Spectra of quantum integrable systems, Langlands duality and category $\mathcal{O}$ for quantum affine algebras

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R-matrices give power tools to study the spectra of quantum integrable systems. A better understanding of transfer-matrices obtained from R-matrices led us to the proof of several conjectures. Our approach is based on the study of a category $\mathcal{O}$ of representations of a Borel subalgebra of a quantum affine algebra.

This is based on several joint works with M. Jimbo, E. Frenkel and B. Leclerc.

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Main motivations (Quantum integrable systems and Langlands duality)

The partition function Z of a (quantum) integrable system is crucial to understand its physical properties. For important examples, it can be expressed as $Z = \mathrm{Tr}(T^M)$ where T is the transfer matrix and M is the size of system. Therefore, one needs to find the spectrum of T : the spectrum of the system.

The ODE/IM correspondence (Ordinary Differential Equations/Integrable models) was discovered at the end of the 90's (Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov) and gives a surprising relation between functions associated to Schrödinger differential operators of the form $-\partial^2 + x^{2M} + \ell x^{-2}$ (with $M > 0$ integer and $\ell \in \mathbb{C}$) and the spectrum of quantum systems called "quantum KdV". The functions for the Schrödinger systems are the spectral determinants defined as coefficients of the expansion of certain solutions with remarkable asymptotical properties (subdominant solutions) towards a natural basis of solutions.

Feigin-Frenkel [FF] have proposed a large generalization and an interpretation of this correspondence in terms of Langlands duality. The Schrödinger operators are generalized to affine opers (without monodromy), associated to Langlands dual of the affine Lie algebra attached to the quantum KdV system. This conjecture is largely open, but it is a fruitful source of researches. In particular, a remarkable system of relations (the $Q\tilde{Q}$-system) was observed [MRV] to be satisfied by spectral determinants of certain solutions of affine opers. These $Q\tilde{Q}$-systems are particularly important as they imply the famous Bethe relations. What is the explanation for such a system and does it hold on the Integrable Model side ?

1. Category $\mathcal{O}$

Let us consider these questions in terms of representation theory. Let $\hat{\mathfrak{g}}$ be an untwisted affine Kac-Moody algebra and $q \in \mathbb{C}^*$ which is not a root of 1. Let $n = rk(\mathfrak{g})$ and for $i \in I = \{1, \cdots, n\}$, $q_i = q^{r_i}$ with $B_{i,j} = r_i C_{i,j}$ where C (resp. B) is the (resp. symmetrized) Cartan matrix of $\mathfrak{g}$. Consider the corresponding quantum affine algebra $U_q(\hat{\mathfrak{g}})$. This is a quantum group of Drinfeld-Jimbo.

Let $U_q(\hat{\mathfrak{b}}) \subset U_q(\hat{\mathfrak{g}})$ be a Borel subalgebra (in the sense of Chevalley). It is a Hopf subalgebra of $U_q(\hat{\mathfrak{g}})$. A simple finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$ is still simple when restricted to $U_q(\hat{\mathfrak{b}})$. $U_q(\hat{\mathfrak{b}})$ has itself a triangular decomposition deduced from the Drinfeld realization of $U_q(\hat{\mathfrak{g}})$. This lead [HJ] to the definition of a category $\mathcal{O}$ of $U_q(\hat{\mathfrak{b}})$-modules whose weight spaces (for the analog of the finite type Cartan algebra) are finite-dimensional and whose weights satisfy the same axiomatic properties as for the usual category $\mathcal{O}$ of $\mathfrak{g}$.

Theorem 1 [Hernandez-Jimbo, 2012] *The simple objects in the category $\mathcal{O}$ are parametrized by n-tuples $(f_i(z))_{i\in I}$ of $f_i(z) \in \mathbb{C}(z)$ regular at the origin.*

For example, for $i \in I$, $a \in \mathbb{C}^*$ we have the prefundamental representation $L_{i,a}$ associated to $\mathbf{\Psi}_{i,a} = (1, \cdots, 1, 1 - za, 1, \cdots, 1)$ with $1 - za$ in position i. It was constructed in [HJ] as (the dual of) a limit of finite-dimensional representations. For $\mathfrak{g} = sl_2$ it was constructed explicitly by Bazhanov-Lukyanov-Zamolodchikov.

The Grothendieck ring $K_0(\mathcal{O})$ has a very rich structure. In [HL] we used this category $\mathcal{O}$ to obtain new monoidal categorifications of cluster algebras. For $\mathfrak{g} = sl_2$ and V simple of dimension 2, we get a Fomin-Zelevinsky mutation relation

$$[V \otimes L_{1,aq}] = [\omega][L_{1,aq^{-1}}] + [-\omega][L_{1,aq^3}]. \tag{1}$$

where $a \in \mathbb{C}^*$ and $[\pm\omega]$ are invertible representations of dimension 1. This is a categorified realization of the Baxter's TQ-relation. Our cluster algebra framework lead to a natural generalization (the QQ^*-system established in [HL]) :

$$[L^*_{i,a}][L_{i,a}] = \prod_{j, C_{i,j}<0} [L_{j,aq^{-B_{j,i}}}] + [-\alpha_i] \prod_{j, C_{i,j}<0} [L_{j,aq^{B_{i,j}}}] \text{ for } i \in I,\, a \in \mathbb{C}^*,$$

with $[-\alpha_i]$ of dimension 1 and $L^*_{i,a}$ simple corresponds to $\mathbf{\Psi}^{-1}_{i,a} \prod_{j,C_{i,j}<0} \mathbf{\Psi}_{j,aq^{-B_{j,i}}}$.

2. Transfer-matrices

A very important property of $U_q(\hat{\mathfrak{g}})$ is the existence of the universal R-matrix

$$\mathcal{R}(z) \in (U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}}))[[z]],$$

solution of the Yang-Baxter equation in a (slight) completion of the tensor square. Given V in the category $\mathcal{F}$ of finite-dimensional representations of $U_q(\mathfrak{g})$, we have

$$t_V(z) = \mathrm{Tr}_V(\pi_{V(z)} \otimes \mathrm{Id})(\mathcal{R}) \in U_q(\hat{\mathfrak{g}})[[z]],$$

the transfer-matrix where $V(z)$ is a twist of V for a natural grading of $U_q(\hat{\mathfrak{g}})$ and Tr_V is the (graded) trace on V. As a consequence of the Yang-Baxter equation the coefficients of transfer-matrices generate a commutative subalgebra of $U_q(\hat{\mathfrak{g}})$. As the first factor of $\mathcal{R}(z)$ lies in $U_q(\hat{\mathfrak{b}})$, $t_V(z)$ can also be defined for V in $\mathcal{O}$.

The transfer-matrix construction gives rise to various families of quantum integrable systems with an action of $K_0(\mathcal{F})$ (and of $K_0(\mathcal{O})$) on a space W. For XXZ-type models W is a tensor product of simple objects in $\mathcal{F}$ and for quantum KdV models W is the Fock space of a quantum Heisenberg algebra.

A representation V in $\mathcal{F}$ has a q-character $\chi_q(V) \in \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i\in I, a\in\mathbb{C}^\times}$ [FR]. For $\mathfrak{g} = \hat{sl}_2$ and V simple of dimension 2, $\chi_q(V) = Y_{1,a} + Y^{-1}_{1,aq^2}$ where $a \in \mathbb{C}^*$.

Theorem 2 [Frenkel-Hernandez 2015, conjectured by Frenkel-Reshetikhin 1998] *Let V, W as for a XXZ-type model above. The eigenvalues λ_k of $t_V(z)$ on*

W are obtained from $\chi_q(V)$ by replacing each variable $Y_{i,a}$ by a quotient

$$\frac{f_i(azq^{-1})Q_{i,k}(zaq_i^{-1})}{f_i(azq_i)Q_{i,k}(zaq)}$$

where the functions $f_i(z)$ do not depend on λ_k and $Q_{i,k}$ is a polynomial.

Our proof [FH1] is based on the study of the category $\mathcal{O}$. We establish relations in $K_0(\mathcal{O})$ generalizing the relation (1) and we prove the transfer-matrix associated to $L_{i,a}$ are polynomial on W up to a scalar. For $\mathfrak{g} = sl_2$ and V of dimension 2, we recover the Baxter formula $\lambda_k = A(z)\frac{Q_k(zq^2)}{Q_k(z)} + D(z)\frac{Q_k(zq^{-2})}{Q_k(z)}$ with A, D universal.

The functions $f_i(z)$ can be computed. What about the polynomials $Q_{i,k}$?

3. $Q\tilde{Q}$-systems and consequences.

The $Q\tilde{Q}$-system of [MRV] for two families of functions $(Q_i(z))_{i\in I}$, $(\tilde{Q}_i(z))_{i\in I}$ is

$$Q_i(zq_i^{-1})\tilde{Q}_i(zq_i) - Q_i(zq_i)\tilde{Q}_i(zq_i^{-1}) = \prod_{j|C_{i,j<0}} Q_j(zq^{C_{i,j}+1})Q_j(zq^{C_{i,j}+3})\cdots Q_j(zq^{-C_{i,j}-1}).$$

Theorem 3 [Frenkel-Hernandez, 2016] *There is a natural family $[\tilde{L}_{i,a}]$ ($i \in I, a \in \mathbb{C}^*$) in the Grothendieck ring $K_0(\mathcal{O})$ such that $Q_i(z) = [L_{i,z}]$, $\tilde{Q}_i(z) = [\tilde{L}_{i,z}]$ satisfy the $Q\tilde{Q}$-system in $K_0(\mathcal{O})$.*

This gives an explanation [FH2] for the results of [MRV]. We also derive informations on the the root of the Baxter's polynomials, the Bethe Ansatz equations conjectured by various authors (see [FR, H]) : for w a (generic) root of $Q_{i,k}$,

$$v_i^{-1}\prod_{j\in I}\frac{Q_{j,k}(wq^{B_{i,j}})}{Q_{j,k}(wq^{-B_{i,j}})} = -1, \tag{2}$$

where the v_i are the parameters of the twisted trace. The genericity condition was dropped by Feigin-Jimbo-Miwa-Mukhin by using the QQ^*-systems as in [HL].

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