Asymptotical representations, spectra of quantum integrable systems and cluster algebras

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We explain how the existence of the asymptotical prefundamental representations ([HJ] *joint with M. Jimbo) can be used to prove a general conjecture on the spectra of quantum integrable systems (*[FH] *joint with E. Frenkel) and to establish new monoidal categorifications of cluster algebras (*[HL2] *joint with B. Leclerc).*

The partition function Z of a (quantum) integrable system is crucial to understand its physical properties. For a quantum model on an $M \times N$ lattice it may be written in terms of the eigenvalues of the transfer matrix T:

$$Z = \mathrm{Tr}(T^M) = \sum_j \lambda_j^M.$$

Therefore, to find Z, one needs to find the spectrum of T. Baxter tackled this question in his seminal 1971 paper [B] for the (ice) 6- and 8-vertex models. Baxter observed moreover that the eigenvalues λ_j of T have a very remarkable form

$$\lambda_j = A(z)\frac{Q_j(zq^2)}{Q_j(z)} + D(z)\frac{Q_j(zq^{-2})}{Q_j(z)}, \tag{1}$$

where q, z are parameters of the model (quantum and spectral), the functions $A(z)$, $D(z)$ are universal (in the sense that there are the same for all eigenvalues), and Q_j is a polynomial. The above relation is now called Baxter's relation (or Baxter's TQ relation) and Q_j are called Baxter's polynomials.

In 1998, Frenkel-Reshetikhin conjectured [FR] that the spectra of more general quantum integrable systems (more precisely, generalizing the XXZ model of quantum spin chains, whose spectrum is the same as that of the 6-vertex model) have an analog form. Let us formulate this conjecture in terms of representation theory. Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra and $q \in \mathbb{C}^*$ which is not a root of 1. For simplicity of notations, we suppose that $\mathfrak{g}$ is simply-laced, but our results hold in general. Consider the corresponding quantum affine algebra $U_q(\mathfrak{g})$. Its completed tensor square contains the universal R-matrix $\mathcal{R}$ satisfying the Yang–Baxter relation. Given a finite-dimensional representation V of $U_q(\mathfrak{g})$, we have the (twisted) transfer-matrix

$$t_V(z) = \mathrm{Tr}_V(\pi_{V(z)} \otimes \mathrm{Id})(\mathcal{R}) \in U_q(\mathfrak{g})[[z]],$$

where $V(z)$ is a twist of V by a "spectral parameter" z for the natural grading of $U_q(\mathfrak{g})$ and Tr_V is the (graded) trace on V. As a consequence of the Yang-Baxter equation we have $[t_V(z), t_{V'}(z')] = 0$ for all V, V' and z, z'. Therefore these transfer-matrices give rise to a family of commuting operators on any finite-dimensional representation W of $U_q(\mathfrak{g})$.

The finite-dimensional representation V has also a q-character defined in [FR] and which is a Laurent polynomial $\chi_q(V) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ where I is the set of vertices of the underlying finite type Dynkin diagram. For example, if $\mathfrak{g} = \hat{sl}_2$ and V is a simple two-dimensional representation, then there is $a \in \mathbb{C}^*$ such that

$$\chi_q(V) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

Conjecture 1 [Frenkel-Reshetikhin, 1998] *The eigenvalues λ_j of $t_V(z)$ on W are obtained from $\chi_q(V)$ by replacing each variable $Y_{i,a}$ by a quotient*

$$\frac{f_i(azq^{-1})Q_{i,j}(zaq^{-1})}{f_i(azq)Q_{i,j}(zaq)}$$

where the functions $f_i(z)$ are universal (in the sense that there are the same for all eigenvalues),and $Q_{i,j}$ is a polynomial.

For $\mathfrak{g} = \hat{sl}_2$ and V of dimension 2, the conjecture is the Baxter's formula. In general, there are more than 2 terms and no hope for explicit computations. However :

Theorem 1 [Frenkel-Hernandez, 2013] *The conjecture 1 is true.*

Our proof [FH] of the conjecture 1 for arbitrary untwisted affine types is based on the study of the "prefundamental representations" which I had previous constructed with M. Jimbo [HJ]. They are obtained as asymptotical limits of the Kirillov-Reshetikhin modules $W_{k,a}^{(i)}$ ($a \in \mathbb{C}^*$, $k \geq 0$, $i \in I$) which form a family of simple finite-dimensional representations of $U_q(\mathfrak{g})$. More precisely, we have constructed an inductive system

$$W_{1,a}^{(i)} \subset W_{2,a}^{(i)} \subset W_{3,a}^{(i)} \subset \cdots$$

By using one of the main result of [H], we prove that the inductive limit $W_{\infty,a}^{(i)}$ has a structure of representation for the Borel subalgebra $U_q(\mathfrak{b}) \subset U_q(\mathfrak{g})$:

Theorem 2 [Hernandez-Jimbo, 2011] *The action of $U_q(\mathfrak{b})$ on the inductive system "converges" to a simple infinite-dimensional $U_q(\mathfrak{b})$-module $L_{i,a} = W_{\infty,a}^{(i)}$.*

Such a representation was constructed explicitly in the case of $\mathfrak{g} = \hat{sl}_2$ by Bazhanov-Lukyanov-Zamolodchikov. In general, we have proved moreover in [HJ] that these prefundamental representations $L_{i,a}$ (and their duals) form a family of fundamental representations for a category $\mathcal{O}$ of $U_q(\mathfrak{b})$-modules extending the category of finite-dimensional representations.

Our proof [FH] of the conjecture 1 has two main steps. First we establish the following relation in the Grothendieck ring of $\mathcal{O}$, generalizing the Baxter relation: for any finite-dimensional representation V of $U_q(\mathfrak{g})$, take its q-character and replace each $Y_{i,a}$ by the ratio of classes $[L_{i,aq^{-1}}]/[L_{i,aq}]$ (times the class a one-dimensional representation, that we omit here for clarity). Then this expression is equal to the class of V in the Grothendieck ring of $\mathcal{O}$ (viewed as a representation of $U_q(\mathfrak{b})$). For example, if $\mathfrak{g} = \hat{sl}_2$ and V of dimension 2, we get a categorified Baxter's relation :

$$[V \otimes L_{1,aq}] = [L_{1,aq^{-1}}] + [L_{1,aq^3}].$$

The second step is that the transfer-matrix associated to $L_{i,a}$ is well-defined and all of its eigenvalues on any irreducible finite-dimensional representation W of $U_q(\mathfrak{g})$ are *polynomials* up to one and the same factor that depends only on W. Combining these two results, we obtain the proof of the conjecture 1.

In a work in progress with B. Leclerc [HL2], we also use this category $\mathcal{O}$ and such asymptotical representations to obtain new monoidal categorification of (infinite rank) cluster algebras. The program of monoidal categorifications of cluster algebras was initiated in [HL1]. The cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative algebra with a distinguished set of generators called cluster variables and obtained inductively by relations called Fomin-Zelevinsky mutations. When the quiver Q is finite, the cluster algebra is said to be of finite rank.

A monoidal category $\mathcal{C}$ is said to be a monoidal categorification of $\mathcal{A}(Q)$ if there exists a ring isomorphism with its Grothendieck ring $K_0(\mathcal{C})$:

$$\phi : \mathcal{A}(Q) \to K_0(\mathcal{C})$$

which induces a bijection between cluster variables and isomorphism classes of simple modules which are prime (without non trivial tensor factorization) and real (whose tensor square is simple). Various examples of monoidal categorifications have been established in terms of quantum affine algebras (Hernandez-Leclerc), perverse sheaves on quiver varieties (Nakajima, Kimura-Qin, Qin) and Khovanov-Lauda-Rouquier algebras (Kang-Kashiwara-Kim-Oh).

Consider $\mathcal{O}^+$ the subcategory of the category $\mathcal{O}$ of $U_q(\mathfrak{b})$-modules generated by finite-dimensional representations and prefundamental representations.

Theorem 3 [Hernandez-Leclerc 2015] *There is an infinite rank cluster algebra $\mathcal{A}(Q)$ such that there is a ring isomorphism $\mathcal{A}(Q) \simeq K_0(\mathcal{O}^+)$, with prefundamental representations corresponding to cluster variables.*

In particular, Baxter's relations get interpreted as mutation relations.

Conjecture 2 [Hernandez-Leclerc 2015] *The category $\mathcal{O}^+$ is a monoidal categorification of $\mathcal{A}(Q)$.*

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