = \partial M_b$ is transverse to $\widetilde{G} = M \times M$. Note that if M_b is connected with nonempty boundary, $\chi = 0$ (in $KK(C_0(T^*\mathring{M}), C_0(M_b))$) so that we obtain a (noncanonically) split short exact sequence:

$$0 \longrightarrow K_*(C_0(M_b)) \longrightarrow K_*(\Sigma_{\mathrm{SBlup}}(G,V)) \longrightarrow K_{*+1}(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}})) \longrightarrow 0.$$

4.16.1. *The index map via relative K-theory.* It follows from [14, Prop. 4.8]:

Proposition 4.17. *Let $\psi_{\mathrm{DNC}}\colon C_0(\mathrm{DNC}_+(M,V)) \to \Psi^*(\mathrm{DNC}_+(G,V))$ be the inclusion map which associates to a (smooth) function f the order 0 (pseudo) differential operator of multiplication by f and $\sigma_{\mathrm{full}} : \Psi^*(\mathrm{DNC}_+(G,V)) \to \Sigma_{\mathrm{DNC}_+}(G,V)$ the full symbol map. Put $\mu_{\mathrm{DNC}} = \sigma_{\mathrm{full}} \circ \psi_{\mathrm{DNC}}$. Then the relative K-group $K_*(\mu_{\mathrm{DNC}})$ is naturally isomorphic to $K_{*+1}(C_0(\mathfrak{A}^*G))$. Under this isomorphism, $\mathrm{ind}_{\mathrm{rel}}\colon K_*(\mu_{\mathrm{DNC}}) \to K_*(C^*(G\times\mathbb{R}_+^*)) = K_{*+1}(C^*(G))$ identifies with ind_G.*

Let us say also just a few words on the relative index map for $\mathrm{SBlup}_{r,s}(G,V)$, *i.e.,* for the map $\mu_{\mathrm{SBlup}}\colon C_0(\mathrm{SBlup}_+(M,V))\to\Sigma_{\mathrm{SBlup}}(G,V)$, which is the composition of the inclusion $\psi_{\mathrm{SBlup}}\colon C_0(\mathrm{SBlup}(M,V)\to\Psi^*(\mathrm{SBlup}_{r,s}(G,V))$ with the full index map $\sigma_{\mathrm{full}}\colon\Psi^*(\mathrm{SBlup}_{r,s}(G,V))\to\Sigma_{\mathrm{SBlup}}(G,V))$, and the corresponding relative index map $\mathrm{ind}_{\mathrm{rel}}\colon K_*(\mu_{\mathrm{SBlup}})\to K_*(C^*(G_{\mathring{M}}^{\mathring{M}}))$. We restrict to the case when V is $\mathfrak{A}G$ small. Equivalently, we wish to compute the relative index map $\mathrm{ind}_{\mathrm{rel}}:K_*(\mu_{\widetilde{\mathrm{DNC}}})\to K_{*+1}(C^*(G_{\mathring{M}}^{\mathring{M}}))$, where $\mu_{\widetilde{\mathrm{DNC}}}: C_0(\widetilde{\mathrm{DNC}}_+(M,V))\to\Sigma_{\widetilde{\mathrm{DNC}}_+}(G,V)$.

We have a short exact sequence

$$0\longrightarrow C_0(\widetilde{\mathrm{DNC}}_+(M,V))\longrightarrow C_0(\mathrm{DNC}_+(M,V))\longrightarrow C_0(V\times\mathbb{R}_+)\longrightarrow 0$$

and it follows that the inclusion $C_0(\widetilde{\mathrm{DNC}}_+(M,V))\to C_0(\mathrm{DNC}_+(M,V))$ is a KK-equivalence.

Since the inclusions

$$\Psi^*(\widetilde{\mathrm{DNC}}_+(G,V))\to\Psi^*(\mathrm{DNC}_+(G,V))\quad\text{and}\quad\Sigma_{\widetilde{\mathrm{DNC}}_+}(G,V)\to\Sigma_{\mathrm{DNC}_+}(G,V)$$

are also KK-equivalences (see Proposition 4.8), it follows that the inclusion $\mathsf{C}_{\mu_{\widetilde{\mathrm{DNC}}}}\to\mathsf{C}_{\mu_{\mathrm{DNC}}}$ of mapping cones is a KK-equivalence, and therefore the relative K-groups $K_*(\mu_{\widetilde{\mathrm{DNC}}})$ and $K_*(\mu_{\mathrm{DNC}})$ are naturally isomorphic. Using this, together with the Connes–Thom isomorphism, we deduce:

Corollary 4.18. *We assume that V is $\mathfrak{A}G$ small.*

(i) *The relative K-group $K_*(\mu_{\widetilde{\mathrm{DNC}}})$ is naturally isomorphic to*

$$K_{*+1}(C_0(\mathfrak{A}^*G)).$$

Under this isomorphism, $\mathrm{ind}_{\mathrm{rel}}:K_*(\mu_{\widetilde{\mathrm{DNC}}})\to K_*(C^*(G\times\mathbb{R}_+^*))=K_{*+1}(C^*(G))$ *identifies with* ind_G.

(ii) *The relative K-group $K_*(\mu_{\mathrm{SBlup}})$ is naturally isomorphic to*

$$K_*(C_0(\mathfrak{A}^*G)).$$

Under this isomorphism, $\mathrm{ind}_{\mathrm{rel}}\colon K_*(\mu_{\mathrm{SBlup}})\to K_*(C^*(G))$ *identifies with* ind_G.

5. Application: A Boutet de Monvel type calculus

Recall (see [4, 5, 15, 39]) that if M is a manifold with boundary ∂M, the Boutet de Monvel algebra consists of matrices of the form $\begin{pmatrix}\Phi_++S & P\\ T & Q\end{pmatrix}$. Without entering details, let us say that

- Φ is a pseudodifferential operator on $\widetilde{M}$ (a smooth neighborhood of M) satisfying a property called the transmission property, and Φ_+ the corresponding operator on smooth functions on M;
- S is a singular Green operator acting on M;
- P is a singular Poisson (or Potential) operator mapping functions on ∂M to functions on M;
- T is a singular trace operator mapping functions on M to functions on ∂M;

- Q is a (usual) pseudodifferential operator on ∂M.

The Boutet de Monvel algebra has two symbol maps:

- A "usual" symbol
$$\begin{pmatrix} \Phi_+ + S & P \\ T & Q \end{pmatrix} \mapsto \sigma(\Phi)$$
(often called "interior symbol") whose kernel is the algebra of operators of the form $\left(\begin{smallmatrix} S & P \\ T & Q \end{smallmatrix}\right)$.
- A non-commutative one of the form
$$\begin{pmatrix} \Phi_+ + S & P \\ T & Q \end{pmatrix} \mapsto \begin{pmatrix} \sigma_{\mathrm{Green}}(\Phi_+ + S) & \sigma_{\mathrm{Poisson}}(P) \\ \sigma_{\mathrm{Trace}}(T) & \sigma_{\partial M}(Q) \end{pmatrix}$$
(where $\sigma_{\partial M}$ is the usual symbol of ∂M).

In this section we consider the SBlup construction in the special case of a transverse submanifold of the unit space of a groupoid. We use the bimodule that we constructed in [12] in order to obtain an algebra resembling the algebra of 2×2 matrices in the Boutet de Monvel pseudodifferential calculus (of order 0) on manifolds with boundary.

From now on, we suppose that V is a *transverse submanifold* of M with respect to the Lie groupoid $G \rightrightarrows M$. In particular, V is $\mathfrak{A}G$-small—of course, we assume that (in every connected component of V), the dimension of V is strictly smaller than the dimension of M.

5.1. **The Poisson-trace bimodule.** As V is transverse to G, the groupoid G_V^V is a Lie groupoid, so that we can construct its "gauge adiabatic groupoid" $(G_V^V)_{ga}$ (see Section 3.7.3).

In [12], we constructed a bimodule relating the C^*-algebra of the groupoid $(G_V^V)_{ga}$ and the C^*-algebra of pseudodifferential operators of G_V^V.

In this section:

- We first show that the groupoid $(G_V^V)_{ga}$, is (sub-) Morita equivalent to $\mathrm{SBlup}_{r,s}(G,V)$ (*cp.* also Section 3.7.4 for a local construction).
- Composing the resulting bimodules, we obtain the "Poisson-trace" bimodule that relates the C^*-algebras $C^*(\mathrm{SBlup}_{r,s}(G,V))$ and $\Psi^*(G_V^V)$.

5.1.1. *The* $\mathrm{SBlup}_{r,s}(G,V)-(G_V^V)_{ga}$*-bimodule* $\mathscr{E}(G,V)$. Define the map $j\colon M\sqcup (V\times\mathbb{R}) \to M$ by letting $j_0\colon M\to M$ be the identity and $j_1\colon V\times\mathbb{R}\to M$ the composition of the projection $V\times\mathbb{R}\to V$ with the inclusion. Let $\mathcal{G} = G_j^j$. As V is assumed to be transverse, the map j is also transverse, and therefore $\mathcal{G}$ is a Lie groupoid.

It is the union of four clopen subsets:

- the groupoids $G_{j_0}^{j_0} = G = \mathcal{G}_M^M$ and $G_{j_1}^{j_1} = G_V^V \times (\mathbb{R}\times\mathbb{R}) = \mathcal{G}_{V\times\mathbb{R}}^{V\times\mathbb{R}}$,
- the *linking spaces* $G_{j_1}^{j_0} = \mathcal{G}_{V\times\mathbb{R}}^M = G_V \times \mathbb{R}$ and $G_{j_0}^{j_1} = \mathcal{G}_M^{V\times\mathbb{R}} = G^V \times\mathbb{R}$.

By functoriality, we obtain a sub-Morita equivalence of $\mathrm{SBlup}_{r,s}(G_V^V\times\mathbb{R}\times\mathbb{R},V)$ and $\mathrm{SBlup}_{r,s}(G,V)$ (see Section 3.5.3).

Let us describe this sub-Morita equivalence in a slightly different way:

Let also $\Gamma = V \times \{0,1\}^2$, sitting in $\mathcal{G}$:

$$\begin{array}{ll} V \times \{(0,0)\} \subset G = G_{j_0}^{j_0}, & V \times \{(0,1)\} \subset G_V \times \{0\} \subset G_{j_1}^{j_0}, \\ V \times \{(1,0)\} \subset G^V \times \{0\} \subset G_{j_0}^{j_1} & V \times \{(1,1)\} \subset G_V^V \times \{(0,0)\} \subset G_{j_1}^{j_1}. \end{array}$$

It is a subgroupoid of $\mathcal{G}$. The blow-up construction applied to $\Gamma \subset \mathcal{G}$ gives then a groupoid $\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)$ which is the union of:

$$\begin{array}{ll} \mathrm{SBlup}_{r,s}(G,V), & \mathrm{SBlup}_{r,s}(G_V \times \mathbb{R}, V), \\ \mathrm{SBlup}_{r,s}(G^V \times \mathbb{R}, V), & \mathrm{SBlup}_{r,s}(G_V^V \times \mathbb{R} \times \mathbb{R}, V). \end{array}$$

Recall that $\mathrm{SBlup}(V \times \mathbb{R}, V \times \{0\}) \simeq V \times (\mathbb{R}_- \sqcup \mathbb{R}_+)$. Thus $\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)$ is a groupoid with objects $\mathrm{SBlup}(M,V) \sqcup V \times \mathbb{R}_- \sqcup V \times \mathbb{R}_+$.

The restriction of $\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)$ to $V \times \mathbb{R}_+$ coincides with the restriction of $\mathrm{SBlup}_{r,s}(G_V^V \times \mathbb{R} \times \mathbb{R}, V)$ to $V \times \mathbb{R}_+$: it is the gauge adiabatic groupoid $(G_V^V)_{ga}$ of G_V^V (*cp.* Section 3.7.3).

Put $\mathrm{SBlup}_{r,s}(G_V \times \mathbb{R}, V)_+ = \mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)_{V \times \mathbb{R}_+}^{\mathrm{SBlup}(M,V)}$. It is a linking space between the groupoids $\mathrm{SBlup}_{r,s}(G,V)$ and $(G_V^V)_{ga}$. Put also $\mathrm{SBlup}_{r,s}(G^V \times \mathbb{R}, V)_+ = \mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)^{V \times \mathbb{R}_+}_{\mathrm{SBlup}(M,V)}$.

With the notation used in Fact 3.4, we define the $C^*(\mathrm{SBlup}_{r,s}(G,V)) - C^*((G_V^V)_{ga})$-bimodule $\mathscr{E}(G,V)$ to be $C^*(\mathrm{SBlup}_{r,s}(G_V \times \mathbb{R}, V)_+)$. It is the closure of $C_c(\mathrm{SBlup}_{r,s}(G_V \times \mathbb{R}, V)_+)$ in $C^*(\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma))$. It is a Hilbert-$C^*(\mathrm{SBlup}_{r,s}(G,V)) - C^*((G_V^V)_{ga})$-bimodule.

The Hilbert-$C^*((G_V^V)_{ga})$-module $\mathscr{E}(G,V)$ is full and $\mathcal{K}(\mathscr{E}(G,V))$ is the ideal $C^*(\mathrm{SBlup}_{r,s}(G_\Omega^\Omega, V))$, where $\Omega = r(G_V)$ is the union of orbits which meet V.

Notice that $\Omega = M \setminus V \sqcup V \times \mathbb{R}^*$ and $F = \mathbb{S}N_V^M \sqcup V \sqcup V$ gives a partition by, respectively, open and closed saturated subsets of the units of $\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)$. Furthermore, $\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)_\Omega^\Omega = \mathcal{G}_\Omega^\Omega$ and $C^*(\mathcal{G}_\Omega^\Omega) = C^*(\mathcal{G})$, according to Proposition 4.7. This decomposition gives rise to an exact sequence of C*-algebras.

$$0 \longrightarrow C^*(\mathcal{G}) \longrightarrow C^*(\mathrm{SBlup}_{r,s}(\mathcal{G},\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^{\mathcal{G}}) \longrightarrow 0.$$

This exact sequence gives rise to an exact sequence of bimodules:

$$\begin{tikzcd} 0 \arrow[r] & C^*(G) \arrow[r] \arrow[d, dash, "\mathring{\mathscr{E}}(G,V)"] & C^*(\mathrm{SBlup}_{r,s}(G,V)) \arrow[r] \arrow[d, dash, "\mathscr{E}(G,V)"] & C^*(\mathcal{S}N_V^G) \arrow[r] \arrow[d, dash, "\mathscr{E}^\partial(G,\Gamma)"] & 0 \\ 0 \arrow[r] & C^*(G_{V \times \mathbb{R}_+^*}^{V \times \mathbb{R}_+^*}) \arrow[r] & C^*((G_V^V)_{ga}) \arrow[r] & C^*(\mathfrak{A}G_V^V \rtimes \mathbb{R}_+^*) \arrow[r] & 0, \end{tikzcd}$$

where

$$\mathring{\mathscr{E}}(G,V) = C^*(\mathcal{G}_{V \times \mathbb{R}_+^*}^{M \setminus V}) \quad \mathscr{E}^\partial(G,\Gamma) = C^*((\mathcal{S}N_\Gamma^{\mathcal{G}})_V^{\mathbb{S}N_V^M}) = \mathscr{E}(G,V)/\mathring{\mathscr{E}}(G,V).$$

5.1.2. *The Poisson-trace bimodule $\mathscr{E}_{PT}$.* In [12], we constructed, for every Lie groupoid H, a $C^*(H_{ga})-\Psi^*(H)$-bimodule $\mathscr{E}_H$.

Recall that the Hilbert $\Psi^*(H)$-module $\mathscr{E}_H$ is full and that $\mathcal{K}(\mathscr{E}_H)\subset C^*(H_{ga})$ is the kernel of a natural $*$-homomorphism $C^*(H_{ga})\to C_0(H^{(0)}\times\mathbb{R})$. We also showed that the bimodule $\mathscr{E}_H$ gives rise to an exact sequence of bimodules as above:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(H\times\mathbb{R}_+^*\times\mathbb{R}_+^*) & \longrightarrow & C^*(H_{ga}) & \longrightarrow & C^*(\mathfrak{A}H\rtimes\mathbb{R}_+^*) & \longrightarrow & 0\\ & & \Big| \, \mathring{\mathscr{E}}_H & & \Big| \, \mathscr{E}_H & & \Big| \, \mathscr{E}_H^\partial & & \\ 0 & \longrightarrow & C^*(H) & \longrightarrow & \Psi^*(H) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*H) & \longrightarrow & 0.\end{array}$$

Then, by putting together the bimodule $\mathscr{E}(G,V)$ and $\mathscr{E}_{G_V^V}$, we obtain a $C^*(\mathrm{SBlup}_{r,s}(G,V))-\Psi^*(G_V^V)$ bimodule $\mathscr{E}(G,V)\otimes_{C^*((G_V^V)_{ga})}\mathscr{E}_{G_V^V}$ that we call the *Poisson-trace* bimodule and we denote it by $\mathscr{E}_{PT}(G,V)$ or just $\mathscr{E}_{PT}$. It leads to the exact sequence of bimodule:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(G) & \longrightarrow & C^*(\mathrm{SBlup}_{r,s}(G,V)) & \longrightarrow & C^*(\mathcal{S}N_V^G) & \longrightarrow & 0\\ & & \Big| \, \mathring{\mathscr{E}}_{PT(G,V)} & & \Big| \, \mathscr{E}_{PT(G,V)} & & \Big| \, \mathscr{E}^\partial_{PT}(G,V) & & \\ 0 & \longrightarrow & C^*(G_V^V) & \longrightarrow & \Psi^*(G_V^V) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G_V^V) & \longrightarrow & 0.\end{array}$$

The Poisson-trace bimodule is a full Hilbert $\Psi^*(G_V^V)$-module; $\mathcal{K}(\mathscr{E}_{PT}(G,V))$ is a two sided ideal of $C^*(\mathrm{SBlup}_{r,s}(G,V))$. Denote by $\mathscr{E}_{PT}(G,V)^*$ its dual module, *i.e.,* the $\Psi^*(G_V^V)-C^*(\mathrm{SBlup}_{r,s}(G,V))$-bimodule $\mathcal{K}(\mathscr{E}_{PT}(G,V),\Psi^*(G_V^V))$.

5.2. **A boundary modeled algebra.** The C^*-algebra $C^*_{BM}(G,V)$ is the algebra $\mathcal{K}(C^*(\mathrm{SBlup}_{r,s}(G,V))\oplus\mathscr{E}_{PT}(G,V)^*)$ of compact operators of the Hilbert $C^*(\mathrm{SBlup}_{r,s}(G,V))$-module $C^*(\mathrm{SBlup}_{r,s}(G,V))\oplus\mathscr{E}_{PT}(G,V)^*$. Its elements are matrices of the form $\left(\begin{smallmatrix}K & P\\ T & Q\end{smallmatrix}\right)$, where

- $K\in C^*(\mathrm{SBlup}_{r,s}(G,V))=\mathcal{K}(C^*(\mathrm{SBlup}_{r,s}(G,V)))$,
- $P\in\mathscr{E}_{PT}(G,V)=\mathcal{K}(\mathscr{E}_{PT}(G,V)^*,C^*(\mathrm{SBlup}_{r,s}(G,V)))$,
- $T\in\mathscr{E}_{PT}(G,V)^*=\mathcal{K}(C^*(\mathrm{SBlup}_{r,s}(G,V),\mathscr{E}_{PT}(G,V)^*))$,
- $Q\in\Psi^*(G_V^V)=\mathcal{K}(\mathscr{E}_{PT}(G,V)^*)$.

We have an exact sequence (where $\mathring{M}\sqcup V\neq M$ denotes the topological disjoint union of $\mathring{M}$ with V):

$$0\longrightarrow C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V})\longrightarrow C^*_{BM}(G,V)\xrightarrow{r_V^{C^*}}\Sigma^{C^*}_{\mathrm{bound}}(G,V)\longrightarrow 0,$$

where the quotient $\Sigma^{C^*}_{\mathrm{bound}}(G,V)$ is the algebra of the *Boutet de Monvel type boundary symbols.* It is the algebra of matrices of the form $\left(\begin{smallmatrix}k & p\\ t & q\end{smallmatrix}\right)$, where $k\in C^*(\mathcal{S}N_V^G)$, $q\in C(\mathbb{S}\mathfrak{A}^*G_V^V)$, $p,t^*\in\mathscr{E}^V_{PT}(G,V):=\mathscr{E}_{PT}(G,V)\otimes_{\Psi^*(G_V^V)}C(\mathbb{S}\mathfrak{A}^*G_V^V)$. The map $r_V^{C^*}$ is of the form

$$r_V^{C^*}\begin{pmatrix}K & P\\ T & Q\end{pmatrix}=\begin{pmatrix}r_V^{\frown}(K) & r_V^{\frown\!\!\!\!\smile}(P)\\ r_V^{\smile\!\!\!\!\frown}(T) & \sigma_V(Q)\end{pmatrix},$$

where:

- the quotient map σ_V is the ordinary order 0 principal symbol map on the groupoid G_V^V;
- the quotient maps $r_V^{\bowtie}, r_V^{\propto}, r_V^{\rotatebox[origin=c]{180}{$\propto$}}$ are restrictions to the boundary N_V^M:
$$r_V^{\bowtie}\colon C^*(\mathrm{SBlup}_{r,s}(G,V)) \to C^*(\mathbb{S}N_V^G) = C^*(\mathrm{SBlup}_{r,s}(G,V))/C^*(G_{\mathring{M}}^{\mathring{M}}),$$
$$r_V^{\propto}\colon \mathscr{E}_{PT}(G,V) \to \mathscr{E}_{PT}^V(G,V) = \mathscr{E}_{PT}(G,V)/C^*(G_V^{\mathring{M}}),$$
and $r_V^{\rotatebox[origin=c]{180}{$\propto$}}(T) = r_V^{\propto}(T^*)^*$.

The map $r_V^{C^*}$ is called the *zero order symbol map of the Boutet de Monvel type calculus.*

5.3. **A boundary modeled pseudodifferential algebra.** We denote by $\Psi_{BM}^*(G,V)$ the algebra of matrices $\left(\begin{smallmatrix}\Phi & P\\ T & Q\end{smallmatrix}\right)$, with $\Phi \in \Psi^*(\mathrm{SBlup}_{r,s}(G,V))$, $P \in \mathscr{E}_{PT}(G,V)$, $T \in \mathscr{E}_{PT}(G,V)^*$ and $Q \in \Psi^*(G_V^V)$.

Such an operator $R = \left(\begin{smallmatrix}\Phi & P\\ T & Q\end{smallmatrix}\right)$ has two symbols:

- the *classical symbol* $\sigma_c\colon \Psi_{BM}^*(G,V) \to C_0(\mathbb{S}\mathfrak{A}^*\mathrm{SBlup}_{r,s}(G,V))$ given by $\sigma_c\left(\begin{smallmatrix}\Phi & P\\ T & Q\end{smallmatrix}\right) = \sigma_c(\Phi)$, with its kernel being $C_{BM}^*(G,V)$.
- the *boundary symbol* $r_V^{BM}\colon \Psi_{BM}^*(G,V) \to \Sigma_{\mathrm{bound}}^{\Psi^*}(G,V)$ defined by
$$r_V\begin{pmatrix}\Phi & P\\ T & Q\end{pmatrix} = \begin{pmatrix} r_V^{\psi}(\Phi) & r_V^{\propto}(P)\\ r_V^{\rotatebox[origin=c]{180}{$\propto$}}(T) & \sigma_V(Q)\end{pmatrix},$$
where $r_V^{\psi}\colon \Psi^*(\mathrm{SBlup}_{r,s}(G,V)) \to \Psi^*(\mathcal{S}N_V^G)$ is the restriction.

Here $\Sigma_{\mathrm{bound}}^{\Psi^*}(G,V)$ denotes the algebra of matrices of the form $\left(\begin{smallmatrix}\phi & p\\ t & q\end{smallmatrix}\right)$, with $\phi \in \Psi^*(\mathcal{S}N_V^G)$, $p, t^* \in \mathscr{E}_{PT}^V(G,V)$ and $q \in C(\mathbb{S}\mathfrak{A}^*G_V^V)$.

The *full symbol* map is the morphism
$$\sigma_{BM}\colon \Psi_{BM}^*(G,V) \to \Sigma_{BM}(G,V),$$
where
$$\Sigma_{BM}(G,V) := C_0(\mathbb{S}\mathfrak{A}^*\mathrm{SBlup}_{r,s}(G,V)) \times_{C_0(\mathbb{S}\mathfrak{A}^*\mathcal{S}N_V^G)} \Sigma_{\mathrm{bound}}^{\Psi^*}(G,V),$$
defined by $\sigma_{BM}(R) = (\sigma_c(R), r_V(R))$.

We have an exact sequence:
$$0 \longrightarrow C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V}) \longrightarrow \Psi_{BM}^*(G,V) \xrightarrow{\sigma_{BM}} \Sigma_{BM}(G,V) \longrightarrow 0. \qquad (E_{BM})$$

We may note that $\Psi^*(\mathrm{SBlup}_{r,s}(G,V))$ (*resp.* $\Psi^*(\mathcal{S}N_V^G)$) identifies with the full hereditary subalgebra of $\Psi_{BM}^*(G,V)$ (*resp.* of $\Sigma_{BM}(G,V)$) consisting of elements of the form $\left(\begin{smallmatrix}x & 0\\ 0 & 0\end{smallmatrix}\right)$.

5.4. ***K*-theory of the symbol algebras and index maps.** In this section we examine the index map corresponding to the Boutet de Monvel type calculus and, in particular, to the exact sequence (E_{BM}). We compute the K-theory of the symbol algebra Σ_{BM} and the connecting element $\widetilde{\mathrm{ind}}_{BM} \in KK^1(\Sigma_{BM}, C^*(G))$.[2]

[2]We use the Morita equivalence of $C^*(G)$ with $C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V})$.

We then extend this computation by including bundles into the picture, *i.e.,* by computing a relative K-theory map.

As the Hilbert $\Psi^*(G_V^V)$ module $\mathscr{E}_{PT}(G,V)$ is full,

- the subalgebra
$$\left\{\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} ;\ K \in C^*(\mathrm{SBlup}_{r,s}(G,V))\right\}$$
is a full hereditary subalgebra of $C^*_{BM}(G,V)$;
- the subalgebra
$$\left\{\begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix} ;\ \Phi \in \Psi^*(\mathrm{SBlup}_{r,s}(G,V))\right\}$$
is a full hereditary subalgebra of $\Psi^*_{BM}(G,V)$;
- the subalgebra
$$\left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} ;\ x \in \Sigma_{\mathrm{SBlup}}(G,V)\right\}$$
is a full hereditary subalgebra of $\Sigma_{BM}(G,V)$;
- the subalgebra
$$\left\{\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} ;\ k \in C^*(\mathcal{S}N_V^G)\right\}$$
is a full hereditary subalgebra of $\Sigma^{C^*}_{\mathrm{bound}}(G,V)$;
- the subalgebra
$$\left\{\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} ;\ \phi \in \Psi^*(\mathcal{S}N_V^G)\right\}$$
is a full hereditary subalgebra of $\Sigma^{\Psi^*}_{\mathrm{bound}}(G,V)$.

We have a diagram of exact sequences, where the vertical inclusions are Morita equivalences:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[d, hook] & \Psi^*(\mathrm{SBlup}_{r,s}(G,V)) \arrow[r, "\sigma_{\mathrm{full}}"] \arrow[d, hook] & \Sigma_{\mathrm{SBlup}}(G,V) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V}) \arrow[r] & \Psi^*_{BM}(G,V) \arrow[r, "\sigma_{BM}"] & \Sigma_{BM}(G,V) \arrow[r] & 0.
\end{tikzcd}$$

We thus deduce immediately from Theorem 4.9 and Proposition 4.14 (with the notation of Proposition 4.14) the following:

Corollary 5.5. *The algebra $\Sigma_{BM}(G,V))$ is KK-equivalent with the mapping cone C_χ and, under this K-equivalence, the index $\widetilde{\mathrm{ind}}_{BM}$ is $q^*([\mathrm{Bott}]\otimes_{\mathbb{C}} \mathrm{ind}_G)$, where $q\colon \mathsf{C}_\chi \to C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*)$ is evaluation at* 0*.*

Appendix A. Bundle groupoids

In this section we describe the structure of the normal groupoid $\mathcal{N}_\Gamma^G$, *i.e.*, the restriction of $\mathrm{DNC}(G,\Gamma)$ to its singular part N_V^M, as well as the projective normal groupoid $\mathcal{P}N_\Gamma^G$, the restriction of $\mathrm{Blup}_{r,s}(G,\Gamma)$ to its singular part $\mathbb{P}N_V^M$. The groupoid $\mathcal{N}_\Gamma^G$ is a $\mathcal{VB}$ groupoid in the sense of Pradines [37, 23]. In the particular case where $\Gamma = V$ is just a space, the groupoids $\mathcal{N}_\Gamma^G$ and $\mathcal{P}N_\Gamma^G$ are bundles of linear and projective groupoids over the base V in a sense defined bellow. In that case, a Thom–Connes isomorphism computes the KK-theory of $C^*(\mathcal{N}_\Gamma^G)$ (Proposition A.9).

A.1. **$\mathcal{VB}$ groupoids.** In [37], Pradines introduced the notion of a $\mathcal{VB}$ *groupoid* (see also [23]). Such groupoids naturally appear in our construction, as well as their projective and spherical analogues.

Recall from [37, 23] that a $\mathcal{VB}$ groupoid is a groupoid which is a vector bundle over a groupoid G. More precisely:

Definition A.2. Let $G \overset{r_G,s_G}{\rightrightarrows} G^{(0)}$ be a groupoid. A $\mathcal{VB}$ *groupoid* over G is a vector bundle $p\colon E \to G$ with a groupoid structure $E \overset{r_E,s_E}{\rightrightarrows} E^{(0)}$ such that all the groupoid maps are linear vector bundle morphisms. This means that $E^{(0)} \subset E$ is a vector subbundle of the restriction of E to $G^{(0)}$ and that r_E, s_E, $x \mapsto x^{-1}$ and the composition are linear bundle maps:

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E^{(0)} \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & G^{(0)}. \end{tikzcd}$$

We also assume that the bundle maps $r_E\colon E \to r_G^*(E^{(0)})$ and $s_E\colon E \to s_G^*(E^{(0)})$ are surjective.

When Γ is a closed Lie subgroupoid of G, the projection $\mathcal{N}_\Gamma^G \to \Gamma$ is a groupoid morphism and it is easily seen that $\mathcal{N}_\Gamma^G$ is a $\mathcal{VB}$ groupoid over Γ. In fact, every $\mathcal{VB}$ groupoid $E \to \Gamma$ can be seen as a normal groupoid: the normal groupoid to the inclusion $\Gamma \to E$.

To any $\mathcal{VB}$ groupoid $p\colon E \to G$, we can associate a *projective bundle groupoid* and a *spherical bundle groupoid*.

Definition A.3. Let $p\colon E \to G$. Denote by $\widetilde{E^{(0)}}$ the complement in $E^{(0)}$ of the zero-section $G^{(0)}$. Let $\widetilde{E}$ be the restriction $E^{\widetilde{E^{(0)}}}_{\widetilde{E^{(0)}}} = r_E^{-1}(\widetilde{E^{(0)}}) \cap_E^{-1} (\widetilde{E^{(0)}})$ of E to its open subset $\widetilde{E^{(0)}}$ of its objects.

The natural scaling action α of the group $\mathbb{R}^*$ on the vector bundle E is free and proper on $\widetilde{E}$; for every λ, α_λ is an automorphism of the groupoid E. The quotient spaces $\mathcal{P}E = \widetilde{E}/\mathbb{R}^*$ and $\mathcal{S}E = \widetilde{E}/\mathbb{R}_+^*$ are Lie groupoids, respectively, called the *projective bundle groupoid*, and the *spherical bundle groupoid* of E their units are, respectively, $\mathbb{P}E^{(0)}$ and $\mathbb{S}E$.

A.4. **Linear groupoids.** An easy case of $\mathcal{VB}$ groupoids $G \to \Gamma$, relevant to our discussion, is when the base groupoid Γ is just a space. In that case, a natural Connes–Thom isomorphism relates the K-theory of $C^*(G)$ with that of the space $C_0(G)$.

In order to understand this case, we examine linear, projective and spherical groupoids in an even simple case, when the base groupoid is just one point. We briefly examine this situation.

Let E be a vector space over a field $\mathbb{K}$ and let F be a vector sub-space. Let $r, s\colon E \to F$ be linear retractions of the inclusion $F \to E$.

A.4.1. *The linear groupoid.* The space E is endowed with a groupoid structure $\mathcal{E}$ with base F. The range and source maps are r and s and the product is $(x, y) \mapsto (x \cdot y) = x + y - s(x)$ for (x, y) composable, *i.e.*, such that $s(x) = r(y)$. One can easily check:

- Since r and s are linear retractions, $r(x \cdot y) = r(x)$ and $s(x \cdot y) = s(y)$.
- If (x, y, z) are composable, then $(x \cdot y) \cdot z = x + y + z - (r + s)(y) = x \cdot (y \cdot z)$.
- The inverse of x is $(r + s)(x) - x$.

Remarks A.5. (i) Note that, given E and linear retractions $r, s\colon E \rightrightarrows F$, the only possible linear groupoid structure on E is the one described above.[3] Indeed, for any $x \in E$, one must have $x \cdot s(x) = x$ and $r(x) \cdot x = x$. By linearity, it follows that for every composable pair $(x, y) = (x, s(x)) + (0, y - s(x))$, we have $x \cdot y = x \cdot s(x) + 0 \cdot (y - s(x)) = x + y - s(x)$.

(ii) The morphism $r - s\colon E/F \to F$ gives an action of E/F on F by addition. The groupoid $\mathcal{E}$ is in fact the groupoid $E/F \rtimes F \rightrightarrows F$ associated with this action.

(iii) Given a linear groupoid structure on a vector space E, we obtain the "dual" linear groupoid structure $\mathcal{E}^*$ on the dual space E^* given by the subspace $F^\perp = \{\xi \in E^*;\ \xi|_F = 0\}$ and the two retractions $r^*, s^*\colon E^* \to F^\perp$ with kernels $(\ker r)^\perp$ and $(\ker s)^\perp$: for $\xi \in E^*$ and $x \in E$, $r^*(\xi)(x) = \xi(x - r(x))$ and, similarly, $s^*(\xi)(x) = \xi(x - s(x))$.

A.5.1. *The projective groupoid.* The multiplicative group $\mathbb{K}^*$ acts on $\mathcal{E}$ by groupoid automorphisms. This action is free on the restriction $\widetilde{\mathcal{E}} = \mathcal{E} \setminus (\ker r \cup \ker s)$ of the groupoid $\mathcal{E}$ to the subset $F \setminus \{0\}$ of $\mathcal{E}^{(0)} = F$.

The projective groupoid is the quotient groupoid $\mathcal{P}E = \widetilde{\mathcal{E}}/\mathbb{K}^*$. It is described as follows.

As a set, $\mathcal{P}E = \mathbb{P}(E) \setminus (\mathbb{P}(\ker r) \cup \mathbb{P}(\ker s))$ and $\mathcal{P}^{(0)} = \mathbb{P}(F) \subset \mathbb{P}(E)$. The source and range maps $r, s\colon \mathcal{P}E \to \mathbb{P}(F)$ are those induced by $r, s\colon E \to F$. The product of $x, y \in \mathcal{P}E$ with $s(x) = r(y)$ is the line $x \cdot y = \{u + v - s(u);\ u \in x,\ v \in y;\ s(u) = r(v)\}$. The inverse of $x \in \mathcal{P}E$ is $(r + s - id)(x)$.

Remarks A.6. (i) When F is just a vector line, $\mathcal{P}E$ is a group. Let us describe it:

[3] A linear groupoid is a groupoid G such that $G^{(0)}$ and G are vector spaces and all structure maps (unit, range, source, product) are linear.

We have a canonical morphism $h\colon \mathcal{P}E \to \mathbb{K}^*$ defined by $r(u) = h(x)s(u)$ for $u \in x$. The kernel of h is $\mathbb{P}(\ker(r-s)) \setminus \mathbb{P}(\ker r)$. Note that $F \subset \ker(r-s)$ and therefore $\ker(r-s) \not\subset \ker r$, whence $\ker r \cap \ker(r-s)$ is a hyperplane in $\ker(r-s)$. The group $\ker h$ is then easily seen to be isomorphic to $\ker(r) \cap \ker(s)$. Indeed, choose a nonzero vector w in F; then the map which assigns to $u \in \ker(r) \cap \ker(s)$ the line with direction $w + u$ gives such an isomorphism onto $\ker h$.

Then:

- If $r = s$, $\mathcal{P}E$ is isomorphic to the abelian group $\ker(r) = \ker(s)$.
- If $r \neq s$, choose x such that r and s do not coincide on x and let P be the plane $F \oplus x$. The subgroup $\mathbb{P}(P) \setminus \{\ker r \cap P, \ker s \cap P\}$ of $\mathcal{P}E$ is isomorphic through h with $\mathbb{K}^*$. It thus defines a section of h. In that case $\mathcal{P}E$ is the group of dilations $(\ker(r) \cap \ker(s)) \rtimes \mathbb{K}^*$.

(ii) In the general case, let $d \in \mathbb{P}(F)$. Put $E_d^d = r^{-1}(d) \cap s^{-1}(d)$.

- The stabilizer $(\mathcal{P}E)_d^d$ is the group $\mathcal{P}E_d^d = \mathbb{P}(E_d^d) \setminus (\mathbb{P}(\ker r) \cup \mathbb{P}(\ker s))$ described above.
- The orbit of a line d is the set of $r(x)$ for $x \in \mathcal{P}E$ such that $s(x) = d$. It is therefore $\mathbb{P}(d + r(\ker s))$.

(iii) The following are equivalent:

(i) $(r, s)\colon E \to F \times F$ is onto,

(ii) $r(\ker s) = F$,

(iii) $(r - s)\colon E/F \to F$ is onto,

(iv) the groupoid $\mathcal{P}E$ has just one orbit.

(iv) When $r = s$, the groupoid $\mathcal{P}E$ is the product of the abelian group E/F by the space $\mathbb{P}(F)$.

When $r \neq s$, the groupoid $\widetilde{\mathcal{E}}$ is Morita equivalent to $\mathcal{E}$, since $F \setminus \{0\}$ meets all the orbits of $\mathcal{E}$.

If $\mathbb{K}$ is a locally compact field and $r \neq s$, the smooth groupoid $\mathcal{P}E$ is Morita equivalent to the groupoid crossed-product $\widetilde{\mathcal{E}} \rtimes \mathbb{K}^*$.

In all cases, when $\mathbb{K}$ is a locally compact field, $\mathcal{P}E$ is amenable.

A.6.1. *The spherical groupoid.* If the field is $\mathbb{R}$, we may just take the quotient by $\mathbb{R}_+^*$ instead of $\mathbb{R}^*$. We then obtain, similarly, the *spherical groupoid* $\mathcal{S}E = \mathbb{S}(E) \setminus (\mathbb{S}(\ker r) \cup \mathbb{S}(\ker s))$, where $\mathcal{S}^{(0)}(E) = \mathbb{S}(F) \subset \mathbb{S}(E)$.

The involutive automorphism $u \mapsto -u$ of E leads to a $\mathbb{Z}/2\mathbb{Z}$ action, by groupoid automorphisms on $\mathcal{S}E$. Since this action is free (and proper!), it follows that the quotient groupoid $\mathcal{P}E$ and the crossed product groupoid $\mathcal{S}E \rtimes \mathbb{Z}/2\mathbb{Z}$ are Morita equivalent. Thus $\mathcal{S}E$ is also amenable.

As for the projective case, if $(r, s)\colon E \to F \times F$ is onto, the groupoid $\mathcal{S}E$ has just one orbit. The stabilizer of $d \in \mathbb{S}(F)$ identifies with the group $(\ker r \cap \ker s) \rtimes \mathbb{R}_+^*$, and therefore the groupoid $\mathcal{S}E$ is Morita equivalent to the group $(\ker r \cap \ker s) \rtimes \mathbb{R}_+^*$.

A.7. **A Connes–Thom isomorphism for families of linear groupoids.** We may of course perform the constructions of Section A.5.1 (with say $\mathbb{K} = \mathbb{R}$)

when $E \to X$ is a (real) vector bundle over a locally compact space X, F is a subbundle and $r, s\colon E \to F$ are linear bundle maps. We obtain, respectively, families $\mathcal{E}$, $(\mathcal{P}E, r, s)$ and $(\mathcal{S}E, r, s)$ of linear, projective and spherical groupoids.

Remarks A.8. (i) A family of linear groupoids is just given by a bundle morphism $\alpha = (r - s)\colon E/F \to F$. It is isomorphic to the semi-direct product $F \rtimes_\alpha E/F$.
(ii) All the groupoids defined here are amenable, since they are continuous fields of amenable groupoids (*cp.* [2, Prop. 5.3.4]).

The groupoid $\mathcal{E}$ is a vector bundle E over a locally compact space X, $\mathcal{E}^{(0)}$ is a vector subbundle F and $\mathcal{E}$ is given by a linear bundle map $(r-s)\colon E/F \to F$.

Proposition A.9 (A Thom–Connes isomorphism)**.** *Let $E \to X$ be a family of linear groupoids. Then $C^*(E)$ is KK-equivalent to $C_0(E)$. More precisely, the index* $\mathrm{ind}_E\colon KK(C_0(\mathfrak{A}^*E), C^*(E))$ *is invertible.*

Proof. Put $F = E^{(0)}$ and $H = E/F$. The groupoid H acts on $C_0(F)$ and $C^*(E) = C_0(F) \rtimes H$.

We use the equivariant KK-theory of Le Gall (*cp.* [22]) $KK_H(A, B)$.

The Thom element of the complex bundle $H \oplus H$ defines an invertible element

$$t_H \in KK_H(C_0(X), C_0(H \oplus H)).$$

We deduce that, for every pair A, B of H algebras, the morphism

$$\tau_{C_0(H)}\colon KK_H(A, B) \to KK_H(A \otimes_{C_0(X)} C_0(H), B \otimes_{C_0(X)} C_0(H))$$

is an isomorphism. Its inverse is $x \mapsto t_H \otimes \tau_{C_0(H)}(x) \otimes t_H^{-1}$.

Note that for every H-algebra A, the H-algebra $C_0(H) \otimes_{C(X)} A$ is the algebra of continuous sections of the form $(x, \xi) \mapsto \varphi(x, \xi) \in A_x$ (where $x \in X$, $\xi \in H_x$ and A_x is the fibre of A at $x \in X$) vanishing at infinity. The fibre of $(C_0(H) \otimes_{C(X)} A)_x$ is $C_0(H_x, A_x)$ and the action of H_x on the fibre is given by $(\xi \cdot \varphi)(\eta) = \xi \cdot (\varphi(\eta - \xi))$.

Denote by A_0 the $C_0(X)$ algebra A endowed with the trivial action of H. We have an isomorphism of H-algebras $u_A\colon C_0(H) \otimes_{C(X)} A_0 \simeq C_0(H) \otimes_{C(X)} A$: put $(u_A(\varphi))(x, \xi) = \xi \cdot (\varphi(x, \xi))$.

It follows that the restriction map $KK_H(A, B)$ to $KK_X(A, B)$ (associated to the groupoid morphism $X \to H$) is an isomorphism—compatible of course with the Kasparov product.

Let $v_A \in KK_H(A_0, A)$ be the element whose image in $KK_X(A_0, A)$ is the identity. Its descent $j_H(v_A) \in KK(C_0(H^*) \otimes_{C(X)} A, A \rtimes H)$ is a KK-equivalence.

The index map is given by letting $G = DNC(E, F)_{[0,1]}$ and putting $\mathrm{ind} = \mathrm{ev}_1 \otimes \mathrm{ev}_0^{-1}$, where $\mathrm{ev}_0\colon C^*(G) \to C^*(\mathfrak{A}G) = C^*(H) \otimes C_0(F)$ is evaluation at 0, which is a KK-equivalence and $\mathrm{ev}_1\colon C^*(G) = C_0(F \times [0, 1]) \rtimes H \to C^*(E) = C_0(F) \rtimes H$ is evaluation at 1. Since the evaluation $F \times [0, 1] \to F$ at 1 is a KK_X equivalence, it is also KK_H invertible. It follows that $[ev_1]$ is also invertible. ☐

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List of Symbols

Groupoids, deformation and blow-up spaces.

$G \overset{r,s}{\rightrightarrows} G^{(0)}$: A Lie groupoid with source s, range r and space of units $G^{(0)}$.

$\mathfrak{A}G$: The Lie algebroid of the groupoid G.

G^A, G_B, G_B^A: If A and B are subsets of $G^{(0)}$, $G^A = \{x \in G;\, r(x) \in A\}$, $G_B = \{x \in G;\, s(x) \in B\}$ and $G_A^B = G_A \cap G^B$, page 11.

G^f, G_g, G_g^f: If $f\colon A \to G^{(0)}$ and $g\colon B \to G^{(0)}$ are maps, $G^f = \{(a,x) \in A \times G;\, r(x) = f(a)\}$, $G_g = \{(x,b) \in G \times B; s(x) = g(b)\}$ and $G_g^f = G^f \cap G_g$, page 11.

N_V^M: The normal bundle of a submanifold V of a manifold M.

$\mathrm{DNC}(Y,X)$: The deformation to the normal cone of the inclusion of a submanifold X in a manifold Y, $\mathrm{DNC}(Y,X) = Y \times \mathbb{R}^* \cup N_X^Y$, page 6.

$\mathrm{DNC}_+(Y,X)$: The restriction $\mathrm{DNC}(Y,X) \setminus Y \times (-\infty, 0)$, page 9.

$\mathrm{Blup}(Y,X)$: The blow-up of the inclusion of a submanifold X in a manifold Y, $\mathrm{Blup}(Y,X) = Y \setminus X \cup \mathbb{P}(N_X^Y)$, page 8.

$\mathrm{SBlup}(Y,X)$: The spherical blow-up of the inclusion of a submanifold X in a manifold Y, $\mathrm{SBlup}(Y,X) = Y \setminus X \cup \mathbb{S}(N_X^Y)$, page 8.

$\mathrm{Blup}_f(Y,X)$: The subspace of $\mathrm{Blup}(Y,X)$ on which $\mathrm{Blup}(f)\colon \mathrm{Blup}_f(Y,X) \to \mathrm{Blup}(Y',X')$ can be defined for a smooth map $f\colon Y \to Y'$ (with $f(X) \subset X'$), page 10.

$\mathrm{DNC}(G,\Gamma) \rightrightarrows \mathrm{DNC}(G^{(0)},\Gamma^{(0)})$: The deformation groupoid, where Γ is a closed Lie subgroupoid of a Lie groupoid G, page 13.

$\widetilde{\mathrm{DNC}}(G,\Gamma)$, $\widetilde{\mathrm{DNC}}_+(G,\Gamma)$: The open subgroupoid of $\mathrm{DNC}(G,\Gamma)$ consisting of elements whose image by $\mathrm{DNC}(r)$ and $\mathrm{DNC}(s)$ is not in $\Gamma^{(0)} \times \mathbb{R}$ and its restriction to $\mathbb{R}_+$, page 13.

$\mathrm{Blup}_{r,s}(G,\Gamma) \rightrightarrows \mathrm{Blup}(G^{(0)},\Gamma^{(0)})$: The blow-up groupoid $\mathrm{Blup}_r(G,\Gamma) \cap \mathrm{Blup}_s(G,\Gamma)$, where Γ is a closed Lie subgroupoid of a Lie groupoid G; it is the quotient of $\widetilde{\mathrm{DNC}}(G,\Gamma)$ under the zooming action, page 13.

$\mathrm{SBlup}_{r,s}(G,\Gamma)$: The spherical version of $\mathrm{Blup}_{r,s}(G,\Gamma)$; it is quotient of $\widetilde{\mathrm{DNC}}_+(G,\Gamma)$ under the restricted zooming action, page 13.

C^*-algebras.

$C^*(G)$: The (either maximal or reduced) C^*-algebra of the groupoid G.

$\Psi^*(G)$: The C^*-algebra of order ≤ 0 pseudodifferential operators on G vanishing at infinity on $G^{(0)}$.

C_f: The mapping cone of a morphism $f\colon A \to B$ of C^*-algebra.

$\Sigma^W(G)$: The quotient $\Psi^*(G)/C^*(G_W)$.

$\Sigma_{\mathrm{DNC}_+}(G,\Gamma)$, $\Sigma_{\widetilde{\mathrm{DNC}}_+}(G,\Gamma)$: Respectively, the algebras $\Psi^*(\mathrm{DNC}_+(G,\Gamma))/C^*(G\times\mathbb{R}+^*)$ and $\Psi^*(\widetilde{\mathrm{DNC}}_+(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*)$, page 19.

$\Sigma_{\mathrm{SBlup}}(G,\Gamma)$: The algebra $\Psi^*(\mathrm{SBlup}_{r,s}(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}})$, page 19.

KK-elements.

$[f]$: The KK-element, in $KK(A,B)$ associated to a morphism of C*-algebra $f\colon A\to B$.

ind_G: The KK-element $[ev_0]^{-1}\otimes[ev_1]$, which belongs to $KK(C_0(\mathfrak{A}^*G),C^*(G))$, associated to the deformation groupoid $G_{ad}^{[0,1]} = G\times(0,1]\cup\mathfrak{A}(G)\times\{0\} \rightrightarrows G^{(0)}\times[0,1]$.

$\widetilde{\mathrm{ind}}_G$: The connecting element, which belongs to $KK^1(C(\mathbb{S}\mathfrak{A}^*\mathcal{G}),C^*(\mathcal{G}))$ associated to the short exact sequence $0\to C^*(G)\to\Psi^*(G)\to C(\mathbb{S}\mathfrak{A}^*G)\to 0$.

∂_G^W: The connecting element, which belongs to $KK^1(C^*(G|_F),C^*(G|_W))$, associated to the short exact sequence $0\to C^*(G|_W)\to C^*(G)\to C^*(G|_F)\to 0$, where W is a saturated open subset of $G^{(0)}$ and $F=G^{(0)}\setminus W$.

$\partial_{\mathrm{SBlup}}^{G,\Gamma}$, $\partial_{\mathrm{DNC}_+}^{G,\Gamma}$, $\partial_{\widetilde{\mathrm{DNC}}_+}^{G,\Gamma}$: Respectively, the element $\partial_{\mathrm{SBlup}_{r,s}(G,\Gamma)}^{\mathring{M}}$, $\partial_{\mathrm{DNC}_+(G,\Gamma)}^{M\times\mathbb{R}_+^*}$ and $\partial_{\widetilde{\mathrm{DNC}}_+(G,\Gamma)}^{\mathring{M}\times\mathbb{R}_+^*}$, page 20.

$\widetilde{\mathrm{ind}}_{\mathrm{full}}^W(G)$: The connecting element, which belongs to $KK^1(\Sigma^W(G),C^*(G_W))$ associated to the short exact sequence $0\to C^*(G_W)\to\Psi^*(G)\to\Sigma^W(G)\to 0$.

$\widetilde{\mathrm{ind}}_{\mathrm{SBlup}}^{G,\Gamma}$, $\widetilde{\mathrm{ind}}_{\mathrm{DNC}_+}^{G,\Gamma}$, $\widetilde{\mathrm{ind}}_{\widetilde{\mathrm{DNC}}_+}^{G,\Gamma}$: Respectively, the elements $\widetilde{\mathrm{ind}}_{\mathrm{full}}^{\mathring{M}}(\mathrm{SBlup}_{r,s}(G,\Gamma))$, $\widetilde{\mathrm{ind}}_{\mathrm{full}}^{M\times\mathbb{R}_+^*}(\mathrm{DNC}_+(G,\Gamma))$ and $\widetilde{\mathrm{ind}}_{\mathrm{full}}^{\mathring{M}\times\mathbb{R}_+^*}(\widetilde{\mathrm{DNC}}_+(G,\Gamma))$.

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