Introduction to Cartan geometry

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1 Introduction

Cartan geometries are a solution to the very general question: *what is a geometric structure?* Riemannian geometry, conformal geometry and projective geometry are examples of geometric situations.

The mindset is the following. A Cartan geometry should first be a manifold with an homogenous space attached to each point. For instance in Riemannian geometry each point has an attached Euclidean space by equipping the tangent space with the

Riemannian metric. This data is then equipped with a Cartan connection explaining how the homogeneous spaces are infinitesimally *connected*.

When one has two different Cartan geometries, one can ask if they are equivalent. For instance, when are two Riemannian manifold isometric or at least locally isometric? This is a deep question known under the general name of the *equivalence problem*. In Riemannian geometry, the differential system $g = \sum \mathrm{d}x_i^2$ asks wether the space is locally euclidean. It is the case if, and only if, a curvature tensor vanishes. Cartan geometries give a similar procedure for all the geometries: a curvature tensor vanishes if, and only if, the space is locally homogeneous.

But when the curvature is not zero, the equivalence problem is harder to solve. What is the meaning of two curvature on two different spaces being equal? Cartan's method for the equivalence problem is a general procedure to study and solve this problem in many situations. An important example is given by the class of the symmetric spaces: those are the Riemannian spaces that are not flat but have a parallel curvature tensor. With Cartan's method one can verify when two spaces with this property are locally equivalent or not.

In this course, we will describe Cartan geometries and introduce the local equivalence problem between geometric structures. The main global problem we will deal with is the classification of smooth Anosov flows on a compact three manifold and, more generally, of non-compact automorphisms groups acting on a compact manifold preserving a contact distribution and two transverse lines contained in the contact plane at each point of the manifold.

2 Frobenius theorem, Pfaff equations and Cartan's method

A distribution is a subbundle of the tangent bundle. We describe in this section criteria in order to obtain submanifolds tangent to a distribution. The main result is Frobenius theorem which is the foundation for all these integration criteria.

2.1 Frobenius theorem

A reference for this section is [**Wa**; **Sp**]. The basic theorem which is the foundation of the theory is the existence of a local flow defined by a vector field. It is a natural generalization of the following example.

Example 1 With mild regularity conditions (for instance C^1) a vector field on the real line can be locally integrated. Let $X = \alpha(x)\frac{\partial}{\partial x}$ such a vector field. A solution of the Cauchy problem $f_t' = X(f_t)$ with initial condition $f_0 = x$ has a maximal solution defined on an open interval (a_x, b_x) where a_x or b_x could be infinite. For instance, the field

$$X = x^2\frac{\partial}{\partial x}$$

can be integrated to $f_t = \frac{x}{1-tx}$. If $x = 0$ the solution is defined on the whole real line. Otherwise solutions are defined on intervals defined by $tx \neq 1$.

Theorem 2.1 (Local flow)**.** *Let X be a C^1 vector field on a manifold M. There exists an open set $A = \{\ (t,x) \mid a_x < t < b_x\ \} \subset \mathbf{R} \times M$ and a function $\phi : A \to M$ (we write $\phi(t,x) = \phi_t(x)$), such that:*

1. $\phi_0 = \mathrm{id}$ *(so, in particular, $a_x < 0 < b_x$),*
2. $\phi_t(x)$*, for $t \in (a_x, b_x)$, is the maximal solution of the equation $\frac{\mathrm{d}\phi_t(x)}{\mathrm{d}t} = X(\phi_t(x))$ with the initial condition $\phi_0(x) = x$.*

We will also use the time-dependent version of the local flow. That is, for a vector field $X_t(x)$ which depends on time defined on an open subset $\Omega \subset \mathbf{R} \times M$ there exists a local solution $\phi_t(t_0, x_0)$ to the equation

$$\frac{\mathrm{d}\phi_t(t_0, x_0)}{\mathrm{d}t} = X_t(\phi_t(t_0, x_0)) \qquad (1)$$

with initial condition $\phi_0(t_0, x_0) = x_0$.

The flow box theorem gives a local normal form for a vector field (we say we linearize the vector field in local coordinates) on a manifold:

Theorem 2.2 (Flow box)**.** *Let X be a C^1 vector field on a manifold M. For each $x \in M$ there exists an open set $U \subset M$ containing x and a chart $\phi : U \to \mathbf{R}^n = \{(x_1, \cdots, x_n) | x_i \in \mathbf{R}\}$ such that $\phi(x) = 0$ and $\phi_*(X) = \frac{\partial}{\partial x_1}$.*

Proof. Observe first that the flow lines do not intersect. The idea then is to follow the flow starting from a hypersurface transverse to the vector field at the point x. The time will be the first coordinate of an adapted chart.

One can always choose a chart $\psi : V \to \psi(V)$ on a neighborhood V of x so that $\psi(x) = 0$ and $\psi_*(X(x)) = \frac{\partial}{\partial x_1}$. Consider the hypersurface containing x defined by $\psi^{-1}((0, x_2, \cdots, x_n))$ with $(0, x_2, \cdots, x_n) \in \psi(V)$. The existence of the flow implies that for a relatively compact $U \subset V$, there exists $\varepsilon > 0$ such that the flow is defined on $(-\varepsilon, \varepsilon) \times U$. Define then $\sigma(x_1, x_2, \cdots, x_n) = \phi_{x_1}(\psi^{-1}((0, x_2, \cdots, x_n)))$, the flow at time x_1 starting at the point $\psi^{-1}((0, x_2, \cdots, x_n))$. On a perhaps smaller neighborhood one can invert σ to obtain a chart satisfying the condition of the theorem. Indeed:

$$\begin{aligned}
\sigma_*\left(\frac{\partial}{\partial x_1}(x_1, \cdots x_n)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\phi_{x_1}(\psi^{-1}(0, x_2, \cdots, x_n)) && (2)\\
&= X(\phi_{x_1}(\psi^{-1}(0, x_2, \cdots, x_n))) && (3)\\
&= X \circ \sigma(x_1, \ldots, x_n). && (4)
\end{aligned}$$

□

Should be an exercise:

In two real dimensions, one can improve the flow box theorem to obtain that two given vector fields can be normalized to be along coordinates of a chart:

Proposition 2.3. *Let X_1 and X_2 be C^1 vector fields on a two dimensional manifold M which are linearly independent at every point. For each $x \in M$ there exists an open set $U \subset M$ containing x and a chart $\phi : U \to \mathbf{R}^2 = \{(x_1, x_2) | x_i \in \mathbf{R}\}$ such that $\phi(x) = 0$ and $\phi_*(X_1) \in \langle\frac{\partial}{\partial x_1}\rangle$ and $\phi_*(X_2) \in \langle\frac{\partial}{\partial x_2}\rangle$.*

Proof. We may suppose that there is a chart $\psi : V \to \psi(V)$ on a neighborhood V of x so that $\psi(x) = 0$ and $\psi_*(X_1(x)) = \frac{\partial}{\partial x_1}$ and $\psi_*(X_2(x)) = \frac{\partial}{\partial x_2}$. The proof of the previous theorem shows that there exists a neighborhood U of x such that each point $y \in U$ is in a unique integral line of X_1 passing through a point $\psi^{-1}(0, x_2(y))$ and a unique integral line of X_2 passing through a point $\psi^{-1}(x_1(y), 0)$. The map $\phi : U \to \mathbf{R}^2$ defined by $y \to (x_1(y), x_2(y))$ is C^1 with $\mathrm{d}\phi(x) = \mathrm{id}$. This defines a coordinate chart in perhaps a smaller neighborhood. □

Distributions on a manifold, that is, subbundles of the tangent bundle are examples of geometric structures. In the following, for simplicity, we assume that a distribution $D \subset TM$ is of constant rank.

Definition 2.4. *Let D be a distribution on a manifold M. We say that a submanifold $\phi : N \to M$ is an integral manifold of D if* $\mathrm{d}\phi(T_x N) \subset D(\phi(x))$ *for all* $x \in N$.

An important problem is to give conditions so that the dimension of the integral manifold coincides with the rank of the distribution. If this is the case, any vector field contained in the distribution will be tangent to an integral manifold and therefore the Lie bracket of any two vector fields contained in the distribution will also be contained in the distribution.

Definition 2.5. *We say a distribution D generated by vector fields $\{X_1, \cdots, X_n\}$ defined on an open set U of a manifold is involutive if for all i and j, $[X_i, X_j]$ is a vector field in the distribution.*

We can state now the main theorem of this section.

Theorem 2.6 (Frobenius)**.** *Let M be an m-dimensional manifold and D a C^1 distribution of rank n. Then D is involutive if and only if for every $x \in M$ there exists a coordinate chart $(x_1, \cdots, x_m)$ such that D is generated by $\frac{\partial}{\partial x_i}$, for $1 \le i \le n$.*

Proof. The case $n = 1$ is precisely the content of the flow-box theorem. The idea of the proof for $n > 1$ is to linearize one of the generating vector fields around x and then chose a hyperplane transversal to this field at x to obtain a distribution of rank $n-1$ on it and then use induction.

Let us start with generating vector fields $(\frac{\partial}{\partial x_1}, X_2, \cdots, X_n)$ where we linearized the first field in a coordinate system $(x_1, y_2, \cdots y_m)$ which we can suppose to be centred at 0. Here, in order to simplify notations we write $\frac{\partial}{\partial x_1}$ for the vector field on the manifold defined by the corresponding vector field in the chart. The distribution D induces a distribution D' of rank $n-1$ on the codimension one submanifold N passing through 0 defined by $x_1 = 0$: the distribution D' is generated by

$$X_i' = X_i - X_i(x_1)\frac{\partial}{\partial x_1} \tag{5}$$

for $2 \le i \le n$. Indeed, these vectors are tangent to the transverse submanifold because $X_i'(x_1) = 0$. One proves that this distribution is an involutive distribution (exercise). Here, for simplicity, we suppose that $n = 2$ and therefore the induced distribution is generated by a vector field in N. Using the flow-box theorem again, there exists a neighborhood of 0 in N with coordinates $(w_2, \cdots, w_m)$ such that $X_2' = \frac{\partial}{\partial w_2}$. We claim the adapted coordinates on a neighborhood of 0 in M are

$$(x_1, \cdots, x_m) = (x_1, w_2 \circ \pi, \cdots, w_m \circ \pi) \tag{6}$$

where π is the projection to N along the orbits of $\frac{\partial}{\partial x_1}$ (in coordinates we have $\pi(x_1, y_2, \cdots y_m) = (y_2, \cdots y_m)$). First observe that, for $i > 1$, $X_2'(x_i) = X_2'(w_i \circ \pi(x_1, y_2, \cdots y_m)) = X_2'(w_i(y_2, \cdots y_m))$ and therefore by definition of the coordinate chart in N, at points in N we have $X_2'(x_i) = 0$ for $i > 2$ along N. We need to show that $X_2'(x_i) = 0$, for $i > 2$, at all points in a whole neighborhood of the origin. For that sake we compute

$$\frac{\partial}{\partial x_1} X_2'(x_i) = X_2' \frac{\partial x_i}{\partial x_1} + \left[\frac{\partial}{\partial x_1}, X_2'\right](x_i) \tag{7}$$

which, because the distribution is involutive, can be written as

$$\frac{\partial}{\partial x_1} X_2'(x_i) = X_2' \frac{\partial x_i}{\partial x_1} + a_1 \frac{\partial}{\partial x_1}(x_i) + a_2 X_2'(x_i), \tag{8}$$

for two functions a_1 and a_2. The first two terms in the right side are clearly null. We obtain then the differential equation

$$\frac{\partial}{\partial x_1} X_2'(x_i) = a_2 X_2'(x_i). \tag{9}$$

For each $i > 2$, this is a first order ordinary differential equation with initial condition $X_2'(x_i) = 0$ at a point $(0, x_2, \cdots x_m)$. By unicity, $X_2'(x_i) = 0$ for all $(x_1, x_2, \cdots x_m)$ in a neighborhood of the origin. $\square$

Remark 2.7. *We proved that a distribution is involutive if and only if for each $y \in M$ there exists an integral manifold of maximal dimension equal to the rank of the distribution passing through y. In local coordinates defined by Frobenius theorem the integral manifolds are given locally by $(x_1, \cdots, x_n) \to (x_1, \cdots, x_n, x_{n+1}^0, \cdots x_m^0)$, where x_i^0, for $n < i \le m$, are constants. In fact, one can prove that there exists a unique maximal connected integral manifold passing through y (see [***Wa***]).*

2.2 Differential ideals and the equivalence problem

2.2.1 Differential ideals and Frobenius theorem

The formulation of Frobenius theorem using differential forms makes computations simpler. For this reason we introduce introduce in this section the notion of differential ideals which will correspond to involutive distributions. Remark, indeed, that if α is a 1-form that annihilates a distribution then, since

$$\mathrm{d}\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]), \tag{10}$$

dα vanishes on the distribution if, and only if, $[X, Y]$ also belongs to the distribution.

Let M be an n-dimensional manifold and $\Omega^*(M)$ be the set of smooth sections of the space $\Lambda T^* M$, the graded algebra of the exterior powers of the cotangent bundle. The space $\Omega^*(M)$ is the space of all the differential forms of M.

Definition 2.8. *An algebraic ideal $I \subset \Omega^*(M)$ is an ideal for the exterior algebra.*

Definition 2.9. *A differential ideal (we will denote it by EDI) $I \subset \Omega^*(M)$ is an homogeneous ideal for the exterior algebra which is closed under exterior derivative.*

Here, homogeneous ideal means that if $\alpha \in I$ and $\alpha = \alpha^0 + \cdots + \alpha^p$ is its decomposition with $\alpha^i \in \Omega^i(M)$ for $0 \le i \le p$ then $\alpha^i \in I$ for all i.

The (algebraic) ideal in $\Omega^*(M)$ generated by a 1-form θ is given by all multiples of this form by functions on the manifold. The differential ideal generated by a 1-form θ consists of all combinations of θ and dθ. Ideals of this type are studied in Pfaff's problem.

A simple case is the ideal generated by a unique closed form. A particular local description of this ideal, which is simply all multiples of the closed form, is obtained invoking Poincaré's lemma.

Lemma 2.10. *For any closed $(p+1)$-form α there exits locally a p-form β such that*

$$\alpha = \mathrm{d}\beta. \tag{11}$$

Definition 2.11. *Let I be a differential ideal. An integral submanifold is an immersion $\phi\colon N \to M$ such that $\phi^*\omega = 0$ for any $\omega \in I$.*

The most natural example of ideals in $\Omega^*(M)$ arises as the ideal I_D of forms which annihilate a distribution D.

There is a correspondence between the distribution D and the ideal I_D. If the distribution is given by k fields, we chose a coordinate system such that at a fixed point the 1-forms $\mathrm{d}x^1, \cdots, \mathrm{d}x^k$ restricted to the distribution are independent, that is, they are dual to a basis of the distribution at that point. They will then be clearly independent on a neighborhood. One can write, restricted to the distribution, for $k+1 \le j \le n$, $\mathrm{d}x^j = \sum_{i=1}^k c_i^j \,\mathrm{d}x^i$. Therefore one gets $(n-k)$ independent forms $\mathrm{d}x^j - \sum_{i=1}^k c_i^j \,\mathrm{d}x^i$ vanishing on the distribution.

The ideal I_D is a differential ideal if, and only, if the distribution is involutive and Frobenius theorem is stated in this language as the following.[1]

[1]See F. Warner, *Foundations of differentiable manifolds and Lie groups*.

Theorem 2.12 (Frobenius)**.** *Let I be a differential ideal locally (algebraically) generated by $(n-p)$ independent 1-forms. Then, for each $x \in M$, there exists a unique maximal (of dimension p) connected integral manifold of I passing through x.*

In fact, it suffices that the 1-forms in the statement be of regularity C^1.

Example 1 If the algebraic ideal is generated by a single 1-form θ, then being a differential ideal means that $\mathrm{d}\theta = \theta \wedge \omega$, for ω a 1-form. (Hence $\mathrm{d}\theta \wedge \theta = 0$.)

Exercise Prove that if $\theta(x) \neq 0$ and $\theta \wedge \mathrm{d}\theta = 0$ then, at a neighborhood of x, there exists a 1-form such that $\mathrm{d}\theta = \theta \wedge \alpha$.

Example 2 If the ideal is generated by the 1-form $\mathrm{d}y - p\,\mathrm{d}x$ and $\mathrm{d}p - F(x, y, p)\,\mathrm{d}x$ in $\mathbf{R}^3$ we obtain one dimensional integral submanifolds which correspond to solutions of a second order differential equation.

Example 3 A partial differential equation of the form

$$F\left(x_i, u, \frac{\partial u}{\partial x_i}\right) = 0 \tag{12}$$

with $1 \leq i \leq n$ and with certain regularity conditions, can be translated into the problem of finding integral submanifolds to the ideal generated by $\mathrm{d}u - p_i\,\mathrm{d}x_i$ restricted to the submanifold defined by the function $F(x_i, u, p_i) = 0$ in $\mathbf{R}^{2n+1}$.

Exercise Consider $M = \mathbf{R}^n \times \mathbf{R}^m$ with coordinates $(x, y) = (x_1, \cdots, x_n, y_1, \cdots, y_m)$. For each fixed $y = (y_1, \cdots, y_m) \in \mathbf{R}^m$, let $\iota : \mathbf{R}^n \to \mathbf{R}^m$ be the canonical embedding. Let ω^i be 1-forms on M such that $\iota^*\omega^i$ algebraically generate a differential ideal for each fixed $y \in \mathbf{R}^m$. Then the (algebraic) ideal generated by ω^i and $\mathrm{d}y_j$ for $1 \leq j \leq m$ also is a differential ideal.

2.2.2 Characteristic distributions

Frobenius theorem for differential ideal, as in 2.12, says that if a differential ideal is algebraically generated by $(n-p)$ independent one forms then one can find a local coordinate system $(x^1, \cdots, x^p, y^1, \cdots, y^{n-p})$ such that it is locally generated by $dy^1, \cdots, dy^{n-p}$. The distribution defined locally by $y^i = constant$ is the Frobenius distribution which is, at each point, dual to the subspace in the cotangent space defined by these forms.

To a differential ideal I one may associate a distribution D_I, called the characteristic distribution (definition 2.13). If it is of constant rank then one proves it is an involutive distribution. We call $\mathscr{C}_I$, the Cartan system, its dual space in the cotangent space at each point. It turns out (retraction theorem) that the exterior algebra of the Cartan system contains generators of the differential ideal and one can write generators of the exterior differential system using forms on the algebraic ideal generated by $\mathscr{C}_I$. This allows us to reduce the number of variables used in the description of the system analogous to the case in Frobenius theorem. Indeed, theorem 2.18 establishes that there exists generators of the ideal I which only depend on the the y-coordinates associated to the characteristic distribution.

Definition 2.13. *Let I be a differential ideal. The characteristic distribution is defined by*

$$D_I(x) = \{v \in \mathrm{T}_x M \,|\, \iota_v I_x \subset I_x\}\,. \tag{13}$$

We say that the differential ideal is non-singular if the distribution is of constant rank.

Here I_x is the ideal in $\Lambda^*_x M$ obtained by evaluating all elements of I at x.

Example In the particular case of a differential ideal generated by only one closed 2-form α the characteristic distribution is given by

$$D = \{v \in \mathrm{T}M \,|\, \iota_v \alpha = 0\}\,. \tag{14}$$

For instance, in $\mathbf{R}^n$, the ideal generated by the two form $\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots + \mathrm{d}x^{2p-1} \wedge \mathrm{d}x^{2p}$ has characteristic distribution of dimension $n-2p$ generated by the vectors $\frac{\partial}{\partial x^i}$, $2p+1 \leq i \leq n$.

More generally, one defines the rank of a 2-form α to be the number p satisfying $\alpha^p \neq 0$ and $\alpha^{p+1} = 0$. If the rank is constant the differential ideal generated by a closed two form is non-singular of dimension $n-2p$. This can be seen using a normal form of the 2-form as above at each point.

We can now state Cartan's result on the integrability of the characteristic distributions.

Lemma 2.14. *The characteristic distribution of a non-singular differential ideal is an involutive distribution.*

Proof. Let I be a differential ideal. From Cartan's formula $L_X = \mathrm{d}\circ\iota_X + \iota_X \circ d$ we obtain that if X is characteristic then $L_X I \subset I$. Suppose now that X and Y are two characteristic vector fields. From the formula (see equation 47 and Proposition 3.10, pg. 35, in [**KoN**])

$$L_X \iota_Y - \iota_Y L_X = \iota_{[X,Y]}\,, \tag{15}$$

we that if X and Y are characteristic then $[X,Y]$ is characteristic. □

Definition 2.15. *The annihilator of the characteristic distribution D_I,*

$$\mathscr{C}_I(x) = \{\theta \in T_x^* M \mid \theta(v) = 0 \ v \in D_I\}$$

is called the Cartan system of I.

The Cartan system describes, at each point, the smallest subspace of $T_x^* M$ whose exterior algebra contains generators of the ideal $I(x)$. Although we state it for differential systems, it is a purely algebraic result valid in the context of an ideal contained in an exterior algebra.

Theorem 2.16 (retraction theorem)**.** *Let I be a non-singular EDI on a manifold M and $\mathscr{C}_I$ its Cartan system. Then there exists a set of elements of $\Lambda^* \mathscr{C}_I$ which algebraically generate I. Moreover, $\mathscr{C}_I$ is the smallest involutive differential system satisfying this property.*

Proof.

1. Suppose $J \subset T_x^* M$ is a subspace such that $\Lambda^* J$ contains generators of $I(x)$. Then we claim that $\mathscr{C}_I(x) \subset J$. Indeed, if $v \in J^\perp$ and $\alpha \in I(x)$ then one can write $\alpha = \sum \alpha_i \wedge \beta_i$ with $\beta_i \in \Lambda^* J$ generating $I(x)$. Therefore $\iota_v \alpha = \sum \iota_v \alpha_i \wedge \beta_i \in I(x)$ so that $v \in D(x)$ so $J^\perp \subset D_I(x)$ and we conclude that $\mathscr{C}_I \subset J$.

2. Suppose that $\Lambda^* \mathscr{C}_I(x)$ does not contains all generators of $I(x)$ and let $\theta \in I(x)$ be a minimal degree element which is not in $\Lambda^* \mathscr{C}_I(x)$. If $v \in D_I(x)$, then $\iota_v \theta \in I(x)$ is of lower degree and therefore it belongs to the ideal generated by $\mathscr{C}_I(x)$.

   Let $(e_i)_{1 \le i \le p}$ be a basis of $D(x)$ and complete it to a full basis of $T_x M$. Let (θ^i) be the dual basis of $T_x^* M$. Observe that the form $\theta - \theta^1 \wedge \iota_{e_1} \theta$, if written in the basis (θ^i), does not have θ^1 in its development[2]. One can eliminate, in this way, all terms containing θ^i for $1 \le i \le p$. We obtain then a form in $\Lambda^* \mathscr{C}_I(x)$ which shows that θ itself is in this space.

□

Example Consider the differential ideal generated by a 1-form θ. The characteristic distribution is given by

$$D = \{v \in \mathrm{T}M \mid \iota_v \theta = 0, \iota_v \, \mathrm{d}\theta \in \langle \theta \rangle\}. \tag{16}$$

For instance, in $\mathbf{R}^n$, the ideal generated by the 1-form $\mathrm{d}x^{2p+1} + x^{p+1}\,\mathrm{d}x^1 + \cdots + x^{2p}\,\mathrm{d}x^p$ has characteristic distribution of dimension $n - 2p - 1$ generated by the vectors $\frac{\partial}{\partial x^i}$,

[2] For instance, if $\theta = \theta^1 \wedge \theta^2 + \theta^2 \wedge \theta^3$ then $\theta^1 \wedge \iota_{e_1} \theta = \theta^1 \wedge \theta^2$

$2p+2 \leq i \leq n$. Indeed, observe that in this case, if $\iota_v\theta = 0$, $\iota_v \mathrm{d}\theta \in \langle\theta\rangle$ if and only if $\iota_v \mathrm{d}\theta = 0$.

Suppose that θ does not vanish on a neighborhood of a point. The condition, for v such that $\iota_v\theta = 0$, $\iota_v \mathrm{d}\theta \in < \theta >$ is then equivalent to $\iota_v \mathrm{d}\theta \wedge \theta = 0$.

More generally we have the following.

Lemma 2.17. *Let θ be a 1-form. Suppose that there exists a number p satisfying $\mathrm{d}\theta^p \wedge \theta \neq 0$ and $\mathrm{d}\theta^{p+1} \wedge \theta = 0$ at each point. Then the differential ideal generated by the 1-form is non-singular with the characteristic distribution of dimension $n-2p-1$.*

Remark that in the case $p = 0$, the differential system is involutive and the characteristic distribution coincides with the distribution $\ker\theta$.

In order to prove the claim, first observe that $\mathrm{d}\theta^p \wedge \theta \neq 0$ implies that the ideal generated by θ and $\mathrm{d}\theta$ has characteristic variety of dimension less than or equal to $n-2p-1$, because $2p+1$ is the degree of a non-vanishing form in the ideal (which has generators in the annihilator of the characteristic distribution by the retraction theorem).

Consider now the map $I : \ker\theta \to T^*_x M / < \theta >$ given by the composition of $\iota : \ker\theta \to T^*_x M$ (defined by $\iota(v) = \iota_v \mathrm{d}\theta$) and the projection $T^*_x M \to T^*_x M/\langle\theta\rangle$. Then, by definition, the characteristic distribution coincides with $D = \ker I$. Also, clearly we have the identity $\dim\ker I + \dim \mathrm{Im}\, I = n-1$ as θ is non-vanishing. Now, observe that $0 = \iota_v(\mathrm{d}\theta^{p+1} \wedge \theta) = (p+1)\iota_v \mathrm{d}\theta \wedge \mathrm{d}\theta^p \wedge \theta$. Considering the product $\iota_v \mathrm{d}\theta \wedge \mathrm{d}\theta^p$ in the exterior algebra of $T^*_x M / < \theta >$ we therefore obtain that $\dim \mathrm{Im}\, I \leq 2p$ as $\mathrm{d}\theta^p$ is non-vanishing of degree $2p$ in the exterior algebra of $T^*_x M / < \theta >$. This implies that $\dim\ker I \geq n-1-2p$ and we conclude that $\dim D = n-2p-1$.

The generalization of Frobenius theorem to EDI is obtained in the following theorem. It identifies the Cartan system associated to an ideal as giving the relevant coordinates of the EDI. It shows that the retraction theorem can be implemented with generators which depend only on a set of variables transverse to the foliation by the characteristic distribution.

Theorem 2.18. *Suppose I is a EDI such that $\mathscr{C}_I$ is of constant dimension $n-p$. There exists local coordinates $(x^1, \cdots, x^p, y^1, \cdots, y^{n-p})$ such that $dy^1, \cdots, dy^{n-p}$ generate $\mathscr{C}_I$ such that I is generated by forms which depend only on the variables y^i, $1 \leq i \leq n-p$.*

Proof. 1. Frobenius theorem guarantees a coordinate system $(x^1, \cdots, x^p, y^1, \cdots, y^{n-p})$ such that, at each point, $\mathscr{C}_I(x)$ has the basis $(dy^i)_{1\leq i\leq n-p}$. The retraction theorem implies that there are generators of I in $\Lambda^*\mathscr{C}_I$. We have to make sure that one can choose generators of I which depend only on the y-coordinates and do not have any dependence on the x-coordinates.

2. Suppose that I is not the full exterior algebra $\Lambda^* M$. We may then write a decomposition of the ideal in homogeneous components, starting with degree one forms, $I = I^1 + I^2 + \cdots$. Let (ϕ^k) be a basis of 1-forms of I^1. Our goal is to find a basis (ϕ'^k) such that $L_{\frac{\partial}{\partial x^j}} \phi'^k = 0$, for $1 \le j \le p$ and all k. From $\iota_{\frac{\partial}{\partial x^j}} \phi^k = 0$ we have, for a fixed j,
$$L_{\frac{\partial}{\partial x^j}} \phi^k = \iota_{\frac{\partial}{\partial x^j}} d\phi^k = \sum_l a^k_{jl} \phi^l,$$
for functions $a^k_{jl}(x, y)$. A new basis is defined by
$$\phi^k = \sum_r z^k_r \phi'^r$$
where the functions z^k_r form a basis (indexed by r) of the space of functions satisfying the first order ordinary differential system (here j is fixed)
$$\frac{\partial z^k}{\partial x^j} = \sum_l a^k_{jl} z^l.$$
Indeed, we compute $\sum_l a^k_{jl} \phi^l =$
$$L_{\frac{\partial}{\partial x^j}} \phi^k = \sum_r \frac{\partial z^k_r}{\partial x^j} \phi'^r + \sum_r z^k_r L_{\frac{\partial}{\partial x^j}} \phi'^r = \sum_r \sum_l a^k_{jl} z^l_r \phi'^r + \sum_r z^k_r L_{\frac{\partial}{\partial x^j}} \phi'^r$$
$$= \sum_l a^k_{jl} \phi^l + \sum_r z^k_r L_{\frac{\partial}{\partial x^j}} \phi'^r.$$
Therefore $\sum_r z^k_r L_{\frac{\partial}{\partial x^j}} \phi'^r = 0$ and the result follows as (z^k_r) is invertible. By the same argument applied consecutively to each $\frac{\partial}{\partial x^j}$ we finally obtain a new basis satisfying $L_{\frac{\partial}{\partial x^j}} \phi^k = 0$ for all j.

3. Now suppose each I^r, for $r < q$, has generators defined with the y-variables. Consider a basis (ψ^k) n of I^q modulo the ideal J^{q-1} generated by all I^r, for $r < q$. From $\iota_{\frac{\partial}{\partial x^j}} \psi^k \in J^{q-1}$ we have, modulo J^{q-1},
$$L_{\frac{\partial}{\partial x^j}} \psi^k \equiv \iota_{\frac{\partial}{\partial x^j}} d\psi^k \equiv \sum_l b^k_{jl} \psi^l.$$

By the same argument as in the case of 1-forms we obtain finally a basis (ψ'^k) which satisfies

$$L_{\frac{\partial}{\partial x^j}} \psi'^k \equiv 0$$

modulo J^{q-1}. This implies that

$$L_{\frac{\partial}{\partial x^j}} \psi'^k = \sum_k \eta_j^k \wedge \omega_j^k,$$

with $\eta_j^k \in J^{q-1}$ forms which depend only on the y-variables.

We define now forms θ_j^k such that $L_{\frac{\partial}{\partial x^j}} \theta_j^k = \omega_j^k$ (see proposition 2.31). Then $\psi''^k = \psi'^k - \sum_k \eta_j^k \wedge \theta_j^k$ satisfy $L_{\frac{\partial}{\partial x^j}} \psi''^k_j = 0$.

We repeat the same argument for each $\frac{\partial}{\partial x^j}$ to obtain then a basis which depends only on the y-coordinates.

□

2.2.3 The equivalence problem

The equivalence problem in its simplest form is the following. Let M_1 and M_2 be manifolds of the same dimension n and $\{\omega_1^i\}$ and $\{\omega_2^i\}$ be coframe sections, that is, n independent 1-forms (at every point of the manifold). Does there exist a diffeomorphism

$$\psi\colon M_1 \to M_2 \text{ such that } \psi^* \omega_2^i = \omega_1^i\,? \tag{17}$$

To answer that question Cartan used the graph method. The idea is to find the map ψ by its graph in $M_1 \times M_2$. The graph is obtained as an integral submanifold of a differential ideal. In the following theorem we might have manifolds M_1 and M_2 of different dimensions.

Theorem 2.19. *Let M_1 and M_2 be manifolds and π_1, π_2 be the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. Let $(\omega_2^i)_{1\le i\le n}$ be a basis of 1-forms of M_2 and $(\omega_1^i)_{1\le i\le n}$ be a family of forms M_1 respectively. If the ideal of forms on $M_1 \times M_2$ generated by*

$$\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i) \tag{18}$$

is a differential ideal then, for each pair $(x, y) \in M_1 \times M_2$, there exists a map $\phi : U \to M_2$, defined on a neighborhood of x, such that $\phi(x) = y$ and

$$\phi^*(\omega_2^i) = \omega_1^i. \tag{19}$$

Proof. The generating 1-forms are linearly independent because ω_2^i are linearly independent. By Frobenius theorem, there exists a unique maximal submanifold G of dimension n containing a point $(x,y) \in M_1 \times M_2$ which is an integral submanifold of the differential ideal.

We show now that the submanifold is locally a graph. Consider a vector $(v_1, v_2) \in \mathrm{T}G \subset \mathrm{T}M_1 \times \mathrm{T}M_2$. If $(\pi_1)_*(v_1, v_2) = 0$ then $v_1 = 0$ and therefore

$$\pi_1^*(\omega_1^i)(v_1, v_2) = \omega_1^i((\pi_1)_*(v_1, v_2)) = 0 \tag{20}$$

which implies (because G is an integral submanifold of the ideal) that $\pi_2^*\omega_2^i(v_1, v_2) = 0$. We conclude that $v_2 = 0$. Therefore $\mathrm{T}_{(x,y)}G$ is isomorphic to $\mathrm{T}_{m_1}M_1$ and π_1 is a local diffeomorphism.

Let $F\colon U \to G$ be a local inverse of π_1. We have that $F(m) = (m, \phi(m))$ for a certain function $\phi\colon U \to M_2$(that is $\phi = \pi_2 \circ F$). Moreover, as $\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i) = 0$ on G, we obtain $F^*(\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i)) = 0$ and therefore $\omega_1^i = \phi^*(\omega_2^i)$. □

Remark In the theorem, if $(\omega_1^i)_{1\le i\le n}$ generates T^*M_1 then ϕ is an immersion. If furthermore the dimension of M_1 is n then the map ϕ is a local diffeomorphism.

Example As a first example we show how Poincaré's lemma can be proved using Theorem 2.19 . We let θ be a closed form defined on M_1 and $\mathrm{d}x$ be the canonical form on $M_2 = \mathbf{R}$. Then $\pi_1^*(\theta) - \pi_2^*(\mathrm{d}x)$ generates a differential system. Therefore for any $(x,y) \in M_1 \times \mathbf{R}$, there exists a map $\phi : U \to \mathbf{R}$, defined on a neighborhood of x, such that $\phi(x) = y$ and $\phi^*(\mathrm{d}x) = \theta$. That is $\theta = \mathrm{d}\phi$.

Example One special case occurs if we suppose that the coframes in M_1 and M_2 (which we suppose of the same dimension) both verify the same differential equation with constant coefficients:

$$\mathrm{d}\omega^i = c^i_{jk}\omega^j \wedge \omega^k, \tag{21}$$

with c^i_{jk} constant numbers shared by both M_1 and M_2. Here we use Einstein convention of sum of repeated indices. In order to show that the coframes are equivalent we verify that the algebraic ideal generated by $\pi_1^*\omega_1^i - \pi_2^*\omega_2^i$ is a differential ideal:

$$\begin{aligned}
\mathrm{d}\Big(\pi_1^*\omega_1^i - \pi_2^*\omega_2^i\Big) &= \pi_1^*(\mathrm{d}\omega_1^i) - \pi_2^*(\mathrm{d}\omega_2^i) && (22)\\
&= \pi_1^*\left(c^i_{jk}\omega_1^j \wedge \omega_1^k\right) - \pi_2^*\left(c^i_{jk}\omega_2^j \wedge \omega_2^k\right) && (23)\\
&= c^i_{jk}\left(\pi_1^*(\omega_1^j \wedge \omega_1^k) - \pi_2^*(\omega_2^j \wedge \omega_2^k)\right) && (24)\\
&= c^i_{jk}\left(\left(\pi_1^*\omega_1^j - \pi_2^*\omega_2^j\right) \wedge \pi_1^*\omega_1^k - \pi_2^*\omega_2^j \wedge \left(\pi_2^*\omega_2^k - \pi_1^*\omega_1^k\right)\right) && (25)
\end{aligned}$$

so that the ideal is differential and M_1 and M_2 are hence locally equivalent.

The case of Lie groups is particularly important. With any left-invariant frame (X_i) and its coframe (ω^i) we get structure constants c^i_{jk} verifying the preceding condition:

$$\mathrm{d}\omega^i = c^i_{jk}\omega^j \wedge \omega^k. \tag{26}$$

A basis of 1-forms (ω^i) on a manifold M is called a parallelism of M. An automorphism of a parallelism (ω^i) defined over a manifold M is a diffeomorphism $\phi\colon M \to M$ such that $\phi^*\omega^i = \omega^i$. From unicity in the theorem above we obtain the following corollary.

Corollary 2.20. *Any automorphism of a parallelism with a fixed point is the identity.*

Observe that an automorphism of a parallelism is an isometry of the manifold equipped with the Riemannian metric defined by imposing that the coframe (ω^i) is orthonormal.

A parallelism on M defined by a coframe (ω^i) can also be described by a map $\omega : \mathrm{T}M \to \mathbf{R}^n$ which is an isomorphism restricted to the tangent space at any point. We note then (M,ω) a manifold equipped with an $\mathbf{R}^n$-valued 1-form defining a parallelism. One can define a ω-constant vector field associated to $X \in \mathbf{R}^n$ as the vector field on M $\tilde{X}(x) = \omega^{-1}(X)$. For each sufficiently small $X \in \mathbf{R}^n$ we define an exponential map

$$\exp(x,X) = \phi_1(x), \tag{27}$$

where $\phi_1(x)$ is the flow of $\tilde{X}$ computed at the time 1. The differential of the exponential map at the origin is the identity and therefore at each point $x \in M$, $exp(x,\cdot) : U \to M$ is a diffeomorphism between a neighborhood of the origin and its image.

Exercise Let $g_k \in \mathrm{Aut}(M,\omega)$ be a sequence of automorphisms of M equipped with a parallelism $\omega : TM \to \mathbf{R}^n$ such that there exists $x \in M$ such that $g_k(x)$ converges. Then g_k converges to an automorphism in the compact-open topology.

Definition 2.21. *A Killing field of (M,ω) is a vector field X on M such that its flow consists of elements of the automorphism group.*

The definition is equivalent to the condition that $L_X\omega^i = 0$ for all i.

Let $\omega : TM \to \mathbf{R}^n$ define a parallelism on an n-dimensional manifold M. Let (X_i) be a dual basis corresponding to ω. The definition of a Killing field X is equivalent to $L_X X_i = [X, X_i] = 0$ for all i. The set of Killing fields $Kill(M,\omega)$ is a Lie subalgebra of the the algebra of vector fields. Observe that it is of dimension less than $n = \dim M$.

Proposition 2.22. $\dim Kill(M,\omega) \leq \dim M$

Proof. Indeed, fix a reference point $p \in M$. For each Killing field X consider the vector $X(p) \in T_pM$. We show that this map is injective and this implies the proposition. But if $X(p) = Y(p)$ then the difference $X - Y$ generates automorphisms which have a fixed point at p and therefore should be trivial. □

The subalgebra $\mathfrak{a} \subset Kill(M,\omega)$ generated by the fields whose flows are globally defined is shown to be the Lie algebra of $Aut(M,\omega)$.

Theorem 2.23. *Let A be a group acting on M by diffeomorphisms and let $\mathfrak{a}$ be the set of vector fields whose flows are globally defined in A . If $\mathfrak{a}$ generates a finite dimensional Lie subalgebra of the Lie algebra of vector fields then A is a Lie group with Lie algebra $\mathfrak{a}$.*

Proof. Note $\exp tX$ the flow generated by a vector field $X \in \mathfrak{a}$. First prove that the Lie algebra $\mathfrak{a}^*$ generated by $\mathfrak{a}$ is equal to $\mathfrak{a}$ (we still don't know $\mathfrak{a}$ is a vector space). Indeed, consider the simply connected Lie group A^* whose Lie algebra is $\mathfrak{a}^*$ (denote by $e^Y \in A^*$ the element defined by $Y \in A^*$). If $X, Y \subset \mathfrak{a}$, then the flow defined by the composition

$$\exp X \exp tY \exp -X,$$

is defined for all t. The corresponding element in the Lie algebra of $\mathfrak{a}^*$, $Ad_{e^X} Y$, belongs then to $\mathfrak{a}$ for all $X, Y \in \mathfrak{a}$. Taking the differential of $Ad_{e^{tX}} Y$ at the origin, this implies that $ad_{\mathfrak{a}}\mathfrak{a} \subset \mathfrak{a}$. It follows that the vector space generated by $\mathfrak{a}$ is $\mathfrak{a}^*$. It remains to show that $\mathfrak{a}$ is itself a vector space. For that sake, consider a set of generators of $\mathfrak{a}^*$ contained in $\mathfrak{a}$, $\{X_1, \cdots, X_n\}$ and the map

$$a_1X_1 + \cdots + a_nX_n \to e^{a_1X_1} \cdots e^{a_nX_n} \in A^*.$$

This is a local diffeomorphism defined on a neighborhood of the zero vector on $\mathfrak{a}^*$. Therefore, for each $Y \in \mathfrak{a}^*$ one can write, for sufficiently small t,

$$\exp tY = \exp a_1(t)X_1 \cdots \exp a_n(t)X_n,$$

where $a_i(t)$ are unique. Now, for any t one can obtain

$$\exp tY = (\exp \frac{t}{n} Y)^n = (\exp a_1(t/n)X_1 \cdots \exp a_n(t/n)X_n)^n,$$

which is well defined for n sufficiently large.

We might suppose therefore that $\mathfrak{a}$ is a Lie algebra and that A^* is the simply connected Lie group (of smooth flows) generated by $\mathfrak{a}$. Clearly, A^* is a normal subgroup of A as the conjugation of a flow is also a flow. This implies that A has a unique structure of a Lie group such that A^* is its identity component (see section I.3 in Kobayashi). □

2.2.4 Pfaff problem and Darboux normal form

Consider a differential ideal on a manifold generated by a 1-form, say θ. One is interested in giving a normal form for θ by choosing appropriate coordinates. Pfaff's problem is the problem of finding integral manifolds of a system $\theta = 0$ where θ is a 1-form. Here one can multiply the 1-form by a nowhere zero function and the solutions will be the same. In other terms, one is interested in finding a coordinate chart where the form has a simple normal form up to a scalar function. The classification of normal forms is simpler if we impose a constant rank condition on $\mathrm{d}\theta$.

Let θ be a 1-form. Recall, from 2.17, that if there exists a number p satisfying $\mathrm{d}\theta^p \wedge \theta \neq 0$ and $\mathrm{d}\theta^{p+1} \wedge \theta = 0$ at each point, then the differential ideal generated by the 1-form is non-singular with the characteristic distribution of dimension $n-2p-1$. From theorem 2.18 one can find local coordinates $(x^1, \cdots, x^{n-2p-1}, y^1, \cdots, y^{2p+1})$ such that θ depends only on the variables y^i, $1 \leq i \leq 2p+1$. Another way to say this is that $\theta = \pi^*(\omega)$ for the projection in the y-coordinates $\pi : \mathbf{R}^n \to \mathbf{R}^{2p+1}$ with $\mathrm{d}\omega^p \wedge \omega \neq 0$ (that is, ω is a contact form).

In order to find a normal form for θ it is sufficient to find a normal form for a contact form. This is the content of Darboux's theorem. We give a proof which uses Moser's trick.

Theorem 2.24. *Suppose ω is a contact form on a neighborhood of the origin in $\mathbf{R}^{2p+1}$. Locally, there exists coordinates such that*

$$\omega = \mathrm{d}y^{2p+1} + y^1\,\mathrm{d}y^2 + \cdots + y^{2p-1}\,\mathrm{d}y^{2p}.$$

Observe then that the Pfaff form θ will have the same normal form. The idea of the proof is to obtain a local isotopy ψ_t fixing the origin for all $t \in [0,1]$ such that ψ_1^* maps ω to the normal form.

Proof. Let ω_0 to be the normal form $\mathrm{d}y^{2p+1} + y^1\,\mathrm{d}y^2 + \cdots + y^{2p-1}\,\mathrm{d}y^{2p}$ and define

$$\omega_t = (1-t)\omega_0 + t\omega. \tag{28}$$

First, without loss of generality, adjust the coordinates y^i, using linear algebra, so that $\mathrm{d}y^{2p+1}$ is equal to ω and $\mathrm{d}\omega = \mathrm{d}\omega_0$ at the origin. With these conditions, it is clear that, for all t, ω_t is contact in a small neighborhood of the origin.

We define the isotopy as the flow defined by a time-dependent vector field $v_t = h_t R_t + y_t$ to be determined, where y_t is horizontal with respect to ω_t, that is $\omega_t(y_t) = 0$ and R_t is the Reeb vector field of the contact form ω_t. We impose that this isotopy satisfies

$$\psi_t^* \omega_t = \omega_0$$

for all $t \in [0,1]$. By the lemma 2.4

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^* \omega_t\right) = \psi_t^* \left(\dot{\omega}_t + \iota(v_t)\,\mathrm{d}\omega_t + \mathrm{d}\iota(v_t)\omega_t\right). \tag{29}$$

The equation is satisfied if and only if

$$\dot{\omega}_t + \iota(v_t)\,\mathrm{d}\omega_t + \mathrm{d}\iota(v_t)\omega_t = 0. \tag{30}$$

Evaluating at R_t we obtain

$$\dot{\omega}_t(R_t) + dh_t(R_t) = 0. \tag{31}$$

Here, for every fixed t we have an equation $R_t(h_t) = -\dot{\omega}_t(R_t)$, which can be solved on a small neighborhood for a function h_t. We want $v_t = 0$ at the origin so that the origin is fixed. Note that for every t, $\dot{\omega}_t = 0$ at the origin. We may impose then the condition $h_t = 0$ and $dh_t = 0$ at the origin for all t.

Now equation 30 determines the horizontal component y_t. Indeed, combined with equation 31 it implies that $\iota(y_t)\,\mathrm{d}\omega_t = -(\dot{\omega}_t + \mathrm{d}\iota(R_t)\omega_t)$ and this equation can be solved for y_t because $\mathrm{d}\omega_t$ is a non degenerate bilinear form restricted to the contact distribution $\ker\omega_t$. The fact that the flow fixes the the origin for all $t \in [0,1]$ implies that the flow is well defined on a small neighborhood of the origin for all $t \in [0,1]$. This concludes the proof. □

An immediate consequence of this result is the normal form for symplectic forms.

Theorem 2.25. *Le Ω be a closed two form of constant rank p. Then there exists local coordinates such that*

$$\Omega = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \cdots + \mathrm{d}x^p \wedge \mathrm{d}y^p. \tag{32}$$

Proof. By Poincaré's theorem one can write locally $\Omega = \mathrm{d}\theta$. We apply then the previous theorem to θ and differentiate back. □

2.3 Global problems: contact structures

Let M be a closed manifold and let ξ be a contact distribution. Darboux's theorem says that there are no local invariants of that structure. The only invariants of such a structure a global. We will prove in this section a global Darboux's theorem giving a normal form of a contact structure along a compact submanifold. Next, we show that any deformation of the contact structure is equivalent to itself. This is a rigidity theorem of contact structures and shows that different contact structures on a given manifold are far apart. Also, we give a description of vector fields whose flows are automorphisms of the contact structure on the manifold. They are in correspondence with functions on the manifold. This description shows that the group of diffeomorphisms preserving a contact structure is infinite dimensional.

Two manifolds equipped with contact structures are called contactomorphic if there exists a diffeomorphism between them which sends one distribution to the other. Let ψ_t be an isotopy (a differentiable family of diffeomorphisms with $\psi_0 = \mathrm{id}$) of a manifold M and let X_t be the time-dependent vector field on M defined by $X_t \circ \psi_t = \dot{\psi}(t)$. That means that ψ_t is the flow of X_t.

The fundamental theorem for global results is the completeness theorem of flows on a compact manifold:

Theorem 2.26. *On a closed manifold the flow of a vector field (time-dependent or not) exists for all times.*

The following theorem contains, as a special case, Darboux's local form theorem for contact structures.

Theorem 2.27 (Local structure around a compact)**.** *Let M be a manifold and $N \subset M$ a smooth compact submanifold. Suppose ξ_0 and ξ_1 are (co-oriented) contact structures on M which coincide on N (or more generaly $\xi_0 \cap TN = \xi_1 \cap TN$). Then there exists a neighborhood of N and an isotopy ψ_t defined over that neighborhood such that $\psi_0 = \mathrm{id}$ and $\psi_1(\xi_0) = \xi_1$ with $\psi_{t|_N} = \mathrm{id}$.*

Proof. The proof follows the same strategy of that of Darboux's theorem. Suppose ξ_0 and ξ_1 are given by the 1-forms α_0 and α_1 respectively which we assume to coincide on N. A weaker condition is that $\alpha_{0|_{TN}} = \alpha_{1|_{TN}}$. Define the 1-form

$$\alpha_t = (1-t)\alpha_0 + t\alpha_1 \tag{33}$$

which is clearly contact in a neighborhood of N by compactness. Moreover, at every point of N, $\alpha_t = \alpha_0$ when restricted to TN.

Define the isotopy as the flow defined by the time-dependent vector field $v_t = h_t R_t + y_t$ where R_t is the Reeb field and y_t is horizontal with respect to α_t, that is $\alpha_t(y_t) = 0$.

We need

$$\psi_t^* \alpha_t = f_t \alpha_0 \tag{34}$$

for all $t \in [0,1]$, where f_t is a strictly positive function. As in the proof of the local Darboux theorem, we use the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^* \alpha_t\right) = \psi_t^* \left(\dot{\alpha}_t + \iota(v_t)\,\mathrm{d}\alpha_t + \mathrm{d}\iota(v_t)\alpha_t\right). \tag{35}$$

Equation 34 is satisfied if and only if

$$\dot{\alpha}_t + \iota(v_t)\,\mathrm{d}\alpha_t + \mathrm{d}\iota(v_t)\alpha_t = \frac{\dot{f}_t}{f_t}\circ\psi_t^{-1}.\alpha_t. \tag{36}$$

Evaluating at R_t we obtain

$$\dot{\alpha}_t(R_t) + dh_t(R_t) = \frac{\dot{f}_t}{f_t}\circ\psi_t^{-1} = \mu_t. \tag{37}$$

For a given function h_t, μ_t is determined and by the previous equation $\mathrm{d}\iota(v_t)\alpha_t$ is determined which in turn determines y_t.

We want $v_t = 0$ on N in order that the isotopy preserves the form along the submanifold N. For that sake we impose the condition

$$\dot{\alpha}_t + dh_t = 0 \tag{38}$$

along N. As $\dot{\alpha}_{t|_{TN}} = 0$, for all t, we can also impose $h_t = 0$ on N and that condition is compatible with the previous equation. □

Theorem 2.28 (Gray)**.** *Let ξ_t be a smooth family of contact structures on a closed manifold. Then there exists an isotopy ψ_t such that $\psi_0 = \mathrm{id}$ and $\psi_1(\xi_0) = \xi_1$.*

Proof. Let α_t be a smooth family of forms corresponding to ξ_t. We need to find a family of diffeomorphisms ψ_t such that $\psi_t^*\alpha_t = f_t\alpha_0$. Let v_t be the vector field generating the isotopy. By Lemma 2.4, this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^*\alpha_t\right) = \dot{f}_t\alpha_0 = \frac{\dot{f}_t}{f_t}\psi_t^*\alpha_t = \psi_t^*\left(\dot{\alpha}_t + \iota(v_t)\,\mathrm{d}\alpha_t + \mathrm{d}\iota(v_t)\alpha_t\right). \tag{39}$$

So that a necessary and sufficient condition for the existence of the isotopy is that

$$\dot{\alpha}_t + \iota(v_t)\,\mathrm{d}\alpha_t + \mathrm{d}\iota(v_t)\alpha_t = \frac{\dot{f}_t}{f_t}\circ\psi_t^{-1}.\alpha_t \tag{40}$$

We impose that v_t is horizontal, that is, $\alpha_t(v_t) = 0$. We obtain the condition

$$\dot{\alpha}_t + \iota(v_t)\,\mathrm{d}\alpha_t = \frac{\dot{f}_t}{f_t}\circ\psi_t^{-1}.\alpha_t. \tag{41}$$

If R_t is the Reeb vector field for α_t we have

$$\dot{\alpha}_t(R_t) = \frac{\dot{f}_t}{f_t}\circ\psi_t^{-1}, \tag{42}$$

Therefore the function $\frac{\dot{f}_t}{f_t} \circ \psi_t^{-1}$ is determined by the family α_t. Going back to equation 41 the vector v_t is determined as the form $\mathrm{d}\alpha_t$, restricted to the distribution, is non-degenerate. As the manifold is closed the vector field v_t can be integrated to obtain an isotopy ψ_t. □

A family of automorphisms ψ_t with $\psi_0 = Id$ of a fixed contact structure ξ defines a vector field $\dot{\psi}_{|_0}$ which is called an infinitesimal automorphism. In order to determine the infinitesimal automorphisms, observe that we need to impose

$$\psi_t^* \alpha = f_t \alpha,$$

for the flow ψ_t of the infinitesimal automorphism v, where α is a form whose kernel is ξ. Again, by Lemma 2.4, this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^* \alpha\right) = \dot{f}_t \alpha = \frac{\dot{f}_t}{f_t}\psi_t^* \alpha = \psi_t^* \left(\iota(v)\,\mathrm{d}\alpha_t + \mathrm{d}\iota(v)\alpha_t\right). \tag{43}$$

A necessary and sufficient condition for the existence of the flow is that

$$\iota(v_t)\,\mathrm{d}\alpha + \mathrm{d}\iota(v)\alpha = \frac{\dot{f}_t}{f_t} \circ \psi_t^{-1}.\alpha. \tag{44}$$

Write $v = \lambda R + v_\xi$ where λR is the component of the vector field in the Reeb direction R and v_ξ is in the distribution. Then

$$\iota(v_\xi)\,\mathrm{d}\alpha + d\lambda = \frac{\dot{f}_t}{f_t} \circ \psi_t^{-1}.\alpha.$$

The condition that v is an infinitesimal automorphism is therefore that, restricted to ξ,

$$d\lambda = -\iota(v_\xi)\,\mathrm{d}\alpha.$$

As $\mathrm{d}\alpha$ is non-degenerate when restricted to ξ we obtained the following description of the Lie algebra of infinitesimal automorphisms.

Theorem 2.29. *Let ξ be a contact structure on a manifold. Then to any $\lambda \in C^\infty(M)$ one associates an infinitesimal automorphism $v = \lambda R + v_\xi$ where v_ξ is defined as the horizontal vector field satisfying $d\lambda_{|_\xi} = -\iota(v_\xi)\,\mathrm{d}\alpha$. This map is a bijection.*

Observe that infinitesimal automorphisms vanishing at a point give rise to a group of contactomorphisms with a fixed point x_0. In fact this group is infinite dimensional. It suffices to observe that it corresponds to functions λ which vanish at that point and such that $d\lambda_{|_\xi}(x_0) = 0$. Compare this with what happens in Riemannian geometry, for instance, where the group of isometries fixing a point is at most $O(n)$ where n is the dimension of the manifold.

2.4 Formulae of exterior differentiation

We recall some definitions and formulae used in these notes.

Definition 2.30. *Let X be a vector field and ω a form defined on a manifold M. Define the Lie derivative as*

$$L_X\omega = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega\bigg|_{t=0}$$

Here ψ_t is the flow defined by X.

The Lie derivative is a derivation of degree 0, that is, $L_X(\omega\wedge\alpha) = (L_X\omega)\wedge\alpha+\omega\wedge(L_X\alpha)$ for any forms ω and α.

Proposition 2.31. *Let X be a vector field and α a form defined on a manifold. Locally, one can find a form ω such that $L_X\omega = \alpha$.*

Proof. In local coordinates $(x^1,\cdots,x^n)$, given by the flow box theorem 2.2, we write $X = \frac{\partial}{\partial x^1}$. Write the forms in the appropriate basis obtained using the generators dx^i, the equation $L_X\omega = \alpha$ decomposes into differential equations in the variable x^1

$$\frac{d\omega_I}{dx^1} = \alpha_I$$

corresponding to each coefficient ω_I and α_I of the forms in the given basis. □

Definition 2.32. *Let X be a vector. The inner product $\iota_X : \Omega^i \to \Omega^{i-1}$, $i > 0$, on the exterior algebra is defined as a derivation satisfying*

- $\iota_X\omega = \omega(X)$ *for ω a 1-form.*
- $\iota_X(\omega\wedge\alpha) = (\iota_X\omega)\wedge\alpha+(-1)^{\deg\omega}\omega\wedge(\iota_X\alpha)$ *for any forms ω and α.*

The following formulae are frequently used:

$$L_X\omega = \iota_X\,\mathrm{d}\omega + \mathrm{d}\iota_X\omega. \tag{45}$$

$$L_{[X,Y]} = [L_X, L_Y]. \tag{46}$$

$$[L_X,\iota_Y] = \iota_{[X,Y]}. \tag{47}$$

Lemma 2.33. *Let ω_t be a time-dependent family of differential forms on M. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^*\omega_t\right) = \psi_t^*\left(\dot\omega_t + L_{X_t}\omega_t\right). \tag{48}$$

Proof. If ω_t is a function then the formula is valid:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^*\omega_t\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_t(\psi_t)\right) = \dot{\omega}_t(\psi_t) + \omega_t(\dot{\psi}_t) = \psi_t^*\left(\dot{\omega}_t + L_{X_t}\omega_t\right). \tag{49}$$

If ω_t is a 1-form then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^*\omega_t\right) = \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_t^*\omega_t}{h} \tag{50}$$

$$= \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_{t+h}^*\omega_t + \psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h} \tag{51}$$

$$= \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_{t+h}^*\omega_t}{h} + \lim_{h\to 0}\frac{\psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h} = \psi_t^*\left(\dot{\omega}_t + L_{X_t}\omega_t\right) \tag{52}$$

□

3 Lie groups and homogenous spaces

3.1 Lie groups and Lie algebras

We start with the definition of a Lie group. General references for this section are [**Wa**; **Kn**; **Il**; **Sharpe**].

Definition 3.1. *A Lie group is a group G that is also a differential manifold and such that the operations of multiplication and inversion are smooth. That is, the maps $G \times G \to G$ and $G \to G$ given by $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth.*

Definition 3.2. *A homomorphism $H \to G$ of Lie groups is a group homomorphism which is a smooth map. The automorphism group of H is the group of bijective homomorphisms of H into H.*

Note that if we ignore continuity in the definition of homomorphisms of Lie groups one might obtain a much larger set.

To each Lie group is associated a Lie algebra which can be thought as the space of tangent vectors at the identity of the group.

Definition 3.3. *A Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is a real vector space of finite dimension equipped with a bilinear map*

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{53}$$

satisfying, for any $x, y, z \in \mathfrak{g}$ the anti-commutativity property $[x, y] = -[y, x]$ and the Jacobi identity:

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]. \tag{54}$$

Definition 3.4. *A homomorphism $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ between Lie algebras is a linear map preserving the Lie bracket, that is, $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{h}$. The automorphism group of $\mathfrak{h}$ is the group of bijective homomorphisms of $\mathfrak{h}$ into $\mathfrak{h}$.*

Let G be a Lie group. If $a \in G$ is fixed, then one can consider the translations $L_a(g) = ag$ and $R_a(g) = ga$ called left and right multiplication respectively.

Definition 3.5. *A vector field X on a Lie group G is* left invariant *if, for any $a \in G$, $(L_a)_*(X) = X$. Similarly, it is* right invariant *if $(R_a)_*(X) = X$.*

Note that this condition means $(L_a)_*(X(g)) = X(ag)$.

An important consequence of this definition is that left (or right) invariant vector fields are determined by their value at the identity of the group and the Lie bracket of two invariant vector fields is again invariant. Therefore the set of left invariant vector fields forms a Lie algebra that can be identified to the tangent space of the group at the identity.

Definition 3.6. *The Lie algebra of a Lie group G is the set*

$$\mathfrak{g} = \left\{X \in C^{\infty}(\mathrm{T}G) \,\middle|\, \forall a \in G,\ (L_a)_*(X) = X\right\} \tag{55}$$

of left invariant vector fields on G equipped with the bilinear map given by the bracket between vector fields.

A subgroup $H \subset G$ which is a Lie group and such that the inclusion map is smooth is a called a Lie subgroup. Imposing that the inclusion is an embedding is equivalent to assuming that the subgroup is closed as a subspace of G (this result is called the closed-subgroup theorem or Cartan theorem).

The relation between Lie algebra homomorphisms and Lie group homomorphisms is described by the following Theorem. Its proof is an application of Cartan's method.

Theorem 3.7. *Let H and G be Lie groups and $\phi\colon H \to G$ a smooth homomorphism. Then $\mathrm{d}\phi_e\colon \mathfrak{h} \to \mathfrak{g}$ is a homomorphism. Conversely, if $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ is a homomorphism and H is simply connected, then there exists a unique smooth homomorphism $\phi\colon H \to G$ such that $\alpha = \mathrm{d}\phi_e$.*

Corollary 3.8. *The automorphism group of a simply connected Lie group is isomorphic to the automorphism group of its Lie algebra.*

Exercice What is the group of automorphism of $\mathbf{R}$? One has to distinguish the automorphisms of Lie group from the automorphisms of the group without the differential structure.

Examples

1. The additive group $\mathbf{R}^n$. The automorphism group coincides with linear isomorphisms of $\mathbf{R}^n$, that is to say $\mathrm{GL}(n,\mathbf{R})$. But note that the full group of group automorphisms (not necessarily continuous) of the group $\mathbf{R}^n$ contains non-linear maps.

2. The set of matrices with determinant one $\mathrm{SL}(n,\mathbf{R})$ and the usual product of matrices as group law.

3. Let G be a Lie group, $N \subset G$ be a normal subgroup and $K \subset G$ a subgroup satisfying $N \cap K = \{e\}$ and $G = NK$. (This last condition means that $g \in G$ can always be written as nk with $n \in N$ and $k \in K$.) With these conditions, we say that G is the semidirect product of K and N and write $G = N \rtimes K$. Observe that if $g_1 = n_1 k_1$ and $g_2 = n_2 k_2$ then $g_1 g_2 = n_1 (k_1 n_2 k_1^{-1}) k_1 k_2$.

An example is given by the affine linear group $\text{Aff}(\mathbf{R}^n) = \mathbf{R}^n \rtimes \text{GL}(n, \mathbf{R})$. Given an affine transformation T acting on the affine space $\mathbf{R}^n$, the choice of a base point $0 \in \mathbf{R}^n$ allows to write

$$T(x) = c + f(x) \tag{56}$$

with $c \in \mathbf{R}^n$ and $f \in \text{GL}(n, \mathbf{R})$. This decomposition is unique. Hence $\text{Aff}(\mathbf{R}^n) = \mathbf{R}^n\,\text{GL}(n, \mathbf{R})$. Note that the change of the base point from $0 \in \mathbf{R}^n$ to $\zeta \in \mathbf{R}^n$ translates to:

$$\zeta + T(x - \zeta) = \zeta + (c - f(\zeta)) + f(x) \tag{57}$$

therefore the linear part f of T is independent of the choice of the base point, but the translational part depends on it.

The composition of two transformations T_1, T_2 is given by:

$$T_1(T_2(x)) = c_1 + f_1(c_2 + f_2(x)) = (c_1 + f_1(c_2)) + f_1 f_2(x) \tag{58}$$

and it proves that $\text{Aff}(\mathbf{R}^n)$ is indeed the semidirect product $\mathbf{R}^n \rtimes \text{GL}(n, \mathbf{R})$.

Note that a convenient representation of the affine group into $\text{GL}(n+1, \mathbf{R})$ is given by

$$(c, f) \mapsto \begin{pmatrix} f & c \\ 0 & 1 \end{pmatrix}. \tag{59}$$

4. Semidirect products $G = N \rtimes K$ are in correspondance with split exact sequences

$$1 \to N \to G \to K \to 1 \tag{60}$$

and in the case of the affine group, we have indeed

$$0 \to \mathbf{R}^n \to \text{Aff}(\mathbf{R}^n) \to \text{GL}(n, \mathbf{R}) \to 1 \tag{61}$$

with the last morphism being independent of the choice of a base point and therefore is indeed restricted to the identity on $\text{GL}(n, \mathbf{R})$.

5. The three dimensional Heisenberg group Heis(3) is defined as

$$\text{Heis}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| (x, y, z) \in \mathbf{R}^3 \right\} \tag{62}$$

The group law is again the matrix product and is described by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + x' & z + z' + x \cdot y' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{pmatrix} \tag{63}$$

Another description of the same group is given by $\mathbf{C}\times\mathbf{R}$ with the group law

$$(x+\boldsymbol{i}y,z)\cdot(x'+\boldsymbol{i}y',z')=\left((x+x')+\boldsymbol{i}(y+y'),z+z'+\frac{1}{2}(xy'-yx')\right). \tag{64}$$

Both descriptions are compatible. One can start with the Lie algebra:

$$\mathfrak{heis}(3)=\left\{\begin{pmatrix}0&x&z\\0&0&y\\0&0&0\end{pmatrix}\right\}. \tag{65}$$

The exponential of an element is

$$\exp\left(\begin{pmatrix}0&x&z\\0&0&y\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}1&x&z+\frac{1}{2}xy\\0&1&y\\0&0&1\end{pmatrix}. \tag{66}$$

Therefore $\exp\colon\mathfrak{heis}(3)\to\mathrm{Heis}(3)$ is a diffeomorphism. The group law defines a group structure on the Lie algebra by taking the logarithm: for $X,Y\in\mathfrak{heis}(3)$ define

$$X\cdot Y=\log(\exp(X)\exp(Y))=X+Y+\frac{1}{2}[X,Y] \tag{67}$$

and this law on $\mathfrak{heis}(3)$:

$$\begin{pmatrix}0&x&z\\0&0&y\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&x'&z'\\0&0&y'\\0&0&0\end{pmatrix}=\begin{pmatrix}0&x+x'&z+z'+\frac{1}{2}(xy'-yx')\\0&0&y+y'\\0&0&0\end{pmatrix} \tag{68}$$

gives the second description.

In the case of the Heisenberg group (which is diffeomorphic to $\mathbf{R}^3$) one can use the group operation on the Lie algebra to determine the automorphisms.

Proposition 3.9. *The automorphism group of* Heis(3) *(described by coordinates* $(x+\boldsymbol{i}y,t)=(z,t)\in\mathbf{C}\times\mathbf{R}$*) is generated by the following transformations.*

(a) Transformations $(z,t)\mapsto(A(z),t)$ *where* $A\colon\mathbf{C}\to\mathbf{C}$ *is symplectic with respect to the form* $\mathrm{Im}(z\overline{z'})=xy'-yx'$.

(b) Dilations $(z,t)\mapsto(az,a^2t)$, *with* $a\in\mathbf{R}_+^*$.

(c) Conjugations by a translation $(a+\boldsymbol{i}b,c)\in\mathrm{Heis}(3)$*:* $(x+\boldsymbol{i}y,t)\mapsto(x+\boldsymbol{i}y,t+ay-bx)$.

(d) The inversion map $(z,t)\mapsto(\overline{z},-t)$.

Proof. We decompose an automorphism $\phi\colon \mathrm{Heis}(3)\to\mathrm{Heis}(3)$ by decomposing its derivative $\mathrm{d}\phi_e\colon \mathfrak{heis}(3)\to\mathfrak{heis}(3)$. With a linear automorphism $\mathrm{d}\phi_e$, we can write $\mathrm{d}\phi_e(x+\boldsymbol{i}y,t)=(A(x,y,t),at+bx+cy)$, where A a linear transformation and a,b,c three real numbers.

We note that an automorphism has to preserve the center of the group: if ζ is in the center, then $0=\mathrm{d}\phi_e[\zeta,\cdot]=[\mathrm{d}\phi_e\zeta,\mathrm{d}\phi_e\cdot]=[\mathrm{d}\phi_e\zeta,\cdot]$. Therefore A can not depend on t. (The center of $\mathfrak{heis}(3)$ is exactly $(0,t)$.)

From $(A(x,y),at+bx+cy)$ one can compose with the conjugation by a translation such that $\mathrm{d}\phi_e$ becomes $(A(x,y),at)$. (Choose the translation $(-c+\boldsymbol{i}b,0)$.)

Next, if a is negative then we compose with an inversion. We obtain $(A'(x,y),|a|t)$ with A' that is either A or $\overline{A}$. Then we can compose by a dilatation by $\lambda=\sqrt{|a|}^{-1}$ so that we obtain $(\lambda A'(x,y),t)$.

Now, because t is fixed, $\lambda A'$ must be a symplectic transformation of $\mathbf{C}$. □

Note Hilbert's 5th problem deals with the question of to what extent a topological group has a differential structure. This problem has many interpretations. One of the most important of them was solved by Gleason, Montgomery-Zippin and Yamabe among other contributions: *every connected locally compact topological group without small subgroups (a neighborhood of the identity does not contain a subgroup other than the trivial subgroup) is a Lie group.*

3.1.1 The Maurer-Cartan form

Given a Lie group G and its Lie algebra $\mathfrak{g}$, one might wonder how $\mathfrak{g}$ controls the full tangent space $\mathrm{T}G$. Since G is a group, we can always translate T_eG to any T_gG by doing a left translation L_g or a right translation R_g. We choose to identify any tangent space T_gG with the left translation $(L_g)_*\mathrm{T}_eG$. This identification defines a map $\mathrm{T}G\to G\times\mathfrak{g}$ which is encoded by the Maurer-Cartan form.

Definition 3.10. *The (left) Maurer-Cartan form on a Lie group G is the $\mathfrak{g}$-valued 1-form θ defined by*

$$\forall X_g\in\mathrm{T}_gG,\ \theta(X_g)=(L_g)_*^{-1}(X_g)\in\mathfrak{g}. \tag{69}$$

Note Let X be a vector field on G, then $\theta(X) = v$ is constant, if and only if, X is left-invariant and $X(g) = (L_g)_* v$. Choosing a basis of $\mathfrak{g}$ defines a parallelism of G.

Cartan's formula is also valid for vector valued 1-forms. That is, for any 1-form $\alpha : TM \to V$ with values on a vector space V, we have

$$\mathrm{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]). \tag{70}$$

Proposition 3.11 (Structural equation)**.** *For any $X, Y \in \mathrm{T}_g G$,*

$$\mathrm{d}\theta(X,Y) + [\theta(X),\theta(Y)] = 0. \tag{71}$$

Proof. We can evaluate $\mathrm{d}\theta(X,Y)$ by assuming that X, Y are extended by left-invariant vector fields X^* and Y^*. For any left-invariant vector field X^*, the image by the Maurer-Cartan form is constant on $X^*(g)$ for any $g \in G$. Therefore $X^*(\theta(Y^*))$ and $Y^*(\theta(X^*))$ are both zero. Moreover, since X^*, Y^* are left-invariant, so is $[X^*, Y^*]$ and therefore $\theta([X^*, Y^*]) = [\theta(X), \theta(Y)]$. □

Maurer-Cartan form in coordinates The choice of a basis $(e_1, \dots, e_n)$ of $\mathfrak{g}$ allows us to write $\theta = (\theta^1, \dots, \theta^n)$ by duality. With X_i the left-invariant vector field verifying $\theta(X_i) = e_i$, we can determine the *structure coefficients*:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k. \tag{72}$$

The structural equation becomes:

$$\mathrm{d}\theta^k(X,Y) = -\sum_{i<j} c_{ij}^k \theta^i \wedge \theta^j \tag{73}$$

and the Maurer Cartan form is:

$$\theta = \sum_i \theta^i e_i.$$

Note Here we use a convention which might be different in some cases (see [**KoN**] pg. 28) and is sometimes the cause of a factor of $\frac{1}{2}$ in the formula. In fact we define

$$\theta^1 \wedge \theta^2(X,Y) = \theta^1(X) \otimes \theta^2(Y) - \theta^1(Y) \otimes \theta^2(X) \tag{74}$$

in contrast with

$$\theta^1 \wedge \theta^2(X,Y) = \frac{1}{2}\left(\theta^1(X) \otimes \theta^2(Y) - \theta^1(Y) \otimes \theta^2(X)\right). \tag{75}$$

Example Consider the group $\mathrm{SO}(2) \subset \mathrm{GL}(2, \mathbf{R})$. This group is parametrized as follows:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{76}$$

In that coordinate, we obtain

$$\mathrm{d}g_\phi = \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \mathrm{d}\phi \tag{77}$$

The Lie algebra is one dimensional and is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{78}$$

The Maurer-Cartan form translates $\mathrm{d}g_\phi$ for any ϕ to $\mathrm{d}g_0$ by a left translation. Therefore it is given by

$$\theta_\phi = g(\phi)^{-1}\,\mathrm{d}g_\phi \tag{79}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \mathrm{d}\phi \tag{80}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\phi. \tag{81}$$

Matrix groups If $G \subset \mathrm{GL}(n, \mathbf{R})$ is a matrix group with Lie algebra $\mathfrak{g} \subset M_{n\times n}$ one can write the Maurer-Cartan form at $g \in G$ and it is given by $\theta_g = g^{-1}\,\mathrm{d}g$.

Here we interpret $\mathrm{d}g$ as the differential of the embedding of G into the space of matrices $M_{n\times n}$. In coordinates, if g_{ij} is the embedding, one has $\theta_g = g^{-1}_{ik}\,\mathrm{d}g_{kj}$, which is a $\mathfrak{g}$-valued 1-form.

Vector space valued forms The Maurer-Cartan form is an example of vector space valued form. We define the wedge product of a V_1-valued 1-form θ_1 and a V_2-valued 1-form θ_2 to be the $V_1 \otimes V_2$-valued form

$$\theta_1 \wedge \theta_2(X, Y) = \theta_1(X) \otimes \theta_2(Y) - \theta_1(Y) \otimes \theta_2(X). \tag{82}$$

If there exists a bilinear map $[\cdot,\cdot] : V \times V \to V$ we note the composition of $\wedge$ (for 1-forms) and $[\cdot,\cdot]$ by

$$[\theta_1 \wedge \theta_2](X, Y) := [\theta_1(X), \theta_2(Y)] - [\theta_1(Y), \theta_2(X)]. \tag{83}$$

Observe then that $[\theta(X), \theta(Y)] = \frac{1}{2}[\theta \wedge \theta](X, Y)$.

Exercice ($\mathfrak{g}$-valued n-forms) Writing, in general, θ_n for a $\mathfrak{g}$-valued n-form we may define the exterior derivative and the product of two forms accordingly. Prove the following formulae:

1. $[\theta_p \wedge \theta_q] = (-1)^{pq}[\theta_q \wedge \theta_p]$,
2. $(-1)^{pr}[[\theta_p \wedge \theta_q] \wedge \theta_r] + (-1)^{qr}[[\theta_r \wedge \theta_p] \wedge \theta_q] + (-1)^{qp}[[\theta_q \wedge \theta_r] \wedge \theta_p]$.

Moreover,

$$\mathrm{d}[\theta_p \wedge \theta_q] = [\mathrm{d}\theta_p \wedge \theta_q] + (-1)^{pq+1}[\theta_p \wedge \mathrm{d}\theta_q]. \tag{84}$$

Darboux derivatives

A Maurer-Cartan form allows the computation of Darboux derivatives.

Definition 3.12. *If $f \colon M \to G$ is smooth and if θ is the Maurer-Cartan form of G then the Darboux derivative of f is:*

$$f^*\theta = \theta \circ f_*. \tag{85}$$

Example In $\mathbf{R}^n$ the Darboux derivative is in a sense closer to the usual derivative than the differential. Indeed, recall that if $f \colon \mathbf{R}^p \to \mathbf{R}^n$ is smooth, then

$$\forall (x, v) \in \mathrm{T}\mathbf{R}^n,\ f_*(x, v) = (f(x), \mathrm{d}f_x(v)). \tag{86}$$

The maps f_* and $\mathrm{d}f$ depend on the base point. But with the Darboux derivative one identifies all tangent spaces to the tangent space at the origin:

$$f^*\theta(x, v) = \theta(f(x), \mathrm{d}f_x(v)) = {T_{-f(x)}}_*(\mathrm{d}f_x(v)) \in \mathrm{T}_0(\mathbf{R}^n) \tag{87}$$

where $T_{-f(x)}$ is the translation $T_{-f(x)}(z) = z - f(x)$.

Theorem 3.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and M a manifold. Suppose there exists a $\mathfrak{g}$-valued 1-form ϕ defined on M satisfying the Maurer-Cartan formula $d\phi + \frac{1}{2}[\phi \wedge \phi] = 0$. Then for any $m \in M$ there exists a map $f : U \to G$ defined on a neighbourhood of m such that $\phi = f^*\theta$ where θ is the Maurer-Cartan form of G. Moreover if $f' : U \to G$ is another map satisfying this condition $f' = L_h \circ f$ for a certain $h \in G$.*

Proof. We consider, in the product $M \times G$, the Lie algebra valued form

$$\omega = \pi_1^*(\phi) - \pi_2^*(\theta),$$

where π_1 and π_2 are the projections of the product on each of the factors. Let I be the ideal generated by the components ω^i_j of ω. This is a differential ideal because

$$2d\omega = 2(\pi_1^*(d\phi) - \pi_2^*(d\theta)) = -\pi_1^*([\phi\wedge\phi]) + \pi_2^*([\theta\wedge\theta])$$

$$= -[(\pi_1^*\phi - \pi_2^*\theta)\wedge\pi_1^*\phi] - [\pi_2^*\theta\wedge(\pi_1^*\phi - \pi_2^*\theta)]$$

and we invoke theorem 2.19 (p. 14) to conclude the existence of the map $f: U \to G$.

A submanifold passing through another point (m_0, hg) is clearly given by $(m, hf(m))$ and by unicity this implies that $f' = L_h \circ f$. □

The exponential map

One parameter subgroups of a group G are defined by elements of the Lie algebra. For any $X \in \mathfrak{g}$ one defines a homomorphism

$$\exp_X : \mathbf{R} \to G, \tag{88}$$

which is the unique homomorphism satisfying $\exp_X^* \theta = X$.

Definition 3.14. *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is defined by*

$$\exp(X) = \exp_X(1). \tag{89}$$

Although exp has several properties analogous to the real exponential, due to the non-commutativity, one has a more complicated formula for the product of two exponentials (it is the Baker-Campbell-Hausdorff formula which is only valid locally):

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \cdots\right). \tag{90}$$

If $\phi\colon H \to G$ is a group homomorphism one has

$$\exp\circ \mathrm{d}\phi_e = \phi\circ\exp_e. \tag{91}$$

Lemma 3.15. *Let X^* be a left-invariant vector field corresponding to an element $X \in \mathfrak{g}$. Then its flow is given as the right multiplication by the exponential map $R_{\exp(tX)}$.*

Proof. Since X^* is left-invariant, so must be its flow. Therefore the integral curve at $g \in G$ is given by $L_g \exp(tX) = R_{\exp(tX)} g$. Hence the flow is given by $R_{\exp(tX)}$. □

3.1.2 The adjoint representation

An action of a Lie group G on a manifold induces a representation of the group on the automorphism group of the tangent space of a fixed point of the action. For, let $\phi\colon G \times M \to M$ be an action with a fixed point $G \cdot p = p$ at $p \in M$. Then for every $g \in G$, define $\phi_g : M \to M$ ($\phi_g(x) = \phi(g, x)$) and then the automorphism $\rho(g) = \phi_{g*|p} : \mathrm{T}_p M \to \mathrm{T}_p M$. One verifies that the map $\rho : G \to \mathrm{Aut}(\mathrm{T}_p M)$ defined by $\rho(g) = \rho_g$ is a representation.

In particular the adjoint action $G \times G \to G$ defined by $(g, h) \mapsto ghg^{-1}$ induces the representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ (observe that $\mathrm{Aut}(\mathrm{T}_e G)$ is isomorphic to $\mathrm{GL}(n, \mathbf{R})$ with $n = \dim_{\mathbf{R}} G$). For $g \in G$, Ad_g is the automorphism

$$\mathrm{Ad}_g(X) = \mathrm{d}(h \mapsto ghg^{-1})_e(X) = (L_g)_*(R_{g^{-1}})_* X \tag{92}$$

The adjoint representation is also exactly what we need to compare the Maurer-Cartan form θ defined by left-invariance with the action by right translations.

Proposition 3.16. *For any $g \in G$, the Maurer-Cartan form θ verifies*

$$R_g^* \theta(X) = \mathrm{Ad}_g^{-1}(\theta(X)). \tag{93}$$

Proof. Assume that $X = (L_x)_* v$. By the preceding definition, we have:

$$\begin{aligned}
R_g^* \theta(X) &= \theta((R_g)_* X) &(94)\\
&= \theta((R_g)_*(L_x)_* v) &(95)\\
&= \theta((L_x)_*(R_g)_* v) &(96)\\
&= \theta((R_g)_* v) &(97)\\
&= (L_g)_*^{-1}(R_g)_* v = \mathrm{Ad}_g^{-1} v. &(98)
\end{aligned}$$

□

The differential of Ad_g at the origin $g = e$ is denoted by $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathrm{T}_e G)$:

$$\mathrm{ad}_X = \mathrm{d}\mathrm{Ad}_e(X). \tag{99}$$

It is in fact given by the bracket of the Lie algebra.

Lemma 3.17. *Let $X, Y \in \mathfrak{g} \cong \mathrm{T}_e G$. Then*

$$\mathrm{d}\mathrm{Ad}_e(X)(Y) = \mathrm{ad}_X(Y) = [X, Y]. \tag{100}$$

The adjoint automorphism by $g \in G$ fits in the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{g(\cdot)g} & G \end{array} \tag{101}$$

and the adjoint representation satisfies

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}) \end{array} \tag{102}$$

More generally, we have:

Proposition 3.18. *The differential of the representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ *at* $g \in G$ *computed at the vector* $X^* = (L_g)_* X \in \mathrm{T}_g G$ *is*

$$\mathrm{dAd}_g(X)(Y) = \mathrm{Ad}_g(\mathrm{ad}_X(Y)). \tag{103}$$

Proof. Writing a path through g as $L_g\gamma(t)$ with $\gamma(0) = e$ and $\dot\gamma(0) = X$ we have $\mathrm{Ad}_{L_g\gamma(t)}(Y) = \mathrm{Ad}_g \circ \mathrm{Ad}_{\gamma(t)}(Y)$. Therefore

$$(\mathrm{dAd}_g(X))(Y) = \left.\frac{\mathrm{dAd}_g \circ \mathrm{Ad}_{\gamma(t)}}{\mathrm{d}t}\right|_{t=0} (Y) = \mathrm{Ad}_g \circ \mathrm{ad}_X(Y). \tag{104}$$

□

Proposition 3.19. *If* θ_G *is the Maurer-Cartan form, then for any function* ψ *with values in G and any* 1*-form* α *with values in* $\mathfrak{g}$*,*

$$\mathrm{Ad}_\psi\, \psi^*\theta_G = -\psi^{-1^*}\theta_G, \tag{105}$$

$$\mathrm{d}\big(\mathrm{Ad}_\psi(\alpha)\big) = \Big[-\psi^{-1^*}\theta_G \wedge \mathrm{Ad}_\psi(\alpha)\Big] + \mathrm{Ad}_\psi\, \mathrm{d}\alpha \tag{106}$$

$$= \mathrm{Ad}_\psi\left(\left[\psi^*\theta_G \wedge \alpha\right] + \mathrm{d}\alpha\right). \tag{107}$$

3.2 Homogeneous spaces

Homogeneous spaces will be the flat model geometries. They appear naturally when there exists a transitive action. Indeed, if $G \times M \to M$ is a transitive action one can identify M with the quotient ${}^{G}\!/_{H_x}$ where H_x is the isotropy subgroup of a chosen element $x \in M$. A different choice $gx \in M$ gives rise to the isotropy $H_{gx} = gH_xg^{-1}$.

Definition 3.20. *A homogeneous space is a differential manifold obtained by the quotient of a Lie group G by a closed Lie subgroup $H \subset G$. We note the set of left cosets gH by ${}^{G}\!/_{H}$.*

The group G acts transitively on the homogeneous space ${}^{G}\!/_{H}$ by left translations, the isotropy subgroup at the identity being H.

Note If H were not closed then the quotient ${}^{G}\!/_{H}$ would not be separated with the quotient topology. In general, an immersion of a Lie group into a Lie group is called a Lie subgroup. If a subgroup of a Lie group is path-connected then it is a Lie subgroup by a theorem of Kuranishi-Yamabe. Closed subgroups are, on the other hand, embedded submanifolds.

Examples

1. *The Euclidean space.*

   The group of the isometries of the Euclidean space is $\mathrm{Eucl} = \mathbf{R}^n \rtimes \mathrm{O}(n)$. It acts on $\mathbf{R}^n$ with isotropy $\mathrm{O}(n)$. Therefore $\mathbf{R}^n = {}^{\mathrm{Eucl}}\!/_{\mathrm{O}(n)}$ as homogeneous space.

2. *The hyperbolic space.*

   Hyperbolic space is the simply connected complete constant negative sectional curvature Riemannian space. Its connected isometry group is $\mathrm{SO}(n,1)$ with isotropy $\mathrm{SO}(n)$. Here $\mathrm{SO}(n,1)$ is the group preserving the quadratic form

   $$\begin{pmatrix} \mathrm{id}_{\mathbf{R}^n} & 0 \\ 0 & -1 \end{pmatrix}. \tag{108}$$

3. *The similarity group acting on $\mathbf{R}^n$.*

   The connected similarity group is the group $\mathrm{Sim}(\mathbf{R}^n) = \mathbf{R}^n \rtimes (\mathbf{R}_+^* \times \mathrm{O}(n))$. It is a subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$. Transformations of $\mathbf{R}_+^* \times \mathrm{O}(n)$ are of the form $\lambda P(x)$ with $\lambda > 0$ and P an orthogonal transformation.

   The similarity group is the conformal group acting on $\mathbf{R}^n$. (Each conformal transformation has to be defined on the full space $\mathbf{R}^n$.) Therefore, it consists of the transformations of $\mathbf{R}^n$ which preserve angles. The isotropy at the origin is $\mathbf{R}_+^* \times \mathrm{O}(n)$.

4. *The conformal sphere.*

   There are more conformal transformations than just $\mathrm{Sim}(\mathbf{R}^n)$. But those are not defined strictly on $\mathbf{R}^n$ but rather on the one-point compactification S^n. The conformal sphere is the homogeneous space $\mathrm{PO}(n+1,1)/\mathrm{Sim}(\mathbf{R}^n)$.

5. *The projective space.*

   The projective space $\mathbf{RP}^n$ is the homogenous space $\mathrm{GL}(n+1,\mathbf{R})/H$ where

$$H = \left\{ \begin{pmatrix} \star & \star \\ 0 & A \end{pmatrix} \middle| \, A \in \mathrm{GL}(n,\mathbf{R}) \right\}. \tag{109}$$

6. *Flag spaces.*

   The projective space is an example of flag spaces. A flag is a sequence $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbf{F}^n$ for any field $\mathbf{F}$. For instance, the projective space $\mathbf{FP}^n$ is the set of lines in $\mathbf{F}^{n+1}$.

   A complete flag is a flag with $\dim V_i = i$. They are maximal in length. When $\mathbf{F} = \mathbf{C}$ we get an homogeneous space structure with the quotient

$$\left. \mathrm{SU}(n) \middle/ \mathrm{S}(\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)) \right.. \tag{110}$$

7. *Stiefel manifolds.*

   The space of orthonormal k-frames in $\mathbf{R}^n$ (with $0 < k < n$) is the Stiefel manifold $S(k,n)$. It is possible to show that

$$S(k,n) = \left. \mathrm{SO}(n) \middle/ \mathrm{SO}(n-k) \right.. \tag{111}$$

8. *Every manifold is a homogeneous space.*

   The full group of the diffeomorphisms of a manifold is not a Lie group but might be described by an analogous structure with infinite dimension.

   The easiest situation is for a compact manifold, say M. The smooth diffeomorphism group $\mathrm{Diff}^\infty(M)$ has a structure of a Fréchet Lie group which is homeomorphic to the space of smooth vector fields. The group $\mathrm{Diff}^\infty(M)$ acts transitively on M. Therefore, any manifold can be considered as a homogeneous space $\mathrm{Diff}^\infty(M)/H$, where H is the isotropy at a point in M, that is to say, the set of diffeomorphisms fixing the point. We will not deal with infinite dimension Lie groups.

3.2.1 The tangent space

With a homogeneous space G/H the tangent space can be described infinitesimally and the action of G (on the left) can be measured.

At eH, the tangent space is naturally isomorphic to $\mathfrak{g}/\mathfrak{h}$ as linear spaces. Therefore, the tangent bundle of the homogenous spaces $\mathrm{T}G/H$ can be seen as a quotient of the trivial bundle $G \times \mathfrak{g}/\mathfrak{h}$ by the right action of H:

$$(g, v) \cdot h = (gh, \mathrm{Ad}(h)^{-1} v). \tag{112}$$

We write the quotient as

$$G \times_H \mathfrak{g}/\mathfrak{h}. \tag{113}$$

Note that at the isotropy $H \subset G$, the action of $h \in H$ on a point pH is $hpH = hph^{-1}H$ and therefore H acts on $\mathrm{T}_{eH} G/H$ by $\mathrm{Ad}(h)$.

Proposition 3.21. *There exists a canonical isomorphism*

$$\mathrm{T}G/H \cong G \times_H \mathfrak{g}/\mathfrak{h}. \tag{114}$$

Proof. Let $\pi\colon G \to G/H$ be the quotient map. Let $\phi\colon G \times \mathfrak{g}/\mathfrak{h} \to \mathrm{T}G/H$ be defined by

$$\phi(g, v) = (gH, \pi_*(L_g)_* v). \tag{115}$$

We prove that this map is well defined in the quotient by the right action of H. Note that $\pi_*(R_h)_* = \pi_*$ since $\pi \circ R_h = \pi$ and $\pi_*(L_g)_* = (L_g)_*\pi_*$.

$$\begin{aligned}
\phi((g, v) \cdot h) &= \phi(gh, \mathrm{Ad}(h)^{-1} v) && (116)\\
&= (ghH, \pi_*(L_{gh})_* \mathrm{Ad}(h)^{-1} v) && (117)\\
&= (gH, (L_g)_* \pi_*(R_h)_* v) && (118)\\
&= (gH, (L_g)_* \pi_* v) = \phi(g, v) && (119)
\end{aligned}$$

We can check that this morphism is injective at every point. If $\phi(g, v) = (gH, 0)$ then $\pi_* v = 0$ and therefore $v \in \mathfrak{h}$. It is surjective by dimensionality. □

3.2.2 Effective pairs

It is important to keep track of both groups G and H and not only their quotient space. On the other hand it is reasonable to consider only connected quotients G/H.

Definition 3.22. *We will refer as a Klein geometry a pair* (G, H) *such that the homogeneous space* G/H *is connected.*

There are two conditions which one can add without much loss of generality, namely, that the action of G be effective and that G be connected.

Note that if $g \in G$ acts trivially on ${}^{G}\!/_{H}$ then $geH = eH$ and therefore $g \in H$. Let $h \in H$ be acting trivially. For any $g \in G$ and any coset pH we would have that $ghg^{-1}pH = g(h(g^{-1}pH))$ is equal to $g(g^{-1}pH)$ since h acts trivially on $g^{-1}pH$ and therefore $ghg^{-1}pH = pH$. So if h acts trivially, then ghg^{-1} does too.

Definition 3.23. *We say that a maximal subgroup $K \subset H$ which is normal in G is the kernel of a Klein geometry. The action of K is trivial and we say that the geometry is effective if $K = \{e\}$.*

If K is the maximal normal subgroup in H (the definition implies that K is a closed subgroup of G) one can consider the effective geometry $({}^{G}\!/_{K}, {}^{H}\!/_{K})$ which describes the same homogeneous space as ${}^{({}^{G}\!/_{K})}\!/_{({}^{H}\!/_{K})}$. It is diffeomorphic to ${}^{G}\!/_{H}$ with an equivariant action by ${}^{G}\!/_{K}$.

Sometimes one might consider non-effective Klein geometries. For instance, ${}^{\mathrm{SL}(2,\mathbf{R})}\!/_{\mathrm{SO}(2)}$ corresponds to the hyperbolic geometry but the subgroup $\mathbf{Z}_2 \subset \mathrm{SL}(2,\mathbf{R})$ generated by $-\mathrm{id}$ is a maximal normal subgroup contained in $\mathrm{SO}(2)$. Nonetheless, this subgroup is discrete and is does not intervene infinitesimally.

If G is not connected one can consider the connected component containing the identity $G_e \subset G$ and we obtain that ${}^{G}\!/_{H}$ is diffeomorphic to ${}^{G_e}\!/_{(H \cap G_e)}$ with an equivariant action by G_e. This follows since if ${}^{G}\!/_{H}$ is connected, one has $G = G_eH$. On the other hand, one can prove that if H is connected then G is also connected.

4 Principal bundles

Consider a smooth right free action

$$\mu\colon P \times H \to P \tag{120}$$

of a Lie group H on a manifold P. We denote R_h the right action of H:

$$\forall h \in H, \forall p \in P,\ R_h(p) = \mu(p, h). \tag{121}$$

Such an action μ is called proper if for any K_1, K_2 compact subsets of P, the set

$$\{h \in H \,|\, R_h(K_1) \cap K_2 \neq \emptyset\} \tag{122}$$

is compact.

Let M be a manifold and H a Lie group. A (right) principal bundle

$$\pi\colon P \to M \tag{123}$$

consists of a manifold P with a right action μ by H which is locally trivial: for each $x \in M$, there exists a trivialization over an open set U containing x

$$\Psi = (\pi, \psi_H)\colon \pi^{-1}(U) \to U \times H \tag{124}$$

that is a diffeomorphism and such that

$$\Psi(\mu(u, h)) = (\pi(u), \psi_H(u)h). \tag{125}$$

A characterization of right actions which gives rise to principal bundles is the following.

Proposition 4.1. *Let $\mu\colon P \times H \to P$ be a proper smooth right free action. Then ${}^{P}\!/_{H}$ is a smooth manifold with the quotient topology and it has a unique smooth structure such that the projection $P \to {}^{P}\!/_{H}$ defines a right H-principal bundle.*

Example Homogenous spaces are an important class of examples

$$\pi\colon G \times H \to {}^{G}\!/_{H} \tag{126}$$

where the right action $\mu\colon G \times H \to G$ is the Lie group law:

$$\mu(g, h) = gh. \tag{127}$$

This action is indeed proper. For if $h_i \in H$ and $K_1, K_2 \subset G$ are compact, assume that $R_{h_i} K_1 \cap K_2 \neq \emptyset$. We need to prove that h_i converge (up to a subsequence). For each i, let $k_i^1 \in K_1$ and $k_i^2 \in K_2$ such that $R_{h_i} k_i^1 = k_i^2$. But both k_i^1 and k_i^2 converge (up to a subsequence) to k_1 and k_2 respectively. Hence $h_i = (k_i^1)^{-1} k_i^2$ converge (up to a subsequence) to $k_1^{-1} k_2$. The limit lies in H since it is closed.

Definition 4.2. *Let $\pi_1 \colon P_1 \to M_1$ and $\pi_2 \colon P_2 \to M_2$ be two right H-principal bundles. A H-bundle diffeomorphism $F \colon P_1 \to P_2$ is a diffeomorphism that preserves the fibers and verifies $F \circ R_h = R_h \circ F$ (it is right equivariant).*

Since an H-bundle diffeomorphism preserves the fibers, it defines a diffeomorphism $f \colon M_1 \to M_2$. Hence, following diagram commutes.

$$\begin{tikzcd} P_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & P_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd} \tag{128}$$

4.1 Frame and coframe bundles

4.1.1 Some linear algebra

The linear group of matrices $\mathrm{GL}(n, \mathbf{R})$ does not act canonically on a vector space. Indeed, an isomorphism $\mathrm{GL}(V) \simeq \mathrm{GL}(n, \mathbf{R})$ relies on a choice of a basis of V. However, $\mathrm{GL}(n, \mathbf{R})$ does act canonically on the spaces of the frames and coframes of V. Let

$$F = \{(e_1, \dots, e_n) \text{ is an ordered basis of } V\}. \tag{129}$$

We say that F is the space of frames of V.

In order to deal with right actions on principle bundles we will consider the right action of $\mathrm{GL}(n, \mathbf{R})$ on the frame bundle F given by

$$e'_i = {g^{-1}}^j_i e_j \tag{130}$$

where (g^j_i) is a matrix $g \in \mathrm{GL}(n, \mathbf{R})$. (We assume the Einstein summation convention.)

This right action on F corresponds to a right action on F^*, the space of coframes:

$$F^* = \left\{(e^1, \dots, e^n) \text{ is an ordered basis of } V^*\right\}. \tag{131}$$

This last action is given by:

$$e^{i'} = e^j b^i_j \tag{132}$$

with (b^i_j) a matrix $b \in \mathrm{GL}(n, \mathbf{R})$. The correspondance with the action on F is determined by the relation $e^{i'}(e_j) = \delta^i_j$:

$$e^{i'}(e_j) = e^k b^i_k (e_m {g^{-1}}^m_j) = b^i_k {g^{-1}}^k_j \tag{133}$$

and the equation $b^i_k {g^{-1}}^k_j = \delta^i_j$ shows that $b = g$ in $\mathrm{GL}(n, \mathbf{R})$.

4.1.2 Bundles and the tautological form

Definition 4.3. *The frame bundle on a smooth manifold M is the set*

$$F = \left\{ v \,\middle|\, v \textit{ is a frame at a point of } \mathrm{T}_M \right\}. \tag{134}$$

And the coframe bundle is:

$$F^* = \left\{ \omega \,\middle|\, \omega \textit{ is a coframe at a point of } \mathrm{T}_M \right\}. \tag{135}$$

By the preceding considerations, we will consider each bundle F and F^* as a right principal $\mathrm{GL}(n,\mathbf{R})$-bundle.

Note A reduction of the principal bundle F and F^* to a subbundle (not necessarily principal) corresponds generally to the choice of a geometric structure on M.

Definition 4.4. *An H-structure on a smooth manifold M is a principal subbundle of F (or F^*) with fiber a closed subgroup $H \subset \mathrm{GL}(n,\mathbf{R})$.*

Examples

1. A Riemannian geometry on M, that is to say a Riemannian metric, corresponds to the choice of a subbundle of orthonormal frames or coframes.

2. A conformal geometry on M, that is to say a conformal class of Riemannian metrics, corresponds to the choice of a subbundle of frames that are orthonormal up to an homogeneous factor.

3. A contact structure on a 3-manifold M, that is to say the data of an everywhere non-integrable plane distribution $D^2 \subset \mathrm{T}M$ corresponds to the choice of a subbundle constituted of vectors (v_1, v_2, v_3) such that (v_1, v_2) generates D^2.

As a matter of fact, those three subbundles are principal. The first for the choice of $\mathrm{O}(n) \subset \mathrm{GL}(n,\mathbf{R})$, the second with $\mathbf{R}_+\mathrm{O}(n) \subset \mathrm{GL}(n,\mathbf{R})$ and the last with $P_2 \subset \mathrm{GL}(3,\mathbf{R})$ the set of the matrices:

$$P_2 = \left\{ \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix} \right\} \subset \mathrm{GL}(3,\mathbf{R}). \tag{136}$$

A coframe bundle over a manifold is a special principal bundle obtained as a subbundle of the bundle of all coframes over a manifold. They occur naturally when a geometric structure is described by fixing a set coframes.

More generally, suppose $(U_1, \{\omega_1^i\})$ and $(U_2, \{\omega_2^i\})$ are two open sets with sections of coframes and H a subgroup of $GL(n, \mathbf{R})$, the group of the coframe bundle. We think of them as two geometric structures defined on these open sets. The equivalence problem is the question of whether there exists a diffeomorphism $\psi : U_1 \to U_2$ such that $\psi^*(\omega_2^i) = \omega_1^j h_j^i$, with $(h_j^i) \in H$. Cartan's method consists of building a canonical connection on a principal bundle associated to the geometric structure.

There are two fundamental observations. The first one is that, given a section of coframes ω^j and a group H, it is natural to consider the principal bundle $\pi : P \to M$ of all coframes $\omega^j h_j^i$. This doesn't solve the problem but makes the computations more intrinsic. The second observation is that one can define n linearly independent 1-forms on P by, at the point $p = \{\omega^i\} \in P$,

$$\theta_p^i(v) = \omega_{\pi(p)}^i(\pi_*(v)).$$

These forms are called *tautological forms*. They do not form a basis of forms of P because all of them vanish on vertical vectors.

Using a local section $\{\omega^j\}$ we can trivialize the fiber bundle and write any other coframe as $p = \{\omega^j h_j^i\}$. Then $\theta_p^i(v) = \omega_{\pi(p)}^j(\pi_*(v)) h_j^i$, or in other words,

$$\theta_p^i = \pi^*(\omega_{\pi(p)}^j) h_j^i.$$

which, by abuse of notations, we write $\theta_p^i = \omega_{\pi(p)}^j h_j^i$. or, writing $\theta = (\theta^1, \cdots, \theta^n)$, $\omega = (\omega^1, \cdots, \omega^n)$ and $h = (h_j^i)$, as

$$\theta = \omega h.$$

The construction of the coframe principal H-bundle P_1 and P_2 above U_1 and U_2 allows us to consider the lifting of the equivalence problem:

Proposition 4.5. *There exists a diffeomorphism $\phi : U_1 \to U_2$ satisfying*

$$\phi^*(\omega_2^i) = \omega_1^j h_j^i$$

for a function $(h_j^i) : U_1 \to H$ if and only if there exists a diffeomorphism $\tilde{\phi} : P_1 \to P_2$ such that

$$\tilde{\phi}^*(\theta_2) = \theta_1.$$

Proof. Suppose $\phi : U_1 \to U_2$ such that $\phi^*(\omega_2) = \omega_1 h$ exists (that is $(\phi^{-1})^*(\omega_1) = \omega_2 h^{-1}$). By abuse of notation we write here h^{-1} for $h^{-1} \circ \psi^{-1}$. Define $\tilde{\phi} : P_1 \to P_2$ by

$$\tilde{\phi}(\omega_1 g) = \phi^{-1^*}(\omega_1 g) = \omega_2 h^{-1} g.$$

Then, in coordinates as above, $\tilde{\phi}^*\theta_{2_{\omega_2 h^{-1} g}} = \tilde{\phi}^*(\omega_2 h^{-1} g) = \omega_1 g = \theta_{1_{\omega_1 g}}$.

On the other hand, if $\tilde{\phi}^*(\theta_2) = \theta_1$, we show first of all that $\tilde{\phi}$ sends fibers to fibers. If X is a vertical vector tangent to a fiber of P_1 then $\theta_2(\tilde{\phi}_*(X)) = \tilde{\phi}^*(\theta_2)(X) = \theta_1(X) = 0$. This shows that $\tilde{\phi}_*(X)$ is vertical. This defines a function $\phi : U_1 \to U_2$. Now, we can obtain the result by observing that there exists a function $h : U_1 \to H$ such that $\phi^*(\omega_2) = \omega_1 h$. □

There exists a related notion of tautological form for a frame bundle $\pi\colon P \to M$. Define the *fundamental form* $\theta\colon \mathrm{T}P \to \mathbf{R}^n$ by:

$$\theta|_v(X) = (\theta^1(X), \cdots, \theta^n(X)) \tag{137}$$

where $\theta^i(X)$ are the coefficients of the vector $\pi_*(X)$ on the basis $v = (v^1, \cdots, v^n)$. That is, $\pi_*(X) = \sum \pi_*(X)_i v^i$.

A section $\sigma\colon M \to P$ corresponds to the choice of a frame at each point $x \in M$. A section σ is also called a *moving frame*. Any other moving frame α is then determined by a right translation by a function $h : M \to H$:

$$\alpha(x) = \sigma(x) h(x). \tag{138}$$

4.2 Ehresmann connections

Invariant vector fields With a right principal bundle P, one can consider a canonical vector field X^* associated to any $X \in \mathfrak{h}$:

$$X^*(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t} R_{\exp(tX)}\, p\right|_{t=0}. \tag{139}$$

An alternative definition of X^* is the following. With $\mu\colon P \times H \to P$ the right action, we have

$$X^*(p) = \mu_*|_{(p,e)}(0, X). \tag{140}$$

For instance, in the case where $P = G$ is a Lie group and $H \subset G$, we get that X^* is again the left-invariant vector field $X^*(p) = L_{p_*}(X)$.

Definition 4.6. *An Ehresmann connection ω on P is an $\mathfrak{h}$-valued 1-form satisfying:*

1. *for any $h \in H$, $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$;*
2. *for any $X \in \mathfrak{h}$, $\omega(X^*) = X$.*

This definition restricts to the Maurer-Cartan form in the case where M collapses to one point. Considering the projection $G \to G/H$ as a homogeneous space, a connection is only a part of the Maurer-Cartan form of G.

Note An equivalent formulation arises if we consider the distribution D defined by the kernel of ω. That distribution is an invariant *horizontal* distribution as $R_{h*}D = D$.

Lifting curves An Ehresmann connection defines a way to make a parallel displacement along curves of M from the fiber at the origin of the curve to the fiber at the end of the curve.

Let $\gamma\colon [0,1] \to M$ be a smooth path in M. Then there exists a unique lift $\widetilde{\gamma}\colon [0,1] \to P$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}(t) \in \ker \omega_{\widetilde{\gamma}(t)} \tag{141}$$

with an initial condition $\widetilde{\gamma}(0) = p$.

Lemma 4.7. *Both conditions $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$ and $\omega(X^*) = X$ are equivalent to the following.*

$$R_\psi^*\omega = \psi^*\theta_H + \mathrm{Ad}(\psi)^{-1}\omega, \tag{142}$$

where θ_H is the Maurer-Cartan form of H and ψ is any smooth function with values in H.

To be entirely precise, if $\psi\colon X \to H$ is a smooth map, then one defines $R_\psi\colon P \times X \to P \times H \to P$ and we state

$$R_\psi^*\omega(u,v)|_{(p,x)} = \psi^*\theta_H(v)|_x + \mathrm{Ad}(\psi(x))^{-1}\omega(u)|_p. \tag{143}$$

Proof. Since $R_\psi^*\omega$ is a differential form, we can consider separately vectors $(u,0)$ and $(0,v)$ at $(p,x) \in P \times X$. Since $R_\psi = \mu \circ (\mathrm{id} \times \psi)$ we only need to show the equivalence with:

$$\mu^*\omega(u,v)|_{(p,h)} = \theta_H(v) + \mathrm{Ad}(h)^{-1}\omega(u) \tag{144}$$

since the precomposition by $(\mathrm{id} \times \psi)_*$ would conclude the proof.

With vectors $(u,0)|_{(p,h)}$, the product $\mu_*(u,0)$ is equal to $R_{h*}(u)$. Hence the preceding formula and the first condition are equivalent.

With vectors $(0,v)|_{(p,h)}$, the product $\mu_*(0,v)$ gives exactly $V^*(\mu(p,h))$ where V^* is the invariant vector field corresponding to $\theta_H(v)$. Hence the preceding formula and the second condition are equivalent. $\square$

5 Cartan geometries

5.1 Definitions

Definition 5.1. *A Cartan geometry modeled on* $(\mathfrak{g}, \mathfrak{h})$ *is a right* H*-principal bundle* $P \to M$*, with* $Lie(H) = \mathfrak{h}$*, together with a* 1*-form* $\omega \colon \mathrm{T}P \to \mathfrak{g}$*, called a Cartan connection, verifying:*

1. *at each* $p \in P$*,* ω *is an isomorphism* $\mathrm{T}_p P \to \mathfrak{g}$*;*
2. *for all* $h \in H$*,* $R_h^* \omega = \mathrm{Ad}(h)^{-1} \omega$*;*
3. *for all* $X \in \mathfrak{h}$ *and* X^* *the corresponding invariant vector field,* $\omega(X^*) = X$*.*

Note A Cartan connection defines a parallelism of P since $\mathrm{T}_p P \simeq \mathfrak{g}$ by ω. Hence $\mathrm{T}P \simeq P \times \mathfrak{g}$.

Note There are many possible choices for models G/H with $Lie(G) = \mathfrak{g}$ and $Lie(H) = \mathfrak{h}$. Usually we deal with an effective pair (G, H), with connected groups H and G.

Definition 5.2. *The homogeneous space* G/H *is reductive if there exists a linear decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{145}$$

such that $\mathfrak{p}$ *is* $\mathrm{Ad}(H)$*-invariant.*

Example A semidirect product $N \rtimes K$ defines a reductive homogeneous space

$$N \rtimes K / K.$$

Note When a homogenous space is reductive, one can decompose a Cartan connection ω that has values in $\mathfrak{g}$ along $\mathfrak{h}$ and $\mathfrak{p}$, that is to say $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{p}}$. The factor $\omega_{\mathfrak{h}}$ is then an Ehresmann connection.

By the same proof as in the case of an Ehresmann connection (see 4.7 (p. 45)), we have:

Lemma 5.3. *The two last conditions of the definition of a Cartan connection are equivalent to:*

$$R_\psi^* \omega = \mathrm{Ad}(\psi)^{-1} \omega + \psi^* \theta_H, \tag{146}$$

where θ_H *is the Maurer-Cartan form of* H *and* $\psi : P \to H$ *is any smooth function with values in* H*.*

Definition 5.4. *The curvature of a Cartan geometry is*

$$\Omega(u,v) = \mathrm{d}\omega(u,v) + [\omega(u),\omega(v)]. \tag{147}$$

If $\Omega = 0$ *on* $\mathrm{T}P$*, then we say that the Cartan geometry is flat.*

Homogeneous spaces The simplest example of a Cartan geometry is the fiber bundle $G \to {}^{G}\!/_{H}$ equipped with its Maurer-Cartan form θ. In this case $\Omega = \mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] = 0$ is the structural equation.

Lemma 5.5. *If* $\psi : P \to H$ *is any smooth function with values in* H *then*

$$R_\psi^* \Omega = \mathrm{Ad}(\psi)^{-1}\Omega. \tag{148}$$

Proof. As proved previously, we have $R_\psi^*\omega = \mathrm{Ad}(\psi)^{-1}\omega + \psi^*\theta_H$. In the case of linear groups the formula can be written

$$R_\psi^*\omega = \psi^{-1}\omega\psi + \psi^{-1}\,\mathrm{d}\psi.$$

We compute the pull-back of the curvature. This is most easily obtained in the case of linear groups and we leave the general case as an exercise.

$$\begin{aligned}
R_\psi^*\Omega &= R_\psi^*\,(\mathrm{d}\omega + \omega \wedge \omega)\\
&= R_\psi^*\,\mathrm{d}\omega + R_\psi^*\omega \wedge R_\psi^*\omega\\
&= R_\psi^*\,\mathrm{d}\omega + (\psi^{-1}\omega\psi + \psi^{-1}\,\mathrm{d}\psi) \wedge (\psi^{-1}\omega\psi + \psi^{-1}\,\mathrm{d}\psi)\\
&= R_\psi^*\,\mathrm{d}\omega + \psi^{-1}\omega\psi \wedge \psi^{-1}\omega\psi + \psi^{-1}\omega\psi \wedge \psi^{-1}\,\mathrm{d}\psi + \psi^{-1}\,\mathrm{d}\psi \wedge \psi^{-1}\omega\psi + \psi^{-1}\,\mathrm{d}\psi \wedge \psi^{-1}\omega\psi.
\end{aligned}$$

We need a formula for $\mathrm{d}(R_\psi^*\omega)$. We obtain:

$$\begin{aligned}
\mathrm{d}(R_\psi^*\omega) &= \mathrm{d}(\psi^{-1}\omega\psi + \psi^{-1}d\psi)\\
&= -\psi^{-1}\,\mathrm{d}\psi \wedge \psi^{-1}\omega\psi + \psi^{-1}\,\mathrm{d}\omega\psi - \psi^{-1}\omega\psi \wedge \psi^{-1}\,\mathrm{d}\psi - \psi^{-1}\,\mathrm{d}\psi \wedge \psi^{-1}\,\mathrm{d}\psi.
\end{aligned}$$

Substituting the formula for $\mathrm{d}(R_\psi^*\omega)$ above we have

$$R_\psi^*\Omega = \psi^{-1}\Omega\psi = \mathrm{Ad}(\psi)^{-1}\Omega. \tag{149}$$

□

Recall that we can identify T_pH with the tangent space of the fiber at $p \in P$. Indeed the function $H \to P$ defined by $h \mapsto ph$ is a diffeomorphism for each fixed p.

Lemma 5.6. *The curvature* $\Omega(u,v)$ *vanishes if* u *or* v *are tangent to the fiber (belong to* $\mathrm{T}_pH \subset \mathrm{T}_pP$*).*

Proof. Assume that $u \in \mathrm{T}_pH$. Let $\psi\colon P \to H$ be such that $\psi(p) = e$ and $\psi_*(u) = -\omega(u)$. Then

$$R_\psi^*\omega\Big|_p(u) = \psi^*\theta_H(u) + \mathrm{Ad}(\psi(p))^{-1}\omega(u) \tag{150}$$

$$= -\omega(u) + \omega(u) = 0 \tag{151}$$

hence $\omega(R_{\psi_*}u) = 0$ implies $R_{\psi_*}u = 0$ and we get

$$\mathrm{Ad}(\psi)^{-1}\Omega(u,v) = \Omega(R_{\psi_*}u, R_{\psi_*}v) = \Omega(0, R_{\psi_*}v) = 0. \tag{152}$$

□

5.2 Bianchi identities

The derivative of the curvature gives the Bianchi identities.

Lemma 5.7. *Let* $P \to M$ *be a Cartan geometry and* ω_P *its connection. We have*

$$\mathrm{d}\Omega = [\Omega \wedge \omega_P]. \tag{153}$$

Proof. We differentiate by definition of the curvature.

$$\mathrm{d}\Omega = \mathrm{d}\left(\mathrm{d}\omega_P + \frac{1}{2}[\omega_P \wedge \omega_P]\right) \tag{154}$$

$$= \frac{1}{2}\mathrm{d}[\omega_P \wedge \omega_P] \tag{155}$$

$$= \frac{1}{2}([\mathrm{d}\omega_P \wedge \omega_P] - [\omega_P \wedge \mathrm{d}\omega_P]) \tag{156}$$

$$= [\mathrm{d}\omega_P \wedge \omega_P] \tag{157}$$

$$= \left[\left(\Omega - \frac{1}{2}[\omega_P \wedge \omega_P]\right) \wedge \omega_P\right] \tag{158}$$

$$= [\Omega \wedge \omega_P] - \frac{1}{2}[[\omega_P \wedge \omega_P] \wedge \omega_P] \tag{159}$$

$$= [\Omega \wedge \omega_P] \tag{160}$$

Indeed, $[[\omega_P \wedge \omega_P] \wedge \omega_P] = 0$ by using the Jacobi identity. □

5.3 Example 1: Riemannian geometry

As one can anticipate, a Cartan connection on a Riemannian geometry will be a certain Ehresmann connection combined with the tautological form. But as usual in Riemannian geometry, one can consider many connections. Only one will have vanishing torsion. It is that property that will determine the corresponding Ehresmann connection.

We start with a description of $\mathrm{Eucl}(n)$, the group of the isometries of the Euclidean space. This space is the model for the Riemannian geometry. By the identification $\mathrm{Eucl}(n) = \mathbf{R}^n \rtimes \mathrm{O}(n)$, one can represent $\mathrm{Eucl}(n) \to \mathrm{GL}(n+1, \mathbf{R})$ by

$$(x, A) \mapsto \begin{pmatrix} A & 0 \\ x & 1 \end{pmatrix}. \tag{161}$$

So the Lie algebra is:

$$\mathfrak{eucl}(n) = \begin{pmatrix} \mathfrak{o}(n) & 0 \\ \mathbf{R}^n & 1 \end{pmatrix}. \tag{162}$$

An important observation is that the adjoint action by an element of the orthogonal subgroup is a right translation on the $\mathbf{R}^n$ coordinate:

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Id & 0 \\ v & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Id & 0 \\ vA & 1 \end{pmatrix}. \tag{163}$$

The Maurer-Cartan form θ_{Eucl} can be written:

$$\theta_{\mathrm{Eucl}} = g^{-1} \mathrm{d}g = \begin{pmatrix} \omega & 0 \\ \theta & 0 \end{pmatrix}, \tag{164}$$

where ω and θ are respectively the Maurer-Cartan forms of $\mathrm{O}(n)$ and $\mathbf{R}^n$. And the structure equation becomes

$$\begin{pmatrix} \mathrm{d}\omega + \omega \wedge \omega & 0 \\ \mathrm{d}\theta + \theta \wedge \omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{165}$$

Given a Riemannian manifold M we obtain the $\mathrm{O}(n)$ coframe bundle P of all orthogonal basis of the dual tangent space. Part of the Cartan connection is given by the tautological form which we denote by θ. We will describe the (Levi-Civita) Cartan connection ω_P decomposed as:

$$\omega_P = \begin{pmatrix} \omega & 0 \\ \theta & 0 \end{pmatrix}. \tag{166}$$

Along any local section $\sigma\colon U \to P$, we get a trivialisation $\pi^{-1}(U) = U \times H$ (Here $H = O(n)$). A first observation is that the Maurer-Cartan form of H, $A^{-1}\,\mathrm{d}A$, can be seen as a $\mathfrak{o}(n)$-valued 1-form on P by pulling it back by the projection on the second factor. We observe then that for all $X \in \mathfrak{o}(n)$ and denoting X^* the corresponding fundamental field we have

$$A^{-1}\,\mathrm{d}A(X^*) = X.$$

Indeed, write a local section as a moving coframe $\underline{\theta} = (\underline{\theta}^i)$. A trivialisation of the coframe bundle is obtained through the map $(x, A) \to \underline{\theta} A$. We have

$$X^* = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{\theta} A e^{tX}\right)_{|t=0}.$$

Therefore $A^{-1}\,\mathrm{d}A(X^*) = A^{-1}\,\mathrm{d}A(\frac{\mathrm{d}}{\mathrm{d}t}\left(Ae^{tX}\right)_{|t=0})$. The forms $\underline{\theta}^i$ are defined on M and therefore we may, write expanding on a basis of two-forms,

$$\mathrm{d}\underline{\theta}^i = -\sum_{jk} a^i_{jk}\underline{\theta}^j \wedge \underline{\theta}^k, \tag{167}$$

with functions a^i_{jk} verifying $a^i_{jk} = -a^i_{kj}$. This can be written as

$$\mathrm{d}\underline{\theta} = -\underline{\theta} \wedge \underline{w},$$

where $\underline{w}$ is the matrix $(a^i_{jk}\underline{\theta}^k)$.

Recall that the tautological form can be written with the help of a moving coframe:

$$\theta_{\underline{\theta}A} = \pi^*\underline{\theta}A,$$

which we write, for simplicity, as $\theta = \underline{\theta}A$. Now we can differentiate the tautological form with the help of a moving frame:

$$\begin{aligned} \mathrm{d}\theta &= (\mathrm{d}\underline{\theta})A - \underline{\theta}\,\mathrm{d}A = -\underline{\theta} \wedge \underline{w}A - \underline{\theta}\,\mathrm{d}A \\ &= -\underline{\theta}A \wedge A^{-1}\underline{w}A - \underline{\theta}AA^{-1}\,\mathrm{d}A. \end{aligned}$$

We obtained therefore a 1-form $\omega = A^{-1}\underline{w}A + A^{-1}\,\mathrm{d}A$ defined on P with values in $\mathfrak{gl}(n, \mathbf{R})$ satisfying

$$d\theta + \theta \wedge \omega = 0.$$

Observe that ω satisfies, for all $X \in \mathfrak{o}(n)$

$$\omega(X^*) = X.$$

This construction depends on the choice of the section. The ambiguity in the definition of the form ω is explicited by Cartan's lemma.

Lemma 5.8. *There exists a unique skew-symmetric matrix of 1-forms ω such that*

$$d\theta + \theta \wedge \omega = 0.$$

Proof. To understand the ambiguity of ω, suppose there is another one ω' satisfying the equation. One obtains that $\theta \wedge (\omega - \omega') = 0$. Writing $\omega - \omega'$ as a matrix $c^i_{jk}\theta^k$ one obtains, using Cartan's lemma, that $c^i_{jk} = c^i_{kj}$.

The next step is to obtain the condition of skew-symmetry on ω. Given ω' satisfying the equation $d\theta + \theta \wedge \omega' = 0$ we write the equations of skew-symmetry : $a^i_{jk} + a^j_{ik} = a'^i_{jk} + a'^j_{ik} + c^i_{jk} + c^j_{ik} = 0$ and solve them for c^i_{jk}:

We have

$$c^i_{jk} + c^j_{ik} = -(a'^i_{jk} + a'^j_{ik})$$

$$-(c^j_{ki} + c^k_{ji}) = -(a'^j_{ki} + a'^k_{ji})$$

$$c^k_{ij} + c^i_{kj} = (a'^k_{ij} + a'^i_{kj})$$

Adding the three equations gives the unique values c^i_{jk} defining a skew-symmetric ω in terms of the 1-form ω'. □

Proposition 5.9. *The $eucl$-valued form on P given by*

$$\omega_P = \begin{pmatrix} \omega & 0 \\ \theta & 0 \end{pmatrix}$$

is a Cartan connection on P.

Proof. . The fact that ω_P is an isomorphism at each point of P follows from the fact that θ is a tautological form and from the condition $\omega(X^*) = X$ for $X \in \mathfrak{o}(n)$. We verify the transformation properties for a Cartan connection. For instance, consider the ω component of the $eucl$-valued form. We compute the action by an element $A \in O(n)$ on the right of the coframe bundle:

$$d(\theta A) = (d\theta)A = -\theta \wedge \omega A =$$

$$= -(\theta A) \wedge (A^{-1}\omega A),$$

which means precisely that

$$R^*_A(\omega_{\theta A}) = Ad_{A^{-1}}\omega_\theta.$$

□

The curvature of the Cartan connection is given by

$$\Omega = d\omega_P + \omega_P \wedge \omega_P = \begin{pmatrix} d\omega + \omega\wedge\omega & 0 \\ 0 & 0 \end{pmatrix},$$

so we obtain that the Cartan geometry is torsion free. Writing $W = d\omega + \omega \wedge \omega$ we may express in coordinates:

$$W^i_j = R^i_{jkl}\theta^k \wedge \theta^l,$$

with $R^i_{jkl} = -R^j_{ikl} = -R^i_{jlk}$.

The Bianchi identity is given by differentiating $\Omega = d\omega_P + \omega_P \wedge \omega_P$, that is

$$d\Omega = d\omega_P \wedge \omega_P - \omega_P \wedge d\omega_P = (\Omega - \omega_P \wedge \omega_P) \wedge \omega_P - \omega_P \wedge (\Omega - \omega_P \wedge \omega_P) = \Omega \wedge \omega_P - \omega_P \wedge \Omega.$$

Or, in matrix form:

$$d\Omega = \begin{pmatrix} dW & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega & 0 \\ \theta & 0 \end{pmatrix} - \begin{pmatrix} \omega & 0 \\ \theta & 0 \end{pmatrix}\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W\wedge\omega - \omega\wedge W & 0 \\ W\wedge\theta & 0 \end{pmatrix}.$$

Therefore $W \wedge \theta = 0$ (first Bianchi identity) and $dW = W \wedge \omega - \omega \wedge W$ (second Bianchi identity). Usually one writes, in coordinates, $W^i_j \wedge \theta^j = 0$ and $dW^i_j = \sum_k (W^i_k \wedge \omega^k_j - \omega^i_k \wedge W^k_j)$. Substituting $W^i_k = R^i_{jkl}\theta^k \wedge \theta^l$ in the first Bianchi identity one has

$$R^i_{jkl}\theta^k \wedge \theta^l \wedge \theta^j = 0$$

which implies $R^i_{jkl} + R^i_{ljk} + R^i_{klj} = 0$.

5.3.1 Gauss-Bonnet theorem

In this section we prove Gauss-Bonnet theorem for compact surfaces. In dimension two the coframe bundle P over the surface Σ is a circle bundle. We consider the tautological forms θ^1, θ^2 and the 2x2 skew symmetric connection form ω^i_j. In this case there is only one relevant form ω^1_2. The curvature form is then

$$W = \begin{pmatrix} 0 & d\omega^1_2 \\ -d\omega^1_2 & 0 \end{pmatrix}.$$

That is, in this case the curvature form on P is exact. Moreover, because the structural group is abelian, from the formula $R^*_g W = Ad_{g^{-1}} W = W$ we obtain that Ω^1_2 is the pull-back of a form defined on Σ which we usually denote by $K dv$ where $dv = \underline{\theta}^1 \wedge \underline{\theta}^2$ is the volume form associated to the metric in Σ and does not depend of the choice of coframe $\{\theta^1, \theta^2\}$.

Theorem 5.10. *Let Σ be a compact oriented surface with Euler characteristic $\chi(\Sigma)$. Fix a Riemannian metric and let $K dv$ the curvature form on M defined as above. then*

$$\frac{1}{2}\int_\Sigma K dv = \chi(\Sigma).$$

Proof. Consider a section of P over the complement of a finite number of points $F \subset M$ (this is always possible even if we chose F to have only one point). This is equivalent to a choice of a global unit vector field on the complement of F. Indeed, choosing a unit vector field fixes, for an oriented surface, an orthogonal vector field which defines a positive basis. Reciprocally, from a coframe one can define the dual basis and choose the first basis vector.

Choose small discs Δ_i around the isolated points and one computes using the section $s : \Sigma \setminus F \to P$;

$$\int_\Sigma K dv = \int_{\Sigma\setminus\bigcup_i \Delta_i} K dv + \int_{\bigcup_i \Delta_i} K dv = \int_{s(\Sigma\setminus\bigcup_i \Delta_i)} \Omega_2^1 + \int_{\bigcup_i \Delta_i} K dv.$$

Now we use the fact that Ω is exact and Stokes theorem:

$$\int_\Sigma K dv = \int_{s(\Sigma\setminus\bigcup_i \Delta_i)} d\omega_2^1 + \int_{\bigcup_i \Delta_i} K dv = \int_{s(\partial\bigcup_i \Delta_i)} \omega_2^1 + \int_{\bigcup_i \Delta_i} K dv.$$

The last integral tends to zero when the radius of the discs vanish while

$$\frac{1}{2\pi}\int_{s(\partial\bigcup_i \Delta_i)} \omega_2^1 \to \text{index of the vector field.}$$

Indeed, one computes this integral using a coframe θ'^i defined on Δ_i, that is, with no singularities. We may write then (by the formula $R_g^*\omega = Ad_{g^{-1}}\omega + g^{-1}dg$) $\omega_2^1 = \omega_2'^1 + d\phi$ where ϕ is the angle between the coframes. Remark that this angle is the same as the angle between the vector fields associated to them. In the limit, the integration of the first term disappears as the circles get smaller and the integration of the second term tends to the index of the singular vector field.

We conclude the proof by invoking Poincaré-Hopf theorem: the index of the vector field $= \chi(\Sigma)$. $\square$

5.4 Example 2: web geometry

Web geometries on $\mathbf{R}^2$ are a way to study the geometry of differential equations

$$\mathrm{d}y = F(x, y)\,\mathrm{d}x. \tag{168}$$

The geometric data correspond to the three distributions defined by the axes and the tangent lines defined by the differential equation.

Definition 5.11. *A web on* $\mathbf{R}^2$ *is the data of three line distributions* $L_1, L_2, L_3 \subset \mathrm{T}\mathbf{R}^2$ *such that any two are linearly independent at each point.*

By duality, a line corresponds to the kernel of a form: $L_1 = \ker\alpha^1$. The forms are defined up to a scalar multiple. That is, α^1 and $\lambda\alpha^1$ generate the same line L_1. Hence, by rescaling, since $\alpha^3 = \lambda\alpha^1 + \mu\alpha^2$, we can assume that $\alpha^3 = \alpha^1 - \alpha^2$.

Definition 5.12. *A coframe of a web on* $\mathbf{R}^2$ *is the data of three* 1*-forms* $\alpha^1, \alpha^2, \alpha^3 \in \mathrm{T}^*\mathbf{R}^2$ *such that any two are linearly independent at each point and* $\alpha^3 = \alpha^1 - \alpha^2$.

Observe that a coframe is in fact the data of only α^1 and α^2. So it is indeed a coframe of $\mathbf{R}^2$. The bundle of all coframes of a web on $\mathbf{R}^2$ is an $\mathbf{R}^*$-principal bundle. Indeed, if α^1 becomes $\lambda_1\alpha^1$ and α^2 becomes $\lambda_2\alpha^2$ then $\lambda_1\alpha^1 - \lambda_2\alpha^2$ must still be proportional to α^3, hence $\lambda_1 = \lambda_2$.

Flat model A flat model for this geometry is given by $\alpha^1 = \mathrm{d}y$, $\alpha^2 = \mathrm{d}x$ and the third form $\alpha^3 = \mathrm{d}y - \mathrm{d}x$ (this corresponds to the differential equation $\mathrm{d}y = \mathrm{d}x$) . We admit that the invariance group is exactly $G = \mathbf{R}^2 \rtimes \mathbf{R}^*$ where $\mathbf{R}^2$ acts by translation and the isotropy $H = \mathbf{R}^*$ by dilation. A representation $G \to \mathrm{GL}(3, \mathbf{R})$ is given by:

$$(x, y, \lambda) \mapsto \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ x & y & 1 \end{pmatrix}. \tag{169}$$

One can check that in this representation,

$$\mathrm{Ad}(\lambda^{-1})(x, y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x\lambda & y\lambda & 0 \end{pmatrix}. \tag{170}$$

The Maurer Cartan-form is

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ \theta^1 & \theta^2 & 0 \end{pmatrix}$$

whose components satisfy the equations

$$\begin{aligned} \mathrm{d}\omega &= 0, \\ \mathrm{d}\theta^1 + \theta^1 \wedge \omega &= 0, \\ \mathrm{d}\theta^2 + \theta^2 \wedge \omega &= 0. \end{aligned}$$

The torsion free Cartan connection Let $\theta = (\theta^1, \theta^2)$ be the tautological form of the coframe bundle P.

Proposition 5.13. *There exists a unique 1-form ω on P such that*

$$d\theta^1 = \omega \wedge \theta^1$$

$$d\theta^2 = \omega \wedge \theta^2$$

Moreover, the $\mathfrak{g}$ valued form

$$\varpi = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ \theta^1 & \theta^1 & 0 \end{pmatrix}$$

is a Cartan connection.

Proof. We let $\underline{\theta}^i$ be a section of the tautological forms so that the tautological forms are described by

$$\theta^i = \lambda \underline{\theta}^i.$$

We obtain then, observing that $\mathrm{d}\underline{\theta}^i$ are basic forms,

$$d\theta^i = \mathrm{d}\lambda \wedge \underline{\theta}^i + \lambda\, \mathrm{d}\underline{\theta}^i = \frac{\mathrm{d}\lambda}{\lambda} \wedge \theta^i + \tau^i \theta^1 \wedge \theta^2,$$

where τ^i are functions on P. Now we can write

$$d\theta^1 = (\frac{d\lambda}{\lambda} - \tau^1\theta^2) \wedge \theta^1,$$

$$d\theta^2 = (\frac{d\lambda}{\lambda} + \tau^2\theta^1) \wedge \theta^2.$$

Therefore $\omega = \frac{d\lambda}{\lambda} + \tau^2\theta^1 - \tau^1\theta^2$. One can also easily verify that the $\mathfrak{g}$ valued form is a Cartan connection. $\square$

The curvature of that Cartan connection is of the form

$$\Omega = d\varpi + \varpi \wedge \varpi = \begin{pmatrix} K\theta^1 \wedge \theta^2 & 0 & 0 \\ 0 & K\theta^1 \wedge \theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where K is a real function defined on P. The component $K\theta^1 \wedge \theta^2$ is known as the Blaschke-Chern curvature. Observe that $K\theta^1 \wedge \theta^2$ is well defined on $\mathbf{R}^2$.

Exercise Let T^2 be a torus equipped with a web structure. Then

$$\int_{T^2} K\theta^1 \wedge \theta^2 = 0.$$

Application Consider the equation

$$\mathrm{d}y = F(x,y)\,\mathrm{d}x \tag{171}$$

and the corresponding web

$$\alpha^1 = \mathrm{d}y, \tag{172}$$
$$\alpha^2 = F(x,y)\,\mathrm{d}x, \tag{173}$$
$$\alpha^3 = \mathrm{d}y - F(x,y)\,\mathrm{d}x = \alpha^1 - \alpha^2 \tag{174}$$

where $F(x,y)$ does not vanish.

By following the method, we differentiate (α^1, α^2). It gives:

$$\mathrm{d}\alpha^1 = 0 \tag{175}$$
$$\mathrm{d}\alpha^2 = \frac{\partial F}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x \tag{176}$$
$$= \frac{1}{F}\frac{\partial F}{\partial y}\,\mathrm{d}y \wedge \alpha^2 \tag{177}$$

and it determines a connection form ω (actually a pull back by the section of the coframe bundle) verifying $\mathrm{d}\alpha = \omega \wedge \alpha$:

$$\omega = \frac{1}{F}\frac{\partial F}{\partial y}\,\mathrm{d}y. \tag{178}$$

Hence, the Blaschke-Chern curvature is:

$$\mathrm{d}\omega = \frac{\partial}{\partial x}\left(\frac{1}{F}\frac{\partial F}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \tag{179}$$
$$= \frac{1}{F}\left(\frac{\partial^2 F}{\partial x \partial y} - \frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \tag{180}$$
$$= \frac{-1}{F^2}\left(\frac{\partial^2 F}{\partial x \partial y} - \frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\right)\alpha^1 \wedge \alpha^2. \tag{181}$$

For instance, it vanishes for any $F(x,y)$ linear in x and y, or any F independent from x or y. It does not vanish with $F(x,y) = \sin(xy)$. Indeed:

$$\frac{\partial^2 F}{\partial x \partial y} = -xy\sin(xy) + \cos(xy), \tag{182}$$

$$\frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} = xy\frac{\cos(xy)^2}{\sin(xy)}. \tag{183}$$

5.5 Example 3: path geometry

The model space for path geometry is ${}^{G}\!/\!_{H}$ where $G = SL(3,\mathbf{R})$ and $H = B$, the so called Borel subgroup of G of upper triangular matrices. It can be realized as a flag manifold F_{12}, the space of complete flags in $\mathbf{R}^3$. For more details see [**IL**] and [**Mm2**].

Definition 5.14. *Le M be a real three dimensional manifold and $\mathrm{T}M$ be its tangent bundle.*

1. *A* path structure $\mathscr{L} = (E^1, E^2)$ *on M is a choice of two line sub-bundles E^1 and E^2 in $\mathrm{T}M$, such that $E^1 \cap E^2 = \{0\}$ and $E^1 \oplus E^2$ is a contact distribution.*

2. *A* strict path structure $\mathscr{T} = (E^1, E^2, \theta)$ *on M is a path structure with a fixed contact form θ such that* $\ker\theta = E^1 \oplus E^2$.

3. *A (local)* automorphism *of $(M, \mathscr{T})$ is a (local) diffeomorphism f of M that preserves E^1, E^2 and θ.*

The condition that $E^1 \oplus E^2$ be a contact distribution means that, locally, there exists a one form θ on M such that $\ker\theta = E^1 \oplus E^2$ and $\theta \wedge d\theta$ is never zero. On the other hand, for strict path structures we impose the existence of a *globally defined* contact form θ. Therefore, strict path structures are *unimodular geometries*: there exists a canonical volume form $\mu_{\mathscr{T}} = \theta \wedge d\theta$ on M, preserved by the automorphism group of $\mathscr{T}$ (in contrast, path structures are not unimodular).

There exists a unique vector field R such that $d\theta(R,\cdot) = 0$ and $\theta(R) = 1$, called the *Reeb vector field* of θ, that we will also call the Reeb vector field of the strict path structure $\mathscr{T}$. In particular, the distribution $E^1 \oplus E^2$ of a strict path structure $\mathscr{T}$ is thus oriented, and the manifold M supporting $\mathscr{T}$ is orientable.[3]

[3] If the contact distribution is oriented, then there exists a global contact form. Indeed, using a global metric on the distribution one can define locally a transversal vector to the distribution taking a Lie bracket of orthonormal vectors in the distribution. This defines a global 1-form.

Flat path model Flat path geometry is the geometry of real flags in $\mathbf{R}^3$. That is the geometry of the space of all couples (p, l) where $p \in \mathbf{R}P^2$ and l is a real projective line containing p. The space of flags is identified to the quotient

$$\mathrm{SL}(3,\mathbf{R})\diagup B \tag{184}$$

where B is the Borel group of all real upper triangular matrices.

Flat strict path model The Heisenberg group Heis(3) is the flat model for the strict path geometry. With

$$\mathrm{Heis}(3) = \left\{(x, y, t) \in \mathbf{R}^3\right\} \tag{185}$$

and the multiplication defined by $(x_1, y_1, t_1)\cdot(x_2, y_2, t_2) = (x_1+x_2, y_1+y_2, t_1+t_2+2(x_1y_2-x_2y_1))$. We consider the left invariant distributions determined by their value at the origin:

$$E_1 = \frac{\partial}{\partial x} \text{ and } E_2 = \frac{\partial}{\partial y} \tag{186}$$

and it has a global corresponding contact form:

$$\theta = \mathrm{d}t - x\,\mathrm{d}y + y\,\mathrm{d}x. \tag{187}$$

5.5.1 Path structures and second order differential equations

A second order differential equation in one variable is described locally as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right). \tag{188}$$

Introducing a new coordinate $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, we define a path structure on a neighborhood of a point in $\mathbf{R}^3$ with coordinates (x, y, p):

$$E_1 = \ker(\mathrm{d}y - p\,\mathrm{d}x) \cap \ker(\mathrm{d}p - F\,\mathrm{d}x), \tag{189}$$

$$E_2 = \ker(\mathrm{d}x) \cap \ker(\mathrm{d}y). \tag{190}$$

The contact structure is defined by the form

$$\theta = \mathrm{d}y - p\,\mathrm{d}x. \tag{191}$$

By defining the forms $Z^1 = \mathrm{d}x$ and $Z^2 = \mathrm{d}p - F\,\mathrm{d}x$, one has that $\mathrm{d}\theta = Z^1 \wedge Z^2$.

One can show that every path structure is, in fact, locally equivalent to a second order equation. That is, there exists local coordinates such that E_1 and E_2 are defined via a second order differential equation as above.

For, one first finds coordinates such that $E_2 = \ker \mathrm{d}x \cap \ker \mathrm{d}y$ by the flowbox theorem. Forms which annihilate $E_2 + E_1$ should be described by $q\,\mathrm{d}x + p\,\mathrm{d}y$, for functions q and p. Without loss of generality, one can assume locally that $\mathrm{d}x + p\,\mathrm{d}y$ and using the contact condition one concludes that x, y, p are local coordinates. Then $E_1 = \ker(\alpha\,\mathrm{d}p + \beta\,\mathrm{d}y + \gamma\,\mathrm{d}x) \cap \ker(\mathrm{d}y - p\,\mathrm{d}x)$ and one let, for $\alpha \neq 0$, without loss of generality, $\beta = 0$ and $\alpha = 1$.

Local equivalence (also called point equivalence) between path structures happens when there exists a local diffeomorphism which gives a correspondence between the lines defining each structure.

One can choose a contact form θ up to a scalar function and interpret this as follows: one has an $\mathbf{R}^*$-bundle over the manifold given by the choice of θ at each point (one might keep only positive multiples for simplicity). Over this line bundle one defines the tautological form $\omega_{\theta\alpha} = \pi^*\theta\alpha$. This bundle is trivial if and only if there exists a global contact form θ.

Let θ and local forms Z^1 and Z^2 defining the lines as above such that $d\theta = Z^1 \wedge Z^2$. There exists global forms Z^1 and Z^2 if and only if there exists global vector fields along the lines. Clearly, if the contact distribution is oriented, it suffices that there exists a global vector field along one of the foliations by lines.

5.5.2 Examples

Example 1 Consider the Heisenberg group

$$\mathrm{Heis}(3) = \{(z, t) \in \mathbf{C} \times \mathbf{R}\} \tag{192}$$

with multiplication defined by $(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}\, z_1\overline{z_2})$. The contact form

$$\theta = \mathrm{d}t + x\,\mathrm{d}y - y\,\mathrm{d}x \tag{193}$$

is invariant under left multiplications (also called Heisenberg translations). If $\Lambda \subset \mathrm{Heis}(3)$ is a lattice then the quotient ${}_{\Lambda}\backslash^{\mathrm{Heis}(3)}$ is a circle bundle over the torus with a globally defined contact form.

A lattice Λ determines a lattice $\Gamma \subset \mathbf{C}$ corresponding to the projection in the exact sequence

$$\{0\} \to \mathbf{R} \to \mathrm{Heis}(3) \to \mathbf{C} \to \{0\}. \tag{194}$$

There are many global vector fields in the distribution defined by θ and invariant under Λ, it suffices to lift a vector field on $\mathbf{C}$ invariant under Γ. All circle bundles obtained in this way are not trivial and the fibers are transverse to the distribution.

Example 2 We consider the torus T^3 with coordinates $(x, y, t) \in {\mathbf{R}/\mathbf{Z}}^3$ and the global contact form

$$\theta_n = \cos(2\pi n t)\,\mathrm{d}x - \sin(2\pi n t)\,\mathrm{d}y. \tag{195}$$

There are two canonical global vector fields on the distribution given by

$$\frac{\partial}{\partial t} \text{ and } \sin(2\pi n t)\frac{\partial}{\partial x} + \cos(2\pi n t)\frac{\partial}{\partial y}. \tag{196}$$

In this example, the fiber given by the coordinate t has tangent space contained in the distribution.

Example 3 An homogeneous example is the Lie group SU(2) with left invariant vector fields X and Y with $Z = [X, Y]$ and cyclic commutation relations. The vector fields X and Y define a path structure on SU(2).

Example 4 Another homogeneous example is the Lie group $\mathrm{SL}(2,\mathbf{R})$ with left invariant vector fields X and Y with $Z = [X, Y]$ with $[Z, X] = 2X$ and $[Z, Y] = -2Y$ given by the generators in $\mathfrak{sl}(2)$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{197}$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{198}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{199}$$

The path structure defined by X and Y induces a path structure on the quotient $\Gamma\backslash\mathrm{SL}(2,\mathbf{R})$ by a discrete torsion free subgroup $\Gamma \subset \mathrm{SL}(2,\mathbf{R})$. This structure is invariant under the flow defined by right multiplication by e^{tZ}.

Example 5 Let Σ be a surface equipped with a Riemannian metric. The geodesic flow on its unit tangent bundle $\mathrm{T}^1\Sigma$ defines a distribution which, together with the distribution defined by the vertical fibers of the projection of the unit tangent bundle on Σ, defines a path structure which is not invariant under the geodesic flow. For $\Sigma = \mathbf{H}^2_{\mathbf{R}}$, the hyperbolic upper plane, we obtain $\mathrm{T}^1\Sigma = \mathrm{PSL}(2,\mathbf{R})$ with distributions defined by the left invariant distributions $X - Y$ and Z (using the same generators of the Lie algebra $\mathfrak{sl}(2)$ as in the previous example).

5.5.3 Path structures with a fixed contact form

We now go back to strict path structures, by considering the specific case of Cartan geometries modeled on Heis(3), the flat model of strict path structures. So G denotes from now on the subgroup of $\mathrm{SL}(3,\mathbf{R})$ defined by

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ x & \frac{1}{a^2} & 0 \\ z & y & a \end{pmatrix} \middle| \, a \in \mathbf{R}^*, (x,y,z) \in \mathbf{R}^3 \right\} \tag{200}$$

and $H \subset G$ the isotropy subgroup of G defined by

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix} \right\}. \tag{201}$$

The Heisenberg group is identified to:

$$\mathrm{Heis}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \right\}. \tag{202}$$

The semidirect structure $G = \mathrm{Heis}(3) \rtimes H$ is described by the action of H on Heis(3) by conjugation:

$$\begin{pmatrix} \frac{1}{a} & & \\ & a^2 & \\ & & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{pmatrix} \begin{pmatrix} a & & \\ & \frac{1}{a^2} & \\ & & a \end{pmatrix} = \begin{pmatrix} 1 & & \\ a^3 x & & \\ z & \frac{1}{a^3} y & 1 \end{pmatrix}. \tag{203}$$

Writing the Maurer-Cartan form of G as the matrix

$$\begin{pmatrix} w & 0 & 0 \\ \theta^1 & -2w & 0 \\ \theta & \theta^2 & w \end{pmatrix} \tag{204}$$

one obtains the structural equations:

$$\begin{cases} \mathrm{d}\theta + \theta^2 \wedge \theta^1 = 0 \\ \mathrm{d}\theta^1 - 3w \wedge \theta^1 = 0 \\ \mathrm{d}\theta^2 + 3w \wedge \theta^2 = 0 \\ \mathrm{d}w = 0. \end{cases} \tag{205}$$

Let M be a three-manifold equipped with a strict path structure $\mathcal{T} = (E^1, E^2, \theta)$ with Reeb vector field R. Now let $X_1 \in E^1$, $X_2 \in E^2$ be such that $\mathrm{d}\theta(X_1, X_2) = 1$. The dual coframe of (X^1, X^2, R) is $(\alpha^1, \alpha^2, \theta)$, with two 1-forms α_1 and α_2 verifying $\mathrm{d}\theta = \alpha^1 \wedge \alpha^2$.

At any point $x \in M$, any coframe $(\theta^1, \theta^2, \theta)$ verifying $\mathrm{d}\theta = \theta^1 \wedge \theta^2$ is of the form

$$\theta^1 = a^3 \alpha^1,\ \theta^2 = \frac{1}{a^3}\alpha^2 \tag{206}$$

with a a function with values in $\mathbf{R}^*$.

Definition 5.15. *We denote by $\pi \colon P \to M$ the right $\mathbf{R}^*$-coframe bundle over M given by the set of coframes $(\theta^1, \theta^2, \theta)$.*

We will denote the tautological forms defined by θ^1, θ^2 and θ by using the same letters. That is, we write $\theta^i = \pi^* \theta^i$.

Proposition 5.16. *There exists a unique Cartan connection on $P \to M$*

$$\omega = \begin{pmatrix} w & 0 & 0 \\ \theta^1 & -2w & 0 \\ \theta & \theta^2 & w \end{pmatrix} \tag{207}$$

such that its curvature form is of the form

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] = \mathrm{d}\omega + \omega \wedge \omega = \begin{pmatrix} \mathrm{d}w & 0 & 0 \\ \theta \wedge \tau^1 & -2\,\mathrm{d}w & 0 \\ 0 & \theta \wedge \tau^2 & \mathrm{d}w \end{pmatrix} \tag{208}$$

with $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$.

Observe that the condition $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$ implies that we may write $\tau^1 = \tau^1_2 \theta^2$ and $\tau^2 = \tau^2_1 \theta^1$.

Proof. We differentiate the tautological forms. One obtains with $\theta^1 = a^3 \alpha^1$:

$$\begin{aligned} \mathrm{d}\theta^1 &= 3a^2\,\mathrm{d}a \wedge \alpha^1 + a^3\,\mathrm{d}\alpha^1 && (209) \\ &= -3\theta^1 \wedge \frac{\mathrm{d}a}{a} + a^3\left(3v^1 \wedge \alpha^1 + b_1 \theta \wedge \alpha^2\right) && (210) \end{aligned}$$

for a certain function b_1 and a 1-form v^1 defined on M. Rearranging terms we obtain

$$\mathrm{d}\theta^1 = -3\theta^1 \wedge \left(\frac{\mathrm{d}a}{a} + v^1\right) + a^6 b_1 \theta \wedge \theta^2. \tag{211}$$

Analogously we have

$$d\theta^2 = 3\theta^2 \wedge \left(\frac{da}{a} + v^2\right) + \frac{b_2}{a^6}\theta \wedge \theta^1 \tag{212}$$

for a certain function b_2 and a 1-form v^2 defined on M. Observe now that by differentiating $d\theta = \alpha^1 \wedge \alpha^2$ one obtains that

$$d^2\theta = 0 = d\alpha^1 \wedge \alpha^2 - \alpha^1 \wedge d\alpha^2 \tag{213}$$

$$= -3\alpha^1 \wedge v^1 \wedge \alpha^2 - 3\alpha^1 \wedge \alpha^2 \wedge v^2. \tag{214}$$

This implies that the terms in θ of v^1 and v^2 are the same. One can therefore define a unique w by adding to $\frac{da}{a} - v^1$ the term in v^2 which is proportional to θ^2.

Unicity of this construction follows easily from Cartan's lemma. The verification that it is actually a Cartan connection is left to the reader. □

The Heisenberg group as a strict path flat space revisited The Heisenberg group

$$\mathrm{Heis}(3) := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ z & x & 1 \end{pmatrix} \middle| (x, y, z) \in \mathbf{R}^3 \right\}$$

is the model of strict path structures. We consider on Heis(3) the left-invariant structure $(\mathbf{R}\tilde{X}, \mathbf{R}\tilde{Y}, \widetilde{Z^*})$, where $\tilde{X}, \tilde{Y}$ are the left invariant vector fields and $\widetilde{Z^*}$ the left invariant 1-form induced by the basis

$$X = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&1&0 \end{pmatrix}, Y = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}, Z = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} \tag{215}$$

of its Lie algebra. To describe the automorphism group of this structure, we introduce the subgroups

$$P = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a \in \mathbf{R}^* \right\} \subset G = \left\{ \begin{pmatrix} a & 0 & 0 \\ y & \frac{1}{a^2} & 0 \\ z & x & a \end{pmatrix} \middle| a \in \mathbf{R}^*, (x, y, z) \in \mathbf{R}^3 \right\}$$

of $\mathrm{SL}(3, \mathbf{R})$. We verify that

$$\psi\colon (h, p) \in \mathrm{Heis}(3) \times P \mapsto hp \in G.$$

is a group isomorphism between G and the semi-direct product $\mathrm{Heis}(3) \rtimes P$, where P acts on Heis(3) by $p \cdot h := php^{-1}$.

Define the left action of $G = \mathrm{Heis}(3) \rtimes P$ on Heis(3):

$$hp \cdot x := h(pxp^{-1})$$

for any $(h, p) \in \mathrm{Heis}(3) \times P$ and $x \in \mathrm{Heis}(3)$.

This action being transitive, it induces an identification of Heis(3) with G/P, by choosing the identity e for base-point.

Lemma 5.17. *G is the automorphism group of the strict path structure on* Heis(3).

Proof. It is easy to verify that $\mathbf{R}X$, $\mathbf{R}Y$ and Z are fixed by the adjoint action of P, so that G acts on Heis(3) by automorphisms of its strict path structure. In order to show that this is full automorphism group we use first the Heisenberg translations to reduce the question to the isotropy group. □

An example with constant curvature Consider $\mathrm{SL}(2,\mathbf{R})$ with its left invariant vector fields defined by a Lie algebra basis (E, F, H) of $\mathfrak{sl}(2)$ with $[E, F] = H$, $[H, E] = 2E$ and $[H, F] = -2F$. Explicitly:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{216}$$

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{217}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{218}$$

The structural equations of $\mathrm{SL}(2,\mathbf{R})$ for a dual basis $\alpha^1, \alpha^2, \theta$ are:

$$\begin{cases} \mathrm{d}\theta + \alpha^1 \wedge \alpha^2 = 0 \\ \mathrm{d}\alpha^1 - 2\alpha^1 \wedge \theta = 0 \\ \mathrm{d}\alpha^2 - 2\theta \wedge \alpha^2 = 0. \end{cases} \tag{219}$$

Indeed, note that:

$$\begin{pmatrix} \theta & \alpha^1 \\ \alpha^2 & -\theta \end{pmatrix} \wedge \begin{pmatrix} \theta & \alpha^1 \\ \alpha^2 & -\theta \end{pmatrix} = \begin{pmatrix} \alpha^1 \wedge \alpha^2 & -2\alpha^2 \wedge \theta \\ -2\theta \wedge \alpha^2 & -\alpha^1 \wedge \alpha^2 \end{pmatrix}. \tag{220}$$

Now, we define a strict path structure on $\mathrm{SL}(2,\mathbf{R})$. At any point, we do a left translation (by $\mathrm{SL}(2,\mathbf{R})$) of $(\mathbf{R}F, \mathbf{R}E, H)$. It defines a path structure. It is strict with the left translation of θ. The tautological forms are θ, $\theta^1 = a^3\alpha^2$ and $\theta^2 = a^{-3}\alpha^1$.

We can now compare with the previous proposition and the structural equations of the strict path geometry. That is to say, we compare the two sets of equations:

$$\begin{cases} \mathrm{d}\theta + \alpha^1 \wedge \alpha^2 = 0 \\ \mathrm{d}\alpha^1 - 2\alpha^1 \wedge \theta = 0 \\ \mathrm{d}\alpha^2 - 2\theta \wedge \alpha^2 = 0 \end{cases} \text{and} \begin{cases} \mathrm{d}\theta + \theta^2 \wedge \theta^1 = 0 \\ \mathrm{d}\theta^2 + 3w \wedge \theta^2 = \theta \wedge \tau^2 \\ \mathrm{d}\theta^1 - 3w \wedge \theta^1 = \theta \wedge \tau^1. \end{cases} \tag{221}$$

We read those equations in the section $(\alpha^1, \alpha^2, \theta)$. The first equation of both systems is indeed verified:

$$\mathrm{d}\theta + \theta^2 \wedge \theta^1 = \mathrm{d}\theta + \alpha^1 \wedge \alpha^2 = 0. \tag{222}$$

The equations in second position:

$$\mathrm{d}\alpha^1 - 2\alpha^1 \wedge \theta = 0 \text{ and } \mathrm{d}\theta^2 + 3w \wedge \theta^2 = \theta \wedge \tau^2 \tag{223}$$

show that $\tau^2 = 0$ and w must be $\frac{2}{3}\theta$ along the section $(\alpha^1, \alpha^2, \theta)$. The last equations shows that $\tau^1 = 0$ and w is again $\frac{2}{3}\theta$.

As a consequence, the strict path structure on $\mathrm{SL}(2,\mathbf{R})$ has curvature:

$$\Omega = \begin{pmatrix} \frac{2}{3}\theta^2 \wedge \theta^1 & 0 & 0 \\ 0 & -\frac{2}{3}\theta^2 \wedge \theta^1 & 0 \\ 0 & 0 & \frac{2}{3}\theta^2 \wedge \theta^1 \end{pmatrix}. \tag{224}$$

One can think of $\mathrm{SL}(2,\mathbf{R})$ with the above strict path structure as a constant curvature model. Observe that one can vary the curvature by choosing different multiples of H. The curvature sign corresponds then to different choices of orientation.

The automorphism group of this structure is $\mathrm{SL}(2,\mathbf{R}) \times \mathbf{R}^*$. The action is through left translations by $\mathrm{SL}(2,\mathbf{R})$ and right translations by $\mathbf{R}^*$ identified to the one parameter subgroup

$$\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| \, t \in \mathbf{R} \right\} \tag{225}$$

Indeed, this group acts simply transitively on the adapted coframe bundle P over $\mathrm{SL}(2,\mathbf{R})$ and preserves the connection.

6 Curvature

6.1 Universal covariant derivative

We consider a Cartan geometry $P \to M$ with connection $\omega : TP \to \mathfrak{g}$. Although the following definition makes sense for general maps $f : P \to V$ we will consider only the more interesting case of equivariant maps and, in particular, the curvature function. Let $\rho : H \to GL(V)$ be a representation of the structural group and $f : P \to V$ map satisfying, for all $h \in H$,

$$R_h^* f = \rho(h^{-1}) f.$$

A special case is the the representation $Ad : H \to Aut(\mathfrak{g})$ and the function being the Cartan connection. Another example is the curvature function $K : P \to V$ where $V = Hom(\Lambda^2\mathfrak{g}, \mathfrak{g})$ defined in 6.3.

Definition 6.1. *Let $f : P \to V$ be an equivariant map. The universal covariant derivative of f with respect to $X \in \mathfrak{g}$ is the map*

$$D_X f := \omega^{-1}(X) f.$$

In other words, the universal covariant derivative of f is the map

$$Df := \omega^{-1}(\cdot) f \in \mathfrak{g}^* \otimes V.$$

Note that it is defined on P and not on M. It is also important to observe that Df is an equivariant function with values in $\mathfrak{g}^* \otimes V$ with the representation on $\mathfrak{g}^* \otimes V$ given by $Ad^* \otimes \rho$. The universal covariant derivative along the fibers is easy to compute in the case of these equivariant functions:

Lemma 6.2. *Let $f : P \to V$ be a function satisfying the transformation $R_h^* f = \rho(h^{-1}) f$. Then, for all $X \in \mathfrak{h}$,*

$$D_X f(p) := -\rho_*(X) f(p),$$

where $\rho_*(X) = \frac{d}{dt}|_{t=0} \rho(e^{tX})$.

Proof. From

$$D_X f(p) = \omega^{-1}(X) f(p) = f_*(\omega^{-1}(X))(p) = \frac{d}{dt}|_{t=0} f(pe^{tX})$$

we obtain

$$D_X f(p) = \frac{d}{dt}|_{t=0} \rho(e^{-tX}) f(p) = -\rho_*(X) f(p).$$

□

6.2 The curvature function

One can define a function with values on a vector space which contains the same information as the curvature 2-form. Let $V = \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$.

Definition 6.3. *Let $P \to M$ be a Cartan geometry with connection ω and curvature Ω. We define its curvature function*

$$K : P \to V.$$

by

$$K(p)(u, v) = \Omega\left(\omega^{-1}(u)|_p, \omega^{-1}(v)|_p\right). \tag{226}$$

It is well defined since Ω vanishes on $\mathfrak{h}$.

Note The curvature function K has values in $\mathfrak{h}$ if, and only if, Ω has vanishing torsion.

In the case the Cartan geometry is reductive, one can write $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as an H-module decomposition. One can decompose the universal covariant derivative by projecting it into $\mathfrak{n}$ and $\mathfrak{h}$. We write $D_\mathfrak{n}$ for its projection into $\mathfrak{n}$ and call it the covariant derivative of the reductive Cartan geometry.

Definition 6.4. *Let $P \to M$ be a reductive Cartan geometry ($\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$) with connection $\omega_\mathfrak{n}$ and curvature Ω. It has constant curvature if K does not depend on P.*

Definition 6.5. *A reductive Cartan geometry modeled on $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ is said to be locally symmetric if the curvature function satisfies $D_\mathfrak{n} K = 0$.*

Let now $W \in TM$ and $\tilde{W}$ be the unique lift to TP such that $\omega_\mathfrak{h}(\tilde{W}) = 0$. Define the covariant derivative $\nabla_W K$ by the formula

$$\nabla_W K = D_{\omega(\tilde{W})} K.$$

6.2.1 Curvature function on a coframe bundle

In the case the Cartan bundle is a coframe bundle one can use the tautological forms θ^i to express the curvature as

$$\Omega = R^i_{jkl} \theta^i \wedge \theta^j.$$

Therefore, the curvature function is given by $K(e^k, e^l) = R^i_{jkl}$. The covariant derivative of the curvature is given by

$$D^1 K(e^r \otimes e^k \otimes e^l) = \varpi^{-1}(e^r) K(e^k, e^l) = R^i_{jkl,r}$$

where $dR^i_{jkl} = R^i_{jkl,r} \theta^r$.

6.3 Mutations

Definition 6.6. *Let $P \to M$ be a Cartan geometry modeled on $(\mathfrak{g}, \mathfrak{h})$. Its torsion is the projection of the curvature Ω by $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.*

Note When the model space is reductive, that is to say there exists $\mathfrak{p}$ that is $\mathrm{Ad}(h)$-invariant and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, then the torsion is the $\mathfrak{p}$ factor of Ω.

Example In Riemannian geometry the torsion vanishes exactly for the Levi-Civita connection. It is indeed what we constructed by asking $\mathrm{d}\theta + \theta \wedge \omega = 0$.

Definition 6.7. *Let $(\mathfrak{g}_1, \mathfrak{h})$ and $(\mathfrak{g}_2, \mathfrak{h})$ be two geometric pairs sharing a same group H corresponding to $\mathfrak{h}$ and having two respective adjoint representations $\mathrm{Ad}_1 \colon H \to \mathrm{Aut}(\mathfrak{g}_1)$ and $\mathrm{Ad}_2 \colon H \to \mathrm{Aut}(\mathfrak{g}_2)$.*

A mutation is a linear isomorphism

$$\lambda \colon \mathfrak{g}_1 \to \mathfrak{g}_2 \tag{227}$$

such that

1. *for all $h \in H$ and $u \in \mathfrak{g}_1$, $\lambda(\mathrm{Ad}_1(h)(u)) = \mathrm{Ad}_2(h)(\lambda(u))$;*
2. *$\lambda|_{\mathfrak{h}}$ is the identity;*
3. *in $\mathfrak{g}_2/\mathfrak{h}$, we have $\lambda([u,v]) = [\lambda(u), \lambda(v)]$.*

Examples The three constant curvature models for the Riemannian geometry are mutations. Let $\mathbf{R}^n = \mathrm{Eucl}_+(n)/\mathrm{SO}(n)$, $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $H^n_{\mathbf{R}} = \mathrm{SO}(n,1)/\mathrm{SO}(n)$. The three Lie algebras $\mathfrak{so}(n+1), \mathfrak{so}(n,1)$ and $\mathfrak{eucl}(n)$ are decomposed into $\mathfrak{so}(n) \oplus \mathbf{R}^n$. Note that $\mathfrak{so}(n)$ is the Lie algebra of a shared isotropy $H = \mathrm{SO}(n)$. Let $A \in \mathfrak{so}(n)$ and $v \in \mathbf{R}^n$. Then the mutations are deduced from the three following representations.

$$\mathfrak{eucl}(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \right\} \tag{228}$$

$$\mathfrak{so}(n+1) = \left\{ \begin{pmatrix} A & v \\ -{}^t v & 0 \end{pmatrix} \right\} \tag{229}$$

$$\mathfrak{so}(n,1) = \left\{ \begin{pmatrix} A & v \\ {}^t v & 0 \end{pmatrix} \right\} \tag{230}$$

Proposition 6.8. *Let* $(\mathfrak{g}_1, \mathfrak{h})$ *and* $(\mathfrak{g}_2, \mathfrak{h})$ *be two geometric pairs with a mutation* $\lambda\colon \mathfrak{g}_1 \to \mathfrak{g}_2$.

If $P \to M$ *is a Cartan geometry modeled on* $(\mathfrak{g}_1, \mathfrak{h})$ *with Cartan connection* ω_1 *then*

$$\omega_2 = \lambda \circ \omega_1 \tag{231}$$

gives a Cartan connection for $P \to M$ *modeled on* $(\mathfrak{g}_2, \mathfrak{h})$*. Furthermore, the curvature* Ω_1 *becomes:*

$$\Omega_2 = \lambda \circ \Omega_1 + \frac{1}{2}\left([\omega_2 \wedge \omega_2] - \lambda[\omega_1 \wedge \omega_1]\right). \tag{232}$$

Proof. Since λ is an isomorphism, ω_2 is a linear isomorphism at each point: $\mathrm{T}_p P \to \mathfrak{g}_2$. It verifies the other properties since on the equivalent property we have:

$$\begin{align} R_\psi^* \omega_2 &= (\lambda \circ \omega_1)(R_{\psi *}) \tag{233}\\ &= \lambda\left(\psi^* \theta_H + \mathrm{Ad}_1(\psi)^{-1} \omega_1\right) \tag{234}\\ &= \psi^* \theta_H + \mathrm{Ad}_2(\psi)^{-1} \lambda \circ \omega_1 \tag{235}\\ &= \psi^* \theta_H + \mathrm{Ad}_2(\psi)^{-1} \omega_2. \tag{236} \end{align}$$

The identity on Ω_2 follows by definition. □

Note If Ω_1 has vanishing torsion then Ω_2 does too since λ preserves $\mathfrak{h}$.

Theorem 6.9. *Let* $P \to M$ *be a Cartan geometry modeled on* $(\mathfrak{g}_1, \mathfrak{h})$ *with connection* ω *and curvature* Ω*. Assume it has constant curvature and vanishing torsion. Then* $\lambda\colon \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathfrak{g}_1$ *defined linearly by* id *(*$\mathfrak{g}_2$ *is a linear copy of* $\mathfrak{g}_1$*) but with bracket*

$$[u, v]_{\mathfrak{g}_2} = [u, v]_{\mathfrak{g}_1} - K(u, v) \tag{237}$$

defines a mutant geometry on which $P \to M$ *is flat.*

Proof. We prove first that $\mathfrak{g}_2$ is well defined. It only depends on wether the bracket is indeed a bracket of Lie algebra. It is certainly anti-symmetric and bilinear since K is a 2-form. The Jacobi identity is comes from the following computation.

$$\begin{align} [u,[v,w]_{\mathfrak{g}_2}]_{\mathfrak{g}_2} &= [u,[v,w]_{\mathfrak{g}_2}]_{\mathfrak{g}_1} - K(u,[v,w]_{\mathfrak{g}_2}) \tag{238}\\ &= [u,[v,w]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} - [u, K(v,w)]_{\mathfrak{g}_1} - K(u,[v,w]_{\mathfrak{g}_1}) + K(u, K(v,w)) \tag{239}\\ &= [u,[v,w]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} - [u, K(v,w)]_{\mathfrak{g}_1} - K(u,[v,w]_{\mathfrak{g}_1}) \tag{240}\\ -[v,[u,w]_{\mathfrak{g}_2}]_{\mathfrak{g}_2} &= -[v,[u,w]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} + [v, K(u,w)]_{\mathfrak{g}_1} + K(v,[u,w]_{\mathfrak{g}_1}) \tag{241}\\ [[u,v]_{\mathfrak{g}_2}, w]_{\mathfrak{g}_2} &= \left[[u,v]_{\mathfrak{g}_1}, w\right]_{\mathfrak{g}_1} - K([u,v]_{\mathfrak{g}_1}, w) - [K(u,v), w]_{\mathfrak{g}_1} \tag{242} \end{align}$$

Hence the Jacobi identity only depends on a circular identity of $K(u,[v,w])$ and $[K(u,v),w]$. For this we use the Bianchi identity.

Let U,V,W be $\omega^{-1}(u),\omega^{-1}(v),\omega^{-1}(w)$. Then, since the curvature is constant, (we now take every bracket in $\mathfrak{g}_1$)

$$\mathrm{d}\Omega(U,V,W) = -\Omega([U,V],W) + \Omega(U,[V,W]) - \Omega(V,[U,W]) \tag{243}$$

and the Bianchi identity say this is equal to

$$[\Omega(U,V),\omega(W)] - [\omega(U),\Omega(V,W)] + [\omega(V),\Omega(U,W)]. \tag{244}$$

With a torsion free curvature we have also:

$$\Omega([U,V],W) = K([u,v]-K(u,v),w) = K([u,v],w). \tag{245}$$

So the Bianchi identity states:

$$-K([u,v],w)+K(u,[v,w])-K(v,[u,w]) = [K(u,v),w]-[u,K(v,w)]+[v,K(u,w)] \tag{246}$$

finishing to prove that $[\cdot,\cdot]_{\mathfrak{g}_2}$ is a bracket.

Now we prove that we have indeed a mutation. We need to prove that $[\mathrm{Ad}(h)u,\mathrm{Ad}(h)v]_{\mathfrak{g}_2} = \mathrm{Ad}(h)[u,v]_{\mathfrak{g}_1}$. This equality will be proved if we show

$$K(\mathrm{Ad}(h)u,\mathrm{Ad}(h)v) = \mathrm{Ad}(h)K(u,v). \tag{247}$$

It is true since by constant curvature:

$$\Omega(U,V) = \mathrm{Ad}(h)^{-1}\Omega(\mathrm{Ad}(h)U,\mathrm{Ad}(h)V). \tag{248}$$

Finally, the new connection is indeed flat by the preceding proposition and a straightforward computation. □

As an example of mutation using Theorem 6.9 we consider the strict path structure with constant curvature described in section 5.5.3. Its curvature is given by

$$\Omega = \begin{pmatrix} \frac{2}{3}\theta^2\wedge\theta^1 & 0 & 0 \\ 0 & -\frac{2}{3}\theta^2\wedge\theta^1 & 0 \\ 0 & 0 & \frac{2}{3}\theta^2\wedge\theta^1 \end{pmatrix}. \tag{249}$$

Recall that the Lie algebra of $G = \mathrm{Heis}(3) \rtimes P$ is the algebra $\mathfrak{g} = \mathfrak{heis}(3) \otimes \mathfrak{p}$ where $\mathfrak{p}$ is the Lie algebra of P, whose adjoint action on $\mathfrak{heis}(3)$ is described by

$$[X,Y] = Z, [D,X] = X, [D,Y] = -Y, [D,Z] = 0, \tag{250}$$

with D the generator of $\mathfrak{p}$ verifying

$$\exp(tD) = \begin{pmatrix} e^{\frac{t}{3}} & 0 & 0 \\ 0 & e^{-2\frac{t}{3}} & 0 \\ 0 & 0 & e^{\frac{t}{3}} \end{pmatrix}.$$

We define, as in Theorem 6.9 a new Lie algebra structure on the vector space $\mathfrak{g}$ by the following relation for $u, v \in \mathfrak{g} \times \mathfrak{g}$:

$$[u, v]' := [u, v] - K(u, v). \tag{251}$$

We denote by $\mathfrak{g}'$ the Lie algebra defined by the vector space $\mathfrak{g}$ endowed with the Lie bracket $[\cdot,\cdot]'$. Now $\mathfrak{g}'$ is described by the following relations:

$$[X, Y]' = Z - 3D, [D, X]' = X, [D, Y]' = -Y, [Z, X]' = [Z, Y]' = [D, Z]' = 0. \tag{252}$$

The conclusion of Theorem 6.9 applies and we obtain that this Cartan geometry is a flat $(\mathfrak{g}', \mathfrak{d})$ Cartan structure. In the following we will show how to identify the algebra $\mathfrak{g}'$ to $\mathrm{SL}(2, \mathbf{R}) \times A$.

The Lie algebra of $\mathrm{SL}(2, \mathbf{R}) \times A$ is the direct sum $\mathfrak{sl}_2 \oplus \mathfrak{a}$, and the copy of $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in the right factor of $\mathfrak{sl}_2 \oplus \mathfrak{a}$ is denoted by T. Note that $[T, \cdot] = 0$ on $\mathfrak{sl}_2 \oplus \mathfrak{a}$.

We define a group isomorphism $\Lambda\colon P \to \Delta$ by

$$\Lambda\left(\begin{pmatrix} a^{\frac{1}{3}} & 0 & 0 \\ 0 & a^{-\frac{2}{3}} & 0 \\ 0 & 0 & a^{\frac{1}{3}} \end{pmatrix}\right) = \left(\begin{pmatrix} a^{\frac{1}{2}} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} a^{\frac{1}{2}} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix}\right). \tag{253}$$

Define a vector space isomorphism $\mathfrak{g}' \to \mathfrak{sl}_2 \oplus \mathfrak{a}$ by:

$$\lambda(X) = \sqrt{\frac{3}{2}} E, \lambda(Y) = -\sqrt{\frac{3}{2}} F, \lambda(Z) = \frac{3}{2} T, \lambda(D) = \frac{1}{2}(H + T). \tag{254}$$

A simple computation shows that λ is a Lie algebra isomorphism from $\mathfrak{g}'$ to $\mathfrak{sl}_2 \oplus \mathfrak{a}$.

One can interpret this isomorphism as a mutation isomorphism $\lambda\colon \mathfrak{heis}(3) \oplus \mathfrak{p} \to \mathfrak{sl}_2 \oplus \mathfrak{a}$. That is:

Lemma 6.10. *1. The differential of Λ at the identity coincide with $\lambda|_{\mathfrak{p}}\colon \mathfrak{p} \to \mathfrak{a}$.*

2. For any $u, v \in \mathfrak{g}$: $[\lambda(u), \lambda(v)] = \lambda([u, v])$ modulo $\mathfrak{a}$ (the brackets being respectively in $\mathfrak{sl}_2 \oplus \mathfrak{a}$ and $\mathfrak{heis}(3) \oplus \mathfrak{p}$).

3. *For any* $p \in P$*:* $\lambda \circ Ad_p = Ad_{\Lambda(p))} \circ \lambda$ *(the adjoint actions being respectively within the Lie groups* $\mathrm{Heis}(3) \rtimes P$ *and* $\mathrm{SL}(2,\mathbf{R}) \times A$*).*

Proof. 1. This directly follows from the definitions of Λ and λ.
2. In fact for $u, v \in \mathfrak{g}$, $K(u,v) \in \mathfrak{p}$ and $[\lambda(u), \lambda(v)] = \lambda([u,v]') = \lambda([u,v]) - \lambda(K(u,v))$.
4. For $p \in P$ written as in (253), the matrix of Ad_p in the basis (X,Y,Z,D) is the diagonal matrix $[a, a^{-1}, 1, 0]$, and the matrix of $Ad_{\Lambda(p)}$ in the basis (E,F,H,T) is the diagonal matrix $[a, a^{-1}, 1, 1]$. The claim directly follows from the definition of λ. □

6.4 Developing map and uniformization of Cartan geometries

6.4.1 Path development

Lemma 6.11. *Let* $f\colon [0,1] \to \mathfrak{g}$ *be a smooth function. Let* $\omega\colon \mathrm{T}X \to \mathfrak{g}$ *be a complete parallelism (its constant vector fields* $\omega^{-1}(v)$ *are complete, i.e. have flows fully defined on* $\mathbf{R}$*) verifying the structural equation. Then the differential equation*

$$\gamma^* \omega = f \, \mathrm{d}t \tag{255}$$

has a solution $\gamma\colon [0,1] \to X$ *that is unique once an initial condition* $\gamma(0) = x \in X$ *is given.*

Proof. Note that $f\,\mathrm{d}t$ verifies the structural equation. By Cartan's method, a local solution does always exist and is unique once an initial condition is given. We have to show that a solution can always be extended to the full interval $[0,1]$.

Suppose that a local solution γ is only defined for $t < 1$. Then $\gamma(t)$ escapes every compact set of X when $t \to 1$. But when $t \to 1$, $f(t) \to v \in \mathfrak{g}$ and a global solution to $\gamma^*\omega = v$ exists by completeness of ω on X. A contradiction. □

The development of paths follows from this lemma. We let $\omega = \theta_G$ be the Maurer-Cartan form of a Lie group G. Any path $\delta\colon [0,1] \to P$ defined on a manifold P equipped with a $\mathfrak{g}$-valued 1-form $\omega\colon \mathrm{T}P \to \mathfrak{g}$ gives by pulling back the 1-form $\delta^*\omega$. Then by what precedes, $\delta^*\omega = \gamma^*\theta_G$ for a path $\gamma\colon [0,1] \to G$.

In our context P will be the total space of a principal bundle $P \to M$ and the form ω will be a Cartan connection.

Definition 6.12. *Let* P *be a smooth manifold equipped with a* $\mathfrak{g}$*-valued* 1*-form* ω*. Any path* $\delta\colon [0,1] \to P$ *determines a path* $D(\delta)\colon [0,1] \to G$ *such that*

$$\delta^*\omega = D(\delta)^*\theta_G \tag{256}$$

and $D(\delta)$ *is unique as soon as* $D(\delta)(0) \in G$ *is prescribed.*

The map giving the endpoint:

$$E(\delta) = D(\delta)(0)^{-1} D(\delta)(1) \tag{257}$$

is well defined and does not depend on the choice of $D(\delta)$.

The map E is defined on the space of the paths of P. Its values are in G. Now, the goal is to obtain a map

$$F\colon P \to G \tag{258}$$

that would be a complete integration of ω:

$$F^* \theta_G = \omega. \tag{259}$$

The most natural way would be to fix $p \in P$ and define $F(z)$ as $E(\delta)$ for any path δ joining p to z. With this goal in mind, one needs to compare the different values of E for different paths joining the same points.

A natural assumption is to compare paths that have the same homotopy class in $\pi_1(P, p)$. Those have indeed same endpoints by E if the space P is flat.

Lemma 6.13. *If there exists F such that* $F^*\theta_G = \omega$ *then* ω *verifies the structural equation*

$$\mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] = 0. \tag{260}$$

Proof. This follows by naturality of the pulling-back and the fact that θ_G itself verifies the structural equation. □

Since η is defined on the whole $\mathrm{T}[0,1]$, the function f is bounded.

Proposition 6.14. *Let* P *be a smooth manifold equipped with a* $\mathfrak{g}$*-valued* 1*-form* ω *that verifies the structural equation. If* $H\colon [0,1]\times[0,1] \to P$ *is an homotopy between* $\delta_1 = H(0,t)$ *and* $\delta_2 = H(1,t)$ *then* $E(\delta_1) = E(\delta_2)$.

Proof. Since ω verifies the structural equation, one can apply Cartan's method. Again, by completeness of the Maurer-Cartan form, it defines a complete integral $H_G\colon [0,1]\times [0,1] \to G$ such that

$$H_G^* \theta_G = H^* \omega. \tag{261}$$

Since H is a homotopy, $H^*\omega$ vanishes on $[0,1]\times\{0,1\}$. Hence H_G does too and it furnishes an homotopy in G. Therefore H_G has two equal endpoints for $H_G(0,t) = D(\delta_1)$ and $H_G(1,t) = D(\delta_2)$. □

Definition 6.15. *Let P be a smooth manifold equipped with a $\mathfrak{g}$-valued 1-form ω verifying the structural equation. The monodromy morphism*

$$\Phi_\omega : \pi_1(P, p) \to G \tag{262}$$

is the value of $E(\delta)$ for any δ realizing a chosen class $[\delta] \in \pi_1(P, p)$. It is a group homomorphism by concatenation of paths. Its image is the monodromy subgroup $\Phi_\omega(\pi_1(P, p)) \subset G$.

Corollary 6.16. *Let P be a smooth manifold equipped with a $\mathfrak{g}$-valued 1-form. There exists a global map $F\colon P \to G$ such that*

$$F^*\theta_G = \omega \tag{263}$$

if, and only if, ω verifies the structural equation and its monodromy is trivial.

6.4.2 Flat Cartan geometries

Now we consider a Cartan geometry $P \to M$. The development F of P in G will allow to define a developing map from $\widetilde{M}$ to ${}^{G}\!/_{H}$. Here we can see $\widetilde{M}$ as the space of the paths of M modulo homotopy.

A first step is to verify that the principal H-bundle structure on P is compatible with the one on G under F.

Lemma 6.17. *Let $P \to M$ be a Cartan geometry modeled on a reductive pair $(\mathfrak{g}, \mathfrak{h})$ with a (non-necessarily flat) Cartan connection ω. Let $\delta\colon [0,1] \to P$ be a path and $\psi\colon [0,1] \to H$ be a smooth function. Then*

$$D(\delta\psi) = D(\delta)\psi \tag{264}$$

if we have the compatibility $D(\delta\psi)(0) = D(\delta)(0)\psi(0)$.

Proof. Both $D(\delta\psi)$ and $D(\delta)\psi$ are paths on G with same initial point. We only need to check that their derivatives are equal since the unicity of the development of paths would conclude. Indeed we have:

$$\begin{aligned} D(\delta\psi)^*\theta_G &= (\delta\psi)^*\omega && (265)\\ &= \mathrm{Ad}(\psi)^{-1}\delta^*\omega + \psi^*\theta_H && (266)\\ &= \mathrm{Ad}(\psi)^{-1}D(\delta)^*\theta_G + \psi^*\theta_H && (267)\\ &= (D(\delta)\psi)^*\theta_G. && (268) \end{aligned}$$

□

Proposition 6.18. *Let $P \to M$ be a Cartan geometry modeled on an effective Kleinian pair $(\mathfrak{g}, \mathfrak{h})$ with a flat Cartan connection ω. Then there exists a local diffeomorphism*

$$D\colon \widetilde{M} \to {}^{G}\!/_{H} \tag{269}$$

called a developing map.

Proof. With the universal cover $\pi_1\colon \widetilde{M} \to M$ we define the pulled-back bundle $\widetilde{P}$ by

$$\widetilde{P} = \left\{(p, x) \in P \times \widetilde{M} \,\middle|\, \pi_P(p) = \pi_1(x)\right\}. \tag{270}$$

We have the projection maps $\widetilde{\pi_1}\colon \widetilde{P} \to P$ and $\pi_{\widetilde{P}}\colon \widetilde{P} \to \widetilde{M}$. The pulled-back Cartan connection $\omega_{\widetilde{P}} = \widetilde{\pi_1}^*\omega$ defined on $\widetilde{P}$ has still flat curvature by naturality.

$$\begin{tikzcd}
\mathrm{T}\widetilde{P} \arrow[r, "\widetilde{\pi_1}_*"] \arrow[d] & \mathrm{T}P \arrow[r, "\omega"] \arrow[d] & \mathfrak{g} & \mathrm{T}G \arrow[l, "\omega_G"'] \arrow[d] \\
\widetilde{P} \arrow[r, "\widetilde{\pi_1}"] \arrow[d, "\pi_{\widetilde{P}}"] & P \arrow[d, "\pi_P"] & & G \arrow[d, "\pi_G"] \\
\widetilde{M} \arrow[r, "\pi_1"] & M & & {}^{G}\!/_{H}
\end{tikzcd} \tag{271}$$

The short exact sequence of the fiber bundle $H \to \widetilde{P} \to \widetilde{M}$ shows that

$$\pi_1(H, e) \to \pi_1(\widetilde{P}, p) \to \pi_1(\widetilde{M}, x) = \{e\}. \tag{272}$$

By composition with the monodromy morphism, we obtain the exact sequence:

$$\{e\} = \Phi_{\omega_{\widetilde{P}}}(\pi_1(H, e)) \to \Phi_{\omega_{\widetilde{P}}}(\pi_1(\widetilde{P}, p)) \to \{e\} \tag{273}$$

showing that the monodromy of $\widetilde{P}$ is trivial. (Note that $\Phi_{\omega_{\widetilde{P}}}(\pi_1(H, e))$ is trivial since $H \subset \widetilde{P}$ is developed by the identity diffeomorphism to $H \subset G$.)

By the preceding corollary, we obtain a development

$$F_{\omega_{\widetilde{P}}}\colon \widetilde{P} \to G. \tag{274}$$

It is necessarily a local diffeomorphism that preserves the fibers since $\omega_{\widetilde{P}}$ identifies the tangent space of each fiber with $\mathfrak{h}$.

Therefore, $F_{\omega_{\widetilde{P}}}$ descends to a developing map

$$D\colon \widetilde{M} \to {}^{G}\!/_{H} \tag{275}$$

that is again a local diffeomorphism. □

Proposition 6.19. *Under the same assumptions, the developing map* $D\colon \widetilde{M} \to {}^{G}\!/_{H}$ *is paired with a holonomy morphism*

$$\rho\colon \pi_1(M,x) \to G \tag{276}$$

that is equivariant:

$$\forall \gamma \in \pi_1(M,x), \forall y \in \widetilde{M},\ D(\gamma y) = \rho(\gamma) D(y). \tag{277}$$

Proof. Recall that with the universal cover $\pi_1\colon \widetilde{M} \to M$ we constructed

$$\widetilde{P} = \left\{(x,p) \in \widetilde{M} \times P \,\middle|\, \pi_1(x) = \pi_P(p)\right\}. \tag{278}$$

The left action of $\pi_1(M,x)$ on $\widetilde{M}$ can be lifted to $\widetilde{P}$ by:

$$\forall \gamma \in \pi_1(M),\ \gamma \cdot (x,p) = (\gamma \cdot x, p). \tag{279}$$

Hence $\widetilde{\pi_1} \circ \gamma = \widetilde{\pi_1}$. We obtain:

$$\omega_{\widetilde{P}} = \widetilde{\pi_1}^*\omega = \gamma^*\widetilde{\pi_1}^*\omega = \gamma^*\omega_{\widetilde{P}}. \tag{280}$$

Since γ is an automorphism of $\widetilde{P}$, it corresponds to a left translation $\rho(\gamma)$ of G. For indeed, with any path η based at $p \in \widetilde{P}$, the forms $\eta^*\omega_{\widetilde{P}}$ and $\eta^*\gamma^*\omega_{\widetilde{P}}$ are equal and hence the endpoints of their developments differ by $\rho(\gamma)$ which does not depends on η. It can be checked that ρ is indeed a morphism by concatenation of loops in $\pi_1(M,x)$. It verifies the equivariance property by what precedes. □

Theorem 6.20. *Let* $P \to M$ *be a Cartan geometry modeled on an effective Kleinian pair* $(\mathfrak{g},\mathfrak{h})$ *with a flat Cartan connection* ω. *If the Cartan connection* ω *is complete, that is to say every* $\omega^{-1}(v)$ *vector field is complete (its flow is defined on* $\mathbf{R}$*), then the developing map*

$$D\colon \widetilde{M} \to {}^{G}\!/_{H} \tag{281}$$

is a covering map. If ${}^{G}\!/_{H}$ *is also simply connected then it follows, with* $\Gamma = \rho(\pi_1(M,x))$ *the image of the holonomy morphism, that* D *is a diffeomorphism and*

$$M \cong {}_{\Gamma}\backslash{}^{G}\!/_{H}. \tag{282}$$

Proof. The developing map D is a cover if, and only if, it has the lifting property. That is to say, we check that D can lift uniquely any path in ${}^{G}\!/_{H}$ with the choice of base points $x \in \widetilde{M}$ and $D(x) \in {}^{G}\!/_{H}$.

Any smooth path $\delta\colon [0,1] \to {}^{G}\!/_{H}$ can be lifted in G by a path $\widetilde{\delta}\colon [0,1] \to G$. Then $\widetilde{\delta}^*\theta_G = f\,\mathrm{d}t$. By lemma 6.17 (p. 74), there exists a unique path $\gamma\colon [0,1] \to \widetilde{P}$ such that $\gamma^*\omega_{\widetilde{P}} = f\,\mathrm{d}t$. Then the projection of γ in $\widetilde{M}$ lifts δ by construction. □

Corollary 6.21. *Let P be smooth manifold equipped with a complete parallelism $\omega\colon \mathrm{T}P \to \mathfrak{g}$ verifying the structural equation. If P is simply connected, then P is diffeomorphic to G the unique simply connected Lie group with Lie algebra $\mathfrak{g}$. Its group law is the concatenation of paths.*

6.4.3 Constant curvature

The preceding construction on flat Cartan geometries can be generalized for Cartan geometries with constant curvature and flat torsion. Indeed, recall that the mutation of a pair $(\mathfrak{g}, \mathfrak{h})$ allows us to obtain a Cartan geometry with a flat curvature.

Corollary 6.22. *Let $P \to M$ be a Cartan geometry modeled on an effective Kleinian pair $(\mathfrak{g}, \mathfrak{h})$ with Cartan connection ω. Assume that ω is complete, has constant curvature K and vanishing torsion. Then there exists a Lie group G_K which has for Lie algebra a mutation of $\mathfrak{g}$ with Lie bracket:*

$$\forall x, y \in \mathfrak{g},\ [x, y]_K = [x, y]_{\mathfrak{g}} - K(x, y). \tag{283}$$

If ${}^{G_K}\!/_{H}$ is simply connected then there exists a subgroup $\Gamma_K \subset G_K$ such that

$$M \cong {}_{\Gamma_K}\backslash{}^{G_K}\!/_{H} \tag{284}$$

by a geometric isomorphism.

7 Strict path geometry and large automorphism group

7.1 Large automorphism groups: some dynamics

The automorphism group of a geometric structure can be equipped with the compact-open topology. One can then consider those structures with large symmetries in the sense that their automorphism group is non-compact. They should be very special and probably can be described in a more precise way (see [**DG**]). One example of this phenomenon is the following theorem by Obata and Lelong-Ferrand ([**O**; **LF**]).

Theorem 7.1. *Let M be a manifold equipped with a conformal structure such that* $\mathrm{Aut}(M,[g])$ *does not act properly. Then either M is the sphere with its standard conformal structure or M is $\mathbf{R}^n$ with its standard conformal structure.*

Observe that if M is compact, the non-compactness of the automorphism group implies that the conformal structure is flat.

7.1.1 Poincaré's recurrence theorem

We will need to pass from a non-compact Lie group action to a recurrent action. This is easily achieved through measure preserving maps.

Definition 7.2. *Let (M,μ) be a finite measure space and $\phi : M \to M$ be a measurable map. We say that ϕ is measure preserving if $\phi_*\mu = \mu$, that is, for every measurable subset $A \subset M$ we have $\mu(\phi^{-1}(A)) = \mu(A)$.*

The simpler examples of measure preserving maps are isometries defined on a Riemannian manifold but more flexible examples are given by symplectic transformations of a symplectic space. We will deal mainly with smooth transformations over manifolds equipped with a volume form.

The definition of recurrent point for a dynamical system involves only the topology of a space:

Definition 7.3. *Let M be a Hausdorff topological space and $\phi : M \to M$ be a map. We say that $x \in M$ is a recurrent point if, for each neighborhood U of x,*

$$\{\, n \in \mathbf{N} \mid f^n(x) \in U \,\}$$

is infinite.

A basic theorem on dynamical systems is the following Poincaré's recurrence theorem for maps which are measure preserving

Theorem 7.4 (Poincaré's recurrence theorem)**.** *Let (M,μ) be a finite measure space and $\phi : M \to M$ be a measure preserving map. Then, for any measurable subset $A \subset M$ and for almost every $x \in A$, the set*

$$\{ n \in \mathbf{N} \mid \phi^n(x) \in A \}$$

is infinite.

Proof. The statement is equivalent to the condition that for any $N \in \mathbf{N}$ the set

$$E_N(\phi) = \left\{ x \in A \mid \phi^n(x) \in M \setminus A \text{ for } n \geq N \right\}$$

satisfies

$$\mu\left(E_N(\phi)\right) = 0.$$

Observe that $E_N(\phi) = \bigcup_{k=N}^{2N-1} E_1(\phi^k)$. We need to show therefore only that $\mu\left(E_1(\phi)\right) = 0$. Observe now that

$$E = E_1(\phi) = A \cap \bigcap_{n=1}^{\infty} \phi^{-n}(M \setminus A)$$

so that $\phi^{-n}(E) \cap E = \emptyset$ for $n \geq 1$ as $E \subset A$ and $\phi^{-n}(E) \subset M \setminus A$. Therefore, more generally, we have $\phi^{-n}(E) \cap \phi^{-m}(E) = \emptyset$ for all $n \neq m$.

Now, as the map is measure preserving, one has

$$\mu(M) \geq \mu\left(\cup_{n=1}^{\infty} \phi^{-n}(E)\right) = \sum_{n=1}^{\infty} \mu\left(\phi^{-n}(E)\right) = \sum_{n=1}^{\infty} \mu(E)\,.$$

We conclude that for a finite measure space $\mu(E) = 0$. □

To obtain recurrent points we use the following theorem which is a consequence of Poincaré's recurrence theorem.

Theorem 7.5. *Let (M,μ) be a finite measure space where M is a Hausdorff second countable space and μ a Borel measure. Let $\phi : M \to M$ be a measure preserving map. Then, almost every $x \in M$ is recurrent.*

Proof. Exercise. □

In order to deal with actions by automorphisms groups and not only with iterations of a single map one introduces the following definitions. Here G is a topological group which will be a Lie group or a countable discrete group. More generally, G is a locally compact second countable topological group.

Definition 7.6. *Let (M,μ) be a measure space and $G \times M \to M$ be a measurable action. We say the action is measure preserving if $g_*\mu = \mu$ for all $g \in G$.*

Definition 7.7. *Let M be a Hausdorff topological space and $G \times M \to M$ be a continuous action. We say that $x \in M$ is a recurrent point for the action if there exists a non-relatively compact sequence $g_k \in G$ satisfying $g_k x \to x$.*

If G is a subset of the space of homeomorphisms of M, there exists a sequence g_k of homeomorphisms which leaves every compact in the compact open topology on the space of homeomorphisms and such that $g_k x$ converges to x. Poincaré's recurrence theorem implies that measure preserving actions are recurrent almost everywhere (see [**FK**]):

Theorem 7.8. *Let (M,μ) be a finite measure space where M is a Hausdorff second countable space and μ a Borel measure. Let $G \times M \to M$ be a measure preserving action. Then almost all points of M are recurrent.*

Proof. We prove first that for every $A \subset M$ and almost all $x \in A$ the set

$$R_A = \{\, g \in G \mid gx \in A \,\}$$

is not relatively compact. We conclude then the proof by the same argument as in 7.5 using a countable base of open subsets. For each compact subset $K \subset G$ define

$$B_K = \{\, x \in M \mid gx \in A^c \text{ for all } g \in K^c \,\}.$$

We claim that $\mu(B_K) = 0$. As G is a countable union of compact subspaces the complement of the set $R_A = \bigcup_K B_K$ will be of zero measure and the theorem is proved.

In order to prove the claim define for $L \subset G$ a countable dense symmetric subset (it contains the inverse of each of its elements), the set:

$$C_K = \{\, x \in A \mid gx \in A^c \text{ for all } g \in L \cap K^c \,\}.$$

We show now that there exists a sequence of elements (g_i) in L such that $g_i^{-1} C_K$ are disjoint and this clearly proves the theorem as M is of finite measure. In order to construct the sequence, start with $g_1 \in L$ and chose g_2 such that $g_2 g_1^{-1} \in L \cap K^c$. Clearly $g_1(g_2^{-1} C_K \cap g_1^{-1} C_K) = g_1 g_2^{-1} C_K \cap C_K = \emptyset$. By induction, suppose $(g_i)_{1 \le i \le r}$ are chosen such that $g_i g_j^{-1} \in L \cap K^c$ for $i, j \le r$. Then $\cap_{i=1}^r (L \cap K^c) g_i = \cap_{i=1}^r L \cap K^c g_i \neq \emptyset$ and one choses g_{r+1} in the intersection. The sequence is constructed and this finishes the proof. $\square$

A non-wandering point $x \in M$ for the action of G is a point such that for every neighborhood U of x the set $\left\{ g \in G \,\middle|\, gU \cap U \neq \emptyset \right\}$ is not relatively compact. Suppose now that the measure μ is a non-trivial Borel measure. By Poincaré recurrence theorem, if the action is measure preserving then all points are non-wandering points.

In what follows, we consider a special non-compact one parameter subgroup of diffeomorphisms: Anosov flows. An Anosov flow preserving a geometric structure imposes a very strong constraint on the geometry. We will see later that even if the group of automorphisms is discrete (in contrast with a flow) the geometry might be equally constrained.

7.1.2 Anosov diffeomorphisms and flows

Definition 7.9. *Let M be a compact manifold and $\phi : M \to M$ be a diffeomorphism. We say ϕ is an Anosov diffeomorphism if there exists an invariant splitting $\mathrm{T}M = E^s \oplus E^u$ of the tangent bundle where E^s and E^u are non-trivial distributions of constant rank verifying the following estimates (with respect to any Riemannian metric on M).*

1. *The* stable distribution E^s *is* uniformly contracted *by φ^n*, i.e. *there are two constants $C > 0$ and $0 < \lambda < 1$ such that for any $n \in \mathbf{N}$ and $x \in M$:*

$$\left\| \mathrm{d}_x\varphi^n\big|_{E^s} \right\| \le C\lambda^n. \tag{285}$$

2. *The* unstable distribution E^s *is* uniformly expanded *by φ^n*, i.e. *uniformly contracted by (φ^{-n}):*

$$\left\| \mathrm{d}_x\varphi^{-n}\big|_{E^u} \right\| \le C\lambda^n. \tag{286}$$

The simplest example is the following determinant one linear map defined on $\mathbf{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which induces a diffeomorphism $\phi : \mathrm{T}^2 \to \mathrm{T}^2$ on the torus quotient space

$$\mathrm{T}^2 = \mathbf{R}^2/\mathbf{Z}^2.$$

Observe that the eigenvalues of A are given by $\lambda = \frac{3+\sqrt{5}}{2}$ and $\lambda^{-1} = \frac{3-\sqrt{5}}{2}$ with corresponding eigenspaces E^s and E^u generated by $\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$. Also, one can identify E^s and E^u on the tangent space of the torus which is identified at each point to $\mathbf{R}^2$. We obtain for the euclidean norm:

$$\left\| \mathrm{d}_x\phi^n\big|_{E^s} \right\| = \left\| A^n\big|_{E^s} \right\| = \lambda^n,$$

and

$$\left\| \mathrm{d}_x\phi^{-n}\big|_{E^u} \right\| = \left\| A^{-n}\big|_{E^u} \right\| = \lambda^n.$$

Important properties of this map are the following (see [**KH**]):

1. The fixed points of ϕ are dense in the torus.

2. ϕ is topologically transitive (there exists a dense orbit).

3. ϕ is topologically mixing (for any two open sets U and V there exists $N \in \mathbf{N}$ such that $\phi^n(U) \cap V$ for all $n \geq N$.

It is not known which compact manifolds admit Anosov diffeomorphisms. It is a general belief that they must be infranil manifolds, that is, compact quotients of nilpotent Lie groups by a discrete subgroup. For the 2-torus one can prove that they are all conjugated by a C^1-diffeomorphism to a linear one corresponding to a determinant one integer matrix as in the example above.

We get now to the definition of Anosov flows:

Definition 7.10. *A non-singular flow* (φ^t) *of class* C^∞ *on a closed Riemannian manifold* M *is called* Anosov, *if its differential preserves a splitting* $\mathrm{T}M = E^s \oplus E^0 \oplus E^u$ *of the tangent bundle, where* $E^0 = \mathbf{R}X^0$ *with* X^0 *the (non-singular) vector field generating* (φ^t), *and where* E^s *and* E^u *are non-trivial distributions of constant rank verifying the following estimates (with respect to any Riemannian metric on* M*).*

1. *The* stable distribution E^s *is* uniformly contracted *by* (φ^t), i.e. *there are two constants* $C > 0$ *and* $0 < \lambda < 1$ *such that for any* $t \in \mathbf{R}$ *and* $x \in M$*:*

$$\left\| \mathrm{d}_x \varphi^t \big|_{E^s} \right\| \leq C\lambda^t. \tag{287}$$

2. *The* unstable distribution E^s *is* uniformly expanded *by* (φ^t), i.e. *uniformly contracted by* (φ^{-t})*:*

$$\left\| \mathrm{d}_x \varphi^{-t} \big|_{E^u} \right\| \leq C\lambda^t. \tag{288}$$

Observe that the definition does not depend on the Riemannian metric because any two metrics are equivalent on a compact manifold. In the definition of Anosov flows, *no regularity* is requested on the stable and unstable distributions. Even if they are automatically Hölder continuous ([]), E^s and E^u have, in general, no reason to be differentiable (even if the flow is C^∞).

A first example of an Anosov flow is obtained through the suspension of an Anosov diffeomorphism:

Suspension of an Anosov diffeomorphism Let M be a manifold and $\phi : M \to M$ a map. The mapping torus is the manifold

$$\Sigma_\phi = M \times [0,1] / (x,0) \backsim (\phi(x),1).$$

We define the suspension flow $\varphi^t_\phi : \Sigma_\phi \to \Sigma_\phi$ to be the flow given locally by the formula $\varphi^t_\phi(x,s) = (x, s+t)$. Observe then that $\varphi^t_\phi(x,1) = (\phi(x), t)$ for $0 \le t \le 1$ and, more generally, $\varphi^{n-1+t}_\phi(x,1) = (\phi^n(x), t)$ for $0 \le t \le 1$.

Exercise: Prove that the suspension flow of an Anosov diffeomorphism is an Anosov flow.

The suspension flow has the property that the distribution generated by the stable and unstable distributions is integrable. Indeed its integral submanifolds are the leaves $M \times \{s_0\}$ where $0 \le s_0 \le 1$.

One can define a strong equivalence between flows by imposing conjugation via a diffeomorphism.

Definition 7.11. *Two flows, (φ^t) on M and (ψ^t) on N, are equivalent if there exists a diffeomorphism $h\colon M \to N$ such that the flow $(h^{-1} \circ \psi^t \circ h)$ is equal to (φ^t).*

One usually allows reparametrizations of the orbits as in the following much weaker notion of equivalence.

Definition 7.12. *Two flows (φ^t) on M and (ψ^t) on N are orbit equivalent if there exists a diffeomorphism $h\colon M \to N$ such that the flow $(h^{-1} \circ \psi^t \circ h)$ on M is a time-change of (φ^t),* i.e. *there exists a time-change function $\tau : \mathbf{R} \times M \to \mathbf{R}$ (satisfying $\tau(t,x) \ge 0$ for $t \ge 0$) such that $\varphi^{\tau(t,x)}(x) = (h^{-1} \circ \psi^t \circ h)(x)$ for all $(t,x) \in \mathbf{R} \times M$.*

A particular class of Anosov flows is the one preserving a contact form θ. This implies that it also preserves a volume form. In dimension three this is $\theta \wedge d\theta$.

An important problem concerns the existence of a dense orbit of a dynamical system or a group action. Indeed, a dense orbit will imply that the geometry is close to being homogeneous.

General Anosov flows are not necessarily topologically transitive. The first examples of nontransitive Anosov flows were constructed in [**FW**]. On the other hand one can prove (see [**KH**]) that the nonwandering set of an Anosov flow admits a spectral decomposition, that is, a disjoint decomposition

$$NW = \bigcup_{i=1}^{N} \Omega_i$$

where each Ω_i is a closed invariant subset where the flow is topologically transitive. In the case of a volume preserving flow, the nonwandering set coincides with the manifold and therefore the decomposition is trivial and one concludes that the flow is topologically transitive.

A corollary of this result is the following

Theorem 7.13. *Let (ϕ^t) be an Anosov flow on a compact manifold M preserving a contact form. Then the flow is topologically transitive (it is in fact topologically mixing, see [**KH**] pg. 576).*

7.1.3 Geodesic flows

Definition 7.14. *For a complete Riemannian manifold (M, g) define the geodesic flow $\varphi^t : \mathrm{T}M \to \mathrm{T}M$ by*

$$\varphi^t(x, v) = \left(\exp_x(tv), \frac{\mathrm{d}}{\mathrm{d}t}\exp_x(tv)\right). \tag{289}$$

Exercise Prove that this is indeed a flow.

Note Since geodesics can be parametrized by unit tangent vectors, the flow itself preserves the unit tangent bundle:

$$\mathrm{T}^1M = \left\{(x, v) \in \mathrm{T}M \,\middle|\, g_x(v, v) = 1\right\}. \tag{290}$$

So we consider the geodesic flow restricted to the unit tangent bundle

$$\varphi^t : \mathrm{T}^1M \to \mathrm{T}^1M \tag{291}$$

and one should note that the fiber of $\mathrm{T}^1M \to M$ is compact and diffeomorphic to the $(n-1)$-sphere.

The hyperbolic half-place Recall that for this model, the metric is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{|\mathrm{d}z|^2}{(\mathrm{Im}\, z)^2}. \tag{292}$$

The induced volume form is

$$\mathrm{d}v = \frac{1}{y^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{293}$$

We recall the description of the geodesics of hyperbolic space and the description of its isometry group:

Proposition 7.15. *The orientation preserving isometry group of the hyperbolic upper-half place is* $\mathrm{PSL}(2,\mathbf{R})$ *which acts by Möbius transformations:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}. \tag{294}$$

Proposition 7.16. *The geodesics of* $\mathbf{H}^2_{\mathbf{R}}$ *are vertical lines or circles perpendicular to the* $\mathbf{R}$*-axis.*

We also recall the fundamental uniformization theorem describing all complete hyperbolic surfaces as quotients of $\mathbf{H}^2_{\mathbf{R}}$.

Theorem 7.17. *Every complete surface of constant curvature* -1 *is a quotient* $\Gamma \backslash \mathbf{H}^2_{\mathbf{R}}$ *for a torsion free discrete subgroup* $\Gamma \subset \mathrm{PSL}(2,\mathbf{R})$

The orientation preserving isometry group $\mathrm{PSL}(2,\mathbf{R})$ acts on $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$ with trivial isotropy. We identify then $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$ with $\mathrm{PSL}(2,\mathbf{R})$. For instance one can fix the point $\boldsymbol{i} \in \mathbf{H}^2_{\mathbf{R}}$ and the unit vertical vector $v = \boldsymbol{i}$ based at $\boldsymbol{i}$ and obtain all the other unit vectors (at every other point) by $h \cdot v$ for $h \in \mathrm{PSL}(2,\mathbf{R})$.

The geodesic passing through $\boldsymbol{i}$ in the direction $v = \boldsymbol{i}$ is given by $\boldsymbol{i}e^t$ and therefore $\boldsymbol{i}e^t$ is again the tangent vector at $\boldsymbol{i}e^t$. In order to identify the flow along that geodesic, one can observe that the family

$$g^t = \left\{ \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} \right\} \tag{295}$$

is such that $g_t \cdot \boldsymbol{i} = \boldsymbol{i}e^t$. One concludes that the orbit of the vector v by the geodesic flow is given by g_t.

Suppose now that $h \in \mathrm{PSL}(2,\mathbf{R})$ gives a different vector $h \cdot v \in \mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$. Then the geodesic determined by this vector is the image of the geodesic determined by v that is hg^t. We obtain therefore that the geodesic flow in $\mathrm{PSL}(2,\mathbf{R})$ is given by right multiplication by g^t.

We describe now the stable and unstable distributions. Given a geodesic $\gamma_v(t)$ (say defined by a vector v based at z), one can consider the set of points equidistant from a point in the geodesic along positive times and containing z. The limit for $t \to +\infty$ is a Euclidean circle tangent to the real line (for geodesics which are half-circles) or horizontal lines (for vertical geodesics) called the horosphere S^+_v. The set of inward unit orthogonal vectors to a horosphere (called the horocycle in $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$) define geodesics which all converge towards the same point at infinity as $\gamma_v(t)$. Analogously we define the horosphere S^-_v obtained by taking limits of circles centered on the geodesic along negative times. The set of inward unit orthogonal vectors to that horosphere define geodesics which all converge towards the same point at infinity as $\gamma_v(-t)$.

The horocycle flow $h_t^* : T^1 H^2_{\mathbf{R}} \to T^1 H^2_{\mathbf{R}}$ is defined to be the map moving vectors along S_v^+ to the left of v with unit speed. Analogously, the horocycle flow $h_t : T^1 H^2_{\mathbf{R}} \to T^1 H^2_{\mathbf{R}}$ is defined to be the map moving vectors along S_v^- to the right of $-v$ with unit speed. We have $h_t = -h^*_{-t}(-v)$.

The horosphere $S^+_{i_i}$ is the horizontal line $y = 1$ in the half plane which is an orbit of the left action by

$$h_t^+ = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

In the identification of PSL(2, $\mathbf{R}$) with $T^1 H^2_{\mathbf{R}}$ we obtain then that the horocycle flow is

$$h_t^*(g) = g \cdot h_t^+.$$

Analogously the other horocycle flow is given by

$$h_t(g) = g \cdot h_t^-.$$

where

$$h_t^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Exercise: Prove that $g_t \circ h_s^* = h^*_{se^t} \circ g_t$ and, analogously, $g_t \circ h_s = h_{se^t} \circ g_t$.

Consider PSL(2, $\mathbf{R}$) identified with $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$. Any metric defined on $\mathrm{T}_e\,\mathrm{PSL}(2,\mathbf{R})$ can be extended to a left-invariant metric which can be seen as an invariant metric on $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$ under the action of PSL(2, $\mathbf{R}$).

Proposition 7.18. *The geodesic flow on* $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$ *is Anosov with respect to any left-invariant metric.*

Proof. At $v = \boldsymbol{i} \in \mathrm{T}^1_{\boldsymbol{i}}\mathbf{H}^2_{\mathbf{R}}$, consider the three vectors in $\mathrm{T}_v\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$ corresponding to the three flows. That is to say:

$$e_0 = \frac{\mathrm{d}}{\mathrm{d}t} g^t(v)\Big|_{t=0} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{296}$$

$$e_1 = \frac{\mathrm{d}}{\mathrm{d}t} h^{*\,t}(v)\Big|_{t=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{297}$$

$$e_2 = \frac{\mathrm{d}}{\mathrm{d}t} h^t(v)\Big|_{t=0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{298}$$

We compute $\|g^t_* e_i\|$ for an invariant metric on $\mathrm{T}^1\mathbf{H}^2_{\mathbf{R}}$. We have :

$$\left\|g^t_* e_0\right\| = \left\|g^{-t}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}g^t\right\| = \left\|\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\right\| = \|e_0\|, \tag{299}$$

$$\left\|g^t_* e_1\right\| = \left\|g^{-t}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}g^t\right\| = \left\|\begin{pmatrix}0 & e^{-t}\\ 0 & 0\end{pmatrix}\right\| = e^{-t}\|e_1\|, \tag{300}$$

$$\left\|g^t_* e_2\right\| = \left\|g^{-t}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}g^t\right\| = \left\|\begin{pmatrix}0 & 0\\ e^t & 0\end{pmatrix}\right\| = e^{t}\|e_2\|. \tag{301}$$

□

On a compact surface Let $\Sigma = {}_{\Gamma_0}\backslash^{\mathbf{H}^2_{\mathbf{R}}}$ be the quotient by a cocompact lattice $\Gamma_0 \subset \mathrm{PSL}(2,\mathbf{R})$ (without torsion). Since $\mathrm{PSL}(2,\mathbf{R})$ acts simply transitively on the unitary tangent bundle of $\mathbf{H}^2_{\mathbf{R}}$, one verifies that the geodesic flow on the unit tangent bundle $\mathrm{T}^1\Sigma$ is smoothly conjugated to the right diagonal flow (by a constant time-change) on the quotient ${}_{\Gamma_0}\backslash^{\mathrm{PSL}(2,\mathbf{R})}$.

Geodesic flows of compact hyperbolic surfaces are very specific.

1. Their stable and unstable distributions both are C^∞ (because they arise from left-invariant distributions on the Lie group $\mathrm{PSL}(2,\mathbf{R})$).

2. The distribution $E^s \oplus E^u$ is moreover a contact distribution and Anosov flows verifying this last property are called *contact-Anosov*.

3. If (φ^t) is a contact-Anosov flow with smooth invariant distributions and X^0 is its infinitesimal generator, then we define the *canonical* 1-form θ of (φ^t) by $\theta|_{E^s\oplus E^u} = 0$ and $\theta(X^0) = 1$. This is a contact form with kernel $E^s \oplus E^u$. By construction, (φ^t) preserves the strict path structure $\mathscr{T} = (E^s, E^u, \theta)$ that we call *canonical*.

Note that the structures obtained in this way have a purely geometrical specificity: the Reeb flow of their contact form is a flow of automorphisms of the structure $\mathscr{T}$ (this has no reason to be true in general). Indeed, the Reeb vector field of the canonical structure of (φ^t) is its generator X^0, so that the Anosov flow itself is encoded in the structure $\mathscr{T}$.

7.2 Strict path geometry with non compact automorphism group

This section is based on [**FMMV**] (see also [**MM**]).

Definition 7.19. *A group acting on a manifold is topologically transitive if it has a dense orbit.*

Theorem 7.20. *Let $(M, E^1 \oplus E^2, \theta)$ be a strict path structure on a closed three-manifold M. We assume that* $\mathrm{Aut}(M)$ *is topologically transitive. Then the canonical curvature of M is constant with vanishing torsion and its connexion is complete. By Theorem 6.20 (p. 76) and Corollary 6.22:*

$$M \cong {}_{\Gamma}\backslash^{\mathrm{Heis} \rtimes \mathbf{R}_+^*}/_{\mathbf{R}_+^*} \text{ or } M \cong {}_{\Gamma}\backslash^{\widetilde{\mathrm{SL}(2,\mathbf{R})} \times \mathbf{R}_+^*}/_{\Delta(\mathbf{R}_+^*)}. \tag{302}$$

where $\Delta(\mathbf{R}_+^*)$ *is the diagonal embedding of* $\mathbf{R}_+^*$ *in the product.*

Lemma 7.21. *Under those assumptions, the curvature is constant and is of the form*

$$\Omega = \begin{pmatrix} R\theta^1 \wedge \theta^2 & & \\ & -2R\theta^1 \wedge \theta^2 & \\ & & R\theta^1 \wedge \theta^2 \end{pmatrix}. \tag{303}$$

Proof. A priori, the curvature of M is of the form

$$\Omega = \begin{pmatrix} \mathrm{d}w & & \\ \tau_2^1 \theta \wedge \theta^2 & -2\,\mathrm{d}w & \\ 0 & \tau_1^2 \theta \wedge \theta^1 & \mathrm{d}w \end{pmatrix} \tag{304}$$

with τ_2^1, τ_1^2 two functions and a decomposition of $\mathrm{d}w$ into:

$$\mathrm{d}w = R\theta^1 \wedge \theta^2 + W^1 \theta \wedge \theta^1 + W^2 \theta \wedge \theta^2 \tag{305}$$

with R, W^1, W^2 three functions. To prove the theorem, we need to prove that only R is non zero.

The idea is that an automorphism preserves the curvature, so the curvature must be stable under a sequence of transformations that is not relatively compact but the existence of recurrent points impose strong constraints.

Recall that, if h the matrix with diagonal (a, a^{-2}, a), $R_h^* \omega = \omega$, $R_h^* \omega^1 = a^3 \omega^1$ and $R_h^* \omega^2 = \frac{1}{a^3} \omega^2$. Also, $R_h^* \Omega = \mathrm{Ad}(h)^{-1} \Omega$ and we get:

$$R_h^* \Omega = \begin{pmatrix} \mathrm{d}w & & \\ a^3 \tau_2^1 \theta \wedge \theta^2 & -2\,\mathrm{d}w & \\ 0 & a^{-3} \tau_1^2 \theta \wedge \theta^1 & \mathrm{d}w \end{pmatrix}. \tag{306}$$

Taking into account the change of ω, ω^1 and ω^2 we obtain the curvature functions change. Indeed from

$$R_h^* \,\mathrm{d}w = (R_h^* R) R_h^* \theta^1 \wedge R_h^* \theta^2 + (R_h^* W^1) \theta \wedge R_h^* \theta^1 + (R_h^* W^2) \theta \wedge R_h^* \theta^2$$

$$= (R_h^* R)\theta^1 \wedge \theta^2 + (R_h^* W^1) a^3 \theta \wedge \theta^1 + (R_h^* W^2)\frac{1}{a^3}\theta \wedge \theta^2$$

and, as this expression is equal to $\mathrm{d}w$, we have

Similarly, we obtain

$$R_h{}^* \tau^1_2 = a^6 \tau^1_2, \;\; R_h{}^* \tau^2_1 = \frac{1}{a^6}\tau^2_1.$$

Also, the Bianchi identities link together τ^1_2, τ^2_1 with W^1 and W^2.

$$\mathrm{d}\Omega = [\Omega \wedge \omega] \tag{307}$$

$$= \begin{pmatrix} \mathrm{d}w & & \\ \tau^1_2 \theta \wedge \theta^2 & -2\,\mathrm{d}w & \\ 0 & \tau^2_1 \theta \wedge \theta^1 & \mathrm{d}w \end{pmatrix} \wedge \begin{pmatrix} w & & \\ \theta^1 & -2w & \\ \theta & \theta^2 & w \end{pmatrix} \tag{308}$$

$$\Longrightarrow \begin{cases} \mathrm{d}(\tau^1_2 \theta \wedge \theta^2) = \tau^1_2 \theta \wedge \theta^2 \wedge w - 2\,\mathrm{d}w \wedge \theta^1 \\ \mathrm{d}(\tau^2_1 \theta \wedge \theta^1) = -2\tau^2_1 \theta \wedge \theta^1 \wedge w + \mathrm{d}w \wedge \theta^2 \end{cases} \tag{309}$$

$$\Longrightarrow \begin{cases} \mathrm{d}(\tau^1_2 \theta \wedge \theta^2) = \tau^1_2 \theta \wedge \theta^2 \wedge w - 2W^2 \theta \wedge \theta^2 \wedge \theta^1 \\ \mathrm{d}(\tau^2_1 \theta \wedge \theta^1) = -2\tau^2_1 \theta \wedge \theta^1 \wedge w + W^1 \theta \wedge \theta^1 \wedge \theta^2 \end{cases} \tag{310}$$

Now, these two equations clearly imply that if $\tau^2_1 = \tau^1_2 = 0$ then $W^1 = W^2 = 0$. We need to show $\tau^2_1 = \tau^1_2 = 0$.

As the group of automorphisms is non-compact and preserves the measure induced by the contact form we apply Poincaré's recurrence theorem: almost all points are recurrent. Let $x \in M$ be a recurrent point and $\varphi_k \in \mathrm{Aut}(M)$ a non relatively compact sequence such that $\varphi_k(x) \to x$.

That means that at x we have $\varphi_k^* \theta^1 = a_k^3 \theta^1 = R^*_{a_k}\theta^1$ and $\varphi_k^*\theta^2 = \frac{1}{a_k^3}\theta^2 = R^*_{a_k}\theta^2$ with $a_k \to +\infty$. On the other hand the curvature is preserved by the automorphism group so $\varphi_k^*\Omega = \Omega$. Comparing with the above equation for $R^*_{a_k}\Omega = \mathrm{Ad}(a_k)^{-1}\Omega$, shows $\tau^1_2 = 0$ and by the same argument $\tau^2_1 = 0$. □

The fact that there exists a dense orbit implies that the function R defined on the base manifold M is constant. The proof of the theorem now relies on the completeness of the structure. By Carrière [**Carriere**] and Klingler [**Klingler**] it is indeed the case. (See also [**DZ**].) In the proof we assume that the structure is of class C^2. Indeed we use the dynamics to obtain information about the curvature. If we suppose the structure is of class C^3 we can give away the hypothesis of the existence of a dense orbit and use the dynamics again to show that the curvature function is constant.

Indeed, observe that R is a function defined on M. If the structure is of class C^3, taking its differential, we write

$$dR = R_0\theta + R_1\theta^1 + R_2\theta^2,$$

where $(\theta,\theta^1,\theta^2)$ is an adapted coframe on M. Now we consider a recurrent point $x \in M$ with ϕ_k a non-relatively compact sequence of automorphisms such that $\phi_k(x) \to x$. R is invariant under automorphisms and therefore we have

$$dR = \phi_k^*(dR) = R_0 \circ \phi_k \phi_k^*\theta + R_1 \circ \phi_k \phi_k^*\theta^1 + R_2 \circ \phi_k \phi_k^*\theta^2.$$

But $\phi_k^*\theta = \theta$, $\phi_k^*\theta^1 = a_k\theta^1$ and $\phi_k^*\theta^2 = \frac{1}{a_k}\theta^2$ with $a_k \to \infty$ at x. As $R_i \circ \phi_k(x) \to R_i(x)$, we conclude that $R_1 = R_2 = 0$. Finally, the fact $R_0 = 0$ is implied by the zero torsion condition.

7.2.1 Heisenberg lattices with non compact automorphism groups

Consider the basis (e_1, e_2, e_3) of $\mathfrak{heis}(3)$ with the brackets $[e_1, e_2] = e_3$ and $[\cdot, e_3] = 0$. The action of $\mathbf{R}_+^*$ on (e_1, e_2, e_3) is given by the diagonal matrices (in this basis)

$$\mathbf{R}_+^* = \left\{ \begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & 1 \end{pmatrix} \middle| \lambda \in \mathbf{R}_+^* \right\}. \tag{311}$$

(Note that (e_1, e_2, e_3) corresponds under the connexion to $(\theta^1,\theta^2,\theta)$.)

Lemma 7.22 ([**DZ**])**.** *Up to finite index, a closed manifold* $\Gamma \backslash \mathrm{Heis}(3) \rtimes \mathbf{R}^* / \mathbf{R}^*$ *is given by a subgroup* $\Gamma \subset \mathrm{Heis}(3)$ *that is a cocompact lattice of* $\mathrm{Heis}(3)$.

A lattice in Heis(3) projects to a lattice in $\mathbf{R}^2$ following the following exact sequence

$$0 \to \mathbf{R} \to \mathrm{Heis}(3) \to \mathbf{R}^2 \to 0,$$

giving an exact sequence

$$0 \to \mathbf{Z} \to \Gamma \to \Gamma_2 \to 0.$$

In the following proposition we characterize hyperbolic transformations of $\mathbf{R}^2$, that is, transformations written in a basis of eigenvectors as

$$\begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix},$$

which preserve a lattice of $\mathbf{R}^2$. Those transformations will give rise to automorphisms of the Heisenberg group which preserve a lattice in Heis(3) which will induce an automorphism on a compact quotient of the Heisenberg group.

Proposition 7.23. *There exists a non trivial automorphism $\varphi \in \mathbf{R}_+^*$ (identified to diagonal matrices in* $\mathrm{SL}(2,\mathbf{R})$*) preserving a cocompact lattice Γ_2 of $\mathbf{R}^2$ if, and only if, ϕ verifies a quadratic equation $\phi^2 = q\phi - 1$ with $q \geq 3$ an integer.*

Note that in such a case, we can explicitly give a lattice:

$$\Gamma_2 = \left\{ \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \middle| (x,y) \in \mathbf{Z}^2 \right\} \tag{312}$$

since indeed the automorphism preserves the lattice:

$$\begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix} \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix} = \begin{pmatrix} \phi & \phi^2 \\ 1 & \phi^{-1} \end{pmatrix} \tag{313}$$

$$= \begin{pmatrix} \phi & q\phi - 1 \\ 1 & q - \phi \end{pmatrix} \tag{314}$$

$$= \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & q \end{pmatrix} \tag{315}$$

for the last matrix belongs to $\mathrm{GL}(2,\mathbf{Z})$.

Proof. If

$$\Gamma_2 = \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \mathbf{Z}^2 \tag{316}$$

then the fact that the automorphism preserves a lattice translates to:

$$\begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{317}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbf{Z}). \tag{318}$$

In particular, the trace of the automorphism must be an integer $q \in \mathbf{Z}$ This proves that $\mathbf{Z}[\phi]$ is a quadratic extension of $\mathbf{Z}$. Indeed, note that $\phi^{-1} = q - \phi$ and therefore $\phi^2 = q\phi - 1$. We also have $q \geq 3$ since $\phi \neq \phi^{-1}$. □

Corollary 7.24. *Such an automorphism φ can not be the time $t = 1$ of a flow because q is an integer.*