Langlands duality, shifted quantum groups and cluster algebras

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In the framework of the study of K-theoretical Coulomb branches, Finkelberg-Tsymbaliuk introduced remarkable new algebras, the shifted quantum affine algebras and their truncations. We establish that the Grothendieck ring of the category of their finite-dimensional representations has a natural cluster algebra structure. We propose a conjectural parameterization of simple modules of a non simply-laced truncation in terms of the Langlands dual quantum affine Lie algebra. We have several evidences,including a general result for finite-dimensional representations.

Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima which are at the center of current important developments. A presentation of (truncated) shifted quantum affine algebras by generators and relations was given by Finkelberg-Tsymbaliuk. Their rational analogs, the shifted Yangians, and their truncations, appeared in type A in the context of the representation theory of finite W-algebras by Brundan-Kleshchev, then in the study of quantized affine Grassmannian slices [KTWWY] for general types and in the study of quantized Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories by Braverman-Finkelberg-Nakajima for simply-laced types and [NW] for non simply-laced types.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra, and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra, central extension of the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ called a quantum affine algebra. Shifted quantum affine algebras $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ can be seen as variations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, but depending on two coweights μ_+, μ_- of the underlying simple Lie algebra $\mathfrak{g}$. These coweights corresponding to shifts of formal power series in the Cartan-Drinfeld elements (that is quantum analogs of the $t^r h \in \mathcal{L}\mathfrak{g}$, with $r \in \mathbb{Z}$ and $h \in \mathfrak{h}$ in the Cartan subalgebra of $\mathfrak{g}$). In particular $\mathcal{U}_q^{0,0}(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. Up to isomorphism, $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ only depends on $\mu = \mu_+ + \mu_-$ and will be denoted simply by $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$.

The truncations are quotients of $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$ and depend on additional parameters, including a dominant coweight λ. In this context, these parameters λ and μ can be interpreted as parameters for generalized slices of the affine Grassmannian $\overline{\mathcal{W}}_\mu^\lambda$ (usual slices when μ is dominant).

For simply-laced types, representations of shifted Yangians and related Coulomb branches have been intensively studied, see [KTWWY] and references therein. For non simply-laced types, representations of quantizations of Coulomb branches have been studied by Nakajima and Weekes [NW].

In [H], we develop the representation theory of shifted quantum affine algebras. We establish several analogies with the representation theory of ordinary quantum affine algebras. But our approach is also based on several techniques, which are new for ordinary as well as for shifted quantum affine algebras : induction and restriction functors to the category $\mathcal{O}$ of representations of the Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, relations of truncations with Baxter polynomiality in quantum integrable models, and parametrization of simple modules via Langlands dual interpolating (q,t)-characters.

We first we relate these representations to the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$. For general untwisted types, the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ was introduced and studied in [HJ]. Some representations in this category extend to a representation of the whole quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, but many do not, including the prefundamental representations constructed in [HJ] and whose transfer-matrices have remarkable properties for the corresponding quantum integrable systems [FH2].

Consider a category $\mathcal{O}_\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$. We obtain induction/restriction functors to the category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules and we establish the following. Let us denote by α_i the simple roots of $\mathfrak{g}$ and let n be the rank of $\mathfrak{g}$.

Theorem 1 [H] *The simple representations in $\mathcal{O}_\mu$ are parametrized by n-tuples $(\Psi_i(z))$ of rational fractions regular at 0 with $deg(\Psi_i(z)) = \alpha_i(\mu)$.*

We define a ring structure on the sum of Grothendieck groups $K_0(\mathcal{O}_\mu)$ from fusion products. It contains the Grothendieck ring $K_0(\mathcal{C}^{sh})$ of finite-dimensional representations as a subring. Recall that the cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative ring with a distinguished set of generators called cluster variables and obtained inductively by a procedure called mutation. Using induction/restriction functors, as well as results in [HL], we obtain the following (the last part of the Theorem relies on recent results in [KKOP]).

Theorem 2 [H] *The Grothendieck ring $K_0(\mathcal{C}^{sh})$ has a structure of a cluster algebra with an initial cluster variables which are classes of prefundamental representations. Moreover, all cluster monomials are classes simple objects.*

Let us now discuss truncated shifted quantum affine algebras, quotients of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. For simply-laced types, simple representations of truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals [KTWWY]. See the Introduction of [H] for comments on related results in [NW].

We will use Baxter polynomiality in quantum integrable systems. Let us recall that to each representation V of $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$, is attached a transfer-matrix $t_V(z)$ which is a formal power series in a formal parameter z with coefficients in $\mathcal{U}_q(\hat{\mathfrak{g}})$. Given another simple finite-dimensional representation W of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we get a family of commuting operators on $W[[z]]$. This is a quantum integrable model generalizing the XXZ-model. It is established in [FH2], the spectrum of this system, that is the eigenvalues of the transfer-matrices, can be described in terms of certain polynomials, generalizing Baxter's polynomials associated to the XXZ-model. These Baxter's polynomials are obtained from the eigenvalues of transfer-matrices associated to prefundamental representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$.

Moreover, this Baxter polynomiality implies the polynomiality of certain series of Cartan-Drinfeld elements acting on finite-dimensional representations [FH2]. We relate this result to the structures of representations of truncated shifted quantum affine algebras. In particular, we give in [H] a uniform proof of the finiteness of the number of simple isomorphism classes for truncations.

In non-simply-laced types, we propose a parametrization of these simple representations. We use a limit obtained from interpolating (q, t)-characters. The latter were defined by Frenkel and the author as an incarnation of Frenkel-Reshetikhin deformed W-algebras interpolating between q-characters of a non simply-laced quantum affine algebra and its Langlands dual. They lead to the definition of an interpolation between the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_t(\hat{\mathfrak{g}}^L))$ of finite-dimensional representations of the Langlands dual algebra quantum affine algebra $\mathcal{U}_t(\hat{\mathfrak{g}}^L)$. We found it is relevant for our purposes to introduce a different specialization of interpolating (q, t)-characters that we call Langlands dual q-characters.

Conjecture [H] *The simple modules of a truncation are explicitly parametrized by monomials in the Langlands dual q-character of a finite-dimensional representation of the Langlands dual quantum affine algebra.*

Recall that the deformed W-algebras were introduced by Frenkel-Reshetikhin in the context of the geometric Langlands program. The parametrization in [KTWWY] for simply-laced types can be understood in the context of symplectic duality. Hence the statement of our conjecture can be seen as motivated by symplectic and Langlands duality. Our main evidence for the Conjecture is the following, obtained as a consequence of the Baxter polynomiality.

Theorem 3 [H] *A finite-dimensional simple representation in $\mathcal{O}_\mu$ descends to a certain explicit truncation as predicted by the Conjecture.*

References

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