CLUSTER ALGEBRAS AND CATEGORY $\mathcal{O}$ FOR REPRESENTATIONS OF BOREL SUBALGEBRAS OF QUANTUM AFFINE ALGEBRAS

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Abstract. Let $\mathcal{O}$ be the category of representations of the Borel subalgebra of a quantum affine algebra introduced by Jimbo and the first author. We show that the Grothendieck ring of a certain monoidal subcategory of $\mathcal{O}$ has the structure of a cluster algebra of infinite rank, with an initial seed consisting of prefundamental representations. In particular, the celebrated Baxter relations for the 6-vertex model get interpreted as Fomin-Zelevinsky mutation relations.

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1. Introduction

Let $U_q(\mathfrak{g})$ be an untwisted quantum affine algebra (we assume throughout this paper that $q \in \mathbb{C}^*$ is not a root of unity). M. Jimbo and the first author introduced [HJ] a category $\mathcal{O}$ of representations of a Borel subalgebra $U_q(\mathfrak{b})$ of $U_q(\mathfrak{g})$. Finite-dimensional representations of $U_q(\mathfrak{g})$ are objects in this category as well as the infinite-dimensional prefundamental representations of $U_q(\mathfrak{b})$ constructed in [HJ]. They are obtained as asymptotic limits of Kirillov-Reshetikhin modules, which form a family of simple finite-dimensional representations of $U_q(\mathfrak{g})$. These prefundamental representations, denoted by $L_{i,a}^+$ and $L_{i,a}^-$, are simple $U_q(\mathfrak{b})$-modules parametrized by a complex number $a \in \mathbb{C}^*$ and $1 \leq i \leq n$, where n is the rank of the underlying finite-dimensional simple Lie algebra. Such prefundamental representations were first constructed explicitly for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ by Bazhanov-Lukyanov-Zamolodchikov [BLZ], for $\widehat{\mathfrak{sl}}_3$ by Bazhanov-Hibberd-Khoroshkin [BHK] and for $\widehat{\mathfrak{sl}}_n$ with $i = 1$ by Kojima [Ko].

The category $\mathcal{O}$ and the prefundamental representations were used by E. Frenkel and the first author [FH] to prove a conjecture of Frenkel-Reshetikhin on the spectra of quantum integrable systems. Let us recall that the partition function Z of a quantum integrable system is crucial to understand its physical properties. It may be written in terms of the eigenvalues λ_j of the transfer matrix T. Therefore, to compute Z one needs to find the spectrum of T. Baxter tackled this question in his seminal 1971 paper [Ba] for the 6-vertex and 8-vertex models. He observed moreover that the eigenvalues λ_j of T have a very remarkable form

$$\lambda_j = A(z)\frac{Q_j(zq^2)}{Q_j(z)} + D(z)\frac{Q_j(zq^{-2})}{Q_j(z)}, \tag{1.1}$$

where q, z are parameters of the model (quantum and spectral), the functions $A(z)$, $D(z)$ are universal (in the sense that they are the same for all eigenvalues), and Q_j is a polynomial. The above relation is now called *Baxter's relation* (or Baxter's TQ relation). In 1998, Frenkel-Reshetikhin conjectured [FR] that the spectra of more general quantum integrable systems constructed from a representation V of a quantum affine algebra $U_q(\mathfrak{g})$ have a similar form. (In this framework, the 6-vertex model is the particular case when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and V is irreducible of dimension 2.) One of the main steps in the proof of this conjecture [FH] is to interpret the expected generalized Baxter relations as algebraic identities in the Grothendieck ring of the category $\mathcal{O}$ for $U_q(\mathfrak{b})$ (see [H4] for a short overview). For example, if $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and V is the 2-dimensional simple representation of $U_q(\mathfrak{g})$ with q-character $\chi_q(V) = Y_{1,a} + Y_{1,aq^2}^{-1}$, one gets the following categorical version of Baxter's relation (1.1):

$$[V \otimes L_{1,aq}^+] = [\omega_1][L_{1,aq^{-1}}^+] + [-\omega_1][L_{1,aq^3}^+]. \tag{1.2}$$

(Here, $[\omega_1]$ and $[-\omega_1]$ denote the classes of certain one-dimensional representations of $U_q(\mathfrak{b})$, see Definition 3.4 below.)

In another direction, the notion of monoidal categorification of cluster algebras was introduced by the authors in [HL1]. The cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative $\mathbb{Z}$-algebra with a distinguished set of generators called cluster variables and obtained inductively via the Fomin-Zelevinsky procedure of mutation [FZ1]. By definition, the rank of $\mathcal{A}(Q)$ is the number of vertices of Q (finite or infinite). A monoidal category $\mathcal{C}$ is said to be a *monoidal categorification* of $\mathcal{A}(Q)$ if there exists a ring isomorphism $\mathcal{A}(Q) \xrightarrow{\sim} K_0(\mathcal{C})$ which induces a bijection between cluster variables and classes of simple modules which are *prime* (without non trivial tensor factorization) and *real* (whose tensor square is simple). Various examples of monoidal categorifications have been established in terms of quantum affine algebras [HL1, HL3], perverse sheaves on quiver varieties [N2, KQ, Q], and Khovanov-Lauda-Rouquier algebras [KKKO1, KKKO2].

In this paper, we propose new monoidal categorifications of cluster algebras in terms of the category $\mathcal{O}$ of a Borel subalgebra $U_q(\mathfrak{b})$ of an untwisted quantum affine algebra $U_q(\mathfrak{g})$. More precisely, in [HL3] we have attached to $\mathfrak{g}$ a semi-infinite quiver G^- and we proved that the cluster algebra $\mathcal{A}(G^-)$ is isomorphic to the Grothendieck ring of a monoidal category $\mathcal{C}_{\mathbb{Z}}^-$ of finite-dimensional representations of $U_q(\mathfrak{g})$. Moreover, the classes of the Kirillov-Reshetikhin modules in $\mathcal{C}_{\mathbb{Z}}^-$ are the images under this isomorphism of a subset of the cluster variables. Let Γ be the doubly-infinite quiver corresponding to G^-, as defined in [HL3, §2.1.2]). The main result of this paper (Theorem 4.2) is that the completed cluster algebra $\mathcal{A}(\Gamma)$ attached to this doubly-infinite quiver is isomorphic to the Grothendieck ring of a certain monoidal subcategory $\mathcal{O}_{2\mathbb{Z}}^+$ of $\mathcal{O}$. This subcategory $\mathcal{O}_{2\mathbb{Z}}^+$ is generated by finite-dimensional representations and positive prefundamental representations whose spectral parameters satisfy an integrality condition (see below Definitions 3.8, 4.1 and Proposition 5.16). Moreover, the classes of the positive prefundamental representations form the cluster variables of an initial seed of $\mathcal{A}(\Gamma)$. In particular, when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ the counterparts (1.2) of Baxter's relations (1.1) get interpreted as instances of Fomin-Zelevinsky mutation relations[1]. For general types, the first step mutation relations are interpreted as other remarkable relations in the Grothendieck ring $K_0(\mathcal{O}^+)$ (Formula (6.14)).

Along the way we get interesting additional results, for instance (i) the construction of new asymptotic representations beyond the case of prefundamental representations (Theorem 7.6), and (ii) the tensor factorization of arbitrary simple modules of $\mathcal{O}^+$ into prime modules when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ (Theorem 7.9). We conjecture that the category $\mathcal{O}_{2\mathbb{Z}}^+$ is a monoidal categorification of the cluster algebra $\mathcal{A}(\Gamma)$ (Conjecture 7.2). We prove this conjecture for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ (Theorem 7.11). An essential tool in several of our proofs is a duality D between the Grothendieck rings of certain subcategories $\mathcal{O}^+$ and $\mathcal{O}^-$ of $\mathcal{O}$ (Theorem 5.17), which maps classes of simple objects to classes of simple objects (Theorem 7.7).

The paper is organized as follows. In Section 2 we give some background on quantum affine (or loop) algebras and their Borel subalgebras. In Section 3 we review the main properties of the category $\mathcal{O}$ introduced in [HJ] and we introduce the subcategories $\mathcal{O}^+$ and $\mathcal{O}^-$ of interest for this paper (Definition 3.9). In Section 4 we state the main result on the isomorphism between $\mathcal{A}(\Gamma)$ and the Grothendieck ring of $\mathcal{O}_{2\mathbb{Z}}^+$. In section 5 we establish relevant properties of $\mathcal{O}^+$, in particular we introduce and study the duality D between $\mathcal{O}^+$ and $\mathcal{O}^-$. The proof of Theorem 4.2 is given in Section 6. In the concluding Section 7 we present the conjecture on monoidal categorifications and we give various evidences supporting it, in particular the existence of asymptotic representations (Section 7.2). To conclude we present additional conjectural relations in $K_0(\mathcal{O}^+)$ extending the generalized Baxter relations of [FH] (Conjecture 7.15).

The main results of this paper were presented in several conferences (Oberwolfach Workshop "Enveloping algebras and geometric representation theory" in May 2015, Conference "Categorical Representation Theory and Combinatorics" in Seoul in December 2015, Conference "A bridge between representation theory and Physics" in Canterbury in January 2016). An announcement was also published in the Oberwolfach Report [H4].

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[1]The generalized Baxter relations of [FH] can also be regarded as relations in the cluster algebra, but not as mutation relations since they involve more than 2 terms in general.

2. Quantum loop algebra and Borel algebras

2.1. **Quantum loop algebra.** Let $C = (C_{i,j})_{0\leq i,j\leq n}$ be an indecomposable Cartan matrix of non-twisted affine type. We denote by $\mathfrak{g}$ the Kac-Moody Lie algebra associated with C. Set $I = \{1,\dots,n\}$, and denote by $\dot{\mathfrak{g}}$ the finite-dimensional simple Lie algebra associated with the Cartan matrix $(C_{i,j})_{i,j\in I}$. Let $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$, $\{\omega_i\}_{i\in I}$, $\{\omega_i^\vee\}_{i\in I}$, $\dot{\mathfrak{h}}$ be the simple roots, the simple coroots, the fundamental weights, the fundamental coweights and the Cartan subalgebra of $\dot{\mathfrak{g}}$, respectively. We set $Q = \oplus_{i\in I}\mathbb{Z}\alpha_i$, $Q^+ = \oplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$, $P = \oplus_{i\in I}\mathbb{Z}\omega_i$. We will also use $P_{\mathbb{Q}} = P\otimes\mathbb{Q}$ with its partial ordering defined by $\omega\leq\omega'$ if and only if $\omega'-\omega\in Q^+$. Let $D = \mathrm{diag}(d_0,\dots,d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and the d_i's are relatively prime positive integers. We denote by $(\ ,\ ) : Q\times Q\to\mathbb{Z}$ the invariant symmetric bilinear form such that $(\alpha_i,\alpha_i) = 2d_i$. We use the numbering of the Dynkin diagram as in [Ka]. Let $a_0,\dots,a_n$ be the Kac labels ([Ka], pp.55-56). We have $a_0 = 1$ and we set $\alpha_0 = -(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$.

Throughout this paper, we fix a non-zero complex number q which is not a root of unity. We set $q_i = q^{d_i}$. We fix once and for all $h\in\mathbb{C}$ such that $q = e^h$, and we define $q^r = e^{rh}$ for any $r\in\mathbb{Q}$. Since q is not a root of unity, for $r,s\in\mathbb{Q}$ we have that $q^r = q^s$ if and only if $r = s$.

We will use the standard symbols for q-integers

$$[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}},\quad [m]_z! = \prod_{j=1}^m [j]_z,\quad \begin{bmatrix} s\\ r\end{bmatrix}_z = \frac{[s]_z!}{[r]_z![s-r]_z!}.$$

The *quantum loop algebra* $U_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra defined by generators $e_i,\ f_i,\ k_i^{\pm 1}$ ($0\leq i\leq n$) and the following relations for $0\leq i,j\leq n$.

$$k_ik_j = k_jk_i,\quad k_0^{a_0}k_1^{a_1}\cdots k_n^{a_n} = 1,\qquad k_ie_jk_i^{-1} = q_i^{C_{i,j}}e_j,\quad k_if_jk_i^{-1} = q_i^{-C_{i,j}}f_j,$$
$$[e_i,f_j] = \delta_{i,j}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-C_{i.j}}(-1)^r e_i^{(1-C_{i,j}-r)}e_je_i^{(r)} = 0\quad (i\neq j),\qquad \sum_{r=0}^{1-C_{i.j}}(-1)^r f_i^{(1-C_{i,j}-r)}f_jf_i^{(r)} = 0\quad (i\neq j)\,.$$

Here we have set $x_i^{(r)} = x_i^r/[r]_{q_i}!$ ($x_i = e_i, f_i$). The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure given by

$$\Delta(e_i) = e_i\otimes 1 + k_i\otimes e_i,\quad \Delta(f_i) = f_i\otimes k_i^{-1} + 1\otimes f_i,\quad \Delta(k_i) = k_i\otimes k_i\,,$$
$$S(e_i) = -k_i^{-1}e_i,\quad S(f_i) = -f_ik_i,\quad S(k_i) = k_i^{-1}\,,$$

where $i = 0,\cdots,n$.

The algebra $U_q(\mathfrak{g})$ can also be presented in terms of the Drinfeld generators [Dr, Be]

$$x_{i,r}^{\pm}\ (i\in I, r\in\mathbb{Z}),\quad \phi_{i,\pm m}^{\pm}\ (i\in I, m\geq 0),\quad k_i^{\pm 1}\ (i\in I).$$

Example 2.1. For $\dot{\mathfrak{g}} = \mathfrak{sl}_2$, $e_1 = x_{1,0}^+$, $e_0 = k_1^{-1}x_{1,1}^-$, $f_1 = x_{1,0}^-$, $f_0 = x_{1,-1}^+k_1$.

We shall use the generating series ($i \in I$):

$$\phi_i^\pm(z) = \sum_{m \geq 0} \phi_{i,\pm m}^\pm z^{\pm m} = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{m>0} h_{i,\pm m} z^{\pm m}\right).$$

We also set $\phi_{i,\pm m}^\pm = 0$ for $m < 0$, $i \in I$.

The algebra $U_q(\mathfrak{g})$ has a $\mathbb{Z}$-grading defined by $\deg(e_i) = \deg(f_i) = \deg(k_i^{\pm 1}) = 0$ for $i \in I$ and $\deg(e_0) = -\deg(f_0) = 1$. It satisfies $\deg(x_{i,m}^\pm) = \deg(\phi_{i,m}^\pm) = m$ for $i \in I$, $m \in \mathbb{Z}$. For $a \in \mathbb{C}^*$, there is a corresponding automorphism $\tau_a : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ such that $\tau_a(x) = a^m x$ for every $x \in U_q(\mathfrak{g})$ of degree $m \in \mathbb{Z}$.

By [C, Proposition 1.6], there is an involutive automorphism $\hat{\omega} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by ($i \in I$, $m, r \in \mathbb{Z}$, $r \neq 0$)

$$\hat{\omega}(x_{i,m}^\pm) = -x_{i,-m}^\mp, \quad \hat{\omega}(\phi_{i,\pm m}^\pm) = \phi_{i,\mp m}^\mp, \quad \hat{\omega}(h_{i,r}) = -h_{i,-r}, \quad (i \in I,\ m, r \in \mathbb{Z},\ r \neq 0).$$

Let $U_q(\mathfrak{g})^\pm$ (resp. $U_q(\mathfrak{g})^0$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the $x_{i,r}^\pm$ where $i \in I, r \in \mathbb{Z}$ (resp. by the $\phi_{i,\pm r}^\pm$ where $i \in I$, $r \geq 0$). We have a triangular decomposition [Be]

$$U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g})^- \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^+. \tag{2.3}$$

2.2. **Borel algebra.**

Definition 2.2. *The Borel algebra $U_q(\mathfrak{b})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i and $k_i^{\pm 1}$ with $0 \leq i \leq n$.*

This is a Hopf subalgebra of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{b})$ contains the Drinfeld generators $x_{i,m}^+$, $x_{i,r}^-$, $k_i^{\pm 1}$, $\phi_{i,r}^+$ where $i \in I$, $m \geq 0$ and $r > 0$.

Let $U_q(\mathfrak{b})^\pm = U_q(\mathfrak{g})^\pm \cap U_q(\mathfrak{b})$ and $U_q(\mathfrak{b})^0 = U_q(\mathfrak{g})^0 \cap U_q(\mathfrak{b})$. Then we have

$$U_q(\mathfrak{b})^+ = \langle x_{i,m}^+ \rangle_{i \in I, m \geq 0}, \quad U_q(\mathfrak{b})^0 = \langle \phi_{i,r}^+, k_i^{\pm 1} \rangle_{i \in I, r > 0}.$$

It follows from [Be, Da] that we have a triangular decomposition

$$U_q(\mathfrak{b}) \simeq U_q(\mathfrak{b})^- \otimes U_q(\mathfrak{b})^0 \otimes U_q(\mathfrak{b})^+. \tag{2.4}$$

3. Representations of Borel algebras

In this section we review results on representations of the Borel algebra $U_q(\mathfrak{b})$, in particular on the category $\mathcal{O}$ defined in [HJ] and on finite-dimensional representations of $U_q(\mathfrak{g})$. We also introduce the subcategories $\mathcal{O}^+$ and $\mathcal{O}^-$ of particular interest for this paper.

3.1. **Highest ℓ-weight modules.** For a $U_q(\mathfrak{b})$-module V and $\omega \in P_\mathbb{Q}$, we have the weight space

$$V_\omega = \{v \in V \mid k_i\, v = q_i^{\omega(\alpha_i^\vee)} v\ (\forall i \in I)\}. \tag{3.5}$$

We say that V is *Cartan-diagonalizable* if $V = \bigoplus_{\omega \in P_\mathbb{Q}} V_\omega$.

For V a Cartan-diagonalizable $U_q(\mathfrak{b})$-module, we define a structure of $U_q(\mathfrak{b})$-module on its graded dual $V^* = \bigoplus_{\beta \in P_\mathbb{Q}} V_\beta^*$ by

$$(x\, u)(v) = u\big(S^{-1}(x) v\big), \qquad (u \in V^*,\ v \in V,\ x \in U_q(\mathfrak{b})).$$

Definition 3.1. *A series $\mathbf{\Psi} = (\Psi_{i,m})_{i\in I,\, m\geq 0}$ of complex numbers such that $\Psi_{i,0} \in q_i^{\mathbb{Q}}$ for all $i \in I$ is called an ℓ-weight.*

We denote by P_ℓ the set of ℓ-weights. Identifying $(\Psi_{i,m})_{m\geq 0}$ with its generating series we shall write

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = \sum_{m\geq 0} \Psi_{i,m} z^m.$$

We will often use the involution $\mathbf{\Psi} \mapsto \overline{\mathbf{\Psi}}$ on P_ℓ, where $\overline{\mathbf{\Psi}}$ is obtained from $\mathbf{\Psi}$ by replacing every pole and zero of $\mathbf{\Psi}$ by its inverse.

Since each $\Psi_i(z)$ is an invertible formal power series, P_ℓ has a natural group structure by component-wise multiplication. We have a surjective morphism of groups $\varpi : P_\ell \to P_{\mathbb{Q}}$ given by $\Psi_i(0) = q_i^{\varpi(\mathbf{\Psi})(\alpha_i^\vee)}$. For a $U_q(\mathfrak{b})$-module V and $\mathbf{\Psi} \in P_\ell$, the linear subspace

$$V_{\mathbf{\Psi}} = \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi_{i,m}^+ - \Psi_{i,m})^p v = 0\} \tag{3.6}$$

is called the ℓ-weight space of V of ℓ-weight $\mathbf{\Psi}$. Note that since $\phi_{i,0}^+ = k_i$, we have $V_{\mathbf{\Psi}} \subset V_\omega$, where $\omega = \varpi(\mathbf{\Psi})$.

Definition 3.2. *A $U_q(\mathfrak{b})$-module V is said to be of highest ℓ-weight $\mathbf{\Psi} \in P_\ell$ if there is $v \in V$ such that $V = U_q(\mathfrak{b})v$ and the following hold:*

$$e_i\, v = 0 \quad (i \in I)\,, \qquad \phi_{i,m}^+ v = \Psi_{i,m} v \quad (i \in I,\ m \geq 0)\,.$$

The ℓ-weight $\mathbf{\Psi} \in P_\ell$ is uniquely determined by V. It is called the highest ℓ-weight of V. The vector v is said to be a highest ℓ-weight vector of V. For any $\mathbf{\Psi} \in P_\ell$, there exists a simple highest ℓ-weight module $L(\mathbf{\Psi})$ of highest ℓ-weight $\mathbf{\Psi}$. This module is unique up to isomorphism.

For $\mathbf{\Psi}$ an ℓ-weight, we set $\widetilde{\mathbf{\Psi}} = (\varpi(\mathbf{\Psi}))^{-1}\mathbf{\Psi}$ and we introduce the simple $U_q(\mathfrak{b})$-module

$$\widetilde{L}(\mathbf{\Psi}) := L\left(\widetilde{\mathbf{\Psi}}\right).$$

This is the simple $U_q(\mathfrak{b})$-module obtained from $L(\mathbf{\Psi})$ by shifting all ℓ-weights by $\varpi(\mathbf{\Psi})^{-1}$ (see [HJ, Remark 2.5]).

The submodule of $L(\mathbf{\Psi}) \otimes L(\mathbf{\Psi}')$ generated by the tensor product of the highest ℓ-weight vectors is of highest ℓ-weight $\mathbf{\Psi}\mathbf{\Psi}'$. In particular, $L(\mathbf{\Psi}\mathbf{\Psi}')$ is a subquotient of $L(\mathbf{\Psi}) \otimes L(\mathbf{\Psi}')$.

Definition 3.3. [HJ] *For $i \in I$ and $a \in \mathbb{C}^\times$, let*

$$L_{i,a}^{\pm} = L(\mathbf{\Psi}_{i,a}^{\pm 1}) \quad \textit{where} \quad (\mathbf{\Psi}_{i,a}^{\pm 1})_j(z) = \begin{cases} (1 - za)^{\pm 1} & (j = i)\,, \\ 1 & (j \neq i)\,. \end{cases} \tag{3.7}$$

We call $L_{i,a}^+$ (resp. $L_{i,a}^-$) a positive (resp. negative) *prefundamental representation.*

Definition 3.4. *For $\omega \in P_{\mathbb{Q}}$, let $[\omega] = L(\mathbf{\Psi}_\omega)$ where $(\mathbf{\Psi}_\omega)_i(z) = q_i^{\omega(\alpha_i^\vee)}$ $(i \in I)$.*

Note that the representation $[\omega]$ is 1-dimensional with a trivial action of $e_0, \cdots, e_n$.

For $a \in \mathbb{C}^\times$, the subalgebra $U_q(\mathfrak{b})$ is stable by τ_a. Denote its restriction to $U_q(\mathfrak{b})$ by the same letter. Then the pullbacks of the $U_q(\mathfrak{b})$-modules $L_{i,b}^{\pm}$ by τ_a is $L_{i,ab}^{\pm}$.

3.2. **Category $\mathcal{O}$.** For $\lambda \in P_{\mathbb{Q}}$, we set $D(\lambda) = \{\omega \in P_{\mathbb{Q}} \mid \omega \leq \lambda\}$.

Definition 3.5. [HJ] *A $U_q(\mathfrak{b})$-module V is said to be in category $\mathcal{O}$ if:*
i) V is Cartan-diagonalizable;
ii) for all $\omega \in P_{\mathbb{Q}}$ we have $\dim(V_\omega) < \infty$;
iii) there exist a finite number of elements $\lambda_1, \ldots, \lambda_s \in P_{\mathbb{Q}}$ such that the weights of V are in $\bigcup_{j=1,\ldots,s} D(\lambda_j)$.

The category $\mathcal{O}$ is a monoidal category.

Remark 3.6. The definition of $\mathcal{O}$ is slightly different from that in [HJ] as we allow only rational powers of q for the eigenvalues of k_i.

Let $P_\ell^{\mathfrak{r}}$ be the subgroup of P_ℓ consisting of $\mathbf{\Psi}$ such that $\Psi_i(z)$ is a rational function of z for any $i \in I$.

Theorem 3.7. [HJ] *Let $\mathbf{\Psi} \in P_\ell$. A simple object in the category $\mathcal{O}$ is of highest ℓ-weight and the simple module $L(\mathbf{\Psi})$ is in category $\mathcal{O}$ if and only if $\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}$. Moreover, for V in category $\mathcal{O}$, $V_{\mathbf{\Psi}} \neq 0$ implies $\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}$.*

Given a map $c : P_\ell^{\mathfrak{r}} \to \mathbb{Z}$, consider its support

$$\mathrm{supp}(c) = \{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}} \mid c(\mathbf{\Psi}) \neq 0\}.$$

Let $\mathcal{E}_\ell$ be the additive group of maps $c : P_\ell^{\mathfrak{r}} \to \mathbb{Z}$ such that $\varpi(\mathrm{supp}(c))$ is contained in a finite union of sets of the form $D(\mu)$, and such that for every $\omega \in P_{\mathbb{Q}}$, the set $\mathrm{supp}(c) \cap \varpi^{-1}(\{\omega\})$ is finite. Similarly, let $\mathcal{E}$ be the additive group of maps $c : P_{\mathbb{Q}} \to \mathbb{Z}$ whose support is contained in a finite union of sets of the form $D(\mu)$. The map ϖ is naturally extended to a surjective homomorphism $\varpi : \mathcal{E}_\ell \to \mathcal{E}$.

As for the category $\mathcal{O}$ of a classical Kac-Moody Lie algebra, the multiplicity of a simple module in a module of our category $\mathcal{O}$ is well-defined (see [Ka, Section 9.6]) and we have its Grothendieck ring $K_0(\mathcal{O})$. Its elements are the formal sums

$$\chi = \sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}} \lambda_{\mathbf{\Psi}} [L(\mathbf{\Psi})]$$

where the $\lambda_{\mathbf{\Psi} \in \mathbb{Z}}$ are set so that $\sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}, \omega \in P_{\mathbb{Q}}} |\lambda_{\mathbf{\Psi}}| \dim((L(\mathbf{\Psi}))_\omega)[\omega]$ is in $\mathcal{E}$.

We naturally identify $\mathcal{E}$ with the Grothendieck ring of the category of representations of $\mathcal{O}$ with constant ℓ-weights, the simple objects of which are the $[\omega]$, $\omega \in P_{\mathbb{Q}}$. Thus as in [Ka, Section 9.7] we will regard elements of $\mathcal{E}$ as formal sums

$$c = \sum_{\omega \in \mathrm{supp}(c)} c(\omega)[\omega].$$

The multiplication is given by $[\omega][\omega'] = [\omega + \omega']$ and $\mathcal{E}$ is regarded as a subring of $K_0(\mathcal{O})$. If $(c_i)_{i \in \mathbb{N}}$ is a countable family of elements of $\mathcal{E}$ such that for any $\omega \in P_{\mathbb{Q}}$, $c_i(\omega) \neq 0$ for finitely many $i \in \mathbb{N}$, then the sum $\sum_{i \in \mathbb{N}} c_i$ is well defined as a map $P_{\mathbb{Q}} \to \mathbb{Z}$. When this map is in $\mathcal{E}$ we say that $\sum_{i \in \mathbb{N}} c_i$ is a countable sum of elements in $\mathcal{E}$. Note that we have the analog notion of a countable sum in $K_0(\mathcal{O})$, compatible with countable sums of characters in $\mathcal{E}$.

3.3. **Finite-dimensional representations.** Let $\mathcal{C}$ be the category of (type 1) finite-dimensional representations of $U_q(\mathfrak{g})$.

For $i \in I$, let $P_i(z) \in \mathbb{C}[z]$ be a polynomial with constant term 1. Set

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}.$$

Then $L(\mathbf{\Psi})$ is finite-dimensional. Moreover the action of $U_q(\mathfrak{b})$ can be uniquely extended to an action of $U_q(\mathfrak{g})$, and any simple object in the category $\mathcal{C}$ is of this form. Hence $\mathcal{C}$ is a subcategory of $\mathcal{O}$ and the inclusion functor preserves simple objects.

For $i \in I$ and $a \in \mathbb{C}^*$, we denote by $V_{i,a}$ the simple finite-dimensional representation associated with the polynomials

$$P_i(z) = 1 - za, \qquad P_j(z) = 1 \quad (j \neq i).$$

The modules $V_{i,a}$ are called the *fundamental representations*.

3.4. **The categories $\mathcal{O}^+$ and $\mathcal{O}^-$.** We introduce two new subcategories $\mathcal{O}^+$ and $\mathcal{O}^-$ of the category $\mathcal{O}$.

Definition 3.8. *An ℓ-weight is said to be positive (resp. negative) if it is a monomial in the following ℓ-weights :*

- *the $Y_{i,a} = q_i \mathbf{\Psi}_{i,aq_i}^{-1} \mathbf{\Psi}_{i,aq_i^{-1}}$ where $i \in I$, $a \in \mathbb{C}^*$,*
- *the $\mathbf{\Psi}_{i,a}$ (resp. $\mathbf{\Psi}_{i,a}^{-1}$) where $i \in I$, $a \in \mathbb{C}^*$,*
- *the $[\omega]$, where $\omega \in P_{\mathbb{Q}}$.*

Definition 3.9. $\mathcal{O}^+$ *(resp. $\mathcal{O}^-$) is the category of representations in $\mathcal{O}$ whose simple constituents have a positive (resp. negative) highest ℓ-weight.*

Remark 3.10. (i) By construction, $\mathcal{O}^+$, $\mathcal{O}^-$ are stable by extensions. We will prove they are also stable by tensor products (Theorem 5.17).

(ii) There are other remarkable subcategories of $\mathcal{O}$, for example the category $\widehat{\mathcal{O}}$ of representations of $U_q(\mathfrak{g})$ which belong to $\mathcal{O}$ as representations of $U_q(\mathfrak{b})$. This category $\widehat{\mathcal{O}}$ was introduced in [H1] and further studied in [MY].

(iii) One motivation of Definition 3.9 is that $\mathcal{O}^\pm$ contains $\mathcal{C}$ as well as the prefundamental representations $L^\pm_{i,a}$. We have the following inclusion diagram :

$$\begin{array}{ccc} \mathcal{O} & \supset & \mathcal{O}^+, \mathcal{O}^- \\ \cup & & \cup \\ \widehat{\mathcal{O}} & \supset & \mathcal{C}. \end{array}$$

Note that $\widehat{\mathcal{O}}$ is not contained in $\mathcal{O}^+$ or $\mathcal{O}^-$, and conversely neither $\mathcal{O}^+$ nor $\mathcal{O}^-$ is contained in $\widehat{\mathcal{O}}$. For instance, for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, the representation $L\left(\frac{1-zq}{q-z}\right)$ is in the category $\widehat{\mathcal{O}}$ by [MY, Theorem 3.6], but not in $\mathcal{O}^+$ or $\mathcal{O}^-$ because its highest ℓ-weight has no factorization as in Equation (5.12) below. On the other hand the prefundamental representations $L^\pm_{1,a}$ are in the category $\mathcal{O}^\pm$ but not in $\widehat{\mathcal{O}}$ (see [HJ, section 4.1] or [MY, Theorem 3.6]).

(iv) All generalized Baxter's relations established in [FH] hold in the Grothendieck rings $K_0(\mathcal{O}^+)$ or $K_0(\mathcal{O}^-)$ (see Theorem 5.5 reminded below).

(v) The factorization of real simple modules in $\mathcal{O}$ into prime representations is not unique, so the full category $\mathcal{O}$ is not a good candidate for the notion of monoidal categorification discussed in the introduction. For example for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, it follows from [MY, Remark 4.3, Theorem 4.6] that

$$L\left(q^{-5}\frac{1-q^4z}{1-q^{-6}z}\right)\otimes L\left(q^{-9}\frac{1-q^8z}{1-q^{-10}z}\right)\simeq L\left(q^{-7}\frac{1-q^4z}{1-q^{-10}z}\right)\otimes L\left(q^{-7}\frac{1-q^8z}{1-q^{-6}z}\right).$$

Moreover the tensor product is simple real and each simple factor has the character of a Verma module. Consequently each factor is not isomorphic to a tensor product of prefundamental representations and so is prime.

4. Cluster algebras

We state the main results of this paper. We refer the reader to [FZ2] and [GSV] for an introduction to cluster algebras, and for any undefined terminology.

4.1. **An infinite rank cluster algebra.** Let us recall the infinite quiver G introduced in [HL3, Section 2.1.3]. Put $\widetilde{V} = I \times \mathbb{Z}$. $\widetilde{\Gamma}$ is the quiver with vertex set $\widetilde{V}$. The arrows of $\widetilde{\Gamma}$ are given by

$$((i,r)\to(j,s)) \quad\Longleftrightarrow\quad (C_{i,j}\neq 0 \quad \text{and} \quad s = r + d_iC_{i,j}).$$

By [HL3], the quiver $\widetilde{\Gamma}$ has two isomorphic connected components. We pick one of the two isomorphic connected components of $\widetilde{\Gamma}$ and call it Γ. The vertex set of Γ is denoted by V. A second labeling of the vertices of Γ is deduced from the first one by means of the function ψ defined by

$$\psi(i,r) = (i, r+d_i), \qquad ((i,r)\in V). \tag{4.8}$$

Let $W \subset I\times\mathbb{Z}$ be the image of V under ψ. We shall denote by G the same quiver as Γ but with vertices labeled by W.

By analogy with [HL3, Section 2.2.1], consider an infinite set of indeterminates

$$\mathbf{z} = \{z_{i,r} \mid (i,r)\in V\}$$

over $\mathbb{Q}$. Let $\mathcal{A}(\Gamma)$ be the cluster algebra defined by the initial seed $(\mathbf{z},\Gamma)$. Thus, $\mathcal{A}(\Gamma)$ is the subring of the field of rational functions $\mathbb{Q}(\mathbf{z})$ generated by all the cluster variables, that is the elements obtained from some element of $\mathbf{z}$ via a finite sequence of seed mutations. Each element of $\mathcal{A}(\Gamma)$ is a linear combination of finite monomials in some cluster variables. By the Laurent phenomenon [FZ1], $\mathcal{A}(\Gamma)$ is contained in $\mathbb{Z}[z_{i,r}^{\pm1}]_{(i,r)\in V}$.

For our purposes in this paper, it is always possible to work with sufficiently large finite subseeds of the seed $(\mathbf{z},\Gamma)$, and replace $\mathcal{A}(\Gamma)$ by the genuine cluster subalgebras attached to them. On the other hand, statements become nicer if we allow ourselves to formulate them in terms of the infinite rank cluster algebra $\mathcal{A}(\Gamma)$.

Define an $\mathcal{E}$-algebra homomorphism $\chi : \mathbb{Z}[z_{i,r}^{\pm1}]\otimes_{\mathbb{Z}}\mathcal{E}\to\mathcal{E}$ by setting

$$\chi(z_{i,r}^{\pm1}) = \left[\left(\frac{\mp r}{2d_i}\right)\omega_i\right], \qquad ((i,r)\in V).$$

For $A \in \mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, we write $\chi(A) = \sum_\omega A_\omega[\omega]$ and $|\chi|(A) = \sum_\omega |A_\omega|[\omega]$. We will consider the completed tensor product

$$\mathcal{A}(\Gamma)\hat{\otimes}_{\mathbb{Z}}\mathcal{E},$$

that is, the algebra of countable sums $\sum_{i\in\mathbb{N}} A_i$ of elements $A_i \in \mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$ such that $\sum_{i\in\mathbb{N}} |\chi|(A_i)$ is in $\mathcal{E}$ as a countable sum (as defined in Section 3.2). Note that in particular we have the analog notion of a countable sum in $\mathcal{A}(\Gamma)\hat{\otimes}_{\mathbb{Z}}\mathcal{E}$.

4.2. **Main Theorem.**

Definition 4.1. *Define the category $\mathcal{O}^+_{2\mathbb{Z}}$ as the subcategory of representations in $\mathcal{O}^+$ whose simple constituents have a highest ℓ-weight $\mathbf{\Psi}$ such that the roots and the poles of $\mathbf{\Psi}_i(z)$ are of the form q^r with $(i, r) \in V$.*

We will write for short $\ell_{i,a} = [L^+_{i,a}]$.

Theorem 4.2. *The category $\mathcal{O}^+_{2\mathbb{Z}}$ is monoidal and the identification*

$$z_{i,r} \otimes \left[\frac{r}{2d_i}\omega_i\right] \equiv \ell_{i,q^r}, \qquad ((i, r) \in V) \tag{4.9}$$

defines an isomorphism of $\mathcal{E}$-algebras

$$\mathcal{A}(\Gamma)\hat{\otimes}_{\mathbb{Z}}\mathcal{E} \simeq K_0(\mathcal{O}^+_{2\mathbb{Z}})$$

compatible with countable sums.

Remark 4.3. (i) The identification (4.9) gives an isomorphism of $\mathcal{E}$-algebras

$$\mathcal{E}[z^{\pm 1}_{i,r}]_{(i,r)\in V} \xrightarrow{\sim} \mathcal{E}[\ell^{\pm 1}_{i,q^r}]_{(i,r)\in V}.$$

which can be extended to countable sums as above. So the main point of the proof of Theorem 4.2 will be to show that the subalgebra $\mathcal{A}(G)\hat{\otimes}_{\mathbb{Z}}\mathcal{E}$ is mapped to $K_0(\mathcal{O}^+_{2\mathbb{Z}})$ by this isomorphism.

(ii) As in the case of finite-dimensional representations, the description of the simple objects of $\mathcal{O}^+$ essentially reduces to the description of the simple objects of $\mathcal{O}^+_{2\mathbb{Z}}$ (the decomposition explained in [HL1, §3.7] can be extended to our more general situation by using the asymptotic approach of §7.2 below). Hence the Grothendieck ring $K_0(\mathcal{O}^+_{2\mathbb{Z}})$ contains all the interesting information on $K_0(\mathcal{O}^+)$.

The proof of Theorem 4.2 will be given in §6, using material presented in §5.

5. Properties of the category $\mathcal{O}^+$

5.1. q-**characters.**

The q-character morphism was first considered in [FR] and is a very useful tool for our proofs.

Recall from §3.2 the notations $\mathcal{E}$ and $\mathcal{E}_\ell$. Because of the support condition, we can endow $\mathcal{E}$ with a ring structure defined by

$$(c \cdot d)(\omega) = \sum_{\omega'+\omega''=\omega} c(\omega')d(\omega''), \qquad (c, d \in \mathcal{E},\ \omega \in P_{\mathbb{Q}}).$$

Similarly, $\mathcal{E}_\ell$ also has a ring structure given by

$$(c\cdot d)(\mathbf{\Psi}) = \sum_{\mathbf{\Psi}'\mathbf{\Psi}''=\mathbf{\Psi}} c(\mathbf{\Psi}')d(\mathbf{\Psi}''), \qquad (c,d\in\mathcal{E}_\ell,\ \mathbf{\Psi}\in P_\ell^{\mathfrak{r}}),$$

and such that ϖ becomes a ring homomorphism.

For $\mathbf{\Psi}\in P_\ell^{\mathfrak{r}}$ and $\omega\in P_{\mathbb{Q}}$, we define the delta functions $[\mathbf{\Psi}]=\delta_{\mathbf{\Psi},.}\in\mathcal{E}_\ell$ and $[\omega]=\delta_{\omega,.}\in\mathcal{E}$, where as usual δ denotes the Krönecker symbol. Note that the above multiplications give

$$[\mathbf{\Psi}']\cdot[\mathbf{\Psi}''] = [\mathbf{\Psi}'\mathbf{\Psi}''], \quad [\omega']\cdot[\omega''] = [\omega'+\omega''].$$

Let V be a $U_q(\mathfrak{b})$-module in category $\mathcal{O}$. We define [FR, HJ] the q-character and the character of V;

$$\chi_q(V) := \sum_{\mathbf{\Psi}\in P_\ell^{\mathfrak{r}}} \dim(V_{\mathbf{\Psi}})[\mathbf{\Psi}]\in\mathcal{E}_\ell, \qquad \chi(V) := \varpi(\chi_q(V)) = \sum_{\omega\in P_{\mathbb{Q}}}\dim(V_\omega)[\omega]\in\mathcal{E}.$$

If $V\in\mathcal{O}$ has a unique ℓ-weight $\mathbf{\Psi}$ whose weight $\varpi(\mathbf{\Psi})$ is maximal, we also consider its normalized q-character $\widetilde{\chi}_q(V)$ and normalized character $\widetilde{\chi}(V)$ defined by

$$\widetilde{\chi}_q(V) := [\mathbf{\Psi}^{-1}]\cdot\chi_q(V), \qquad \widetilde{\chi}(V) := \varpi(\widetilde{\chi}_q(V)).$$

Note that

$$\chi_q\left(\widetilde{L}(\mathbf{\Psi})\right) = [\widetilde{\mathbf{\Psi}}]\cdot\widetilde{\chi}_q(L(\mathbf{\Psi})) \neq \widetilde{\chi}_q(L(\mathbf{\Psi})).$$

Proposition 5.1. [HJ] *The q-character morphism*

$$\chi_q : K_0(\mathcal{O})\to\mathcal{E}_\ell, \quad [V]\mapsto\chi_q(V),$$

is an injective ring morphism.

Following [FR], consider the ring of Laurent polynomials $\mathcal{Y}=\mathbb{Z}[Y_{i,a}^{\pm1}]_{i\in I,a\in\mathbb{C}^*}$ in the indeterminates $\{Y_{i,a}\}_{i\in I,a\in\mathbb{C}^*}$. Let $\mathcal{M}$ be the multiplicative group of Laurent monomials in $\mathcal{Y}$. For example, for $i\in I$ and $a\in\mathbb{C}^*$ define $A_{i,a}\in\mathcal{M}$ by

$$A_{i,a} = Y_{i,aq_i^{-1}}Y_{i,aq_i}\left(\prod_{j:C_{j,i}=-1}Y_{j,a}\prod_{j:C_{j,i}=-2}Y_{j,aq^{-1}}Y_{j,aq}\prod_{j:C_{j,i}=-3}Y_{j,aq^{-2}}Y_{j,a}Y_{j,aq^2}\right)^{-1}.$$

For a monomial $m=\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{u_{i,a}}\in\mathcal{M}$, we consider its 'evaluation at $\phi^+(z)$'. By definition it is the element $m(\phi(z))\in P_\ell^{\mathfrak{r}}$ given by

$$m\big(\phi(z)\big) = \prod_{i\in I,a\in\mathbb{C}^*}(Y_{i,a}(\phi(z)))^{u_{i,a}} \text{ where } \Big(Y_{i,a}\big(\phi(z)\big)\Big)_j = \begin{cases} q_i\dfrac{1-aq_i^{-1}z}{1-aq_iz} & (j=i),\\ 1 & (j\neq i).\end{cases}$$

This defines an injective group morphism $\mathcal{M}\to P_\ell^{\mathfrak{r}}$. We identify a monomial $m\in\mathcal{M}$ with its image in $P_\ell^{\mathfrak{r}}$. This is compatible with the notation $Y_{i,a}$ used in definition 3.8. Note that $\varpi(Y_{i,a})=\omega_i$ and $\varpi(A_{i,a})=\alpha_i$.

It is proved in [FR] that a finite-dimensional $U_q(\mathfrak{g})$-module V satisfies $V=\bigoplus_{m\in\mathcal{M}}V_{m(\phi(z))}$. In particular, $\chi_q(V)$ can be viewed as an element of $\mathcal{Y}$.

A monomial $M \in \mathcal{M}$ is said to be dominant if $M \in \mathbb{Z}[Y_{i,a}]_{i\in I, a\in\mathbb{C}^*}$. Given a finite-dimensional simple $U_q(\mathfrak{g})$-module $L(\mathbf{\Psi})$, there exists a dominant monomial $M \in \mathcal{M}$ such that $\mathbf{\Psi} = M\big(\phi(z)\big)$. We will also denote $L(\mathbf{\Psi}) = L(M)$.

For example, for $i \in I$, $a \in \mathbb{C}^*$ and $k \geq 0$, we have the Kirillov-Reshetikhin module

$$W_{k,a}^{(i)} = L\left(Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,aq_i^{2(k-1)}}\right). \tag{5.10}$$

Example 5.2. In the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, we have ($k \geq 0$, $a \in \mathbb{C}^*$) [FR]:

$$\chi_q(W_{k,aq^{1-2k}}^{(1)}) = Y_{aq^{-1}}Y_{aq^{-3}}\cdots Y_{aq^{-2k+1}}(1 + A_{1,a}^{-1} + A_{1,a}^{-1}A_{1,aq^{-2}}^{-1} + \cdots + A_{1,a}^{-1}\cdots A_{1,aq^{-2(k-1)}}^{-1}).$$

Theorem 5.3. *(i)* [HJ, FH] *For any $a \in \mathbb{C}^*$, $i \in I$ we have*

$$\chi_q(L_{i,a}^+) = [\mathbf{\Psi}_{i,a}]\,\chi(L_{i,a}^+) = [\mathbf{\Psi}_{i,a}]\,\chi(L_{i,a}^-),$$

where $\chi(L_{i,a}^+) = \chi(L_{i,a}^-)$ does not depend on a.

(ii) [HJ] *For any $a \in \mathbb{C}^*$, $i \in I$ we have*

$$\chi_q(L_{i,a}^-) \in \left[\mathbf{\Psi}_{i,a}^{-1}\right](1 + A_{i,a}^{-1}\mathbb{Z}[[A_{j,b}^{-1}]]_{j\in I, b\in\mathbb{C}^*}).$$

Example 5.4. In the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, we have:

$$\chi_q(L_{1,a}^+) = [(1-za)]\sum_{r\geq 0}[-2r\omega_1]\ ,\ \chi_q(L_{1,a}^-) = \left[\frac{1}{(1-za)}\right]\sum_{r\geq 0}A_{1,a}^{-1}A_{1,aq^{-2}}^{-1}\cdots A_{1,aq^{-2(r-1)}}^{-1}.$$

Thus, although positive and negative prefundamental representations have the same character

$$\chi(L_{1,a}^+) = \chi(L_{1,a}^-) = \sum_{r\geq 0}[-2r\omega_1],$$

their q-characters are very different: the normalized q-character $\widetilde{\chi}_q(L_{1,a}^+)$ is independent of the spectral parameter a, whereas $\widetilde{\chi}_q(L_{1,a}^-)$ does depend on a.

5.2. **Baxter relations.**

We can now state the *generalized Baxter relations*:

Theorem 5.5. [FH, Theorem 4.8, Remark 4.10[2]] *Let V be a finite-dimensional representation of $U_q(\mathfrak{g})$. Replace in $\chi_q(V)$ each variable $Y_{i,a}$ by $[\omega_i]\frac{\left[L_{i,aq_i^{-1}}^+\right]}{\left[L_{i,aq_i}^+\right]}$ (resp. $[-\omega_i]\frac{\left[L_{i,aq_i^{-1}}^-\right]}{\left[L_{i,aq_i}^-\right]}$) and $\chi_q(V)$ by $[V]$ (resp. $[V^*]$). Then, multiplying by a common denominator, we get a relation in the Grothendieck ring $K_0(\mathcal{O})$.*

Example 5.6. Taking $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $V = L(Y_{1,a})$, we obtain the above-mentioned relation (1.2), namely,

$$[L(Y_{1,a})][L_{1,aq}^+] = [L_{1,aq^{-1}}^+][\omega_1] + [L_{1,aq^3}^+][-\omega_1],$$

and

$$[L(Y_{1,aq^2})][L_{1,aq}^-] = [L_{1,aq^{-1}}^-][-\omega_1] + [L_{1,aq^3}^-][\omega_1].$$

---

[2] The result in [FH] is stated in terms of the $L_{i,a}^+$ and in terms of the $R_{i,a}^+$ such that $(R_{i,a}^+)^* \simeq L_{i,a}^-$. In the last case, the variable $Y_{i,a}$ has to be replaced by $[\omega_i][R_{i,aq_i^{-1}}]^+/[R_{i,aq_i}^+]$ to get $[V]$. That is why in the statement of Theorem 5.5 in terms of the $L_{i,a}^-$ the representations $[-\omega_i] \simeq [\omega_i]^*$ and V^* appear.

Remark 5.7. By our main Theorem 4.2 (and its dual version, see Remark 5.18), these generalized Baxter relations are interpreted as relations in the cluster algebra $\mathcal{A}(G)$. Moreover when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ the original Baxter relation (1.2) gets interpreted as a Fomin-Zelevinsky mutation relation.

The right-hand side of a generalized Baxter relation is an $\mathcal{E}$-linear combination of classes of tensor products of prefundamental representations. As shown in [FH], this can be seen as a tensor product decomposition in $K(\mathcal{O})$. Indeed we have:

Theorem 5.8. [FH] *Any tensor product of positive (resp. negative) prefundamental representations $L^+_{i,a}$ (resp. $L^-_{i,a}$) is simple.*

5.3. **ℓ-characters of representations and duality.** Let $K_0^{\pm}$ be the $\mathcal{E}$-subalgebra of $K_0(\mathcal{O}^{\pm})$ generated by the $[V_{i,a}]$, $[L^{\pm}_{i,a}]$ ($i \in I, a \in \mathbb{C}^*$).

It follows from Theorem 5.5 that the fraction field of $\mathcal{E}[\ell_{i,a}]_{i\in I, a\in\mathbb{C}^*}$ contains $K_0^{\pm}$. More precisely, each element of $K_0^{\pm}$ is a Laurent polynomial in $\mathcal{E}[\ell_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$. Note that the $\ell_{i,a}$ are algebraically independent (this is for example a consequence of Theorem 5.8 which implies that the monomials in the $\ell_{i,a}$ are linearly independent in $K_0(\mathcal{O}^+)$). In particular the expansion in $\mathcal{E}[\ell_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$ is unique and we can define the injective ring morphism

$$\chi_\ell : K_0^+ \to \mathcal{E}[\ell_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$$

which is called the ℓ-character morphism.

In the same way, by using the relations in Theorem 5.5 in terms of the prefundamental representations $[L^-_{i,a}]$ and by setting $\ell'_{i,a} = [L^-_{i,a^{-1}}]$, we get an injective ring morphism

$$\chi'_\ell : K_0^- \to \mathcal{E}[(\ell'_{i,a})^{\pm 1}]_{i\in I,a\in\mathbb{C}^*}$$

Example 5.9. We can reformulate Theorem 5.5 as follows. For a finite-dimensional representation V of $U_q(\mathfrak{g})$, the ℓ-character $\chi_\ell(V)$ (resp. $\chi'_\ell(V^*)$) is obtained from the q-character $\chi_q(V)$ by replacing each variable $Y_{i,a}$ by $[\omega_i]\ell_{i,aq_i^{-1}}\ell_{i,aq_i}^{-1}$ (resp. $[-\omega_i]\ell'_{i,a^{-1}q_i}(\ell'_{i,a^{-1}q_i^{-1}})^{-1}$). In fact, this can also be seen as a change of variables analogous to that of [HL3, Section 5.2.2]. For example for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, using Example 5.6 we have

(5.11)
$$\chi_\ell([L(Y_{1,a})]) = \frac{[\omega_1]\ell_{1,aq^{-1}} + [-\omega_1]\ell_{1,aq^3}}{\ell_{1,aq}} \text{ and } \chi'_\ell([L(Y_{1,a})]) = \frac{[-\omega_1]\ell'_{1,a^{-1}q^3} + [\omega_1]\ell'_{1,a^{-1}q^{-1}}}{\ell'_{i,a^{-1}q}}.$$

Example 5.10. Set $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$. Let $\mathbf{\Psi} = (\frac{1-zq^{-2}}{1-z}, (1-qz)) = [-\omega_1]Y_{1,q^{-1}}\mathbf{\Psi}_{2,q}$. Let us compute $\chi_\ell(L(\mathbf{\Psi}))$. For $k \geq 0$, let $W_k = \widetilde{L}(M_k)$ where $M_k = (Y_{2,q^{2k}} \cdots Y_{2,q^4}Y_{2,q^2})Y_{1,q^{-1}}$. We can prove as in [HJ, Section 7.2] that $(W_k)_{k\geq 0}$ gives rise to a limiting $U_q(\mathfrak{b})$-module W_∞ whose q-character is

$$\chi_q(W_\infty) = \mathbf{\Psi}(1 + A_{1,1}^{-1})\chi(L^+_{2,q}) = \chi_q(L^+_{2,q})[-\omega_1]Y_{1,q^{-1}} + [-\omega_1 + \omega_2]Y_{1,q}^{-1}\chi_q(L^+_{2,q^{-1}}).$$

But it follows from Theorem 7.6 (ii) below that $L(\mathbf{\Psi})$ and W_∞ have the same character $\lim_{k\to+\infty}\chi(\widetilde{L}(M_k)) = \lim_{k\to+\infty}\chi(\widetilde{L}(\overline{M_k}^{-1}))$. As $L(\mathbf{\Psi})$ is a subquotient of W_∞, they are isomorphic. Consequently

$$\chi_\ell(L(\mathbf{\Psi})) = \frac{\ell_{2,q}\ell_{1,q^{-2}} + [-\alpha_1]\ell_{1,q^2}\ell_{2,q^{-1}}}{\ell_{1,1}}.$$

Proposition 5.11. *A representation V in $\mathcal{O}^\pm$ satisfying $[V] \in K_0^\pm$ has finite length.*

Remark 5.12. An object in the category $\mathcal{O}$ has not necessarily finite length. The subcategory of objects of finite length is not stable by tensor product [BJMST, Lemma C.1].

Proof. Suppose $[V] \in K_0^+$. Then there is a monomial M in the $\ell_{i,a}$ such that $M\chi_\ell(V)$ is a polynomial in the $\ell_{i,a}$. There is a tensor product L of positive prefundamental representations such that $\chi_\ell(L) = M$. Then $L \otimes V$ has finite length, hence the result. $\square$

Proposition 5.13. *The identification of the variables $\ell_{i,a}$ and $\ell'_{i,a}$ induces a unique isomorphism of $\mathcal{E}$-algebras*

$$D : K_0^+ \to K_0^-.$$

Proof. The identification gives a well-defined injective ring morphism

$$D' : K_0^+ \to \mathcal{E}[(\ell'_{i,a})^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}.$$

It suffices to prove that its image is $\chi'_l(K_0^-)$. For $V = L_{i,b}^+$ a prefundamental, its image by D' is $\ell'_{i,b} = \chi'_\ell([L^-_{i,b^{-1}}])$. For $V = L(Y_{i,b})$ a fundamental, its image by D' is obtained from its q-character $\chi_q(V)$ by replacing each variable $Y_{i,a}$ by $[\omega_i]\frac{[L^-_{i,aq_i^{-1}}]}{[L^-_{i,aq_i}]}$, that is by $[\omega_i]\frac{\ell'_{i,a^{-1}q_i}}{\ell'_{i,a^{-1}q_i^{-1}}}$. In the construction of χ', this corresponds to $Y^{-1}_{i,a^{-1}}$ (see the formula in Theorem 5.5). By [H2, Lemma 4.10], there is a finite-dimensional representation V' whose q-character is obtained from $\chi_q(V)$ by replacing each $Y_{i,a}$ by $Y^{-1}_{i,a^{-1}}$. Hence $D'(V) = \chi'_\ell((V')^*) \in \chi'_\ell(K_0^-)$. We have proved $\mathrm{Im}(D') \subset \chi'_\ell(K_0(\mathcal{O}^-_{f,\mathbb{Z}}))$. Similarly, for W such that $V = W^*$, we have $D'(W') = \chi'_\ell(V)$ and we get the other inclusion. $\square$

Example 5.14. (i) For any $(i,a) \in I \times \mathbb{C}^*$, we have

$$D([L^+_{i,a}]) = [L^-_{i,a^{-1}}].$$

(ii) For any dominant monomial m, we have

$$D([L(m)]) = [L(m_1)]$$

where m_1 is obtained from m by replacing each $Y_{i,a}$ by $Y_{i,a^{-1}}$. Indeed $L(m_1)$ is isomorphic to $((L(m))')^*$ whose highest weight monomial is the inverse of the lowest weight monomial $(m_1)^{-1}$ of $L(m)'$. For example for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, we have $D([L(Y_{1,a})] = [L(Y_{1,a^{-1}})]$ as this can be observed by comparing the two formulas in (5.11).

Remark 5.15. The duality D is compatible with characters by (i) in Theorem 5.3. However it is not compatible with q-characters (for example negative and positive prefundamental representations have very different q-characters as explained above).

Proposition 5.16. *An element in the Grothendieck group $K_0(\mathcal{O}^\pm)$ is a (possibly countable) sum of elements in $K_0^\pm$.*

Proof. Let us prove it for $K_0(\mathcal{O}^+)$ (the proof is analog for $K_0(\mathcal{O}^-)$). By definition, the positive ℓ-weights label the simple modules in $\mathcal{O}^+$. Moreover, an ℓ-weight is positive if and only if it is a product of highest ℓ-weights of representations $L^+_{i,a}$, $V_i(a)$ and $[\omega]$. This implies that for each positive ℓ-weight $\mathbf{\Psi}$, we can choose (and fix) a monomial in the $[L^+_{i,a}]$, $[V_i(a)]$, $[\omega]$ such that the corresponding representation has highest ℓ-weight equal to $\mathbf{\Psi}$. Hence the positive ℓ-weights also label the linearly independent family of these monomials in K_0^+. Expanding these monomials we get finite sums of classes of simple modules by Proposition 5.11. We get an (infinite) transition matrix from the classes of simple objects in $\mathcal{O}^+$ to such products in K_0^+, and this matrix is unitriangular (for the standard partial ordering with respect to weight). Hence the result. □

This implies immediately the following.

Theorem 5.17. *$\mathcal{O}^+$ and $\mathcal{O}^-$ are monoidal and the morphism D extends uniquely to an isomorphism of $\mathcal{E}$-algebras*

$$D : K_0(\mathcal{O}^+) \to K_0(\mathcal{O}^-).$$

Remark 5.18. Consequently, our main result in Theorem 4.2 may also be written in terms of the subcategory $\mathcal{O}^-_{2\mathbb{Z}}$ of $\mathcal{O}^-_{\mathbb{Z}}$ whose Grothendieck ring is $D(K_0(\mathcal{O}^+_{2\mathbb{Z}}))$.

Proof. For L, L' simple in $\mathcal{O}^+$, we may consider a decomposition of $[L]$, $[L']$ as a countable sum of elements in K_0^+ as in Proposition 5.16. Then $[L][L']$ is also such a countable sum and is in $K_0(\mathcal{O}^+)$. Hence $\mathcal{O}^+$ is monoidal. This is analog for $\mathcal{O}^-$.

The isomorphism of Proposition 5.13 is extended by linearity to $K_0(\mathcal{O}^+)$ by using Proposition 5.16. This map $D : K_0(\mathcal{O}^+) \to K_0(\mathcal{O}^-)$ is an injective ring morphism. The ring morphism $D^{-1} : K_0(\mathcal{O}^-) \to K_0(\mathcal{O}^+)$ is constructed in the same way and so D is a ring isomorphism. □

Proposition 5.19. *A simple object in $\mathcal{O}^\pm$ is a subquotient of a tensor product of two simple representations $V \otimes L$ where V is finite-dimensional and L is a tensor product of positive (resp. negative) prefundamental representations.*

Proof. Let $L(\mathbf{\Psi})$ be simple in $\mathcal{O}^\pm$. By definition, its highest ℓ-weight is a product of highest ℓ-weights of representations $[\omega]$, $L^\pm_{i,a}$, $V_i(a)$ where $\omega \in P_{\mathbb{Q}}$, $i \in I$, $a \in \mathbb{C}^*$. So $\mathbf{\Psi}$ can be factorized in

$$\mathbf{\Psi} = [\omega] \times m \times \prod_{i\in I, a\in\mathbb{C}^*} \mathbf{\Psi}_{i,a}^{u_{i,a}}, \tag{5.12}$$

where $\omega \in P_{\mathbb{Q}}$, $\pm u_{i,a} \geq 0$ and $m \in \mathcal{M}$ is a dominant monomial. The result follows by taking $V = [\omega] \otimes L(m)$ and $L = \bigotimes_{i\in I, a\in\mathbb{C}^*} (L^\pm_{i,a})^{\otimes |u_{i,a}|}$ which is simple by Theorem 5.8. □

Proposition 5.20. *The normalized q-character of a simple object in $\mathcal{O}^-$ belongs to the ring $\mathbb{Z}[[A^{-1}_{i,a}]]_{i\in I, a\in\mathbb{C}^*}$.*

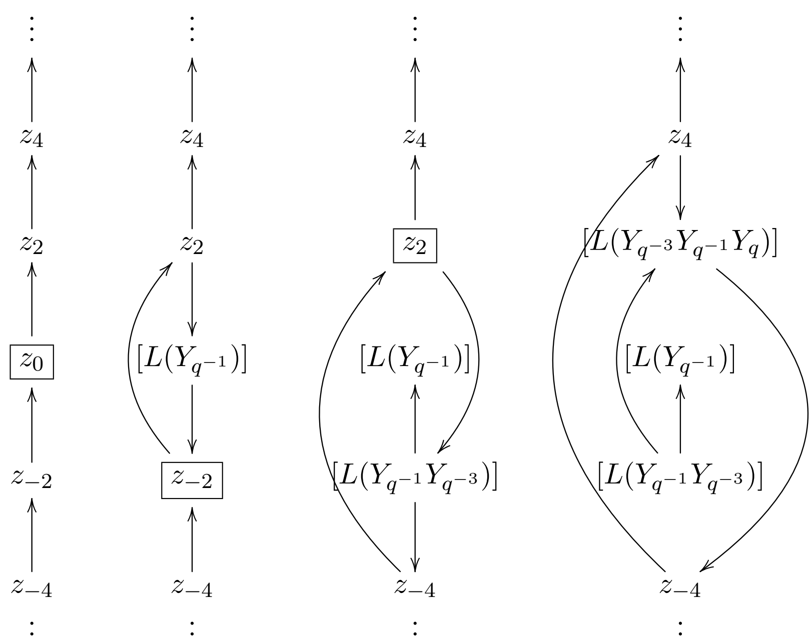
Proof. The result is known for the category $\mathcal{C}$ by [FM]. For negative prefundamental representations, the result is known by [HJ, Theorem 6.1]. Then the general result follows from Proposition 5.19. □

Note that this property is not satisfied in $\mathcal{O}^+$, see Example 5.4.

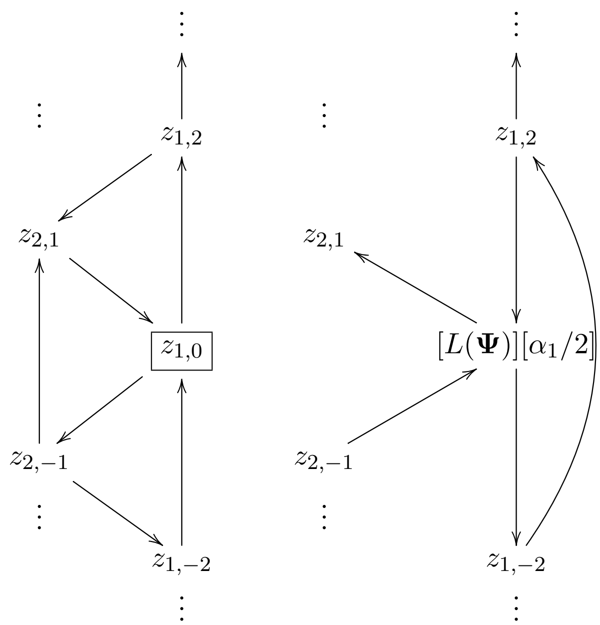
6. Proof of the main Theorem

6.1. **Examples of mutations.**

6.1.1. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. We display a sequence of 3 mutations starting from the initial seed of $\mathcal{A}(G)$. The mutated cluster variables are indicated by a framebox.



6.1.2. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$. Example 5.10 can be reformulated as a one-step mutation from the initial seed, as follows:



Recall that here $\mathbf{\Psi} = [-\omega_1] Y_{1,q^{-1}} \mathbf{\Psi}_{2,q}$, as in Example 5.10.

6.1.3. For an arbitrary $\mathfrak{g}$, let us calculate the first mutation relation for each cluster variable $z_{i,r}$ of the initial seed, generalizing 6.1.2. We denote by $z_{i,r}^*$ the new cluster variable obtained by mutating $z_{i,r}$. Then we have

$$z_{i,r}^* z_{i,r} = \prod_{j, C_{j,i} \neq 0} z_{j, r - d_j C_{j,i}} + \prod_{j, C_{j,i} \neq 0} z_{j, r + d_j C_{j,i}}.$$

We claim that $z_{i,r}^* = [\lambda][L(\mathbf{\Psi})]$ where

$$\mathbf{\Psi} = [-\omega_i] Y_{i, q^{r-d_i}} \prod_{j, C_{j,i} < 0} \mathbf{\Psi}_{j, q^{r - d_j C_{j,i}}}, \text{ and } \lambda = \frac{\alpha_i}{2} - r \sum_{j, C_{j,i} < 0} \frac{\omega_j}{2 d_j}.$$

As in 6.1.2, this is derived from the explicit q-character formula

$$\chi_q(L(\mathbf{\Psi})) = [\mathbf{\Psi}]\,(1 + A_{i,q^r}^{-1}) \prod_{j, C_{j,i} < 0} \chi_j, \tag{6.13}$$

where $\chi_j = \chi(L_{j,a}^+)$ does not depend on a. (By considering $L(\mathbf{\Psi}) \otimes L(Y_{i,q^{r-d_i}} \mathbf{\Psi}^{-1})$ and $L(\mathbf{\Psi}) \otimes L_{i,q^r}^+$ we prove that the multiplicities in $\chi_q(L(\mathbf{\Psi}))$ are larger than in the right-hand side of (6.13). The reverse inequality is established by considering $L(\overline{M_R}^{-1}) \otimes L(\mathbf{\Psi} \overline{M_R})$ where the monomials M_R are defined for $\overline{\mathbf{\Psi}}^{-1}$ as in Theorem 7.6.) The mutation relation thus becomes the following relation in the Grothendieck ring $K_0(\mathcal{O}^+)$:

$$\left[L(\mathbf{\Psi}) \otimes L^+_{i,q^r}\right] = \left[\bigotimes_{j, C_{j,i}\neq 0} L^+_{j,q^{r-d_jC_{j,i}}}\right] + [-\alpha_i]\left[\bigotimes_{j, C_{j,i}\neq 0} L^+_{j,q^{r+d_jC_{j,i}}}\right]. \tag{6.14}$$

By Theorem 5.8, the two terms on the right-hand side are simple. Hence this is the decomposition of the class of the tensor product into simple modules.

6.2. **Proof of Theorem 4.2.** We identify the $\mathcal{E}$-algebras $\mathcal{E}[z_{i,r}^{\pm 1}]_{(i,r)\in V}$ and $\mathcal{E}[\ell_{i,q^r}^{\pm 1}]_{(i,r)\in V}$ as in Remark 4.3.

Proposition 6.1. *We have:* $K_0^+ \cap K_0(\mathcal{O}^+_{2\mathbb{Z}}) \subseteq \mathcal{A}(G) \otimes_{\mathbb{Z}} \mathcal{E}$.

Note that by Proposition 5.16, this implies $K_0(\mathcal{O}^+_{2\mathbb{Z}}) \subseteq \mathcal{A}(G)\hat{\otimes}_{\mathbb{Z}}\mathcal{E}$

Proof. Clearly, $\ell_{i,q^s} = z_{i,s}[(s/2d_i)\omega_i]$ belongs to $\mathcal{A}(G)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{E}$ for any $(i,s) \in V$. By Proposition 5.16, it remains to show that $[V_{i,q^r}]$ belongs to $\mathcal{A}(G)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{E}$ for any $(i,r) \in W$.

Remember from §4.1 that we denote by G the same quiver as Γ with vertices labeled by W instead of V. In the next discussion, we divide the vertices of G and Γ into *columns*, as in [HL3, Example 2.3], and we denote by k the number of columns. As in [HL3, §2.1.3], consider the full subquiver G^- of G whose vertex set is $W^- = \{(i,r) \in W \mid r \leq 0\}$. The definition of G shows that there is only one vertex of $G \setminus G^-$ in each column which is connected to G^- by some arrow. Let H^- denote the ice quiver obtained from G^- by adding these k vertices together with their connecting arrows, and by declaring the new vertices frozen.

Consider the cluster algebras $\mathcal{A}(H^-)$ (with k frozen variables) and $\mathcal{A}(G^-)$ (with no frozen variable). It follows from the definitions that $\mathcal{A}(H^-)$ can be regarded as a subalgebra of $\mathcal{A}(G)$, and $\mathcal{A}(G^-)$ is the coefficient-free counterpart of $\mathcal{A}(H^-)$, studied in [HL3]. Let f_j $(1 \leq j \leq k)$ be the frozen variable of $\mathcal{A}(H^-)$ sitting in column j. The cluster of the initial seed of $\mathcal{A}(H^-)$ thus consists of the frozen variables f_j $(1 \leq j \leq k)$ and the ordinary cluster variables $z_{i,s}$ $((i,s) \in V^-)$, where $V^- = \{(i,s) \in V \mid s + d_i \leq 0\}$. Let us denote by $u_{i,s}$ $((i,s) \in V^-)$ the cluster variables of the initial seed of $\mathcal{A}(G^-)$. We can use a similar change of variables as in [HL3, §2.2.2]:

$$y_{i,r} = u_{i,r-d_i} \quad \text{if } r + d_i > 0, \qquad y_{i,r} = \frac{u_{i,r-d_i}}{u_{i,r+d_i}} \quad \text{otherwise.}$$

Let

$$F\colon \mathbb{Z}[u_{i,s}^{\pm 1} \mid (i,s) \in V^-] \longrightarrow \mathbb{Z}[f_j^{\pm 1}, z_{i,s}^{\pm 1} \mid 1 \leq j \leq k,\ (i,s) \in V^-]$$

be the ring homomorphism defined by

$$F(u_{i,s}) = \frac{z_{i,s}}{f_j} \quad \text{if } (i,s) \text{ sits in column } j.$$

Thus

$$F(y_{i,r}) = \frac{z_{i,r-d_i}}{f_j} \text{ if } r + d_i > 0 \text{ and } (i,r) \text{ sits in column } j, \quad F(y_{i,r}) = \frac{z_{i,r-d_i}}{z_{i,r+d_i}} \text{ otherwise.}$$

We introduce a $\mathbb{Z}^k$-grading on $\mathbb{Z}[f_j^{\pm 1}, z_{i,s}^{\pm 1} \mid 1 \leq j \leq k,\ (i,s) \in V^-]$ by declaring that

$$\deg(f_j) = e_j, \quad \deg(z_{i,s}) = e_j \text{ if } (i,s) \text{ sits in column } j,$$

where $(e_j, 1 \leq j \leq k)$ denotes the canonical basis of $\mathbb{Z}^k$.

Let x be the cluster variable of $\mathcal{A}(G^-)$ obtained from the initial seed $(\{u_{i,s}\}, G^-)$ via a sequence of mutations σ, and let y be the cluster variable of $\mathcal{A}(H^-)$ obtained from the initial seed $(\{z_{i,s}, f_j\}, H^-)$ via the *same* sequence of mutations σ. We want to compare the Laurent polynomials y and $F(x)$. Since $\deg(F(u_{i,s})) = (0, \dots, 0)$ for every (i,s), we see that $F(x)$ is multi-homogeneous of degree $(0, \dots, 0)$ for the above grading. On the other hand, it is easy to check that for every non-frozen vertex (i,s) of the ice quiver H^- the sum of the multi-degrees of the initial cluster variables and frozen variables sitting at the targets of the arrows going out of (i,s) is equal to the sum of the multi-degrees of the initial cluster variables and frozen variables sitting at the sources of the arrows going into (i,s). Therefore, $\mathcal{A}(H^-)$ is a multi-graded cluster algebra, in the sense of [GL]. It follows that y is also multi-homogeneous, of degree $(a_1, \dots, a_k)$. Now, by construction, we have

$$F(x)|_{f_1=1,\dots,f_k=1} = y|_{f_1=1,\dots,f_k=1}.$$

Therefore,

$$y = F(x) \prod_{j=1}^{k} f_j^{a_j}.$$

Taking a cluster expansion with respect to the initial cluster of $\mathcal{A}(H^-)$, we write $y = N/D$ where D is a monomial in the non-frozen cluster variables and N is a polynomial in the non-frozen and frozen variables. Moreover N is not divisible by any of the f_j's. It follows that $\prod_{j=1}^k f_j^{a_j}$ is the smallest monomial such that $F(x)\prod_{j=1}^k f_j^{a_j}$ contains only nonnegative powers of the variables f_j's.

Now we can conclude using [HL3, Theorem 3.1], which implies that for all $(i,r) \in W^-$ with $r \ll 0$, the q-character of V_{i,q^r} (expressed in terms of the variables $y_{i,s} \equiv Y_{i,q^s}$) is a cluster variable x of $\mathcal{A}(G^-)$. By [FM, Corollary 6.14], for $r \ll 0$ this cluster variable does not contain any variable $y_{i,s}$ with $s + 2d_i > 0$, hence $F(x)$ does not contain any frozen variable f_j. Therefore $y = F(x)$, and the q-character of V_{i,q^r} (expressed in terms of the variables $z_{i,s}$) is a cluster variable of $\mathcal{A}(H^-)$, that is, a cluster variable of $\mathcal{A}(\Gamma)$. This proves the claim for every fundamental module V_{i,q^r} with $r \ll 0$. But by definition of the cluster algebra $\mathcal{A}(\Gamma)$, the set of cluster variables is invariant under the change of variables $z_{i,s} \mapsto z_{i,s+2d_i}$. Thus we are done. $\square$

Proposition 6.2. *We have:* $\mathcal{A}(G)\hat{\otimes}_{\mathbb{Z}}\mathcal{E} \subseteq K_0(\mathcal{O}^+_{2\mathbb{Z}})$.

Consider an element χ in $\mathcal{A}(G)$. By the Laurent phenomenon [FZ1], χ is a Laurent polynomial in the initial cluster variables :

$$\chi = P(\{z_{i,r}\}_{(i,r)\in V}).$$

Hence $\mathcal{A}(G)$ is a subalgebra of the fraction field of $K_0(\mathcal{O}^+_{2\mathbb{Z}})$ and the duality D of Proposition 5.13 can be algebraically extended to $\mathcal{A}(G)$. In particular we have

$$D(\chi) = P(\{D(z_{i,r})\}_{(i,r)\in V}) \in \mathrm{Frac}(K_0(\mathcal{O}^-_{2\mathbb{Z}}))$$

in the fraction field of $K_0(\mathcal{O}_{2\mathbb{Z}}^-)$. The q-character morphism can also be algebraically extended to $\mathrm{Frac}(K_0(\mathcal{O}_{2\mathbb{Z}}^-))$. Then $\chi_q(D(\chi))$ is obtained by replacing each $z_{i,r}$ by the corresponding q-character

$$\chi_q(D(z_{i,r})) = \left[\left(\frac{-r}{2d_i}\right)\omega_i\right]\chi_q(L_{i,q^{-r}}^-) = \left[\left(\frac{-r}{2d_i}\right)\omega_i\right]\mathbf{\Psi}_{i,q^{-r}}^{-1}(1+\mathcal{A}_{i,r}), \tag{6.15}$$

where by Theorem 7.6, $\mathcal{A}_{i,r}$ is a formal power series in the $A_{j,a}^{-1}$ without constant term. In particular, we have an analog formula for the inverse which is

$$(\chi_q(D(z_{i,r})))^{-1} = \left[\left(\frac{r}{2d_i}\right)\omega_i\right]\mathbf{\Psi}_{i,q^{-r}}(1+\mathcal{B}_{i,r})$$

where

$$\mathcal{B}_{i,r} = \sum_{k\geq 1}(-\mathcal{A}_{i,r})^k$$

is a formal power series in the $A_{j,a}^{-1}$ without constant term. In particular $\chi_q(D(\chi))$ is in $\mathcal{E}_\ell$ and we get a sum of the form

$$\chi_q(D(\chi)) = \sum_{1\leq\alpha\leq R}\lambda_\alpha[\omega_\alpha]m_\alpha(1+\mathcal{A}_\alpha)\in\mathcal{E}_\ell \tag{6.16}$$

where ω_α is a weight, m_α a Laurent monomial in the $\mathbf{\Psi}_{i,q^{-r}}$, $\mathcal{A}_\alpha$ is a formal power series in the $A_{j,a}^{-1}$ without constant term and $\lambda_\alpha\in\mathbb{Z}$.

We recall the notion of negative ℓ-weight is introduced in Definition 3.8.

Remark 6.3. We say that a sequence $(\mathbf{\Psi}^{(m)})_{m\geq 0}$ of ℓ-weights congerges pointwise as a rational fraction to an ℓ-weight $\mathbf{\Psi}$ if for every $i\in I$ and $z\in\mathbb{C}$, the ratio $\mathbf{\Psi}_i^{(m)}(z)/\mathbf{\Psi}_i(z)$ converges to 1 when $N\to+\infty$ and $|q|>1$. For example, defining the monomials $M_{i,r,N}$ as in Equation (6.17) below, the sequence $(\widetilde{M_{i,r,N}})_{N\geq 0}$ converges pointwise as a rational fraction to $\mathbf{\Psi}_{i,q^{-r}}^{-1}$.

Lemma 6.4. *Let $\chi\in\mathcal{A}(G)$ non zero. Then at least one negative ℓ-weight occurs in $\chi_q(D(\chi))$.*

Proof. We will use the following partial ordering $\preceq$ on the set of ℓ-weights $\mathbf{\Psi}$ satisfying $\varpi(\mathbf{\Psi})=1$: for such ℓ-weights $\mathbf{\Psi}$, $\mathbf{\Psi}'$, we set $\mathbf{\Psi}\succeq\mathbf{\Psi}'$ if

$$\mathbf{\Psi}'(\mathbf{\Psi})^{-1} = \prod_{i,r\geq -M}\widetilde{A_{i,q^r}}^{-v_{i,r}}$$

is a possibly infinite product (that is pointwise the limit of the partial products) with the $v_{i,r}\geq 0$. If $\mathbf{\Psi}=\widetilde{m}$ and $\mathbf{\Psi}'=\widetilde{m'}$ with m,m' monomials in $\mathcal{M}$, $\mathbf{\Psi}\preceq\mathbf{\Psi}'$ is equivalent to $m\preceq m'$ for the partial ordering considered in [N1].

As the sum (6.16) is finite, there is α_0 such that m_{α_0} is maximal for $\preceq$. We prove that m_{α_0} is a negative ℓ-weight.

Let N be such that all the cluster variables $z_{i,r}$ of the initial seed occurring in the Laurent monomials of Equation (6.16) satisfy $r>-2d(N+2)$ where $d=\mathrm{Max}_{i\in I}(d_i)$ is the lacing number of $\dot{\mathfrak{g}}$. We consider as above the semi-infinite cluster algebra $\mathcal{A}(H_N^+)$ obtained from

$\mathcal{A}(G)$ where the cluster variables sitting at $(i,r)\in V$, $r\leq -2d(N+2)$ have been removed. As explained in the proof of Proposition 6.1, $\mathcal{A}(H_N^+)$ can be regarded as a subalgebra of $\mathcal{A}(G)$. We replace every cluster variable $z_{i,r}$ of the initial seed by the class of the Kirillov-Reshetikhin module $W_{i,r,N}$ of highest monomial

$$M_{i,r,N}=\prod_{k\geq 0\ ,\ r+2kd_i\leq 2dN} Y_{i,q^{-r-d_i-2kd_i}}. \tag{6.17}$$

(Here $M_{i,r,N}$ is set to be 1 if $r>2dN$). We obtain

$$\phi_N(\chi)\in \mathrm{Frac}(K_0(\mathcal{C}))$$

the image of χ in the fraction field of $K_0(\mathcal{C})$. By using the duality D, we reverse all spectral parameters (by (ii) in Example 5.14 illustrating Proposition 5.13, or by [HL1, Section 3.4]). We obtain the same[3] as in [HL3, Section 2.2.2]. Then by [HL3, Theorem 5.1], $D(\phi_N(\chi))$ belongs to the Grothendieck ring of $\mathcal{C}$. So by applying D again, $\phi_N(\chi)$ is in $K_0(\mathcal{C})$.

Now we get as for Equation 6.16

$$\chi_q(\phi_N(\chi))=\sum_{1\leq\alpha\leq R}\lambda_\alpha m_\alpha^{(N)}(1+\mathcal{A}_\alpha^{(N)}) \tag{6.18}$$

where $m_\alpha^{(N)}$ is a monomial and $\mathcal{A}_\alpha^{(N)}$ is a formal power series in the $A_{i,a}^{-1}$. Note that $\mathcal{A}_\alpha^{(N)}$ converges when $N\to+\infty$ to $\mathcal{A}_\alpha$ as a formal power series in the $A_{i,a}^{-1}$. Let $\omega_{i,r,N}$ be the highest weight of $W_{i,r,N}$. Now if the initial cluster variable $z_{i,r}$ is replaced by

$$\left[\left(\frac{r}{2d_i}\right)\omega_i-\omega_{i,r,N}\right][W_{i,r,N}] \tag{6.19}$$

instead of $[W_{i,r,N}]$, we just have to replace in (6.18) each $m_\alpha^{(N)}$ by $[\omega_\alpha]\widetilde{m_\alpha^{(N)}}$ (the $\mathcal{A}_\alpha^{(N)}$ are unchanged). Then $\widetilde{m_\alpha^{(N)}}$ converges to m_α when $N\to+\infty$ pointwise as a rational fraction.

Let us show that there are infinitely many N so that $\widetilde{m_{\alpha_0}^{(N)}}$ is maximal among the $\widetilde{m_\alpha^{(N)}}$ for $\preceq$. Otherwise, since Equation 6.16 has finitely many summands, there is α such that $\widetilde{m_{\alpha_0}^{(N)}}\prec\widetilde{m_\alpha^{(N)}}$ for infinitely many N. In the limit, we get that $m_{\alpha_0}\prec m_\alpha$, contradiction.

For N such that $\widetilde{m_{\alpha_0}^{(N)}}$ is maximal for $\preceq$, $m_{\alpha_0}^{(N)}$ is necessarily dominant as $\phi_N(\chi)\in K_0(\mathcal{C})$ (see [FM, Section 5.4]). Then the limit m_{α_0} of the $\widetilde{m_{\alpha_0}^{(N)}}$ is negative as it is easy to check that a limit of dominant monomials is a negative ℓ-weight. Finally, since m_{α_0} is maximal for $\preceq$, it necessarily occurs with a nonzero coefficient in the expansion of $\chi_q(D(\chi))$. □

Lemma 6.5. *Let $\mathbf{\Psi}$ be a negative ℓ-weight such that the roots and the poles of $\mathbf{\Psi}_i(z)$ are of the form q^r with $(i,r)\in V$. Then there is a unique $F(\mathbf{\Psi})\in\chi_q(D(K_0(\mathcal{O}_{2\mathbb{Z}}^+)))$ such that $\mathbf{\Psi}$ is the unique negative ℓ-weight occurring in $F(\mathbf{\Psi})$ and its coefficient is* 1.

[3] Note that we have a term $-d_i$ which does not occur in [HL3, Section 2.2.2] as we use the labeling by V and not by W.

Moreover $F(\mathbf{\Psi})$ is of the form

$$F(\mathbf{\Psi}) = [\mathbf{\Psi}] + \sum_{\mathbf{\Psi}', \varpi(\mathbf{\Psi}') \prec \varpi(\mathbf{\Psi})} \lambda_{\mathbf{\Psi}'}[\mathbf{\Psi}'], \tag{6.20}$$

for the usual partial ordering on weights and with the $\lambda_{\mathbf{\Psi}'} \in \mathbb{Z}$.

Proof. The uniqueness follows from Proposition 6.1 and Lemma 6.4. For each negative ℓ-weight $\mathbf{\Psi}$ as in the Lemma, there is a representation $M(\mathbf{\Psi})$ in $\mathcal{O}_{2\mathbb{Z}}^+$ such that $\chi_q(D([M(\mathbf{\Psi})]))$ is of the form

$$\chi_q(D([M(\mathbf{\Psi})])) = [\mathbf{\Psi}] + \sum_{\mathbf{\Psi}', \varpi(\mathbf{\Psi}') \prec \varpi(\mathbf{\Psi})} \mu_{\mathbf{\Psi}', \mathbf{\Psi}}[\mathbf{\Psi}'].$$

Indeed it suffices to consider a tensor product of fundamental and positive prefundamental representations. Now if the $F(\mathbf{\Psi})$ do exist, we have an infinite triangular transition matrix from the $(F(\mathbf{\Psi}))$ to the $(\chi_q(D([M(\mathbf{\Psi})]))$ with 1 on the diagonal and whose off-diagonal coefficients are the $\mu_{\mathbf{\Psi}', \mathbf{\Psi}}$ for $\mathbf{\Psi}, \mathbf{\Psi}'$ negative. So to prove the existence, it suffices to consider the inverse of this matrix (which is well-defined as for given $\mathbf{\Psi}, \mathbf{\Psi}'$, there is a finite number of $\mathbf{\Psi}''$ satisfying $\varpi(\mathbf{\Psi}') \preceq \varpi(\mathbf{\Psi}'')$ and $\mu_{\mathbf{\Psi}'', \mathbf{\Psi}} \neq 0$). □

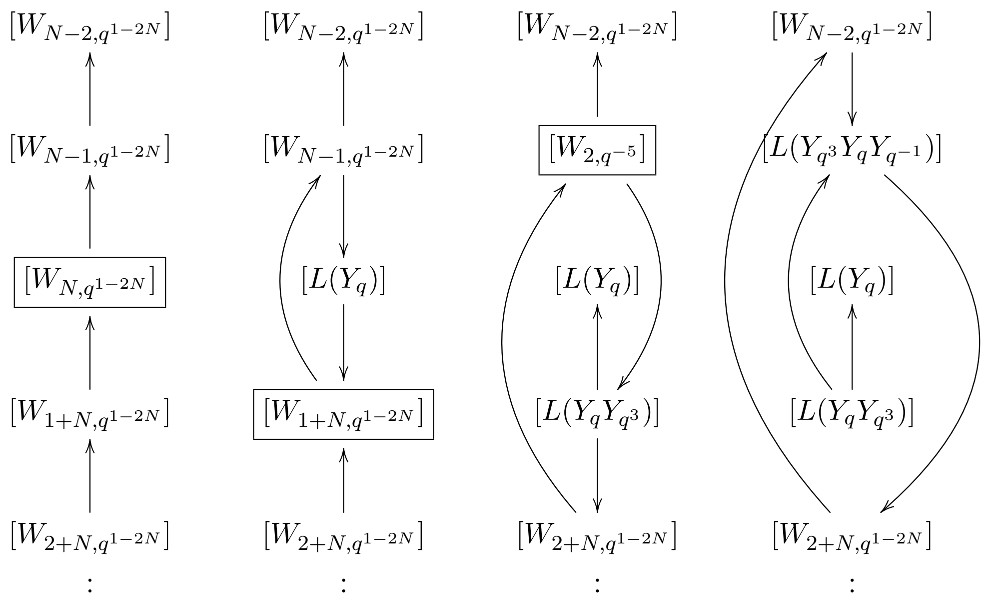
We can now finish the proof of Proposition 6.2 :

Proof. Let χ be in $\mathcal{A}(G)$. For $\mathbf{\Psi}$ a negative ℓ-weight, we denote by $\lambda_{\mathbf{\Psi}}$ the coefficient of $\mathbf{\Psi}$ in $\chi_q(D(\chi))$. Then by lemma 6.4 we have

$$\chi_q(D(\chi)) = \sum_{\mathbf{\Psi} \text{ negative}} \lambda_{\mathbf{\Psi}} F(\mathbf{\Psi}).$$

As the $F(\mathbf{\Psi})$ are of the form (6.20), this sum is well-defined in $\chi_q(D(K_0(\mathcal{O}_{2\mathbb{Z}}^+)))$ and we get $\chi \in K_0(\mathcal{O}_{2\mathbb{Z}}^+)$. □

Example 6.6. Let us illustrate the proof of Lemma 6.4 which is the crucial technical point for the proof of Proposition 6.2. Consider the sequence of mutations of 6.1.1. Let us write the cluster variables $\phi_N(\chi)$:



The cluster variable corresponding to $[L(Y_q)]$ has its q-character which can be written in the form of Equation (6.16) :

$$\begin{aligned}
\chi_q(L(Y_q)) &= Y_q + Y_{q^3}^{-1} = \frac{\chi_q(W_{N-1,q^{1-2N}}) + \chi_q(W_{N+1,q^{1-2N}})}{\chi_q(W_{N,q^{1-2N}})} \\
&= Y_{q^{-1}}^{-1} \frac{1 + A_{q^{-2}}^{-1}(1 + A_{q^{-4}}^{-1}(1 + \cdots (1 + A_{q^{2(1-N)}}^{-1}))\cdots)}{1 + A_1^{-1}(1 + A_{q^{-2}}^{-1}(1 + \cdots (1 + A_{q^{2(1-N)}}^{-1}))\cdots)} \\
&\quad + Y_q \frac{1 + A_{q^2}^{-1}(1 + A_1^{-1}(1 + \cdots (1 + A_{q^{2(1-N)}}^{-1}))\cdots)}{1 + A_1^{-1}(1 + A_{q^{-2}}^{-1}(1 + \cdots (1 + A_{q^{2(1-N)}}^{-1}))\cdots)}.
\end{aligned}$$

We see as in the statement of Lemma 6.4 that the monomial Y_q maximal for $\preceq$ is dominant and so negative in the sense of Definition 3.8. In the limit $N \to +\infty$ (with the renormalized weights) we get the representations in $K_0(\mathcal{O}^-)$

$$\begin{tikzcd}
{[-2\omega][L^-_{q^{-4}}]} & {[-2\omega][L^-_{q^{-4}}]} & {[-2\omega][L^-_{q^{-4}}]} & {[-2\omega][L^-_{q^{-4}}]} \arrow[d] \\
{[-\omega][L^-_{q^{-2}}]} \arrow[u] & {[-\omega][L^-_{q^{-2}}]} \arrow[u] \arrow[d] & \boxed{[-\omega][L^-_{q^{-2}}]} \arrow[u] \arrow[dd, bend left] & {[L(Y_{q^3}Y_qY_{q^{-1}})]} \arrow[ddd, bend left] \\
\boxed{[L^-_1]} \arrow[u] & {[L(Y_q)]} \arrow[d] & {[L(Y_q)]} & {[L(Y_q)]} \\
{[\omega][L^-_{q^2}]} \arrow[u] & \boxed{[\omega][L^-_{q^2}]} \arrow[uu, bend left] & {[L(Y_qY_{q^3})]} \arrow[u] \arrow[d] & {[L(Y_qY_{q^3})]} \arrow[u] \arrow[uu, bend left] \\
{[2\omega][L^-_{q^4}]} \arrow[u] & {[2\omega][L^-_{q^4}]} \arrow[u] & {[2\omega][L^-_{q^4}]} \arrow[uuu, bend left] & {[2\omega][L^-_{q^4}]} \arrow[uuuu, bend left] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

The duality again gives the representations in $K_0(\mathcal{O}^+)$:

$$\begin{tikzcd}
{[-2\omega][L^+_{q^{4}}]} & {[-2\omega][L^+_{q^{4}}]} & {[-2\omega][L^+_{q^{4}}]} & {[-2\omega][L^+_{q^{4}}]} \arrow[d] \\
{[-\omega][L^+_{q^{2}}]} \arrow[u] & {[-\omega][L^+_{q^{2}}]} \arrow[u] \arrow[d] & \boxed{[-\omega][L^+_{q^{2}}]} \arrow[u] \arrow[dd, bend left] & {[L(Y_{q^{-3}}Y_{q^{-1}}Y_q)]} \arrow[ddd, bend left] \\
\boxed{[L^+_1]} \arrow[u] & {[L(Y_{q^{-1}})]} \arrow[d] & {[L(Y_{q^{-1}})]} & {[L(Y_{q^{-1}})]} \\
{[\omega][L^+_{q^{-2}}]} \arrow[u] & \boxed{[\omega][L^+_{q^{-2}}]} \arrow[uu, bend left] & {[L(Y_{q^{-1}}Y_{q^{-3}})]} \arrow[u] \arrow[d] & {[L(Y_{q^{-1}}Y_{q^{-3}})]} \arrow[u] \arrow[uu, bend left] \\
{[2\omega][L^+_{q^{-4}}]} \arrow[u] & {[2\omega][L^+_{q^{-4}}]} \arrow[u] & {[2\omega][L^+_{q^{-4}}]} \arrow[uuu, bend left] & {[2\omega][L^+_{q^{-4}}]} \arrow[uuuu, bend left] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Example 6.7. We now illustrate the proof of Lemma 6.4 by means of the mutation of 6.1.2. Let us write the cluster variables $\phi_N(\chi)$:

$$\begin{tikzcd}
 & \vdots & & & \vdots \\
\vdots & {[W^{(1)}_{N-1,q^{1-2N}}]} \arrow[u] \arrow[dl] & & \vdots & {[W^{(1)}_{N-1,q^{1-2N}}]} \arrow[u] \arrow[dd] \\
{[W^{(2)}_{N-1,q^{2(1-N)}}]} \arrow[dr] & & & {[W^{(2)}_{N-1,q^{2(1-N)}}]} & \\
 & {[W^{(1)}_{N,q^{1-2N}}]} \arrow[uu] \arrow[dl] & & & {[L(m^{(N)})]} \arrow[ul] \arrow[dd] \\
{[W^{(2)}_{N,q^{2(1-N)}}]} \arrow[uu] \arrow[dr] & & & {[W^{(2)}_{N,q^{2(1-N)}}]} \arrow[ur] & \\
\vdots & {[W^{(1)}_{N+1,q^{1-2N}}]} \arrow[uu] & & \vdots & {[W^{(1)}_{N+1,q^{1-2N}}]} \arrow[uuuu, bend right] \\
 & \vdots & & & \vdots
\end{tikzcd}$$

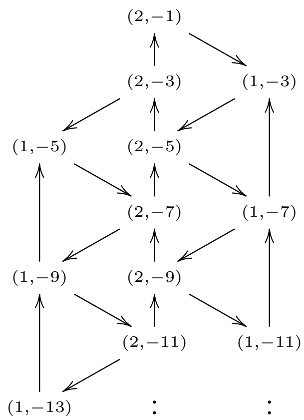
Here $m^{(N)} = Y_{1,q}(Y_{2,q^{2(1-N)}}Y_{2,q^{4-2N}}\cdots Y_{2,q^{-2}})$. The q-character corresponding to the cluster variable $[L(m^{(N)})]$ can be written in the form of Equation (6.16) :

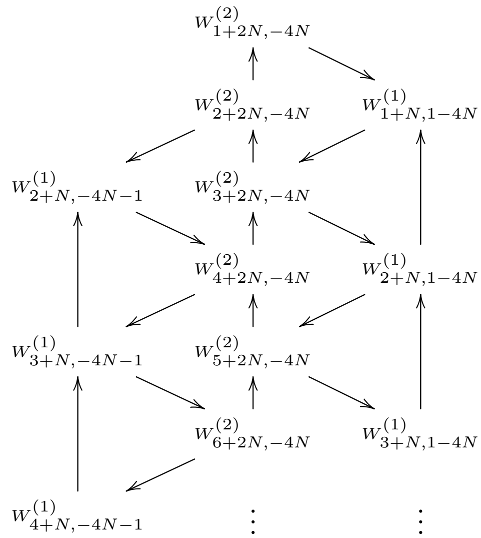
$$\chi_q(L(m^{(N)})) = \frac{\chi_q(W^{(1)}_{N-1,q^{1-2N}})\chi_q(W^{(2)}_{N,q^{2(1-N)}})}{\chi_q(W^{(1)}_{N,q^{1-2N}})} + \frac{\chi_q(W^{(1)}_{N+1,q^{1-2N}})\chi_q(W^{(2)}_{N-1,q^{2(1-N)}})}{\chi_q(W^{(1)}_{N,q^{1-2N}})}$$

$$= Y^{-1}_{1,q^{-1}}(Y_{2,q^{2(1-N)}}Y_{2,q^{4-2N}}\cdots Y_{2,1})(1+\mathcal{A}_1^{(N)}) + Y_{1,q}(Y_{2,q^{2(1-N)}}Y_{2,q^{4-2N}}\cdots Y_{2,q^{-2}})(1+\mathcal{A}_2^{(N)}),$$

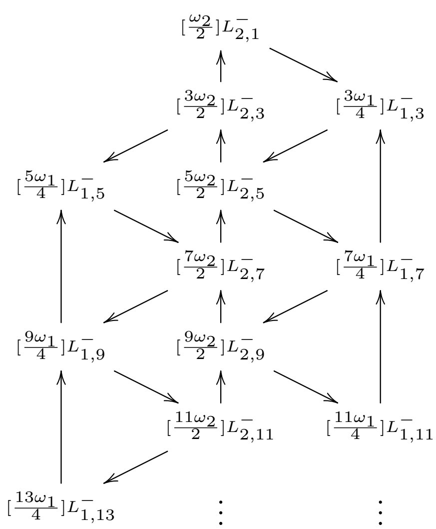
where $\mathcal{A}_1^{(N)}$ and $\mathcal{A}_2^{(N)}$ are formal power series in the A^{-1}_{1,q^a}, A^{-1}_{2,q^a} $(a \in \mathbb{C}^*)$ without constant term. The monomial $\widetilde{m^{(N)}}$ is maximal for $\preceq$. Its limit for $N \to +\infty$ is $[-\omega_1]Y_{1,q}\mathbf{\Psi}^{-1}_{2,q^{-1}}$, which is negative in the sense of Definition 3.8.

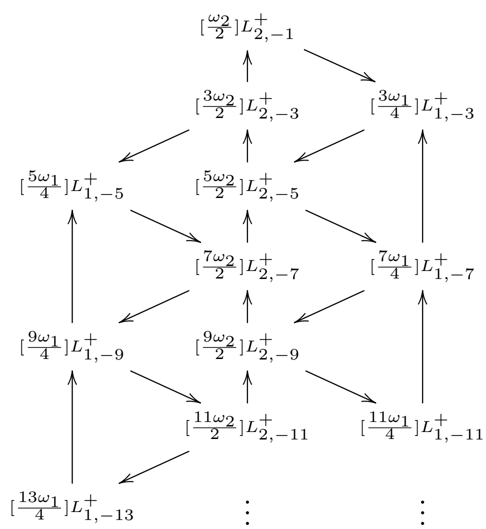
Example 6.8. In this example we check that the images of the initial cluster variables considered in the proof of Lemma 6.4 do match. Let us consider type B_2 with the following initial seed and the initial cluster variables replaced by the $W_{i,r}$:





In the limit $N \to +\infty$ (with the renormalized weights) we get the images of the initial cluster variables :





7. Conjectures and evidences

7.1. **A conjecture.** The concept of a monoidal categorification of a cluster algebra was introduced in [HL3, Definition 2.1]. We say that a simple object S of a monoidal category is *real* if $S \otimes S$ is simple. Let us recall that a cluster monomial is a monomial in the cluster variables of a single cluster.

Definition 7.1. *Let $\mathcal{A}$ be a cluster algebra and let $\mathcal{M}$ be an abelian monoidal category. We say that $\mathcal{A}$ is a monoidal categorification of $\mathcal{A}$ if there is an isomorphism between $\mathcal{A}$ and the Grothendieck ring of $\mathcal{M}$ such that the cluster monomials of $\mathcal{A}$ are the classes of all the real simple objects of $\mathcal{M}$ (up to invertibles).*

See [HL2, Section 2] for a discussion on applications of monoidal categorifications. In view of Theorem 4.2, it is natural to formulate the following conjecture.

Conjecture 7.2. *The isomorphism of Theorem 4.2 defines a monoidal categorification, that is, the cluster monomials in $\mathcal{A}(\Gamma)$ get identified with real simple objects in $\mathcal{O}^+_{2\mathbb{Z}}$ up to invertible representations.*

Remark 7.3. By using the duality in Proposition 5.13, the statement of Theorem 4.2 and of Conjecture 7.2 can also be formulated in terms of the category $\mathcal{O}^-_{2\mathbb{Z}}$.

Note that Theorem 5.8 implies that all cluster monomials of the initial seed are identified with real simple objects, more precisely with simple tensor products of positive prefundamental representations, in agreement with Conjecture 7.2. To give other evidences supporting Conjecture 7.2, we will use the results in the next subsection.

7.2. **Limiting characters.** We will be using the dual category $\mathcal{O}^*$ considered in [HJ] whose definition we now recall.

Definition 7.4. *Let $\mathcal{O}^*$ be the category of Cartan-diagonalizable $U_q(\mathfrak{b})$-modules V such that V^* is in category $\mathcal{O}$.*

A $U_q(\mathfrak{b})$-module V is said to be of lowest ℓ-weight $\mathbf{\Psi} \in P_\ell$ if there is $v \in V$ such that $V = U_q(\mathfrak{b})v$ and the following hold:

$$U_q(\mathfrak{b})^- v = \mathbb{C}v\,, \qquad \phi^+_{i,m} v = \Psi_{i,m} v \quad (i \in I,\ m \geq 0)\,.$$

For $\mathbf{\Psi} \in P_\ell$, there exists up to isomorphism a unique simple $U_q(\mathfrak{b})$-module $L'(\mathbf{\Psi})$ of lowest ℓ-weight Ψ. This module belongs to $\mathcal{O}^*$. More precisely, we have:

Proposition 7.5. [HJ] *For $\mathbf{\Psi} \in P_\ell$ we have $(L'(\mathbf{\Psi}))^* \simeq L(\mathbf{\Psi}^{-1})$.*

We can also define as in §5.1 notions of characters and q-characters for $\mathcal{O}^*$.

We now explain that characters (resp. q-characters) of certain simple objects in the category $\mathcal{O}^-$ can be obtained as limits of characters (resp. q-characters) of finite-dimensional representations. This is known for negative prefundamental representations [HJ].

Let $L(\mathbf{\Psi})$ be a simple module whose highest ℓ-weight can be written as a finite product

$$\mathbf{\Psi} = [\omega] \times m \times \prod_{i \in I} \left(\prod_{r \geq -R_0} \mathbf{\Psi}_{i,q^r}^{u_{i,q^r}} \right)$$

where $\omega \in P_{\mathbb{Q}}$, $R_0 \geq 0$, $u_{i,q^r} \leq 0$ and $m \in \mathcal{M}$ is a dominant monomial. For $R \geq R_0$, set

$$M_R = m \prod_{i \in I} \left(\prod_{r \geq -R_0,\ r' \geq 0,\ r - 2d_i r' \geq -R} Y_{i,q^{r-2d_i r' - d_i}}^{-u_{i,q^r}} \right). \tag{7.21}$$

Theorem 7.6. *(1) We have the limit as formal power series :*

$$\widetilde{\chi}_q(L(M_R)) \underset{R\to+\infty}{\to} \widetilde{\chi}_q(L(\mathbf{\Psi})) \in \mathbb{Z}\left[\left[A_{i,a}^{-1}\right]\right]_{i\in I, a\in\mathbb{C}^*}.$$

(2) We have $\widetilde{\chi}(L(\overline{\mathbf{\Psi}}^{-1})) = \widetilde{\chi}(L(\mathbf{\Psi}))$ and so we have the limit as formal power series :

$$\widetilde{\chi}(L(M_R)) \underset{R\to+\infty}{\to} \widetilde{\chi}(L(\overline{\mathbf{\Psi}}^{-1})) \in \mathbb{Z}[[-\alpha_i]]_{i\in I}.$$

The proof of Theorem 7.6 is essentially the same as that of [HJ, Theorem 6.1], so we just give an outline.

Proof. First let us prove that the dimensions of weight spaces of $\widetilde{L}(\mathbf{\Psi})$ are larger than those of $\widetilde{L}(M_R)$. Consider the tensor product

$$T = \widetilde{L}(\mathbf{\Psi}) \otimes \widetilde{L}(M_R\mathbf{\Psi}^{-1}).$$

By definition of M_R, the ℓ-weight $M_R\mathbf{\Psi}^{-1}$ is a product of $\mathbf{\Psi}_{i,q^r}^{+}$ times $[\lambda]$ for some $\lambda \in P_{\mathbb{Q}}$, so by Theorem 5.8, the module $\widetilde{L}(M_R\mathbf{\Psi}^{-1})$ is a tensor product of positive fundamental representations. Moreover T and $\widetilde{L}(M_R)$ have the same highest ℓ-weight, so $\widetilde{L}(M_R)$ is a subquotient of T. By [FM, Theorem 4.1], each ℓ-weight of $\widetilde{L}(M_R)$ is the product of the highest ℓ-weight $M_R(\varpi(M_R))^{-1}$ by a product of $A_{j,b}^{-1}$, $j \in i$, $b \in \mathbb{C}^*$. Hence, by Theorem 5.3, an ℓ-weight of T is an ℓ-weight of $\widetilde{L}(M_R)$ only if it is of the form

$$\mathbf{\Psi}'\varpi(\mathbf{\Psi})^{-1}(\widetilde{M_R}\widetilde{\mathbf{\Psi}}^{-1})$$

where $\mathbf{\Psi}'$ is an ℓ-weight of $L(\mathbf{\Psi})$ and $\widetilde{M_R}\widetilde{\mathbf{\Psi}}^{-1}$ is the highest ℓ-weight of $\widetilde{L}(M_R\mathbf{\Psi}^{-1})$. We get the result for the dimensions.

Then we prove as in [HJ, Section 4.2] that we can define an inductive linear system

$$L(M_0) \to L(M_1) \to \cdots \to L(M_R) \to L(M_{R+1}) \to \cdots$$

from the $L(M_R)$ so that we have the convergency of the action of the subalgebra $\widetilde{U}_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ generated by the $x_{i,r}^{+}$ and the $k_i^{-1}x_{i,r}^{-}$. We get a limiting representation of $\widetilde{U}_q(\mathfrak{g})$ from which one can construct a representation L_1 of $U_q(\mathfrak{b})$ in the category $\mathcal{O}$ and a representation L_2 of $U_q(\mathfrak{b})$ in the category $\mathcal{O}^*$ [HJ, Proposition 2.4]. Moreover, L_1 (resp. L_2) is of highest (resp. lowest) ℓ-weight $\mathbf{\Psi}$ (resp. $\overline{\mathbf{\Psi}}$).

By construction, the normalized q-character of L_1 is the limit of the normalized q-characters $\widetilde{\chi}_q(L(M_R))$ as formal power series. Combining with the result of the first paragraph of this proof, the representation L_1 is necessarily simple isomorphic to $L(\mathbf{\Psi})$. We have proved the first statement in the Theorem.

Now, by construction L_2^* is in the category $\mathcal{O}$ with highest ℓ-weight $\overline{\mathbf{\Psi}}^{-1}$ and satisfies $\widetilde{\chi}(L_2^*) = \widetilde{\chi}(L_1)$. To conclude, it suffices to prove that L_2^* is irreducible. This is proved as in [HJ, Theorem 6.3]. □

We have the following application:

Theorem 7.7. *Let $L(\mathbf{\Psi})$ be a simple module in the category $\mathcal{O}^-$ such that $\widetilde{\mathbf{\Psi}} = \mathbf{\Psi}$. Then its image by D^{-1} in $K_0(\mathcal{O}^+)$ is simple equal to $D^{-1}([L(\mathbf{\Psi})]) = [L(\overline{\mathbf{\Psi}}^{-1})]$.*

Proof. From Example 5.14 (i), the property is satisfied by negative prefundamental representations. Since these representations generate the fraction field of $K_0(\mathcal{O}^-)$, it suffices to show that the assignment $[L(\mathbf{\Psi})] \mapsto [L(\overline{\mathbf{\Psi}}^{-1})]$ for ℓ-weights $\mathbf{\Psi}$ satisfying $\mathbf{\Psi} = \widetilde{\mathbf{\Psi}}$ is multiplicative (recall that D is a morphism of $\mathcal{E}$-algebras). Let us use the same notation as in the proof of Theorem 7.6 above. For $L(\mathbf{\Psi}) \simeq L_1$ and $L(\mathbf{\Psi}') \simeq L_1'$ simple modules in $\mathcal{O}^-$ with $\widetilde{\mathbf{\Psi}} = \mathbf{\Psi}$ and $\widetilde{\mathbf{\Psi}'} = \mathbf{\Psi}'$, we have the corresponding modules L_2, L_2' in $\mathcal{O}^*$. We consider the decomposition

$$[L(\mathbf{\Psi}) \otimes L(\mathbf{\Psi}')] = \sum_{\mathbf{\Psi}'', \widetilde{\mathbf{\Psi}''} = \mathbf{\Psi}''} m_{\mathbf{\Psi}''}[L(\mathbf{\Psi}'')]$$

in $K_0(\mathcal{O}^-)$ with the $m_{\mathbf{\Psi}''} \in \mathcal{E}$. Each $\chi_q(L(\mathbf{\Psi}''))$ is obtained as a limit as in Theorem 7.6, and by construction the corresponding modules $L'(\mathbf{\Psi}'')$ in the category $\mathcal{O}^*$ satisfy

$$[L_2 \otimes L_2'] = \sum_{\mathbf{\Psi}'', \widetilde{\mathbf{\Psi}''} = \mathbf{\Psi}''} m'_{\mathbf{\Psi}''}[L'(\overline{\mathbf{\Psi}''})].$$

where each $m'_{\mathbf{\Psi}''} \in \mathcal{E}$ is obtained from $m_{\mathbf{\Psi}''}$ via the substitution $[\omega] \mapsto [-\omega]$. This implies $[L_2^* \otimes (L_2')^*] = \sum_{\mathbf{\Psi}'', \widetilde{\mathbf{\Psi}''} = \mathbf{\Psi}''} m_{\mathbf{\Psi}''}[(L'(\overline{\mathbf{\Psi}''}))^*]$, that is, in view of Proposition 7.5,

$$[L(\overline{\mathbf{\Psi}}^{-1})] \otimes [L(\overline{\mathbf{\Psi}'}^{-1})] = \sum_{\mathbf{\Psi}'', \widetilde{\mathbf{\Psi}''} = \mathbf{\Psi}''} m_{\mathbf{\Psi}''}[L(\overline{\mathbf{\Psi}''}^{-1})],$$

as required. □

Example 7.8. Applying D to (6.14), we get the following relation in $K_0(\mathcal{O}^-)$:

$$\left[L(\overline{\mathbf{\Psi}}^{-1}) \otimes L^-_{i,q^{-r}}\right] = \left[\bigotimes_{j, C_{j,i} \neq 0} L^-_{j,q^{-r+d_jC_{j,i}}}\right] + [-\alpha_i]\left[\bigotimes_{j, C_{j,i} \neq 0} L^-_{j,q^{-r-d_jC_{j,i}}}\right],$$

where

$$\overline{\mathbf{\Psi}}^{-1} = [-\omega_i] Y_{i,q^{-r+d_i}} \prod_{j, C_{j,i} < 0} \mathbf{\Psi}^{-1}_{j,q^{-r+d_jC_{j,i}}}.$$

7.3. **Proof of Conjecture 7.2 for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.** In this section we give an explicit description of all simple modules in $\mathcal{O}^+$ and in $\mathcal{O}^-$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.

A q-set is a subset of $\mathbb{C}^*$ of the form $\{aq^{2r} \mid R_1 \leq r \leq R_2\}$ for some $a \in \mathbb{C}^*$ and $R_1 \leq R_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$. The KR-modules $W_{k,a}$, $W_{k',b}$ are said to be in *special position* if the union of $\{a, aq^2, \cdots, aq^{2(k-1)}\}$ and $\{b, bq^2, \cdots, bq^{2(k'-1)}\}$ is a q-set which contains both properly. The KR-module $W_{k,a}$ and the prefundamental representation L_b^+ are said to be in special position if the union of $\{a, aq^2, aq^4, \cdots, aq^{2(k-1)}\}$ and $\{bq, bq^3, bq^5, \cdots\}$ is a q-set which contains both properly. Two positive prefundamental representations are never in special position. Two representations are in *general position* if they are not in special position.

The invertible elements in the category $\mathcal{O}^+$ are the 1-dimensional representations $[\omega]$.

Theorem 7.9. *Suppose that $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. The prime simple objects in the category $\mathcal{O}^+$ are the positive prefundamental representations and the KR-modules (up to invertibles). Any simple object in $\mathcal{O}^+$ can be factorized in a unique way as a tensor product of prefundamental representations and KR-modules (up to permutation of the factors and to invertibles). Moreover, such a tensor product is simple if and only all its factors are pairwise in general position.*

Proof. As in the classical case of finite-dimensional representations, it is easy to check that every positive ℓ-weight has a unique factorization as a product of highest ℓ-weights of KR-modules and positive prefundamental representations in pairwise general position. Hence it suffices to prove the equivalence in the last sentence. By [CP], the result is known for finite-dimensional representations. Now, by using Section 7.2, this result implies that a tensor product with factors which are in general position is simple. Conversely, it is known that a tensor product of KR-modules which are in special position is not simple. Also, it is easy to see that the tensor product of a KR-module and a positive prefundamental representation which are in special position is not simple. □

Remark 7.10. (i) This is a generalization of the factorization of simple representations in $\mathcal{C}$ when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ [CP].

(ii) This result for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ implies that all simple objects in $\mathcal{O}^+$ are real and that their factorization into prime representations is unique.

(iii) In [MY], a factorization is proved for simple modules in $\hat{\mathcal{O}}$ when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. But the factorization is not unique in the category $\hat{\mathcal{O}}$, see Remark 3.10.

(iv) By Proposition 5.13, our result implies a similar factorization in the category $\mathcal{O}^-$.

(v) The combinatorics of q-sets in pairwise general position is very similar to the combinatorics of triangulations of the ∞-gone studied in [GG], in relation with certain cluster structures of infinite rank. However, in [GG] only arcs (m, n) joining two integers m and n are considered, whereas we also allow arcs of the form $(m, +\infty)$ corresponding to positive prefundamental representations. Also, we are only interested in one mutation class, namely the mutation class of the initial triangulation $\{(m, +\infty) \mid m \in \mathbb{Z}\}$.

Theorem 7.11. *Conjecture 7.2 is true in the $\mathfrak{sl}_2$-case.*

Proof. Theorem 7.9 provides an explicit factorization of simple objects in $\mathcal{O}^+_{2\mathbb{Z}}$ into positive prefundamental representations and finite-dimensional KR-modules. In particular, we get an explicit q-character formula for such a simple object and so a complete explicit description of the Grothendieck ring $K_0(\mathcal{O}^+_{2\mathbb{Z}})$. Besides the cluster algebra $\mathcal{A}(\Gamma)$ can be also explicitly described by using triangulations of the ∞-gone (see Remark 7.10). Hence we can argue as in [HL1, §13.4]. □

7.4. **Equivalence of conjectures.** In general, we have the following:

Theorem 7.12. *Conjecture 7.2 is equivalent to* [HL3, Conjecture 5.2].

Combining with the recent results in [Q], this would imply a part of Conjecture 7.2 for ADE types, namely that all cluster monomials are classes of real simple objects.

As reminded in the introduction, [HL3, Conjecture 5.2] states that $\mathcal{C}^-_{\mathbb{Z}}$ is the monoidal categorification of a cluster algebra $\mathcal{A}(G^-)$. Note that $\mathcal{C}^-_{\mathbb{Z}}$ is a subcategory of $\mathcal{O}^-_{2\mathbb{Z}}$ and $\mathcal{O}^+_{2\mathbb{Z}}$.

Proof. For $N > 0$, let $\mathcal{C}_N$ be the category of finite-dimensional $U_q(\mathfrak{g})$-modules V satisfying

$$[V] \in \mathbb{Z}[[V_{i,q^m}]]_{i\in I,-2dN-d_i\leq m<d(N+2)-d_i} \subset K_0(\mathcal{C}).$$

It is a monoidal category similar to the categories considered in [HL1]. It contains the KR-module $W_{i,r,N}$ with highest monomial $M_{i,r,N}$ given in Equation (6.17), where $i \in I$ and $-d(N+2) < r \leq 2dN$. The Grothendieck ring $K_0(\mathcal{C}_N)$ has a cluster algebra structure with an initial seed consisting of these KR-modules $W_{i,r,N}$ (here we use the initial seed as in [HL3]). We have established in the proof of Theorem 4.2 that $K_0(\mathcal{C}_N) \otimes \mathcal{E}$ may be seen as a subalgebra of $K_0(\mathcal{O}^+_{2\mathbb{Z}})$ by using the identification of $z_{i,r}$ with the element defined in equation (6.19). This induces embeddings $K_0(\mathcal{C}_N) \subset K_0(\mathcal{C}_{N+1})$:

$$K_0(\mathcal{C}_1) \subset K_0(\mathcal{C}_2) \subset K_0(\mathcal{C}_3) \subset \cdots \subset K_0(\mathcal{O}^+_{2\mathbb{Z}})$$

which are not the naive embeddings obtained from the inclusion of categories $\mathcal{C}_N \subset \mathcal{C}_{N+1}$. The cluster monomials in $K_0(\mathcal{C}_N)$ corresponds now to cluster monomials in $K_0(\mathcal{O}^+_{2\mathbb{Z}})$.

Note that by Theorem 7.7, we may consider indifferently the statement of Conjecture 7.2 for $\mathcal{O}^+_{2\mathbb{Z}}$ or for $\mathcal{O}^-_{2\mathbb{Z}}$.

Suppose that [HL3, Conjecture 5.2] is true. This implies that the cluster monomials in $K_0(\mathcal{C}_N)$ are the real simple modules for any $N > 0$. Consider a cluster monomial in $\mathcal{O}^+_{2\mathbb{Z}}$. Then for N large enough, we have a corresponding real representation V_N in $K_0(\mathcal{C}_N)$. The highest monomial of V_N is a Laurent monomial in the $m_{i,r,N}$ of the form considered in Theorem 7.6. By Theorem 7.6, $\chi_q(V_N)$ converges to the q-character of a simple module V in $\mathcal{O}^-_{2\mathbb{Z}}$ when $N \to +\infty$. Moreover V is real as the q-character of $V \otimes V$ is obtained as a limit of simple q-character $\chi_q(V_N \otimes V_N)$ by Theorem 7.6. Conversely, every real simple module V in $\mathcal{O}^-_{2\mathbb{Z}}$ is obtained as such a limit simple modules V_N. Moreover since V is real V_N is real (for $\chi_q(V_N \otimes V_N)$ is an upper q-character of $V \otimes V$ in the sense of [H3, Corollary 5.8]). For N large enough the modules V_N correspond to the same cluster monomial which is therefore identified with V.

Conversely suppose that Conjecture 7.2 is true. Consider a cluster monomial χ in $K_0(\mathcal{C}^-_{\mathbb{Z}})$. The cluster variables occuring in χ are produced via sequences of mutations from a finite number of KR-modules in the initial seed. By the proof of Proposition 6.1, there is a seed in $K_0(\mathcal{O}^-_{2\mathbb{Z}})$ containing these KR-modules (and the quiver of this seed has the same arrows joining the corresponding vertices). By our hypothesis χ is the class of a simple real module as an element of $K_0(\mathcal{O}^-_{2\mathbb{Z}}) \supset K_0(\mathcal{C}^-_{\mathbb{Z}})$. Hence it is also real simple in $K_0(\mathcal{C}^-_{\mathbb{Z}})$. Now consider a real simple module $[V]$ in $K_0(\mathcal{C}^-_{\mathbb{Z}})$. It corresponds to a cluster monomial in $K_0(\mathcal{O}^-_{2\mathbb{Z}})$ which is a cluster monomial in $K_0(\mathcal{C}^-_{\mathbb{Z}})$ by the same arguments. □

7.5. **Web property theorem.** Let us prove the following generalization of the main result of [H3]. If Conjecture 7.2 holds, then the statement of the next theorem is a necessary condition for simple modules in the same seed.

Theorem 7.13. *Let $S_1, \ldots, S_N$ be simple objects in $\mathcal{O}^+$ (resp. in $\mathcal{O}^-$). Then $S_1 \otimes \cdots \otimes S_N$ is simple if and only if the tensor products $S_i \otimes S_j$ are simple for $i \leq j$.*

Proof. By Proposition 5.13, it is equivalent to prove the statement in the category $\mathcal{O}^-$. Note that the "only if" part is clear. For the "if" part of the statement, we may assume without loss of generality that the zeros and poles of the highest ℓ-weights of the S_i are in

$q^{\mathbb{Z}}$ (see (ii) in Remark 4.3). For each simple module S_i, consider a corresponding simple finite-dimensional module $L(M_{R,i})$ as in Section 7.2. Since $S_i \otimes S_j$ is simple, there exists R_1 such that for $R \geq R_1$ the tensor product $L(M_{R,i}) \otimes L(M_{R,j})$ is simple. Indeed, by Theorem 7.6, $\widetilde{\chi}_q(S_i \otimes S_j)$ is the limit of the $\widetilde{\chi}_q(L(M_{R,i}M_{R,j}))$. More precisely, there is R_1 such that for $R \geq R_1$, the image of $\widetilde{\chi}_q(S_i \otimes S_j)$ in $\mathbb{Z}[A_{i,q^r}^{-1}]_{i\in I, r\geq -R_1+r_i}$ is equal to $\widetilde{\chi}_q(L(M_{R,i}M_{R,j}))$. This implies that $\widetilde{\chi}_q(L(M_{R,i}M_{R,j})) = \widetilde{\chi}_q(L(M_{R,i}))\widetilde{\chi}_q(L(M_{R,j}))$.

Now, by [H3], $L(M_{R,1}) \otimes \cdots \otimes L(M_{R,N})$ is simple isomorphic to $L(M_{R,1} \cdots M_{R,N})$. This implies that the character of $S_1 \otimes \cdots \otimes S_N$ is the same as the character of the simple module with the same highest ℓ-weight. Hence they are isomorphic.

□

Remark 7.14. This provides an alternative proof of Theorem 7.9.

7.6. **Another conjecture.** To conclude, let us state another general conjecture. Although the cluster algebra structure presented in this paper does not appear in the statement, this conjecture arises naturally if we compare Theorem 4.2 with the results of [HL3].

We consider a simple finite-dimensional representation $L(m)$ whose dominant monomial m satisfies $m \in \mathbb{Z}[Y_{i,aq^r}]_{(i,r)\in W, r\leq R}$ for a given $R \in \mathbb{Z}$. We have the corresponding truncated q-character [HL1]

$$\chi_q^{\leq R}(L(m)) \in m\mathbb{Z}[A_{i,q^r}^{-1}]_{i\in I, r\leq R-d_i}$$

which is the sum (with multiplicity) of the monomials m' occurring in $\chi_q(L(m))$ satisfying $m(m')^{-1} \in \mathbb{Z}[A_{i,q^r}]_{i\in I, r\leq R-d_i}$. As in the statement of Theorem 5.5, we consider

$$\chi_\ell^{\leq R}(L(m)) \in \mathrm{Frac}(K_0(\mathcal{O}^+))$$

obtained from $\chi_q^{\leq R}(L(m))$ by replacing each variable $Y_{i,a}$ by $[\omega_i]\ell_{i,aq_i^{-1}}\ell_{i,aq_i}^{-1}$.

Let us set

$$W_R = \{(i,r) \in W | R \geq r > R - 2d_i\}.$$

For $(i,r) \in W_R$, we set $u_{i,r}$ to be the maximum of 0 and of the powers $u_{i,q^r}(m')$ of Y_{i,q^r} in all monomials m' occurring in $\chi_q^{\leq R}(L(m))$. Let

$$\mathbf{\Psi}_R = \prod_{(i,r)\in W_R} \mathbf{\Psi}_{i,q^{r+d_i}}^{u_{i,r}} \text{ and } \mathbf{\Psi} = m\mathbf{\Psi}_R.$$

The representations $L(\mathbf{\Psi})$, $L(\mathbf{\Psi}_R)$ are in the category $\mathcal{O}^+$. By Theorem 5.8, the representation $L(\mathbf{\Psi}_R)$ is a simple tensor product of positive prefundamental representations.

Conjecture 7.15. *We have the relation in Frac*$(K_0(\mathcal{O}^+))$ *:*

$$\chi_\ell(L(\mathbf{\Psi})) = \chi_\ell^{\leq R}(L(m)) \prod_{(i,r)\in W_R} \ell_{i,q^{r+d_i}}^{u_{i,r}} = \chi_\ell^{\leq R}(L(m))\chi_\ell(L(\mathbf{\Psi}_R)).$$

Remark 7.16. By taking the q-character, the statement is equivalent to the following q-character formula :

$$\chi_q(L(\mathbf{\Psi})) = \chi_q^{\leq R}(L(m))\chi_q(L(\mathbf{\Psi}_R)).$$

In some cases the conjecture is already proved :

Example 7.17. (i) In the case $\chi_q^{\leq R}(L(m)) = \chi_q(L(m))$, we have $u_{i,r} = 0$ for $(i,r) \in W_R$. Here the conjecture reduces to the generalized Baxter relations of Theorem 5.5.

(ii) In the case $R = r + d_i$ and $m = Y_{i,q^{r-d_i}}$, we have $\chi_q(L(m))^{\geq R} = m(1 + A^{-1}_{i,q^r})$. The conjecture reduces to the relations we have established in Formula (6.14).

(iii) As discussed above, it follows from Theorem 7.12 and from the main result of [Q], that for ADE-types, all cluster monomials are classes of real simple objects. In particular for ADE-types, Conjecture 7.15 holds for all simple modules $L(m)$ which are cluster monomials (for the cluster algebra structure defined in [HL3]). Indeed we may assume that $R = 0$. It is proved in [HL3] that for any dominant monomial m, the truncated q-character $\chi_q^{\leq 0}(L(m))$ is an element of the cluster algebra $\mathcal{A}(G^-)$ defined in the proof of Proposition 6.1. In $\chi_q^{\leq 0}(L(m))$ we perform the same substitution as in Theorem 5.5 above (that is we apply the ring homomorphism F of the proof of Proposition 6.1). If we assume that $\chi_q^{\leq 0}(L(m))$ is a cluster variable of $\mathcal{A}(G^-)$, then it follows from the proof of Proposition 6.1 that

$$y := F\left(\chi_q^{\leq 0}(L(m))\right) \prod_{(i,r) \in W_0} z_{i,q^{r+d_i}}^{u_{i,r}}$$

is a cluster variable in $\mathcal{A}(H^-)$. Using Theorem 7.12, we deduce that y is the ℓ-character of a simple module. Since F is multiplicative, the argument readily extends to simple modules $L(m)$ such that $\chi_q^{\leq 0}(L(m))$ is a cluster monomial.

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