Grothendieck ring isomorphims, cluster algebras and Kazhdan-Lusztig polynomials

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1. Main question

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra and $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ its loop algebra. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ called a quantum group. Its representation theory is important, in particular from the point of view of quantum integrable systems. Though it has been intensively studied from several geometric, algebraic, combinatorial perspectives, some basic questions are still open, such as the dimension and character of simple finite-dimensional modules.

Inspired by the general framework of monoidal categorification of cluster algebras [HL1], we get new results in this direction.

2. Quantum Grothendieck ring

Let $\mathcal{C}$ be the monoidal category of finite-dimensional modules of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$. We assume $q \in \mathbb{C}^*$ is not a root of unity. The category $\mathcal{C}$ is non semi-simple, and non braided. It has a very intricate structure. The simple objects in $\mathcal{C}$ have been classified by Chari-Pressley in terms of Drinfeld polynomials. The fundamental modules in $\mathcal{C}$ are distinguished simple modules whose classes generate the Grothendieck ring $K(\mathcal{C})$.

For simply-laced types, Nakajima [N] established a remarkable Kazhdan-Lusztig algorithm to compute the multiplicity $P_{m,m'}$ of a simple module $L(m')$ in a standard module $M(m)$, that is a tensor product of fundamental modules. Here m, m' belong to an ordered set of monomials $(\mathcal{M}, \leq)$ which parametrizes both simple and standard objects. This allows to calculate the classes $[L(m)]$ in terms of the classes of standard modules whose dimensions and characters are known. This gives an answer to the initial problem : the multiplicities $P_{m,m'}$ are proved to be the evaluation at $t = 1$ of analogues $P_{m,m'}(t)$ of Kazhdan-Lusztig polynomials. These polynomials are constructed from the structure of the quantum Grothendieck ring $K_t(\mathcal{C})$ which is a t-deformation of $K(\mathcal{C})$ in a certain quantum torus [N, VV]. The polynomials $P_{m,m'}(t)$ are defined as the transition matrix from a basis $[M(m)]_t$ obtained as a t-deformation in $K_t(\mathcal{C})$ of $[M(m)]$, to a basis $[L(m)]_t$ which is characterized as being canonical. Besides the coefficients of the polynomials $P_{m,m'}(t)$ are positive [N]. These results are based on the geometric realization of standard modules in terms of quiver varieties known only for simply-laced types (note however that geometric characters formulas for standard modules have been obtained in [HL3] for all types).

For general types, the question is still open. A conjectural answer was proposed by the speaker in [H] by giving a different construction of the quantum Grothendieck $K_t(\mathcal{C})$ and the corresponding polynomials $P_{m,m'}(t)$ and canonical basis $[L(m)]_t$ which can be extended to general types. The ambient quantum torus is not obtained from a convolution product for quiver varieties, but by considering properties of vertex operators appearing in the theory of q-characters of Frenkel-Reshetikhin [FR].

This leads to a general precise conjectural formula for the multiplicity of simple modules in standard modules :

Conjecture [Hernandez, 2004]

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m,m'}(1)[L(m')], \tag{1}$$

and the polynomials $P_{m,m'}(t)$ are positive.

The first point of the conjecture above means that $[L(m)]_t$ is $[L(m)]$ at $t = 1$.

In the ADE-cases, a submonoidal category $\mathcal{C}'$ of $\mathcal{C}$ is introduced in [HL2] so that $K_t(\mathcal{C}') \simeq \mathcal{U}_t(\mathfrak{n})$, the canonical bases being identified with the Lusztig dual canonical bases. The corresponding polynomials $P_{m,m'}(t)$ are the actual Kazhdan-Lusztig polynomials expressing dual PBW-bases in terms of the dual canonical bases.

3. Isomorphisms and cluster algebras

We denote by $\mathcal{C}_X$ the category $\mathcal{C}$ for $\mathfrak{g}$ of type X.

Theorem 1 [Hernandez-Oya, 2018] *There is a ring isomorphism*

$$K_t(\mathcal{C}_{B_n}) \simeq K_t(\mathcal{C}_{A_{2n-1}})$$

preserving the canonical bases. Moreover the polynomials $P_{m,m'}$ are positive in type B_n.

A crucial point in the proof is the input from cluster algebra theory. Indeed the quantum Grothendieck rings are subrings of quantum tori, respectively $\mathcal{Y}_{t,A_{2n-1}}, \mathcal{Y}_{t,B_n}$, which a priori are different. However, using cluster algebra structures introduced in [HL3], these quantum tori can be mutated in the sense of the theory of quantum cluster algebras. After a distinguished sequence of mutations that we introduce, the ambient quantum tori can be identified and the quantum Grothendieck rings are proved to be isomorphic :

$$\begin{tikzcd}
\mathcal{Y}_{t,A_{2n-1}} \arrow[r, squiggly, "mutation"] & \mathcal{Y}'_{t,A_{2n-1}} \arrow[r] & \mathcal{Y}_{t,B_n} \\
& K_t(\mathcal{C}_{A_{2n-1}}) \arrow[ul, hook] \arrow[u, hook] \arrow[r] & K(\mathcal{C}_{B_n}) \arrow[u, hook]
\end{tikzcd}$$

4. Application to the Kazhdan-Lusztig algorithm

Simultaneously, Kashiwara-Kim-Oh [KKO] introduced functors

$$\mathcal{C}_{B_n} \longleftarrow \text{KLR-algebra modules} \longrightarrow \mathcal{C}_{A_{2n-1}}$$

from a category of module of a quiver-Hecke (KLR) algebras of type A_∞ and the categories of finite-dimensional modules under study. The functors are obtained as Schur-Weyl dualities generalizing quantum affine Schur-Weyl dualities [CP]. It implies the existence of an isomorphism between *classical* Grothendieck rings

$$K(\mathcal{C}_{B_n}) \simeq K(\mathcal{C}_{A_{2n-1}})$$

preserving the basis of simple modules.

We prove that our isomorphism of quantum Grothendieck rings specializes at $t = 1$ to the isomorphism of [KKO] (note that, as far as the speaker knows, the isomorphism of quantum Grothendieck rings can not be deduced directly from the result of [KKO]). The following diagram is commutative :

$$\begin{tikzcd} K_t(\mathcal{C}_{B_n}) \arrow[r, "{[HO]}"] \arrow[d, "t=1"'] & K_t(\mathcal{C}_{A_{2n-1}}) \arrow[d, "t=1"] \\ K(\mathcal{C}_{B_n}) \arrow[r, "{[KKO]}"] & K(\mathcal{C}_{A_{2n-1}}) \end{tikzcd}$$

Hence, combining all these results, from geometric representation theory, cluster algebras isomorphism and quiver Hecke functors, we obtain that for the category $\mathcal{C}_{B_n}$ the classes $[L(m)]_t$ are specialized to the classes $[L(m)]$:

Theorem 2 [Hernandez-Oya, 2018] *The conjecture of* [H] *is true in type B : a Kazhdan-Lusztig algorithm gives the dimensions and characters of simple finite-dimensional modules.*

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