Introduction to Riemann surfaces

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1 Preliminaries

General references for these notes are [Don11; For81; Mir95].

Differential manifolds are usually equipped with other structures. For instance, every manifold can be given a Riemannian metric. These structures might have local invariants, as the curvature in the Riemannian case, which distinguishes arbitrarily small neighborhoods. Other structures, on the contrary, are not distinguished by local invariants. An example is the structure of a complex manifold.

Definition 1.1. *A complex manifold of dimension n is a connected differential manifold (which we suppose Hausdorff and second countable) equipped with a cover $X = \cup U_\alpha$ by open sets U_α and homeomorphisms (charts) $z_\alpha : U_\alpha \to z_\alpha(U_\alpha) \subset \mathbf{C}^m$ such that the maps $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ are biholomorphisms.*

Remark 1.2. *Sometimes we will consider complex manifolds without the connectedness hypothesis.*

Once a chart cover (we call it an atlas) is defined, one usually considers a maximal family of charts compatible with the given cover. We are thus allowed to introduce new charts whenever we need. Maps between complex manifolds are defined as for real manifolds:

Definition 1.3. *A continuous map $F : X \to Y$ between complex manifolds is holomorphic if, for charts $z_\alpha : U_\alpha \to \mathbf{C}^n$ and $w_\beta : V_\beta \to \mathbf{C}^n$ of M and N respectively such that $F(U_\alpha) \subset V_\beta$, we have that $w_\beta \circ F \circ z_\alpha^{-1} : z_\alpha(U_\alpha) \to w_\beta(V_\beta)$ is holomorphic.*

We say then that two complex manifolds are biholomorphic if there exists a diffeomorphism between them which is a holomorphic map.

Definition 1.4. *A Riemann surface is a one dimensional complex manifold.*

Remark 1.5. *The second countability hypothesis in the definition of a complex manifold can be put aside in the case of dimension one: having an atlas of one dimensional complex charts on a Hausdorff space implies second countability (Radó's theorem 1925).*

Example 1.6. *The Riemann sphere $\mathbf{CP}^1$ is a Riemann surface whose underlying topological manifold is the two dimensional sphere S^2. We write $S^2 = \mathbf{C} \cup \{\infty\}$. There are two natural charts:*

1. $z_1 : \mathbf{C} \cup \{\infty\} \setminus \{0\} = U_1 \to \mathbf{C}$ *defined by* $z_1(z) = 1/z$ *if* $z \neq 0$ *and* $z_1(\infty) = 0$

2. $z_2 : \mathbf{C} = U_2 \to \mathbf{C}$ *defined by* $z_2(z) = z$

In the intersection $U_1 \cap U_2 = \mathbf{C} \setminus \{0\} = z_1(\mathbf{C} \setminus \{0\}) = z_2(\mathbf{C} \setminus \{0\})$ *we obtain*

$$z_2 \circ z_1^{-1} : \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$$

given by $z_2 \circ z_1^{-1}(z) = 1/z$ *which is a biholomorphism.*

This is the most symmetric example of a Riemann surface. It has the largest group of automorphisms (the group of diffeomorphisms which are holomorphic) namely, the group of Möbius transformations. The Riemann sphere contains as an open subset the complex plane whose automorphism group (the similarity group) is a subgroup of the Möbius group.

Complex manifolds of higher dimensions appear naturally in the theory of Riemann surfaces. In particular, we will show that every Riemann surface is embedded in a complex projective space.

Remark 1.7. *We will see that any orientable topological surface has a complex structure making it a Riemann surface. On the contrary, there are higher dimensional manifolds which don't admit any complex structure. For instance, a basic open problem is to decide if the sphere* S^6 *admits a complex structure, the other spheres of dimension bigger than 2 are known not to admit a complex structure.*

Particularly important is the study of holomorphic maps of a Riemann surface X into $\mathbf{C}$ (holomorphic functions). That is, continuous functions $f : X \to \mathbf{C}$ such that for every chart $\phi : U \to \mathbf{C}$ of M the map $f \circ \phi^{-1} : \phi(U) \to \mathbf{C}$ is holomorphic. On a (connected) compact Riemann surface, holomorphic functions are constant. Indeed, there would be a maximum of the function at a point and the maximum principle applied to $f \circ \phi^{-1}$ on a chart $\phi : U \to \mathbf{C}$ containing that point will force the function to be constant.

A much richer class of functions defined on a Riemann surface are holomorphic maps from M to the Riemann sphere. Indeed a basic theorem in the theory is that there exists at least one non-constant meromorphic function. We will see that Riemann-Roch theorem gives a quantitative description of the space of meromorphic functions. One may define them in a way which makes appear the algebraic structure of a field on the space of all meromorphic functions: Meromorphic functions are holomorphic functions defined on the complement of a closed and discrete subset of points (called poles) such that viewed through the charts are meromorphic.

Definition 1.8. *Let* X *be a Riemann surface and* $D \subset X$ *a closed and discrete subset. A meromorphic function is a holomorphic function* $f : X \setminus D \to \mathbf{C}$ *such that for all charts the composition* $f \circ \phi^{-1}$ *is meromorphic. The set of meromorphic functions on* X *is denoted by* $\mathcal{M}(X)$.

A point $p \in X$ is a pole of a meromorphic function f if $\lim_{z\to p} f(z) = \infty$. Meromorphic functions on a Riemann surface may be interpreted as holomorphic maps into $\mathbf{CP}^1$: it is a consequence of the Riemann removable singularity theorem and we leave it as an exercise.

Proposition 1.9. *Let X be a Riemann surface and $f \in \mathcal{M}(X)$ be meromorphic function. Let $D \subset X$ be the set of poles of f. Define an extension of f, $\tilde{f} : \mathbf{CP}^1 \to \mathbf{CP}^1$, by $\tilde{f}(p) = \infty \in \mathbf{CP}^1$ for all $p \in D$. Then $\tilde{f}$ is a holomorphic map. Conversely, any holomorphic map $\tilde{f} : \mathbf{CP}^1 \to \mathbf{CP}^1$ (which is not identically ∞) defines a meromorphic map on $\mathbf{CP}^1$ which is holomorphic on the complement of $D = \tilde{f}^{-1}(\infty)$.*

The first part is a consequence of the Riemann removable singularity theorem. For the converse one needs to show that D is discrete and this follows from the fact that a holomorphic function defined on a connected domain which is constant on a set having an accumulation point must be constant.

Remark 1.10. *Clearly $\mathcal{M}(X)$ is a field (we supposed that a Riemann surface is connected). One can prove that if X is a compact Riemann surface, $\mathcal{M}(X)$ is a transcendental extension of $\mathbf{C}$ of degree one. Moreover, any transcendental extension of $\mathbf{C}$ of degree one is the field of meromorphic functions of a Riemann surface.*

A holomorphic function defined on some neighborhood of a point in the plane is determined by the coefficients (an infinite sequence of numbers) of its power series developement. By contrast, a holomorphic function defined on a Riemann surface does not have a meaningful power series associated to it, meaning that, up to a change of chart, one has always the same local form depending only on a natural number described in the following lemma.

Lemma 1.11. *Let $\phi : Y \to X$ be a non-constant holomorphic map between Riemann surfaces with $\phi(y_0) = x_0$. There exist charts p_Y and p_X around y_0 and x_0 respectively such that $p_Y(y_0) = p_X(x_0) = 0$ and $p_X \circ \phi \circ {p_Y}^{-1}(z) = z^n$ for some $n \geq 1$.*

Proof. Clearly we can assume that there exists local coordinates p'_Y (we will change that coordinate next) and p_X around y_0 and x_0, respectively, such that $p_Y(y_0) = p_X(x_0) = 0$. Now, if $p_X \circ \phi \circ {p'_Y}^{-1}$ is non-constant we may suppose that there exists a holomorphic function $f(w)$ such that $p_X \circ \phi \circ {p'_Y}^{-1}(w) = w^n f(w)$ with $n \geq 1$ and $f(0) \neq 0$. Therefore, on some neighborhood of the origin, there exists a holomorphic function $h(w)$ such that $h^n(w) = f(w)$. Observe that the map $p : w \to wh(w)$ is a biholomorphism in a neighborhood of the origin so that $p_Y = p \circ p'_Y$ is a new chart around y_0. For $z = wh(w)$ we obtain $p_X \circ \phi \circ {p_Y}^{-1}(z) = p_X \circ \phi \circ {p'_Y}^{-1}(w) = w^n f(w) = (wh(w))^n = z^n$. $\square$

Observe that in the case $n=1$ the map ϕ is a local biholomorphism at $y_0 \in Y$.

Definition 1.12. *A point $y_0 \in Y$ with $n \geq 2$ in the above lemma is called a ramification point and the point $x_0 \in X$ as above is a branching point of order n of the map ϕ.*

Definition 1.13. *Let $f \in \mathcal{M}(X)$ be a meromorphic function. One defines the order of f at $p \in X$*

$$ord_p(f) = n$$

if, on a local chart $\phi : U \to \mathbf{C}$ with $p \in U$ and $\phi(p) = 0$, one can write

$$f \circ \phi^{-1}(z) = \sum_{k=n}^{\infty} c_k z^k,$$

with $n \in \mathbf{Z}$ and $c_n \neq 0$.

If f is the null function we usually define $ord_p(f) = \infty$ for all $p \in X$. Note that this definition does not depend on the chosen chart. Observe also that if one considers the meromorphic function as a holomorphic function from X to $\mathbf{CP}^1$ then p is a ramification point of f when $n \neq 0$ and the order of ramification is then $|n|$. The function ord_p defines a valuation on the field $\mathcal{M}(X)$.

Exercises

1. Prove Liouville's theorem: every bounded holomorphic function defined on $\mathbf{C}$ is constant.
2. Let $\phi : Y \to X$ be a non-constant holomorphic map between Riemann surfaces. Show that ϕ is an open map.
3. Let $\phi : X \to \mathbf{C}$ be a non-constant holomorphic map. Show that $|\phi|$ does not attain its maximum. Conclude that every holomorphic function on a compact Riemann surface is constant.
4. Let $\phi : X \to \mathbf{C}$ be a non-constant holomorphic map. Show that $\mathrm{Re}\,\phi$ does not attain its maximum.
5. Show that the meromorphic functions on $\mathbf{C}P^1$ are quotients of two polynomials.
6. Let $\phi : Y \to X$ be a non-constant holomorphic map between compact Riemann surfaces. Show that ϕ is surjective. Prove the fundamental theorem of algebra by considering a polynomial as a holomorphic map between $\mathbf{C}P^1$.

Meromorphic functions on $\mathbf{C}P^1$ are very simple to describe:

Definition 1.14. *A rational function $f \in \mathcal{M}(\mathbf{C}P^1)$ is a meromorphic function of the form*

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials with no common factors.

By the exercise above $\mathcal{M}(\mathbf{C}P^1)$ is the set of rational functions. Here, one can also define $f : \mathbf{C}P^1 \to \mathbf{C}P^1$, defining $\frac{p(z)}{q(z)} = \infty \in \mathbf{C}P^1$ if $q(z) = 0$ and $p(z) \neq 0$.

Writing a meromorphic function in the neighborhood of a point a as $f(z) = (z-a)^k g(z)$ where $g(a) \neq 0$, we define the order of $f(z)$ at a to be k. The order of the function defined on $\mathbf{C}P^1$ at ∞ is computed using the chart $w = 1/z$. So that if $p(z)$ has degree n and $q(z)$ degree m, then ∞ will have order $-(n-m)$. For a rational function, this implies that the sum of the orders of the zeros and poles is zero. Conversely, an easy construction gives

Proposition 1.15. *Let z_i and w_j be two finite disjoint families of points in $\mathbf{C}P^1$ with the same number of elements. Then there exists a rational function vanishing precisely at a_i and having poles precisely at w_j (unique up to a non-vanishing scalar multiplication). The order of the function at each point being the number of times the point appears in the families.*

Proof. Suppose that there are n points in each family and none of them is ∞. Define

$$\frac{\prod_1^n (z - z_i)}{\prod_1^n (z - w_i)}.$$

In the case $z_i = \infty$ we substitute the factor by $z - z_i$ by $1/z$. On the other hand, if $w_i = \infty$ we substitute $z - w_i$ by z. This clearly gives a rational function with the desired properties. □

This is not true for other Riemann surfaces. One cannot fix arbitrarily the structure of zeros and poles of a meromorphic function. By the way, it is much more difficult to prove that there exists a meromorphic function at all.

Observe that the proposition implies that all meromorphic functions on $\mathbf{CP}^1$ are rational. Indeed, it suffices to multiply the meromorphic function by the inverse of the rational function obtained using the zeros and poles of the meromorphic function to obtain a constant. One way to describe this result is to say that fixing a formal sum on $\mathbf{CP}^1$ of the form

$$D = \sum_i^n z_i - \sum_i^n w_i$$

one can find a meromorphic function with poles and zeros as above. D is called a divisor and it determines up to a non-vanishing scalar the meromorphic function

Another approach to the description of meromorphic functions on $\mathbf{CP}^1$ is based on prescribing the principal part at poles. Suppose we fix n points (z_i) in $\mathbf{CP}^1$ (we suppose here that ∞ is not a pole). each z_i being a pole of order at most n_i with principal part

$$\sum_{k=1}^{n_i} \frac{c_{k,i}}{(z-w_i)^k}.$$

Clearly the function

$$\sum_{i=1}^{n} \sum_{k=1}^{n_i} \frac{c_{k,i}}{(z-w_i)^k}$$

has the principal part at each neighbourhood and is a rational function. Observe that we can count the number of such functions: it forms a vector space of dimension $n+1$ (n coefficients of the principal part plus constants). Observe also that in this count we loose any information on the number of zeros of the meromorphic function.

The proposition above has a generalisation to other Riemann surfaces as Abel's theorem. On the other hand, the counting of meromorphic functions described in the last paragraph is the simplest form of the Riemann-Roch formula.

As a preparation for the general formulation of Riemann-Roch theorem, suppose we define a divisor given n points in $\mathbf{CP}^1$ as the formal sum

$$D = \sum n_i z_i,$$

where $n_i \in \mathbf{Z}$ satisfy $\sum n_i = d \in \mathbf{Z}$. The integer d is called the degree of the divisor D. We are interested in the dimension of the space, $L(D)$, of meromorphic functions f such that $ord_{z_i} f \geq -n_i$ and $ord_z f = 0$ if $z \neq z_i$ for all i together with the null function. That is, if $n_i < 0$, f has a zero of order at least n_i at z_i and if $n_i > 0$, f has a pole of order at most n_i at z_i. Observe that If $d < 0$ then $L(D)$ is empty. Indeed if f is meromorphic we showed that $\sum ord_{z_i} f = 0$ (Proposition 1.15). The same proposition shows that if $d = 0$, $\dim L(D) = 1$. Suppose now that $d > 0$. We can get rid of all z_i with $n_i < 0$ by adding an appropriate meromorphic function g with poles at those points. We obtain a new divisor D' and the fundamental observation is that the space $L(D)$ is isomorphic to $L(D')$ ($f \in D$ if and only if $f/g \in D'$) . We can simplify the computation of the dimension further by adding a meromorphic function with poles at all points of the divisor D' except one, say z_0 which will have a zero of order d. We obtain in this way a divisor of the form $d.z_0$. We can suppose in local coordinates that $z_0 = \infty$ and then $L(D)$ is isomorphic to the space of polynomial of degree less that d. That is $\dim L(D) = d+1$.

Appendix: Projective space

Complex projective space $\mathbf{C}P^n$ is the quotient of $\mathbf{C}^{n+1}-0$ by the $\mathbf{C}^*$-action $\lambda(z_1,\cdots,z_{n+1}) = (\lambda z_1,\cdots,\lambda z_{n+1})$. The orbit containing the point $(z_1,\cdots,z_{n+1})$ is denoted $[z_1,\cdots,z_{n+1}]$ (the homogeneous coordinates).

Natural charts are given by defining the open sets $U_i = \{\,[z_1,\cdots,z_{n+1}] \mid z_i \neq 0\,\}$ and $\phi_i : U_i \to \mathbf{C}^n$ as

$$\phi_i([z_1,\cdots,z_{n+1}]) = (\frac{z_1}{z_i},\cdots,1,\cdots,\frac{z_{n+1}}{z_i})$$

where the coordinate 1 corresponding to z_i/z_i should be deleted in the identification with $\mathbf{C}^n$. The transition functions are given by

$$\phi_j \circ \phi_i^{-1}(w_1,\cdots,w_{n+1}) = (\frac{w_1}{w_j},\cdots,\frac{w_{n+1}}{w_j})$$

where we think $(w_1,\cdots,w_{n+1})$ as having the i-coordinate equal to 1 and $(\frac{w_1}{w_j},\cdots,\frac{w_{n+1}}{w_j})$ having the j-coordinate equal to 1.

We denote $\Pi : \mathbf{C}^{n+1}\setminus\{0\} \to \mathbf{C}P^n$ the projection. $\mathbf{C}P^n$ is a compact manifold as the projection Π is continuous and its restriction to the sphere $S^{2n-1} \subset \mathbf{C}^{n+1}$ is surjective.

The group $GL(n+1,\mathbf{C})$ of invertible $(n+1)\times(n+1)$ matrices acts on $\mathbf{C}P^n$: just use the action on C^{n+1} and observe that it passes to the quotient. The subgroup $\mathbf{C}^* \subset GL(n+1,\mathbf{C})$ of multiples of the identity acts trivialy on the quotient. In fact one can prove the following.

Proposition 1.16. *The group of biholomorphism of* $\mathbf{C}P^n$ *is*

$$PGL(n+1,\mathbf{C}) = GL(n+1,\mathbf{C})/\mathbf{C}^*.$$

Appendix: Complex manifolds are orientable

Definition 1.17. *A differential manifold is orientable if one can choose a covering by charts* (U_α,ϕ_α) *such that, for any two charts, the Jacobian of* $\phi_\alpha\circ\phi_\beta^{-1}$ *is positive. That is, writing*

$$\phi_\alpha\circ\phi_\beta^{-1}(x_1,\cdots,x_n) = (y_1,\cdots,y_n),$$

we have

$$det\frac{\partial y_i}{\partial x_j} > 0.$$

Observe that, using differential forms, one can write

$$dy_1\wedge\cdots\wedge dy_n = det(\frac{\partial y_i}{\partial x_j})dx_1\wedge\cdots\wedge dx_n.$$

Proposition 1.18. *Any complex manifold is orientable.*

Proof. Write
$$\phi_\alpha \circ \phi_\beta^{-1}(z_1, \cdots, z_n) = (w_1, \cdots, w_n)$$
for $z_i = x_i + i x_{i+n}$ and $w_j = y_j + i y_{j+n}$ so that $dz_i \wedge d\bar{z}_i = d(x_i + i x_{i+n}) \wedge d(x_i - i x_{i+n}) = -2i dx_i \wedge dx_{i+n}$ and analogously $dw_i \wedge d\bar{w}_i = -2i dy_i \wedge dy_{i+n}$. We have therefore

$$dy_1 \wedge \cdots \wedge dy_{2n} = \det \frac{\partial w_i}{\partial z_i} \det \frac{\partial \bar{w}_j}{\partial \bar{z}_j} dx_1 \wedge \cdots \wedge dx_{2n}$$

so that
$$det \frac{\partial y_i}{\partial x_j} = \left| \det \frac{\partial w_i}{\partial z_i} \right|^2 > 0.$$

□

Appendix: The implicit function theorem

Examples of Riemann surfaces are easily obtained as submanifolds of complex manifolds by using the implicit function theorem. Here is its simplest version with two complex coordinates. We will mainly use that version.

Proposition 1.19. *Let f be a holomorphic function in two complex variables defined on $\{ (z, w) \mid |z| < \varepsilon_1,\ |w| < \varepsilon_2 \}$. Suppose that $f(0,0) = 0$ and $\frac{\partial f}{\partial w}(0,0) \neq 0$. Then, there exists $0 < \delta_1 \leq \varepsilon_1, 0 < \delta_2 \leq \varepsilon_2$ and a unique function ϕ defined on $|z| < \delta_1$ such that $\{(z, \phi(z)) \mid |z| < \delta_1 \} = f^{-1}(0) \cap \{ (z, w) \mid |z| < \delta_1,\ |w| < \delta_2 \}$. Moreover, ϕ is holomorphic.*

Proof. As $\frac{\partial f}{\partial w}(0,0) \neq 0$, there exists $\delta_2 > 0$ such that $f(0, w) \neq 0$ for $|w| = \delta_2$. There exists therefore, by compactness, $\delta_1 > 0$ such that $f(z, w) \neq 0$ for $|z| < \delta_1, |w| = \delta_2$. Writing $f_w(z, w) = \frac{\partial f(z,w)}{\partial w}$, for each z, the number of zeros of $f(z, w)$ in $|w| < \delta_2$ is given by the holomorphic function
$$N(z) = \frac{1}{2\pi i} \int_{|w|=\delta_2} \frac{f_w(z, w)}{f(z, w)} dw$$
which is therefore constant equal to one. The explicit solution is given by the residue theorem (writing $f(z, w) = (w - \phi(z)) h(z, w)$ for a non-vanishing function $h(z, w)$):

$$\phi(z) = \frac{1}{2\pi i} \int_{|w|=\delta_2} w \frac{f_w(z, w)}{f(z, w)} dw$$

which is holomorphic in z. □

Corollary 1.20. *Suppose that* $P(w) = w^n + a_1 w^{n-1} + \cdots + a_n$ *(with* a_i *holomorphic functions defined on a neighborhood of* z*) has* n *distinct solutions* $w_1, \cdots, w_n$ *at* z*. Then there exists unique holomorphic functions* $f_1, \cdots, f_n$ *(defined on perhaps smaller neighborhood of* z*) with* $f_i(z) = w_i$ *satisfying* $P(f_i) = 0$ *so that* $P(w) = \Pi_1^n (w - f_i)$*.*

A more general statement of the implicit function theorem is the following:

Proposition 1.21. *Let* $f = (f_1, \cdots, f_m) : P \to \mathbf{C}^m$ *be a holomorphic function defined on* $P = \{ (z, w) \mid |z| < \varepsilon_1, |w| < \varepsilon_2, \}$ *where* $z = (z_1, \cdots, z_n)$ *and* $w = (w_1, \cdots, w_m)$*. Suppose that* $f(0) = 0$ *and for* $1 \leq i, j \leq m$

$$det \frac{\partial f_i}{\partial w_j}(0,0) \neq 0.$$

Then, there exists $0 < \delta_1 \leq \varepsilon_1, 0 < \delta_2 \leq \varepsilon_2$ *and a unique function* ϕ *defined on* $|z| < \delta_1$ *such that* $\{(z, \phi(z)) \mid |z| < \delta_1 \} = f^{-1}(0) \cap \{ (z, w) \mid |z| < \delta_1, \ |w| < \delta_2 \}$*. Moreover,* ϕ *is holomorphic.*

Proof. Apply the real version of the implicit function theorem first to obtain that there exists $\delta_1 > 0$ and a unique function ϕ defined on $|z| < \delta_1$ such that $f(z, \phi(z)) = 0$. It remains to show that the function is holomorphic. We compute

$$0 = \frac{\partial f_i(z, \phi(z))}{\partial \bar{z}_l} = \frac{\partial f_i}{\partial \bar{z}_l} + \frac{\partial f_i}{\partial \bar{w}_j} \frac{\partial \bar{\phi}_j}{\partial \bar{z}_l} + \frac{\partial f_i}{\partial w_j} \frac{\partial \phi_j}{\partial \bar{z}_l}.$$

The first two terms in the right hand side are null because f_i is holomorphic. Therefore

$$\frac{\partial f_i}{\partial w_j} \frac{\partial \phi_j}{\partial \bar{z}_l} = 0.$$

Because $det \frac{\partial f_i}{\partial w_j} \neq 0$ we conclude that

$$\frac{\partial \phi_j}{\partial \bar{z}_l} = 0.$$

□

We mention a simple application of the proposition: if $f : \mathbf{C}^{n+1} \to \mathbf{C}^n$ is holomorphic of constant rank n then the set

$$\{z \in \mathbf{C}^{n+1} \mid f(z) = 0 \}$$

is a Riemann surface (maybe with several connected components). In particular if

$$F(z,w)=0$$

has no solution with simultaneously vanishing derivatives $\frac{\partial F(z,w)}{\partial z}$ and $\frac{\partial F(z,w)}{\partial w}$, it defines a Riemann surface. Indeed, the charts are given by

$$\phi^{-1}(z)=(z,g(z))$$

or

$$\psi^{-1}(w)=(h(w),w).$$

In the intersection of two charts as above we obtain

$$\psi\circ\phi^{-1}(z)=g(z)$$

which is holomorphic with non-vanishing derivative (because $\psi\circ\phi^{-1}$ is a bijection in the intersection).

Appendix: holomorphic differential forms

Holomorphic differential forms are locally defined on every coordinate neighborhood $\phi_\alpha(U_\alpha)\subset\mathbf{C}$ as

$$g_\alpha(z)dz$$

where the variable z lives in $\phi_\alpha(U_\alpha)$ and the function $g_\alpha:\phi_\alpha(U_\alpha)\to\mathbf{C}$ is holomorphic. They satisfy the expected compatibility condition for each intersection $U_\alpha\cap U_\beta$ which reads

$$g_\beta(w(z))\,w'(z)=g_\alpha(z),$$

where $w(z)=\phi_\beta\circ\phi_\alpha^{-1}(z)$.

Holomorphic functions defined on a compact Riemann surface are constants but the space of holomorphic forms on a compact Riemann surface is a finite dimensional complex vector space whose dimension depends only on the topology of the surface. We will see that it is precisely the genus of the surface. For instance, for an elliptic curve the holomorphic differentials are all multiples of the 'constant' form dz. The Riemann sphere has only the trivial holomorphic form which is identically zero. We can prove that right now:

Suppose we use the covering of $\mathbf{C}P^1$ by the two open sets U_1 and U_2 as before. Then $w(z)=1/z$ and therefore

$$-\frac{1}{z^2}g_1(\frac{1}{z})=g_2(z).$$

The equation $g_1(1/z) = -z^2 g_2(z)$ has only one solution for holomorphic g_2 in $\mathbf{C}$ and g_1 in $\mathbf{CP}^1 \setminus \{0\}$ (in the coordinate w, with ∞ given by $w = 0$, $g_1(w) = c_0 + c_1 w + \cdots + c_k w^k$). It is zero.

Example 1.22. *Let X be given by an equation $F(z, w) = 0$ with partial derivatives non-vanishing simultaneously. Suppose $\frac{\partial F}{\partial w} \neq 0$ and solve for z. A holomorphic differential can be obtained as*

$$\frac{\partial F}{\partial z} dz$$

on the coordinate z. On the other hand, using the coordinate w we define

$$-\frac{\partial F}{\partial w} dw.$$

The equation

$$\frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial w} dw = 0$$

shows that the form is well defined on the whole surface. Also, let

$$\frac{dz}{\frac{\partial F}{\partial w}}$$

and

$$-\frac{dw}{\frac{\partial F}{\partial z}}$$

be defined in the corresponding coordinates. As the partial derivatives don't vanish at the same time the expressions define a global holomorphic form.

2 First examples, Elliptic functions

Examples of open Riemann surfaces are open subsets of $\mathbf{C}$. In particular, the disc is the most important one being biholomorphic to any simply connected bounded open domain by the uniformization theorem. Among domains which are not simply connected, the cylinder is one of the simplest. A cylinder can be realized as a Riemann surface through any of the open subsets of $\mathbf{C}$ where $r > 1$:

$$C_r = \{ z \in \mathbf{C} \mid r > |z| > 1 \}.$$

One can prove that for $1 < r_1 \neq r_2$, C_{r_1} is not biholomorphic to C_{r_2}.

2.1 The infinite cylinder

We call the infinite cylinder the set $\mathbf{C}^*$. The infinite cylinder is not biholomorphic to any C_r with $1 < r < \infty$. One can obtain $\mathbf{C}^*$ by taking the group of translations Γ generated by $z \to z+1$ and considering the quotient $\mathbf{C}/\Gamma$. The biholomorphism between the spaces is given by the exponential function

$$z \to e^{2\pi i z}.$$

We will justify later these assertions and use the following description of meromorphic functions on $\mathbf{C}^*$: they are in correspondence with meromorphic functions on $\mathbf{C}$ which are periodic with respect to the translation $z \to z+1$. Clearly the holomorphic functions defined on $\mathbf{C}^*$ are the convergent power series

$$\sum_{-\infty}^{+\infty} a_n w^n.$$

Convergence is equivalent to the condition $\lim_{w \to \pm\infty} |a_n|^{1/n} = 0$. In terms of the coordinates in $\mathbf{C}$, that gives functions of the form $\sum_{-\infty}^{+\infty} a_n \exp 2\pi n i z$.

A usefull idea to obtain a periodic function on $\mathbf{C}$ is to define an infinite sum

$$\sum_{n \in \mathbf{Z}} f(z-n)$$

where $f(z)$ is any meromorphic function. The problem here is that it is not clear that the sum will converge. A successful example is obtained by taking the meromorphic function $f(z) = \frac{1}{z^2}$ for $z \in \mathbf{C}$. Define

$$P(z) = \sum_{n \in \mathbf{Z}} \frac{1}{(z-n)^2}$$

which is normally convergent on every compact subset of $\mathbf{C}\setminus\mathbf{Z}$. Indeed, on each compact subset of $\mathbf{C}\setminus\mathbf{Z}$ contained in a disc of radius R, for all $n \geq 2R$ we have

$$|z-n| \geq n - |z| \geq n - \frac{n}{2} = \frac{n}{2}.$$

Therefore

$$\left|\frac{1}{(z-n)^2}\right| \leq \frac{4}{n^2}$$

and by Montel's theorem we conclude that $P(z)$ is a meromorphic function on $\mathbf{C}$ holomorphic on $\mathbf{C}\setminus\mathbf{Z}$ which is clearly periodic.

One can also obtain $P(z)$ starting with a function given by an infinite product. In this way we control the zeros of the function. Namely, define

$$S(z) = z\prod_1^\infty (1 - \frac{z^2}{n^2}) = z\prod_1^\infty (1-\frac{z}{n})(1+\frac{z}{n}).$$

The product is normally convergent on compacts as $\sum \log(1-\frac{z^2}{n^2})$ is normally convergent on compacts. It's logarithmic derivative is

$$Z(z) = \frac{S'(z)}{S(z)} = \frac{1}{z} + \sum_1^\infty \left(\frac{1}{z-n} + \frac{1}{z+n}\right) = \frac{1}{z} + \sum_1^\infty \frac{2z}{z^2-n^2}$$

which converges normally on compact subsets of $\mathbf{C}\setminus\mathbf{Z}$. Therefore $Z'(z)$ is also meromorphic on $\mathbf{C}\setminus\mathbf{Z}$. Finally we get back to the meromorphic function

$$P(z) = -Z'(z).$$

Exercise 2.1. *Prove the following identities:*

1. $P(z) = \pi^2 \frac{1}{\sin^2(\pi z)}$.

2. $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

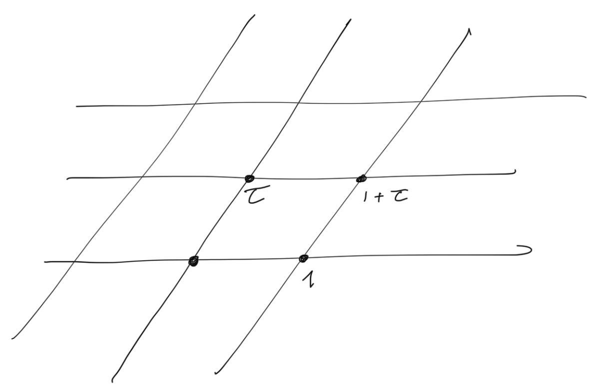
3. $Z(z) = \pi \cot(\pi z)$.

4. $S(z) = \pi \sin(\pi z)$.

Observe that $P(z)$ is defined on the cylinder but $Z(z)$ and $S(z)$ are not invariant functions under the translation $z \to z+1$.

2.2 Elliptic Functions

The next examples of compact Riemann surfaces, after $\mathbf{C}P^1$, consists of complex structures on a torus. We will show later that any such structure arises as a quotient of $\mathbf{C}$ by a translation group generated by two independent directions one of each we may suppose (by a conjugation by a similarity transformation $z \to az+b$) to be $z \to z+1$ and the other one $z \to z+\tau$ with $\tau \in \mathbf{C}$. More precisely, we will show that any compact Riemann surface whose underlying manifold is a torus is biholomorphic to an elliptic curve:



Definition 2.2. *Let $\tau \in \{ z \in \mathbf{C} \mid \operatorname{Im} z > 0 \}$ and $\Gamma_\tau = \mathbf{Z} + \mathbf{Z}\tau$ be the additive group generated by $1, \tau \in \mathbf{C}$. We say that $E_\tau = \mathbf{C}/\Gamma_\tau$ is the complex torus associated to Γ_τ.*

The set of points inside the parallelogram defined by 1 and τ is called a fundamental region. Its closure, with some identifications on the boundary, is homeomorphic to a torus. Observe that any translation of that parallelogram also is a fundamental domain in the sense that any two points in its interior are contained in different orbits and each orbit has a point in the domain or its closure.

2.2.1 General properties

Meromorphic functions defined on E_τ are identified with meromorphic functions defined on $\mathbf{C}$ which are invariant under Γ_τ (called elliptic functions) but holomorphic functions which are invariant reduce to constants due to the maximum principle. It is not obvious that a non-constant function exists but several of its properties, assuming existence, are simple to state. The following is a basic property.

Proposition 2.3. *Let $f \in \mathcal{M}(E_\tau)$ be a meromorphic function without poles on the boundary of a fundamental region. Then, the sum of its residues in the fundamental region is zero.*

Proof. The sum of residues in the interior is given by $\frac{1}{2\pi i}\int_{\partial P} f(z)dz$ where P is a parallelogram which is a fundamental domain. By translation invariance the integrals on opposite sides cancel. □

This shows that, in order to construct a meromorphic function on E_τ with only one pole, its order has to be at least two. A related proposition counts the number of zeros.

Proposition 2.4. *Suppose there are no poles or zeros in the boundary of a fundamental domain. Then the number of zeros is the same as the number of poles counting multiplicities.*

Proof. The proof is simply a corollary to the previous proposition applied to the the function f'/f. In fact the sum of the residues is equal to the number of zeros minus the number of poles counting multiplicity by the following

Exercise 2.5. *If f has no poles nor zeros in ∂P, prove that*

$$\frac{1}{2\pi i}\int_{\partial P}\frac{f'(z)}{f(z)}dz = \textit{number of zeros in } P - \textit{number of poles in } P$$

where P is a domain with boundary ∂P.

□

Another necessary condition on the zeros and poles of a meromorphic function is given in the following proposition. It turns out that these necessary conditions are also sufficient (Abel's theorem).

Proposition 2.6. *Suppose there are no poles or zeros in the boundary of a fundamental domain P. Let a_i and b_j be two finite disjoint families of points inside P and f an elliptic function vanishing precisely at a_i and having poles precisely at b_j (we repeat the points according to the multiplicity of the zero or pole). Then*

$$\sum a_i - \sum b_j \in \Gamma_\tau.$$

Proof. The same as above using the following

Exercise 2.7.

$$\frac{1}{2\pi i}\int_{\partial P}\frac{zf'(z)}{f(z)}dz=\sum a_i-\sum b_i.$$

Indeed, taking into account the invariance of $f(z)$ under translations and supposing that P is the parallelogram with corners $0,1,1+\tau,\tau$, we get

$$\frac{1}{2\pi i}\int_{\partial P}\frac{zf'(z)}{f(z)}dz=\frac{1}{2\pi i}\int_0^1\frac{(-\tau)f'(z)}{f(z)}dz+\frac{1}{2\pi i}\int_0^\tau\frac{f'(z)}{f(z)}dz$$

and observing that $\frac{1}{2\pi i}\int_0^\tau\frac{f'(z)}{f(z)}dz$ is the number of turns that $f(z)$ describes around the origin when z follows the segment from 0 to τ and analogously for $\frac{1}{2\pi i}\int_0^1\frac{f'(z)}{f(z)}dz$, we obtain

$$=\frac{1}{2\pi i}\left(-\tau\log(f(z))|_0^1-\log(f(z))|_0^\tau\right)=n_1\tau+n_2.$$

□

2.2.2 Weierstrass function

The next goal is to construct meromorphic functions on E_τ. In the following discussion we fix a translation τ and let Γ_τ be the lattice generated by 1 and τ. Several objects will depend on τ although we will not make it explicit. A direct construction of elliptic functions is obtained by means of the series

$$F_n(z)=\sum_{\gamma\in\Gamma_\tau}\frac{1}{(z-\gamma)^n}.$$

One can prove that the series converges absolutely and uniformly on compact sets for $n\geq 3$ so that $F_n(z)$ is meromorphic. To see that, we start with the following

Lemma 2.8. *The series*

$$\sum_{\gamma\in\Gamma_\tau-\{0\}}\frac{1}{|\gamma|^s}$$

is convergent for $s>2$.

Proof. Consider the description of the lattice by the layers $n_1+n_2\tau\in\Gamma_\tau$ with $max(|n_1|,|n_2|)=n$. There are $8n$ elements of Γ_τ in that layer. If we let r be the radius of an inscribed circle inside the first layer (that is the parallelogram defined by $\pm(1+\tau), \pm(\tau-1)$), then $|n_1+n_2\tau| \geq r\, max(|n_1|, |n_2|)$. Therefore

$$\sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{|\gamma|^s} \leq \sum_{n\geq 1} \frac{8n}{r^s n^s} = \sum_{n\geq 1} \frac{8}{r^s n^{s-1}}$$

which is convergent for $s>2$. □

Lemma 2.9. *The series*

$$\sum_{\gamma\in\Gamma_\tau} \frac{1}{(z-\gamma)^s}$$

is uniformly convergent on compact sets of $\mathbf{C}-\Gamma_\tau$ *for any integer* $s>2$.

Proof. If $K\subset\mathbf{C}$ is a compact subset we can assume that, except for finitely many γ, $|\gamma|\geq 2|z|$ for $z\in K$. In that case $|z-\gamma|\geq|\gamma|-|z|\geq|\gamma|-\frac{|\gamma|}{2}=\frac{|\gamma|}{2}$. Therefore for all $z\in K$ and γ on the complement of a finite subset in Γ_τ,

$$\sum_\gamma \frac{1}{|z-\gamma|^s} \leq 2^s \sum_\gamma \frac{1}{|\gamma|^s}$$

which is convergent for $s>2$. Together with the previous lemma, this implies the series is uniformly convergent by Weierstrass M-test. □

Having proved convergence, for each $\omega\in\Gamma_\tau$ we obtain

$$F_n(z+\omega) = \sum_{\gamma\in\Gamma_\tau} \frac{1}{(z+\omega-\gamma)^n} = \sum_{\gamma\in\Gamma_\tau} \frac{1}{(z-\gamma)^n}$$

so that $F_n(z)$ is elliptic. In particular the function $F_3(z)$ is elliptic. It has a pole of order 3 at 0. To obtain a meromorphic function with a pole of order 2 we solve the equation

$$\mathscr{P}'(z) = -2F_3(z).$$

A solution is given by the Weierstrass function

$$\mathscr{P}(z) = \frac{1}{z^2} + \sum_{\gamma\in\Gamma_\tau-\{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right).$$

The naive idea would be to start with a function with a pole of order two, namely $\frac{1}{z^2}$, and define $\sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{(z-\gamma)^2}$ which would make it invariant under Γ_τ but unfortunately this sum is not convergent.

Lemma 2.10. *The series*

$$\mathscr{P}(z)=\frac{1}{z^2}+\sum_{\gamma\in\Gamma_\tau-\{0\}}\left(\frac{1}{(z-\gamma)^2}-\frac{1}{\gamma^2}\right).$$

defines an elliptic function with only one pole of order two modulo the lattice.

Proof. To show convergence, the argument is the same as in the previous lemma. The general term of the series $\mathscr{P}$ satisfies, as in the previous lemma, for $|\gamma|\geq 2|z|$ with z in a compact subset of $\mathbf{C}\setminus\Gamma_\tau$.

$$\left|\frac{1}{(z-\gamma)^2}-\frac{1}{\gamma^2}\right|=\left|\frac{z(z-2\gamma)}{\gamma^2(z-\gamma)^2}\right|\leq\frac{4|z|(5/2|\gamma|)}{|\gamma|^2|\gamma|^2}\leq\frac{10|z|}{|\gamma|^3}.$$

Therefore, as before, we conclude that the series converges absolutely and uniformly on compact sets of $\mathbf{C}\setminus\Gamma_\tau$.

The periodicity is not clear from the formula. But we can use the periodicity of its derivative to conclude that $\mathscr{P}(z)-\mathscr{P}(z+1)$ and $\mathscr{P}(z)-\mathscr{P}(z+\tau)$ are constants. The value of the constants are seen to be zero. In fact, $\mathscr{P}(-1/2)-\mathscr{P}(-1/2+1)=0$ and $\mathscr{P}(-\tau/2)-\mathscr{P}(-\tau/2+\tau)=0$ because $\mathscr{P}(z)$ is clearly even. □

A meromorphic function on the elliptic curve can be interpreted as a function $E_\tau\to\mathbf{C}P^1$. In general, the meromorphic function is locally a bijection but it has ramification points when its derivatives vanish. It is important then to determine the zeros of $\mathscr{P}'$:

Lemma 2.11. *The zeros of $\mathscr{P}'$ in a fundamental parallellogram with vertices 0, 1, τ and $1+\tau$ are*

$$\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}$$

Proof. As $\mathscr{P}'$ has order 3, it has only three zeros in the fundamental domain. We have $\mathscr{P}'(z)=-\mathscr{P}'(-z)$ because $\mathscr{P}'$ is odd. On the other hand, because $\mathscr{P}'$ is periodic, $\mathscr{P}'(z)=\mathscr{P}'(z-\gamma)$. Therefore, for $z=\gamma/2$, $\mathscr{P}'$ vanishes. □

One can prove that the Weierstrass function defined on E_τ assumes each value on the Riemann sphere exactly twice except for 4 points; three corresponding to the vanishing of its derivative $\mathscr{P}(\frac{1}{2})$, $\mathscr{P}(\frac{\tau}{2})$, $\mathscr{P}(\frac{1+\tau}{2})$ and the last one corresponding to the unique pole of order 2, ∞. That gives an interpretation of the Weierstrass function as a branched covering of the Riemann sphere by the torus.

The following existence theorem of meromorphic functions on an elliptic curve should be contrasted to the corresponding existence theorem of rational functions on the Riemann sphere. The only if part was proven in a previous proposition.

Theorem 2.12. *(Abel's theorem) Let E_τ be a complex torus with corresponding group Γ_τ. Let a_i and b_j be two finite disjoint families of points in a fundamental domain P with the same number of elements (greater or equal to 2). Then there exists an elliptic function vanishing (inside P) precisely at a_i and having poles (inside P) precisely at b_j if and only if*

$$\sum a_i - \sum b_j \in \Gamma_\tau.$$

Proof. A constructive proof of this theorem can be given by considering the Weierstrass sigma functions (see below)

$$\sigma(z) = z\Pi_{\gamma\in\Gamma'}(1-\frac{z}{\gamma})e^{\frac{z}{\gamma}+\frac{z^2}{2\gamma^2}},$$

which has only simple zeros at points of Γ. They are not functions defined on the quotient but their behavior with respect to the lattice is quite simple. In fact

$$\sigma(z+\gamma) = (-1)^{n_\gamma}\sigma(z)e^{\alpha_\gamma(z+\frac{1}{2}\gamma)}$$

where α_γ and n_γ depend only on $\gamma\in\Gamma$. We define the meromorphic function in the theorem as

$$f(z) = \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}.$$

It is easy to verify that $f(z)$ is indeed defined on the quotient. □

Let us analyse more precisely the Weierstrass sigma-function. It is an analog of the function $S(z)$ introduced above.

Lemma 2.13. $\sigma(z) = z\Pi_{\gamma\in\Gamma'}(1-\frac{z}{\gamma})e^{\frac{z}{\gamma}+\frac{z^2}{2\gamma^2}}$ *converges normally on compact subsets of* $\mathbf{C}$.

Proof. We obtain for large γ (for instance $|\gamma|\geq 2|z|$ pour z dans un compact):

$$\left|\log\left((1-\frac{z}{\gamma})e^{\frac{z}{\gamma}+\frac{z^2}{2\gamma^2}}\right)\right| = |\log(1-\frac{z}{\gamma})+\frac{z}{\gamma}+\frac{z^2}{2\gamma^2}| = |\frac{z^3}{3\gamma^3}+\frac{z^4}{4\gamma^4}+\cdots| \leq C|\frac{z^3}{\gamma^3}|$$

for a constant C, which proves normal convergence. □

Define the logarithmic derivative of the σ-function:

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\gamma\in\Gamma_\tau-\{0\}}\left(\frac{1}{z-\gamma}-\frac{1}{\gamma}+\frac{z}{\gamma^2}\right).$$

And observe that

$$\mathscr{P}(z) = -\zeta'(z).$$

In order to obtain the transformation law for σ we start first to obtain the one for ζ. Indeed, as $\mathscr{P}(z)$ is doubly periodic, we obtain that, for all z,

$$\zeta(z+1) = \zeta(z) + \eta_1$$

and

$$\zeta(z+\tau) = \zeta(z) + \eta_2.$$

Using the definition of ζ we obtain that there are constants c_1 and c_2 such that

$$\log\sigma(z+1) - \log\sigma(z) = \eta_1 z + c_1$$

and

$$\log\sigma(z+\tau) - \log\sigma(z) = \eta_2 z + c_2,$$

therefore

$$\sigma(z+1) = \sigma(z)e^{\eta_1 z + c_1}, \;\; \sigma(z+\tau) = \sigma(z)e^{\eta_2 z + c_2}.$$

For $z = -1/2$ we have $\sigma(1/2) = \sigma(-1/2)e^{\eta_1/2 + c_1}$ so $-e^{\eta_1/2} = e^{c_1}$ because σ is odd. Analogously we obtain $-e^{\eta_2/2} = e^{c_2}$. We conclude with then that

$$\sigma(z + n_1 + n_2\tau) = (-1)^{n_1+n_2+n_1n_2}\sigma(z)e^{(n_1\eta_1+n_2\eta_2)(z+\frac{n_1+n_2\tau}{2})}.$$

The two constants η_1 and η_2 are not independent. We will also need the following lemma describing an explicit relation between them:

Lemma 2.14. *Let Λ_τ be the lattice $\langle 1, \tau\rangle$ and ζ the meromorphic function on $\mathbf{C}$ defined above. Then*

$$\eta_1\tau - \eta_2 = 2\pi i.$$

Proof. ζ has a single pole on the interior of a fundamental domain P containing 0. Therefore

$$\begin{aligned}
2\pi i = \int_{\partial P} \zeta(z)dz &= \int_{\Gamma_1} \zeta(z)dz + \int_{\Gamma_2} \zeta(z)dz + \int_{\Gamma_3} \zeta(z)dz + \int_{\Gamma_4} \zeta(z)dz \\
&= \int_{\Gamma_1} \zeta(z)dz - \int_{\Gamma_1} \zeta(z+\tau)dz + \int_{\Gamma_2} \zeta(z)dz - \int_{\Gamma_2} \zeta(z-1)dz \\
&= \int_{\Gamma_1} -\eta_2 dz + \int_{\Gamma_2} \eta_1 dz = \eta_1\tau - \eta_2.
\end{aligned}$$

□

This lemma implies that there are particular combinations of the function ζ that are periodic:

Lemma 2.15. *Fix $k \geq 1$ and a collection of k points $(z_i)_{1\leq i\leq k}$. The function $g(z) = \sum_1^k a_i\zeta(z-z_i)$ is elliptic if and only if $\sum_1^k a_i = 0$.*

Proof. Compute $g(z+1) = \sum_1^k a_i\zeta(z+1-z_i) = \sum_1^k a_i\zeta(z-z_i) + \sum_1^k a_i\eta_1 = g(z)+\eta_1\sum_1^k a_i$ and, analogously, $g(z+\tau) = \sum_1^k a_i\zeta(z+\tau-z_i) = \sum_1^k a_i\zeta(z-z_i)+\sum_1^k a_i\eta_2 = g(z)+\eta_2\sum_1^k a_i$. □

2.2.3 Divisors on a complex torus

Another description of the set of meromorphic functions is given through divisors on a Riemann surface X. More precisely we fix a divisor, that is, a formal linear combination

$$D = \sum_z n_z z$$

where $n_z \in \mathbf{Z}$ are different from zero only for a finite number of $z \in X$. We think of a divisor as giving the order n_z of a possible function at z, except that a function with precisely these orders might not exist. A divisor defines a function $D : X \to \mathbf{Z}$ of finite support, so we denote by $D(z) = n_z$. The degree of a divisor will be the total order $deg(D) = \sum_z n_z$. In particular we call divisor of f the divisor (called a principal divisor)

$$div(f) = \sum_{z\in X} ord_z(f)z,$$

and we will show that it has zero degree for any meromorphic function defined on any compact Riemann surface. There exists an order relation between disors: we say $D_1 \geq D_2$ if for all $z \in X$, $D_1(z) \geq D_2(z_2)$. Define the vector space

$$L(D) = \{f \in \mathcal{M}(X) \mid f = 0 \text{ or } div(f) \geq -D\}$$

where $div(f) \geq -D$ means that, for each $z \in X$, the order of f at z is greater than or equal to $-n_z$. That is, a meromorphic function in $L(D)$ has poles at z_i of order at most n_i if $n_i > 0$ and zeros of order at least n_i if $n_i < 0$. For instance, if $D = 0$, then $div(f) \geq -D = 0$ means that f is holomorphic. Therefore $L(0) = \mathbf{C}$, the constant functions, and $\dim D = 1$. But also, if $D = z$, that is, just one point, we obtain that $L(D) = \mathbf{C}$ because there are no meromorphic functions with a single simple pole at one point. As another example consider $L(div(g)) = \{f \in \mathcal{M}(X) \mid f = 0 \text{ or } div(f) \geq -div(g)\}$. Observe then that $div(f/g) = div(f) - div(g) \geq 0$ and therefore f/g is a constant. We conclude that $L(div(g)) = \mathbf{C}g$.

More generally, two divisors D_1 and D_2 which differ by a principal divisor (g) ($D_2 = D_1 + (g)$) are called equivalent divisors and have isomorphic spaces $L(D_1)$ and $L(D_2)$. Clearly $f \to f/g$ defines an the isomorphism.

If a divisor is strictly negative, that is, $n_i \leq 0$ with at least one n_i non-vanishing, we clearly have $L(D) = \{0\}$. If one add a point $[z]$ to a divisor D one obtains that $L(D) \subset L(D+z)$ with codimension at most one. Indeed, if the coefficient of D at z is n, then define $L(D+z) \to \mathbf{C}$ as the coefficient of order $n+1$ in the Laurent expansion of a meromorphic function at z. Clearly, $L(D)$ is the kernel of this map.

Exercise 2.16. *Let D be an effective divisor on E with $d = deg(D) \geq 1$. Then*

$$dim(L(D)) \leq d.$$

A deeper theorem describing precisely the dimension of $L(D)$ is the following:

Theorem 2.17. *(Riemann-Roch for elliptic functions) Let D be a divisor on E with $d = deg(D) \geq 1$. Then*

$$dim(L(D)) = d.$$

Proof. Suppose that $D = \sum n_i[z_i]$, $1 \leq i \leq n$ is the divisor. We will only prove the theorem in the case $n_i \geq 1$ for all i. The general case follows from proposition 2.19. For each i consider a family of n_i complex numbers $(c_{ki})_{1 \leq k \leq n_i}$ We write

$$f(z) = c_0 + \sum c_{1i}\zeta(z - z_i) + \sum c_{2i}\mathscr{P}(z - z_i) + \sum c_{3i}F_3(z - z_i) + \cdots + \sum c_{n_i i}F_n(z - z_i).$$

The only problem in that expression being that $\zeta(z - z_i)$ is not an elliptic function. The theorem follows because of Lemma 2.15. Indeed, for each i one can choose n_i coefficients of the Laurent expansion and there is a constraint given by the lemma. The dimension is given then by $\sum_i n_i - 1$ where we have to add one dimension because fixing all Laurent tails determines a function up to a constant. □

2.2.4 The Jacobian map

One can state Abel's theorem in a way more adapted to further generalizations introducing the Jacobian map $J : Div(E_\tau) \to E_\tau$ defined by

$$J(\sum_{n_i}[z_i]) = [\sum_{n_i} z_i],$$

where we use the notation $[z]$ to denote the projection of the point $z \in \mathbf{C}$ into E_τ. We state now Abel's theorem in the following version:

Theorem 2.18. *A divisor D on E_τ is principal if and only if* $\deg(D) = 0$ *and* $J(D) = 0$.

An important consequence of Abel's theorem is the observation that one can always deal with effective divisors on a complex torus in the case $\deg D \geq 1$.

Proposition 2.19. *Any divisor with strictly positive degree in E_τ is equivalent to an effective divisor.*

Proof. Suppose $\deg(D) = d > 0$. Define a new divisor of degree 0:

$$D' = D - d[z]$$

where $[z] \in E$. Choose z such that $J(D') = 0$, that is $[dz] = J(D)$. Then, by Abel's theorem, there exists a meromorphic function f such that $(f) = D - d[z]$, that is D is equivalent to $d[z]$, an effective divisor.

□

2.2.5 The field of meromorphic functions

The field of meromorphic sections is described in the following

Theorem 2.20. $\mathscr{M}(E_\tau) = \mathbf{C}(\mathscr{P}, \mathscr{P}')$*, that is, the field of meromorphic functions is generated by* $\mathbf{C}$*, the Weierstrass function and its derivative.*

Proof. Suppose first that $f \in \mathscr{M}(E_\tau)$ is even of degree n. We choose $a, b \in \mathbf{C}$ such $f(z) - a$ and $f(z) - b$ have only simple roots and none of them a zero of $\mathscr{P}(z)$. Therefore

$$\frac{f(z) - a}{f(z) - b}$$

has zeros in a family $\pm a_i$, with $1 \leq i \leq n$, and poles in a disjoint family $\pm b_i$. The function

$$\frac{\prod(\mathscr{P}(z) - \mathscr{P}(a_i))}{\prod(\mathscr{P}(z) - \mathscr{P}(b_i))}$$

has the same zeros and poles as $\frac{f(z)-a}{f(z)-b}$ and therefore they are equal up to a multiplicative constant. This proves that f is in the field generated by $\mathscr{P}$. If f is odd we use the same argument with the function $f/\mathscr{P}'$ and for a general function we consider the decomposition into its even and odd part. □

One can understand further the field extension $\mathbf{C}(\mathscr{P}, \mathscr{P}')$ over the field $\mathbf{C}(\mathscr{P})$ via the study of a differential equation satisfied by $\mathscr{P}(z)$ which, in fact, establishes an algebraic relation between $\mathscr{P}(z)$ and $\mathscr{P}'(z)$.

Proposition 2.21. *The Weierstrass function satisfies the equation*

$$\mathscr{P}'(z)^2 = 4\mathscr{P}^3(z) - g_2(\tau)\mathscr{P}(z) - g_3(\tau)$$

where

$$g_2(\tau) = 60 \sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{\gamma^4}$$

and

$$g_3(\tau) = 140 \sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{\gamma^6}.$$

Proof. A simple proof can be given by computing the Laurent series at the origin of $\mathscr{P}(z)$ and $\mathscr{P}'(z)$. One must show that the two sides of the equality have equal Laurent series up to the constant term. In that case their difference would be a a bounded holomorphic function vanishing at the origin and therefore, by Liouville, vanishing everywhere.

In order to obtain the Laurent series of $\mathscr{P}(z)$ it is useful to consider the series below satisfying $\zeta'(z) = -\mathscr{P}(z)$.

$$\zeta(z) = \frac{1}{z} + \sum_{\gamma\in\Gamma_\tau-\{0\}} \left(\frac{1}{(z-\gamma)} + \frac{1}{\gamma} + \frac{z}{\gamma^2}\right)$$

Exercise 2.22. *The Laurent series of $\zeta(z)$ at the origin is*

$$\zeta(z) = \frac{1}{z} - G_4 z^3 - G_6 z^5 + \cdots$$

where

$$G_n = \sum_{\gamma\in\Gamma_\tau-\{0\}} \frac{1}{\gamma^n}.$$

We obtain the following developments

$$\mathscr{P}(z) = -\zeta'(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots$$

$$4\mathscr{P}(z)^3 = \frac{4}{z^6} - \frac{36G_4}{z^2} - 60G_6 + \cdots$$

$$\mathscr{P}'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \cdots$$

$$\mathscr{P}'(z)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \cdots$$

and then a simple computation shows that the Laurent series of each side of the equation is equal up to zero order. □

Writing $t = \mathscr{P}(z)$ and the differential equation as $(\frac{dt}{dz})^2 = 4t^3 - g_2 t - g_3$ we see that the inverse function of $\mathscr{P}(z)$, $\mathscr{P}^{-1}(t)$, would be formally given by

$$\int \frac{1}{\sqrt{4t^3 - g_2 t - g_3}} dt.$$

But those integrals are not well defined in general. The problem is that the function $\sqrt{4t^3 - g_2 t - g_3}$ is not well defined in $\mathbf{C}$. For each path of integration (which does not meet the roots) one can define the integral by analytically extending the function along the path, but different paths will give different integrals.

In fact, the study of integrals of the form

$$\int \frac{1}{\sqrt{p(t)}} dt$$

were the motivation for the whole theory. In particular one can think of the elliptic functions as generalizations of the circular functions. For instance

$$\int \frac{1}{\sqrt{1 - t^2}} dt$$

is $\mathrm{Arcsin}(t)$ and the inverse function of that integral is a periodic function. The elliptic functions are inverse functions of the integrals as above with $p(t)$ of degree three and they have the remarkable property of being doubly periodic.

The map $E_\tau - [0] \to \mathbf{C}^2$ given by $z \to (\mathscr{P}(z), \mathscr{P}'(z))$ defined on the complement of the pole ($[0]$ is the projection of the lattice on the quotient space) is a holomorphic embedding whose image is the curve

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).$$

But one can extend that embedding to complex projective space.

Theorem 2.23. *The map $z \to (\mathscr{P}(z), \mathscr{P}'(z), 1)$ for $z \in \mathbf{C} - \Gamma_\tau$ and $z \to (0,1,0)$ for $z \in \Gamma_\tau$ defines a holomorphic embedding $E_\tau \to \mathbf{C}P^2$ whose image is the algebraic curve*

$$y^2 z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3.$$

Several results about elliptic curves are generalized for any compact Riemann surface. In particular, we will

1. Describe any Riemann surface as a quotient of $\mathbf{C}$, D, the unit disc, or the Riemann sphere by a discrete group Γ.

2. Prove that there exist meromorphic functions on any compact surface and, more generally, give a generalization of Abel's theorem, Riemann-Roch theorem and describe the structure of its field of meromorphic functions.
3. Prove that there exists an embedding of a compact Riemann surface as a submanifold of a complex projective space.

3 Review of topology

3.1 Triangulations and classification of surfaces

A two dimensional topological manifold is called a surface. That is a Hausdorff topological space M having a cover by open sets U_α and a collection of homeomorphisms $\phi_\alpha : U_\alpha \to \mathbf{R}^2$ which are compatible in the sense that the transition functions

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are homeomorphisms. We will suppose that it is connected most of the times.

Any two dimensional topological manifold is also a differentiable manifold. That is, one can find in the same maximal atlas defined as above, a covering U_α and charts ϕ_α such that $\phi_\alpha \circ \phi_\beta^{-1}$ are diffeomorphisms.

Riemann surfaces being orientable, the surfaces we need to consider are the orientable ones. We exclude for instance the real projective plane. It does not have a complex structure.

It is convenient to have a combinatorial description of surfaces by means of a triangulation. This allows a direct computation of some topological invariants of the surface as the Euler characteristic.

To be more precise define first the standard 2-simplex Δ given by the convex envelope of the points (vertices) $(0,0), (1,0), (0,1)$ in $\mathbf{R}^2$. Each boundary segment is called an edge. If $\phi : \Delta \to \phi(\Delta) \subset M$ is a homeomorphism, we call $\phi(\Delta)$ a triangle and the images of the vertices and edges of the standard simplex are also called vertices and edges of the triangle.

Definition 3.1. *A triangulation of a compact surface M is a finite set of homeomorphisms $\phi_i : \Delta \to \phi_i(\Delta) \subset M$ covering M, that is, $\bigcup_i \phi_i(\Delta) = M$, and such that the intersection of two triangles is either*

- *empty,*
- *a vertex or*
- *an edge of each of the triangles.*

In particular the interior of the triangles are disjoint. We can now state the theorem whose first rigorous proof was given by Radó in 1924.

Theorem 3.2. *Any compact surface has a triangulation.*

Remark 3.3. *1. In fact, Radó proved that any surface which has a countable basis of open sets can be triangulated. For non-compact surfaces, as the number of triangles is not finite, we need to impose that each point has a neighborhood intersecting only a finite number of triangles.*

2. The existence of a triangulation for a compact manifold dimension 3 was established by Moise in 1952, but in dimensions higher than three a topological manifold might not have a triangulation.

3. One can define orientability for triangulated surfaces by saying that there exists a compatible orientation on all triangles (they induce opposite orientations on common edges).

4. Any triangulation of a compact surface may be obtained from another one by a continuous deformation and a finite sequence of the following elementary moves:

- *the creation of a vertex inside a triangle and thereby introducing three new triangles in the place of the original one and the corresponding inverse operation,*
- *replacing the common side of two adjacent triangles of the triangulation by the other diagonal of the quadrilateral formed by these two triangles (this is called a flip).*

A reference for the classification of compact surfaces is the first chapter of [Mas77] and we state the main result without proof. Riemann surfaces being orientable surfaces we state the theorem of classification only for orientable surfaces. A basic surgery construction is that of connected sum. We start with two surfaces and remove one disc from each and glue the two surfaces along the boundary of the discs. In fact we can obtain any surface, apart the sphere, by this surgery procedure applied to tori.

Theorem 3.4. *A compact orientable surface is homeomorphic to a sphere or to a connected sum of tori.*

Proof. Sketch: Once we know the surface is triangulated, one can prove the theorem of classification of compact surfaces by spreading the triangulation of the surface in the plane to form a polygon with boundary identifications. More precisely, given a triangulated surface we enumerate its triangles $T_1, T_2, \cdots, T_n$ in a way that each T_i has an edge in common with one of the previous triangles in the sequence. If T_i has two edges in common, we choose one of them to identify to one of the edges on the plane but leave the other one as a boundary of the polygon thus obtained. The union of the first two triangles along the common edge gives a parallelogram with possible boundary identifications. Adding each triangle makes the number of sides of this polygon jump by two. At the end we obtain a polygon with a number of sides identifications.

The idea now is to find a normal form for this polygon describing the surface. A usual normal form is the one which describes the surface as a connected sum of tori. A torus corresponds to a sequence $aba^{-1}b^{-1}$ and a handle to sequence $aba^{-1}b^{-1}c$. Clearly $a_1b_1{a_1}^{-1}{b_1}^{-1}a_2b_2{a_2}^{-1}{b_2}^{-1}$ corresponds to a connected sum of two tori. In other words, adding a handle to a torus. The normal form we look for a surface with g handles is therefore

$$a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$$

as in Figure 1) with $g \geq 1$ or aa^{-1} which is a sphere.

This is done using a sequence of operations which simplify the structure of the identifications on the boundary.

□

One can show that the following definition does not depend on the triangulation.

Definition 3.5. *The Euler characteristic of a triangulated surface is defined by the formula $\chi = T - E + V$, where T is the number of triangles, E is the number of edges and V is the number of vertices of a triangulation.*

The genus of a surface is related to the Euler characteristic through the formula

$$\chi = 2 - 2g.$$

3.2 The fundamental group

In this section we recall some basic concepts of algebraic topology necessary to describe the topology of a surface. We will not give proofs but, instead, refer to Hatcher for a complete treatment.

A curve in a topological space X is a continuous map $c : [0,1] \to X$. Two curves c_1 and c_2 with $c_1(0) = c_2(0)$ and $c_1(1) = c_2(1)$ are homotopic (with fixed end points) if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that

1. $F_{|\{0\}\times[0,1]} = c_1(0)$ and $F_{|\{1\}\times[0,1]} = c_1(1)$
2. $F_{|[0,1]\times\{0\}} = c_1$ and $F_{|[0,1]\times\{1\}} = c_2$.

A loop in X is a curve c with $c(0) = c(1)$. We can define the product of two loops c_1 and c_2 such that $c_1(0) = c_2(0) = x_0$ (we say the loops are based at x_0) as the loop $c_2c_1 : [0,1] \to X$

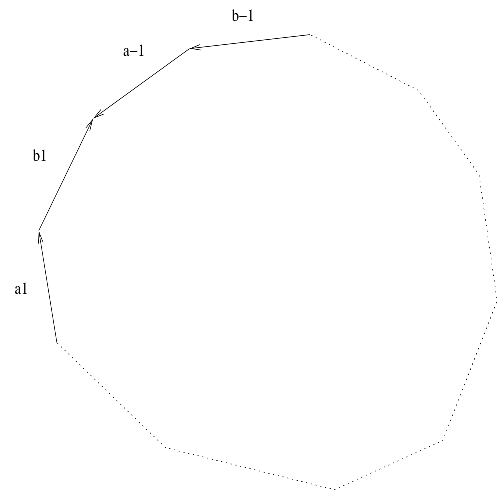


Figure 1: A surface obtained by boundary identifications on a disc.

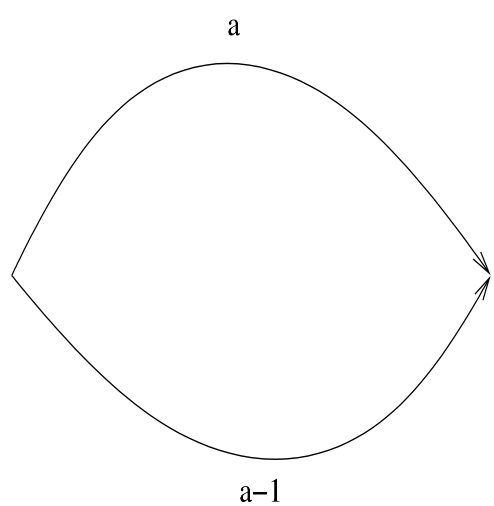


Figure 2: A sphere obtained by boundary identifications on a disc.

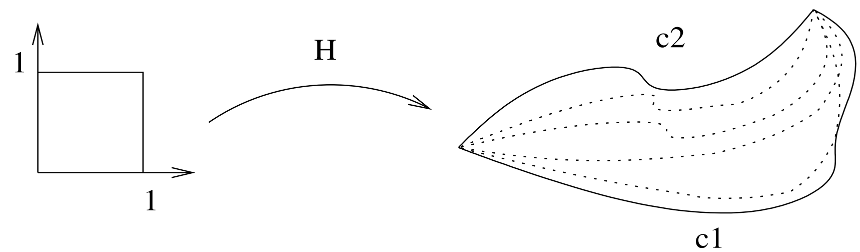


Figure 3: A homotopy between two curves c_1 and c_2.

given by $c_2c_1(t) = c_1(2t)$ for $0 \le t \le 1/2$ and $c_2c_1(t) = c_2(2(t-1/2))$ for $1/2 \le t \le 1$. The constant loop is defined to be $c(t) = x_0$ for all t, and the inverse of a loop c is the loop c^{-1} defined by $c^{-1}(t) = c(1-t)$. We say that two loops are freely homotopic if there exists a homotopy $F : [0,1] \times [0,1] \to X$ such that the first condition is not imposed. That is, the base point may change during the homotopy.

Let X be a manifold and $x_0 \in X$ a base point. We denote by $\pi_1(X, x_0)$, *the fundamental group*, the space of homotopy classes of loops based at x_0. It has a group structure induced by the multiplication on loops. Usually we denote by $[\gamma]$ the class containing the loop γ.

If x_0' is another base point, $\pi_1(X, x_0')$ is isomorphic to $\pi_1(X, x_0)$. In fact, let c be a curve with $c(0) = x_0$ and $c(1) = x_0'$. Then, one can define an isomorphism of groups $\pi_1(X, x_0) \to \pi_1(X, x_0')$ by $\gamma \to c\gamma c^{-1}$.

Example 3.6. *The fundamental group of S^1 is isomorphic to* $\mathbf{Z}$.

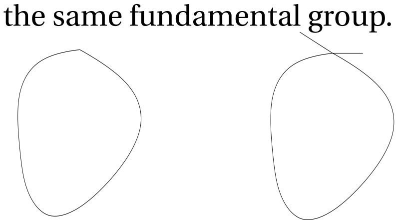
A continuous function $f : X \to Y$ between topological spaces such that $f(x_0) = y_0$

induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. A homeomorphism induces an isomorphism but an isomorphism between fundamental groups does not imply that the corresponding topological spaces are homeomorphic. A typical situation of isomorphic fundamental groups arises in the case of deformation retracts. They are very useful for computations.

Definition 3.7. *A subset $K \subset X$ of a topological space is a deformation retract of X if there exists a homotopy $F : X \times [0,1] \to X$ such that*

- *For all $x \in X$, $F(x,0) = x$.*
- *For all $x \in K$, $F(x,.) = x$.*
- *$F(.,1)(X) \subset K$.*

As in the following picture we can retract the two small segments on the right to obtain an object with the same fundamental group.



Proposition 3.8. *If $K \subset X$ is a deformation retract and $x_0 \in K$ then $\pi_1(X, x_0) = \pi_1(K, x_0)$.*

3.2.1 Group presentations and computations of the fundamental group.

A presentation of a group Γ is given by

$$\Gamma = \langle \gamma_1, \cdots \mid r_1, \cdots \rangle.$$

The γ_i are the generators and the r_i reduced words on the generators (words constructed with γ_i or γ_i^{-1} which don't contain the sequence $\gamma_i \gamma_i^{-1}$). By definition, Γ is the quotient of the free group on the generators γ_i by the normal subgroup generated by the relators. We say that Γ is finitely presented if there exists a presentation with a finitely number of generators and relators.

Example 3.9.

$$\mathbf{Z} \oplus \mathbf{Z} = \langle\, \gamma_1, \gamma_2 \mid [\gamma_1, \gamma_2]\, \rangle.$$

To give the fundamental group by a presentation is very useful for computations. An application of that description is the following theorem which we quote without proof.

Theorem 3.10 (Seifert-Van Kampen Theorem)**.** *Let $M = M_1 \cup M_2$ be the union of two path-connected open sets with $I = M_1 \cap M_2$ path-connected. Suppose the fundamental groups of M_1 and M_2 at a base point $x_0 \in I$ are $\Gamma_1 = \langle \gamma_1, \cdots \ | r_1, \cdots \rangle$. and $\Gamma_2 = \langle \delta_1, \cdots \ | s_1, \cdots \rangle$. Suppose $\pi_1(I, x_0)$ is generated by the elements η_i. Write each η_i as ϕ_{i1} and ϕ_{i2} using the generators of Γ_1 and Γ_2 respectively. Then*

$$\pi_1(M, x_0) = \langle \gamma_1, \cdots \delta_1 \cdots | r_1, \cdots, s_1, \cdots, \phi_{i1}\phi_{i2}^{-1} \rangle.$$

As a first application of the theorem we compute

Exercise 3.11. *The fundamental group of the infinity symbol ∞ is the free group with two generators. More generally, the fundamental group of a bouquet of g circles is the free group with g generators.*

We use the theorem of Seifert-Van Kampen to provide presentations for surface groups.

Exercise 3.12. *The fundamental group of a compact Riemann surface of genus g with a point deleted is the free group with $2g$ generators.*

We say a surface is of finite type if it is homeomorphic to a compact surface (genus g) with a finite number t of points (or disjoint discs) deleted.

Theorem 3.13. *The fundamental group of an orientable surface of finite type has a presentation of the form*

$$\left\langle \ a_1, b_1, \cdots, a_g, b_g, h_1, \cdots h_t \mid \Pi_{j=1}^{g}[a_j, b_j] h_1 \cdots h_t = 1 \ \right\rangle.$$

The elements h_i correspond to loops around the boundaries. In particular, from the presentation, we see that if $t \neq 0$ the fundamental group is free of rank $2g + t - 1$.

Exercise 3.14. *Prove the theorem using the classification of surfaces in the previous section.*

Can we have isomorphic fundamental groups for non-homeomorphic surfaces?

3.3 Covering spaces

In the following we suppose that the topological spaces are all arc connected and locally arc connected. In fact we are interested in connected surfaces which are manifolds and are therefore locally arc connected.

We denote by $\phi : (Y, y_0) \to (X, x_0)$ a continuous map $\phi : Y \to X$ such that $\phi(y_0) = x_0$. Recall that it induces the homomorphism $\phi_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ defined by $[\gamma] \to [\phi \circ \gamma]$.

Definition 3.15. *A map $p : \tilde{X} \to X$ between topological spaces is a covering if each point $x \in X$ has a neighborhood U_x such that $p^{-1}(U_x)$ is a disjoint union of open sets homeomorphic to U_x under p.*

We say that two coverings $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are equivalent if there exists a homeomorphism $p : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ p = p_1$. Coverings have the fundamental path lifting property:

Proposition 3.16. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. A path $\phi : ([0,1], 0) \to (X, x_0)$ can be lifted to a unique path $\tilde{\phi} : ([0,1], 0) \to (\tilde{X}, \tilde{x}_0)$ satisfying $p \circ \tilde{\phi} = \phi$.*

Proof. Let $L = \{\, t \in [0,1] \mid \phi_{|[0,t]}$ can be lifted $\}$. We show that this set is open and closed. It is clearly non-empty as $0 \in L$. If $t_0 \in L$ then $\tilde{\phi}(t_0)$ is contained in a unique component U of $p^{-1}(V)$ homeomorphic to V, a sufficiently small neighborhood of $\phi(t_0)$. There exists therefore a lift of the curve in a neighborhood of t_0 by taking $(p_{|U})^{-1} \circ \phi$. Similarly if t_0 is a limit of points t_n in L we observe that there exists a sufficiently small neighborhood of $\phi(t_0)$ such that $\tilde{\phi}(t_n)$ are contained in a component U of $p^{-1}(V)$. As U is a homeomorphism we can define $\tilde{\phi}(t_0)$. Uniqueness follows by a similar argument. □

Using a similar proof we may lift homotopies on X to homotopies on a covering $\tilde{X}$:

Proposition 3.17. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. A homotopy $F : [0,1] \times [0,1] \to X$ between two paths $\phi_1 : ([0,1], 0) \to (X, x_0)$ and $\phi_2 : ([0,1], 0) \to (X, x_0)$ has a lift to a unique homotopy $\tilde{F} : [0,1] \times [0,1] \to \tilde{X}$ between $\tilde{\phi}_1 : ([0,1], 0) \to (\tilde{X}, \tilde{x}_0)$ and $\tilde{\phi}_2 : ([0,1], 0) \to (\tilde{X}, \tilde{x}_0)$. In particular, $\tilde{\phi}_1(1) = \tilde{\phi}_2(1)$.*

Remark 3.18.

1. *The proposition above shows that $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

2. *If $\tilde{x}_0'$ is another base point for $\tilde{X}$ over x_0 then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\tilde{X}, \tilde{x}_0'))$ are conjugate.*

Definition 3.19. *The subgroup $p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$ is called the defining subgroup of the covering.*

Definition 3.20. *The universal covering of a topological space (arc connected and locally arc connected) is the covering having trivial defining group.*

The definite article above means that two coverings having trivial defining group are equivalent. It follows from the following basic result about coverings:

Theorem 3.21. *There exists a bijection between conjugacy classes of subgroups of $\pi_1(X, x_0)$ and equivalence classes of coverings.*

The construction of the covering space associated to a given subgroup $\Gamma \subset \pi_1(X, x_0)$ can be accomplished by considering the set of equivalence classes of paths $c : [0,1] \to X$ with $c(0) = x_0$. Equivalence between paths c_1 and c_2 meaning that $c_1(1) = c_2(1)$ and that $[{c_2}^{-1} c_1] \in \Gamma$. The map $p : \tilde{X} \to X$ is given by $p([c]) = c(1)$. For details see [Massey].

Remark 3.22. *If X is simply connected any covering is homeomorphic to X.*

The covering transformations (or deck transformations) of a covering $p : \widetilde{X} \to X$ are those homeomorphisms $\phi : \widetilde{X} \to \widetilde{X}$ satisfying $\pi \circ \phi = \pi$. The description of the covering group is given in the following theorem.

Theorem 3.23. *The group of covering transformations is isomorphic to*

$$N(p_* \pi_1(\widetilde{X}, \tilde{x}_0)) / p_* \pi_1(\widetilde{X}, \tilde{x}_0)$$

where N denotes the normalizer of the group in $\pi_1(X, x_0)$.

A covering whose defining subgroup is normal is called a *regular* or *normal* covering. In particular the universal covering is regular and $\pi_1(X, x_0)$ is the group of covering transformations.

Exercises

1. Recall that a map $\phi : X \to Y$ is proper if for any compact $K \subset Y$, $\phi^{-1}(K)$ is compact. Show that a local homeomorphism between manifolds is a finite covering if and only if ϕ is proper.

2. The punctured unit disc D^* has the upper half-plane as a universal covering. An explicit map is given by $e^{2\pi i t}$. The fundamental group is **Z** acting on the half-plane by integer translations. The regular covering corresponding to the subgroup generated by $e^{2\pi i m}$ also is the disc with covering group isomorphic to $\mathbf{Z}/m\mathbf{Z}$. The finite coverings of the punctured unit disc are equivalent to the maps $\phi_m : D^* \to D^*$ given by $z \to z^m$.

3. The torus $S^1 \times S^1$ is covered by the plane. Find its regular coverings.

4. Let the annulus $A = \{\, r < |w| < 1 \,\}$. The map $z \to \exp(2\pi i \log z / \log \lambda)$, where $r = \exp(-2\pi^2 / \log \lambda)$ defines a covering $D \to A$ of A by the unit disc D. The covering group is generated by $z \to \lambda z$.

5. Give an example of a surjective map which is a local homeomorphism but which is not a covering.

6. Let X be a simply connected Riemann surface and $f : X \to \mathbf{C}^*$ a holomorphic function. Prove that there exists a function $\tilde{f} : X \to \mathbf{C}$ such that $exp \circ \tilde{f} = f$.

7. Let M_1 and M_2 be two manifolds which have the same universal covering $\tilde{M}$ with projections $p_1 : \tilde{M} \to M_1$ and $p_2 : \tilde{M} \to M_2$ and covering transformations group G_1 and G_2 respectively. If $\phi : M_1 \to M_2$ is a homeomorphism, then we can lift it to a homeomorphism $\tilde{\phi} : \tilde{M} \to \tilde{M}$. Prove that $G_2 = \tilde{\phi} \circ G_1 \circ \tilde{\phi}^{-1}$.

3.3.1 Monodromy representation

Let $\Gamma = \pi_1(X, x_0)$ be the fundamental group of a manifold X. Theorem 3.21 states that finite coverings of $X \setminus S$, up to equivalence, are classified by conjugacy classes of subgroups of Γ of finite index. Fixing a subgroup $H \subset \Gamma$ of index d, the group Γ acts on the set of cosets Γ / H (a finite set with d elements) transitively. We obtain therefore a representation of Γ into the permutation group of the set Γ / H, call it $S_d(\Gamma / H)$:

$$\rho_H : \Gamma \to S_d(\Gamma / H).$$

Changing H by a conjugation to $H' = zHz^{-1}$ induces a bijection $c_z : S_d(\Gamma / H) \to S_d(\Gamma / H')$. Denoting by $C_z : \Gamma \to \Gamma$ the conjugation by z we have then two intertwined representations:

$$c_z \circ \rho_H = \rho_{H'} \circ C_z.$$

Observe also that the stabilizer of the coset H is the subgroup H itself and the stabilizer of gH is the conjugate $C_g(H)$.

3.4 Group actions

Let G be a group and X a topological manifold.

Definition 3.24. *G acts by homeomorphisms on X if there exists a map $G \times X \to X$ such that*

1. *for fixed $g \in G$, the induced map $g : X \to X$ is a homeomorphism.*

2. *$(gh)x = g(hx)$ for all $x \in X$ and $g, h \in G$*

3. *$1x = x$ for all $x \in X$*

If $G \times X \to X$ is an action we call the set $G_x = \{g \in G \mid gx = x\}$ the *stabilizer* or *isotropy of the action at* x. The *orbit* of $x \in X$ is the set Gx. The action is said to be *transitive* if the orbit of every point coincides with the whole space. The set of all orbits is denoted X/G and we define a topology on it by imposing that $U \subset X/G$ is open if and only if $\pi^{-1}(U) \subset X$ is open, where $\pi : X \to X/G$ is the canonical projection. A very special action is related to covering spaces. We need the following definitions:

Definition 3.25. *Let $G \times X \to X$ be an action.*

1. *The action of G is free if no point of X is fixed by an element of G different from the identity (that is, the isotropy of each element of X is trivial).*

2. *The action is properly discontinuous if for any compact $K \subset X$ the set of all $\gamma \in G$ such that $\gamma K \cap K \neq \emptyset$ is finite.*

Proposition 3.26. *Let $G \times X \to X$ be an action on a manifold X. The quotient X/G is a manifold with projection $X \to X/G$ a covering if the action is free and properly discontinuous.*

Proof. Suppose $x \in X$ and U_x is a relatively compact neighborhood. As the action is properly discontinuous there exists only a finite number of elements in G such that $g\bar{U}_x \cap \bar{U}_x \neq \emptyset$. As the action is free, for each one of those elements, $gx \neq x$. As the space is Hausdorff, we can choose a neighborhood $V_x \subset U_x$ such that for all $g \in G$, $g\bar{V}_x \cap \bar{V}_x = \emptyset$. This proves that the projection $X \to X/G$ is a covering.

The quotient is Hausdorff: suppose $x, y \in X$ are two points in distinct orbits. As X is a manifold, there exists two relatively compact neighborhoods U_x and U_y with $\bar{U}_x \cap \bar{U}_y = \emptyset$. As before, because the action is properly discontinuous and free, we may suppose $g\bar{U}_x \cap \bar{U}_x = \emptyset$ and $g\bar{U}_y \cap \bar{U}_y = \emptyset$. Consider $K = \bar{U}_x \cup \bar{U}_y$. As the action is properly discontinuous, the set of elements $g \in G$ such that $gK \cap K = (g\bar{U}_x \cap \bar{U}_y) \cup (\bar{U}_x \cap g\bar{U}_y) \neq \emptyset$ is finite, and by the same argument as before (using the fact that the action is free), we can choose U_x and U_y smaller such that $gK \cap K = \emptyset$ for all g. □

In fact the fundamental group of a manifold X acts freely and properly discontinuously in the universal cover $\tilde{X}$ such that the quotient map $\tilde{X} \to \tilde{X}/\pi_1(X, x_0)$ is equivalent to the covering $\tilde{X} \to X$.

Exercise 3.27. *A discrete subgroup Γ of a topological group G acts freely properly discontinuously on G by the natural action $\Gamma \times G \to G$ given by $(\gamma, g) \to \gamma g$.*

Example 3.28. *A subgroup of $\mathbf{R}^n$ is discrete if and only if it is generated by a set of linearly independent vectors.*

Proof. Suppose that the group is generated by a set of linearly independent vectors. By a linear transformation we can transform the set into a subset of the canonical base vectors. It is clear that the group is discrete as 0 is an isolated point of the group.

Conversely, suppose that the subgroup $\Gamma \subset \mathbf{R}^n$ is discrete and use induction on the dimension. For $n = 1$, let v be the smallest positive vector. Without loss of generality, suppose $\gamma \in \Gamma$ is positive and let k be the largest integer such that $kv \leq \gamma$. Then $\gamma - kv \in \Gamma$ and is smaller then v. A contradiction unless $\gamma = kv$. We conclude that Γ is generated by v.

Suppose now that any discrete subgroup in $\mathbf{R}^{n-1}$ is generated by a set of linearly independent vectors. Let $\Gamma \subset \mathbf{R}^n$ be discrete and v a vector with minimum norm. Because of the first step of the induction $\Gamma \cap \mathbf{R}v = \mathbf{Z}v$. Let $\pi : \mathbf{R}^n \to \mathbf{R}^n/\mathbf{R}v$ be the quotient map. We claim that $\pi(\Gamma)$ is discrete. Suppose v_i is a sequence in Γ such that $\pi(v_i) \to 0$, that is, $v_i - r_i v \to 0$ (where we can suppose that $r_i \leq 1/2$). Then for large i, $v_i < v$. This implies that $v_i = 0$ for large i so that $\pi(\Gamma)$ is discrete. By the induction hypothesis we can find linearly independent vectors $\{\pi(w_1), \cdots, \pi(w_{m-1})\}$ generating $\pi(\Gamma)$. $\{v, w_1, \cdots, w_{m-1}\}$ are linearly independent and generate Γ.

□

4 Riemann surfaces as branched covers

4.1 Branched coverings

Recall that a non-constant holomorphic map $\phi : Y \to X$ can be written locally, in adapted charts, as $p_X \circ \phi \circ {p_Y}^{-1}(z) = z^n$ for some $n \geq 1$. In the following definition we generalize this behaviour for maps between two dimensional real manifolds. Here we use complex coordinates $z = x + iy$ but we don't assume that there exists complex structures on the manifolds.

Definition 4.1. *A map $\phi : Y \to X$ between surfaces is a branched covering if*

1. *The restriction $\phi|_{\phi^{-1}(X-S)}$, where S is a discrete subset of X, is a covering.*
2. *For each point in $y_0 \in \phi^{-1}(S)$ there are coordinates p_Y around y_0 and p_X around $x_0 = \phi(y_0)$ such that $p_X \circ \phi \circ {p_Y}^{-1}(z) = z^n$. The integer n is called the ramification order of the ramification point y_0 or the multiplicity of ϕ at y_0.*

Observe that the inclusion map $\Delta^* \to \Delta$ satisfies the definition with the choice $S = \{0\}$. This will not happen if we impose that the map ϕ is proper (that is, for each compact $K \subset X$, $\phi^{-1}(K)$ is compact). Indeed we mostly use in applications the following proposition which we leave as an exercise.

Proposition 4.2. *Let $\phi : Y \to X$ be a proper branched cover. Then there exists $n \in \mathbf{N}^*$ such that for all $x \in X$*

$$\sum_{y \in \phi^{-1}(x)} mult_y \phi = n.$$

We say then that the degree of ϕ is n and write $deg\phi = n$.

Definition 4.3. *Let $\phi : Y \to X$ be a branched covering of Riemann surfaces. The ramification divisor is the formal sum*

$$R_\phi = (\sum n_i - 1) y_i$$

where y_i are the ramification points and n_i their ramification order.

4.1.1 Riemann-Hurwitz formula

Any compact Riemann surface can be described as a branched covering of $\mathbf{C}P^1$ once we admit the existence of at least one non-constant meromorphic function. From that description we can easily compute the genus of the surface. We state a more general version of that computation valid for a covering between compact surfaces.

Theorem 4.4. *Let $Y \to X$ be a branched covering of degree d between compact surfaces. For each ramification point $y \in Y$, let $o(y)$ be its ramification order. Then*

$$\chi(Y) = d\chi(X) - \sum (o(y) - 1).$$

Proof. The proof of the theorem follows from the existence of a triangulation with vertices containing the branching locus, that is, the image of all ramification points by the covering map. We will assume the existence of that triangulation of X. If the simplices of this triangulation are sufficiently small, the inverse image of the triangulation is a triangulation of Y. The number of its simplices is d times the number of original simplices, except for the vertices. Each ramification point diminishes by $(o(y) - 1)$ the maximum number of d times the number of vertices of the original triangulation. □

4.2 Riemann existence theorem

Topological coverings of Riemann surfaces inherit a unique complex structure such that the covering map is holomorphic. The equivalence between two coverings with their induced complex structure is a biholomorphism. This implies that the classification of coverings up to equivalence is in fact a classification of holomorphic coverings up to holomorphic equivalence.

A finite covering of a Riemann surface with a number of points deleted can always be extended to a branched covering. This follows from the following:

Exercise: The finite coverings, up to equivalence, of the punctured disc $D \setminus \{0\}$ are given by $\phi_n : D \setminus \{0\} \to D \setminus \{0\}$ where $\phi_n(z) = z^n$.

The following theorem is sometimes called the Riemann existence theorem. It constructs a Riemann surface from a finite covering of a Riemann surface (usually the Riemann sphere) with a number of points deleted. In this version it can be viewed as a purely topological property of the existence of extensions of coverings of punctured surfaces.

Theorem 4.5. *If X is a Riemann surface and $S \subset X$ is a closed discrete subset, then any finite covering $\phi' : Y' \to X' = X \setminus S$ (which we suppose connected) can be extended to a proper holomorphic map $\phi : Y \to X$, where Y is a Riemann surface containing Y' such that $Y \setminus Y'$ is a closed discrete subset.*

Proof. At a point $s \in S$ there exists a neighborhood U_s with $U_s \cap S = \{s\}$ and a coordinate chart $\phi_s : U_s \to D$ where D is the unit disc centered at the origin. As ϕ' is a finite covering, there exists a finite number of components $\phi'^{-1}(U_s \setminus \{s\})$. In fact, $\phi' : \phi'^{-1}(U_s \setminus \{s\}) \to U_s \setminus \{s\}$ is a covering. Let V' be one of the components. As $\phi'_{|V'}$ is a finite covering of the unit punctured disc, there exists a map $\psi' : V' \to D \setminus \{0\}$ so that $\phi_s \circ \phi \circ \psi'^{-1} : D \setminus \{0\} \to D \setminus \{0\}$ and such that $\phi_s \circ \phi \circ \psi'^{-1}(z) = z^k$ and therefore we can add the point 0 to $D \setminus \{0\}$ and obtain a holomorphic map from D to D. Let V be the set obtained by adding an abstract point to V' so that $\psi : V \to D$ is a homeomorphism and defines a holomorphic chart. $\phi_{|V}$ becomes a branched holomorphic covering. Repeating the procedure for each component above every $U_s \setminus \{s\}$ for $s \in S$ we obtain the Riemann surface Y. □

Remark 4.6. *Observe that a covering space of $X \setminus S$ is determined, up to equivalence, by its monodromy. That is a representation of $\rho : \pi_1(X \setminus S) \to S_d$ where S_d is the permutation group of d elements such that the image $\rho(\pi_1)$ acts transitively.*

4.3 Algebraic functions and the transcendence degree of the field of meromorphic functions

Let $\phi : Y \to X$ be a non-constant branched holomorphic covering of degree n between Riemann surfaces. The map $\phi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $g \to g \circ \phi$ is clearly a monomorphism. Considering the field extension $\phi^*(\mathcal{M}(X)) \subset \mathcal{M}(Y)$ we show the following

Theorem 4.7. *Let $\phi : Y \to X$ be a branched holomorphic covering of degree n between Riemann surfaces. Then $\phi^*(\mathcal{M}(X)) \subset \mathcal{M}(Y)$ is an algebraic field extension of degree n.*

Proof. We prove here that the degree is less than or equal to n In order to prove that the degree is precisely n we need a result (see Section 8.7) that guarantees the existence of a meromorphic function which assumes pairwise different values at points of a generic fiber (that is, whose points are not ramification points).

Let $f \in \mathcal{M}(Y)$. Let $S \subset X$ be a closed discrete subset such that $\phi : Y \setminus \phi^{-1}(S) \to X \setminus S$ is a covering. Consider the restriction of f to the meromorphic function $f \in \mathcal{M}(Y \setminus \phi^{-1}(S))$. We can define meromorphic functions on $X \setminus S$ by taking the elementary symmetric functions $s_1, \cdots s_n$ of the n functions $f \circ \phi_i^{-1} : U \to \mathbf{C}$ where $\phi_i = \phi_{|U_i} : U_i \to Y$ and U_i is a component of $\phi^{-1}(U)$ (supposing that each component of $\phi^{-1}(U)$ is homeomorphic to U). Observe that, by construction, f is a solution of the equation

$$\Pi_{i=1}^{n}(w - \phi^*(f \circ \phi_i^{-1})) = w^n - \phi^* s_1 w^{n-1} + \cdots + (-1)^n \phi^* s_n = 0.$$

To conclude that the extension is algebraic we need to show that the coefficients s_i extend to meromorphic functions on X. We divide the proof in two steps:

1. If f is holomorphic then s_i are bounded holomorphic functions on a neighborhood of a point $s \in S$. By Riemann's removable singularity theorem we can extend s_i to a holomorphic function.

2. If f is meromorphic at a point in $\phi^{-1}(s)$, consider a coordinate chart $z : U \to D$ such that $z(s) = 0$. Then $(\phi^* z)^m f$ is holomorphic if m is large and therefore the elementary symmetric functions of $(\phi^* z)^m f$ can be extended to holomorphic functions of the form $z^{m_i} s_i$ and therefore the s_i can be extended to meromorphic functions.

Suppose $f_0 \in \mathcal{M}(Y)$ is an element such that the minimal polynomial is of maximal degree n_0. We show now that $\mathcal{M}(X)(f_0) = \mathcal{M}(Y)$, thereby proving that the degree of the extension is less than n. In fact if $f \in \mathcal{M}(Y)$ is another element we have, by the existence of a primitive element ($\mathcal{M}(X)$ is of characteristic 0), $\mathcal{M}(X)(f_0, f) = \mathcal{M}(X)(g)$ and then

$$n_0 = dim_{\mathcal{M}(X)}\mathcal{M}(X)(f_0) \leq dim_{\mathcal{M}(X)}\mathcal{M}(X)(f_0, f) = dim_{\mathcal{M}(X)}\mathcal{M}(X)(g) \leq n_0$$

so that $\mathcal{M}(X)(f_0) = \mathcal{M}(X)(f_0, f)$. □

In the following we will prove a converse to that theorem. One of the origins of Riemann surface theory concerns the study of algebraic equations of the form

$$w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0,$$

where the coefficients $a_i(z)$ are rational functions on $\mathbf{CP}^1$. The idea is that the solution to that equation is, in fact, defined on a Riemann surface Y which is a branched covering $Y \to \mathbf{C}P^1$. We state the theorem in a more general form substituting $\mathbf{CP}^1$ for a general Riemann surface X.

Theorem 4.8. *Let X be a Riemann surface and*

$$P(w) = w^n + a_1 w^{n-1} + \cdots + a_n$$

an irreducible polynomial in $\mathcal{M}(X)[w]$ of degree n. Then there exists a Riemann surface Y, a branched holomorphic covering $p : Y \to X$ of degree n and a meromorphic function $F \in \mathcal{M}(Y)$ such that

$$P(F) = F^n + p^* a_1 F^{n-1} + \cdots + p^* a_n = 0.$$

Definition 4.9. *We say that Y is the Riemann surface associated to the irreducible polynomial P.*

Remark 4.10. *1. As $\mathcal{M}(X)$ is a field of characteristic 0, we know that the irreducible polynomial $P(w) \in \mathcal{M}(X)[w]$ is separable. That is, its roots in the algebraic closure of $\mathcal{M}(X)$ are all distinct.*

2. Recall that the elementary symmetric polynomial $s_i(t_1, \cdots, t_n)$ $(1 \le i \le n)$ of the variables t_i generate the algebra of symmetric polynomials of those variables. Observe that the functions $a_i \in \mathcal{M}(X)$ are the elementary symmetric functions of the roots of the polynomial $P(w)$. That is

$$\Pi_{1\le i\le n}(w - t_i) = w^n - s_1 w^{n-1} + \cdots + (-1)^n s_n.$$

Therefore, the polynomial $\Delta = \Pi_{i<j}(t_i - t_j)^2$ which is clearly symmetric belongs to $\mathcal{M}(X)$. It is called the discriminant of $P(w)$. In particular, by the previous remark, the discriminant vanishes identically only if $P(w)$ is reducible.

Proof. The discriminant Δ of $P(w)$ vanishes at points of X where there are multiple roots. Therefore, because $P(w)$ is irreducible, Δ vanishes only on a closed discrete set of points S which we also suppose contains the poles of a_i. Let $X' = X \setminus S$ and define Y' to be the set of all points in $(z, w) \in (X \setminus S) \times \mathbf{C}$ satisfying the equation $P(w) = 0$. By the implicit function theorem (Proposition 1.19) and its corollary, $\phi' : Y' \to X'$ is a covering map. We extend then this covering to a branched covering $\phi : Y \to X$. The meromorphic function is defined first as a holomorphic function on Y' as $(z, w) \to w$ and then by extension (with a similar argument as in the previous theorem) to the whole of Y. To show that Y is connected, suppose that $Y = Y_1 \cup \cdots \cup Y_k$ is a decomposition in connected components with $\phi_i : Y_i \to X$ branched coverings. Then, for each ϕ_i the meromorphic function F restricted to Y_i defines a polynomial $P_i(w) \in \mathcal{M}(X)$ such that $P(w) = P_1(w)\cdots P_k(w)$ contradicting the irreducibility of $P(w)$. □

Theorem 4.11. *Let k be a finitely generated field of transcendence degree one over $\mathbf{C}$. Then, there exists a compact Riemann surface X such that $\mathcal{M}(X) = k$.*

Proof. Let $z \in k$ generating a purely transcendental extension. Then $k/\mathbf{C}(z)$ is a finite extension (say of degree d) which we can write, by choosing a primitive element $f \in k$ as $k = \mathbf{C}(z, f)$ By the hypothesis, one can write $k = \mathbf{C}(z)[w]/P$, as the quotient ring by the ideal generated by P (the minimal polynomial in $\mathbf{C}(z)[w]$, of degree d, satisfied by f).

Identify $\mathbf{C}(z)$ to the field of rational functions on $X = \mathbf{CP}^1$. Now, we construct the Riemann surface Y associated to P as in theorem 4.8. Let $\mathcal{M}(Y)$ be its field of meromorphic functions. We may consider $z \in \mathcal{M}(Y)$. As P has degree d in w one obtains that $[\mathcal{M}(Y), \mathbf{C}(z)] = d = [k, \mathbf{C}(z)]$ and therefore $k \cong \mathcal{M}(Y)$. □

4.4 Hyperelliptic Riemann surfaces

Let $f(z) = (z-a_1)\cdots(z-a_k) \in \mathcal{M}(\mathbf{C}P^1)$ with distinct roots $a_i \in \mathbf{C}$. The algebraic function defined by $P(z,w) = w^2 - f$ is a Riemann surface together with a branched covering of degree two which is branched on $a_1, \cdots a_k$ if k is even and on $a_1, \cdots a_k, \infty$ if k is odd. These Riemann surfaces are called hyperelliptic.

Observe that in that case the algebraic curve $\{ (z,w) \in \mathbf{C}^2 \mid P(z,w) = 0 \}$ is a Riemann surface by the implicit function theorem as at each solution (z,w) we have $P_z \neq 0$ or $P_w \neq 0$.

To understand the topology of hyperelliptic Riemann surfaces, consider the Riemann-Hurwitz formula to compute their genera. Let X_f be the Riemann surface as defined by $P(z,w) = w^2 - f$. If k is even we obtain

$$\chi(X_f) = 2\chi(\mathbf{C}P^1) - k = 4 - k$$

and as the Euler characteristic is given by $\chi = 2 - 2g$, we obtain $g = \frac{-2+k}{2} = k/2 - 1$. In the case k is odd we obtain

$$\chi(X_f) = 2\chi(\mathbf{C}P^1) - (k+1) = 3 - k$$

so that $g = \frac{-1+k}{2} = (k-1)/2$. In particular, for $k = 3$ we obtain an elliptic curve.

Exercises

1. Determine the Riemann surface defined by $P(z,w) = z^2 - w^3$ over $\mathbf{C}P^1$.
2. Determine the genus of the Riemann surface defined by $P(z,w) = z^n + w^n - 1$ over $\mathbf{C}P^1$.
3. The field $\mathcal{M}(\mathbf{C}P^1)$ is $\mathbf{C}(z)$, a purely transcendental extension of $\mathbf{C}$.

4.5 Belyi's theorem

As an application of the construction of a Riemann surface of an algebraic function we will describe a relation between the field of definition of an algebraic function and the number of branching points of the covering over $\mathbf{CP}^1$.

We say that the Riemann surface X is defined over $\bar{\mathbf{Q}}$ if it is constructed as above starting with an irreducible polynomial in $\bar{\mathbf{Q}}[z,w]$, where $\bar{\mathbf{Q}}$ is the field of algebraic numbers.

Theorem 4.12 (Belyi)**.** *A compact Riemann surface X is defined over $\bar{\mathbf{Q}}$ if and only if there exists a holomorphic covering $\pi : X \to \mathbf{C}P^1$ branched on three points.*

Proof. We will prove the "only if" part. The other implication being outside our scope because it needs basic algebraic geometry. We start with a polynomial $P \in \bar{\mathbf{Q}}[z, w]$. By theorem 4.7 there exists $\phi : X \to \mathbf{C}P^1$ which is branched over a finite set S of algebraic points. We divide the proof in two steps:

1. We first modify this branched covering to a covering which is branched over rational points. Take $s \in S$ and let $h \in \mathbf{Q}[X]$ be its minimal polynomial. The map $h \circ \phi : X \to \mathbf{C}P^1$ is a branched covering with branching points contained in $h(S) \cup \{h(z) \mid h'(z) = 0\}$. Observe that $h(s) = 0$ so we made one of the branching points in S rational at the cost of introducing new branching points. But the minimal polynomial of a point $z_0 \in \{z \mid h'(z) = 0\}$ is of degree strictly smaller than the degree of h and therefore the minimal polynomial of $h(z_0) \in \{h(z) \mid h'(z) = 0\}$ has strictly smaller degree too (being in the same field extension as $\mathbf{Q}(z_0)$). We repeat this procedure with each element in S and obtain, by composing with each minimal polynomial, a branched covering where the new branching points have minimal polynomials of strictly smaller degrees. Eventually the degree is one and we obtain only rational branching points.

2. By the previous step, we may suppose that $\phi : X \to \mathbf{C}P^1$ is branched on rational points. Now we reduce the number of branching points to at most three. Supposing it is greater than three, we can always assume that $\{0, 1, \infty\}$ are among those points by composing with an automorphism of $\mathbf{C}P^1$. For $m, n \in \mathbf{Z}^*$ such that $m + n \neq 0$, consider the map $f_{mn} : \mathbf{C}P^1 \to \mathbf{C}P^1$ defined by

$$f_{mn}(z) = \frac{(m+n)^{m+n}}{m^n n^n} z^m (1-z) n.$$

   The critical values are computed solving $f'_{mn}(z) = 0$ and we obtain that they are contained in $\{0, 1, \infty, \frac{m}{m+n}\}$. But the branching points are contained in $\{0, 1, \infty\}$. We conclude that for each rational branching point of ϕ outside $\{0, 1, \infty\}$ we can find a map f_{mn} so that $f_{mn} \circ \phi$ transforms this branching point to one of $\{0, 1, \infty\}$. This concludes the proof.

□

5 Riemann surfaces as quotients

One of the most challenging problems concerning Riemann surfaces is their classification. A natural classification is up to equivalence under biholomorphisms. Fortunately simply connected Riemann surfaces have a simple classification. We will state this fundamental theorem without proof.

Theorem 5.1 (Riemann uniformization theorem)**.** *A simply connected Riemann surface is biholomorphic to either*

1. $\mathbf{C}P^1$
2. $\mathbf{C}$
3. $H^1_{\mathbf{C}} = \{z \in \mathbf{C}, |z| < 1\}$.

This theorem implies that the study of Riemann surfaces is very related to the study of discrete subgroups of the automorphism groups of the simply connected Riemann surfaces: Any Riemann surface is biholomorphic to the quotient of one of these simply connected models by a discrete subgroup of its automorphism group.

Remark 5.2. *In higher dimensions the classification of simply connected complex manifolds does not have a clear answer. For instance,it is easy to construct deformations of the complex two dimensional ball such that any two of those deformed balls are not biholomorphic.*

5.1 Automorphism groups

It will be important to determine for each manifold M its group of biholomorphisms Aut(M). For the proof of the following theorem we need to recall Schwarz lemma:

Lemma 5.3. *Let $f : H^1_{\mathbf{C}} \to H^1_{\mathbf{C}}$ be a holomorphic map such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in H^1_{\mathbf{C}}$ and $|f'(0)| \leq 1$. If $|f'(0)| = 1$ or if $f(z) = z$ for some $z \neq 0$ then $f(z) = e^{i\theta} z$.*

Theorem 5.4. *The automorphism groups of the simply connected Riemann surfaces are*

1. $Aut(\mathbf{C}P^1) = PSL(2,\mathbf{C}) = SL(2,\mathbf{C})/\{\pm I\}$*, all Möbius transformations.*
2. $Aut(\mathbf{C}) = \{az + b \mid a \neq 0\ ,\ b \in \mathbf{C}\}$.
3. $Aut(H^1_{\mathbf{C}}) = PSU(1,1) = SU(1,1)/\{\pm I\}$*, Möbius transformations preserving the disc.*

Proof. We first describe $f \in Aut(\mathbf{C})$, which is an entire function. We have $f(z) = a_0 + a_1 z + \cdots$. As f is an automorphism, the image of a neighborhood of infinity is a neighborhood of infinity. Therefore it can be extended to a holomorphic function at infinity. We conclude that $f(z)$ is a polynomial and by the fundamental theorem of algebra, it must be linear.

To show 1. observe that we can write, in homogeneous coordinates, $\mathbf{C}P^1 = \{[z_0, z_1]\}$, where z_0, z_1 are not both null. Any transformation of the form $[z_0, z_1] \to [az_0 + bz_1, cz_0 + dz_1]$, with $ad - bc \neq 0$ is an automorphism. So we have an action $PSL(2,\mathbf{C}) \times \mathbf{C}P^1 \to \mathbf{C}P^1$. Given an element $\gamma \in Aut(\mathbf{C}P^1)$ we can find an element $\gamma_1 \in PSL(2,\mathbf{C})$ such that $\gamma \circ \gamma_1(\infty) = \infty$. So $\gamma \circ \gamma_1 \in Aut(\mathbf{C})$ and we conclude using the description of $Aut(\mathbf{C})$.

To show 3. we observe first that $PSU(1,1) \subset PSL(2,\mathbf{C})$. That is, $SU(1,1) = \{A \in SL(2,\mathbf{C}) \mid h(Az, Az) = h(z,z)\}$, where $h(z,w) = z_0\overline{w}_0 - z_1\overline{w}_1$ is a hermitian form. So $PSU(1,1)$ preserves the disc $H^1_{\mathbf{C}} = \{z \in \mathbf{C}P^1 \mid h(z,z) < 0\}$. If $\gamma \in Aut(H^1_{\mathbf{C}})$, there exists an element $\gamma_1 \in PSU(1,1)$ such that $\gamma \circ \gamma_1(0) = 0$. By Schwarz's lemma we obtain $|f'(0)| \leq 1$ and, as f is a biholomorphism, the same inequality for the inverse function gives $|f'(0)| = 1$. By Schwarz's lemma we conclude that $\gamma \circ \gamma_1(z) = e^{i\theta} z$ and that concludes the proof. □

Corollary 5.5. *A Riemann surface covered by* $\mathbf{C}P^1$ *is biholomorphic to* $\mathbf{C}P^1$.

Proof. This follows from the fact that any Möbius transformation has a fixed point. It implies that there is no subgroup of the Möbius group acting freely on $\mathbf{C}P^1$. □

On the other hand observe that the involution $\iota : z \to -\frac{1}{\bar{z}}$ defined on $\mathbf{C}P^1$ does not have fixed points. The quotient space $\mathbf{C}P^1/\langle\iota\rangle$ is the real projective plane which is not a Riemann surface.

Exercise 5.6. *A meromorphic function on* $\mathbf{C}P^1$ *is a holomorphic map of* $\mathbf{C}P^1$ *on itself. They are all rational functions, that is* $f(z) = \frac{p(z)}{q(z)}$ *where* $p(z)$ *and* $q(z)$ *are polynomials.*

Exercise 5.7. *The disc and the half plane* $H_{\mathbf{R}} = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ *are biholomorphic.* $Aut(H_{\mathbf{R}}) = PSL(2,\mathbf{R})$.

Exercise 5.8. *If* K *is a field* $PSL(n,K) = PGL(n,K)$ *if and only if every element of* K *has an* n*-th root. For instance* $PSL(2,\mathbf{R}) \neq PGL(2,\mathbf{R})$.

Exercise 5.9. $PU(1,1)$ *acts doubly transitively on the boundary. That is given* $x_1, y_1, x_2, y_2 \in \partial H^1\mathbf{C}$ *with* $x_i \neq y_i$, *there exists an element* $\gamma \in PU(1,1)$ *such that* $\gamma x_1 = x_2$ *and* $\gamma y_1 = y_2$.

5.1.1 Conjugacy classes

It is important to understand the conjugacy classes of elements in the automorphism groups. Elements in the same conjugacy class act in an "equivalent" way.

Lemma 5.10. *An element in $PSL(2,\mathbf{C})$ has one or two fixed points. We have*

1. *If it has only one fixed point then it is conjugate to $z \to z+1$.*

2. *If it has only two fixed points it is conjugate to $z \to \lambda z$, $\lambda \neq 1, 0$.*

Proof. Given any (non-trivial) Möbius transformation we solve the equation

$$\frac{az+b}{cz+d} = z.$$

It has one or two solutions. If it has only one solution, by conjugating with an element of $PSL(2,\mathbf{C})$, we can suppose that ∞ is that fixed point. In that case the element must be of the form $z \to az+b$. We immediately see that $a = 1$ otherwise there would be a second fixed point. Moreover, by conjugating with $z \to \frac{1}{b}z$ we obtain $z \to z+1$. To show the second part we observe that we can conjugate an element with two fixed points to one fixing 0 and ∞. That gives clearly the form $z \to \lambda z$. $\square$

We can further refine that lemma to obtain the orbit space by the conjugation action of $PSL(2,\mathbf{C})$. The proof of the following proposition is a simple consequence of the lemma.

Proposition 5.11. *The conjugacy classes of $PSL(2,\mathbf{C})$ are uniquely represented by the following elements*

1. $z \to z+1$ *called parabolic.*

2. $z \to e^{i\theta}z$, $0 \leq \theta \leq \pi$, *called elliptic.*

3. $z \to \lambda z$, $\lambda \in \mathbf{C}$ $|\lambda| > 1$, *called loxodromic. In the case* $\lambda \in \mathbf{R}$ *we call it a hyperbolic transformation.*

Proof. The first part is contained in the previous lemma. For the second and third part we observe that if $\gamma(z) = \lambda z$, in order to preserve the fixed points, we are allowed to conjugate by elements of the form $z \to az$, which commute with γ (so irrelevant), or $z \to a/z$. In that case γ is transformed to $g\gamma g^{-1}(z) = \frac{1}{\lambda}z$. This shows the result. $\square$

Considering only elements in $PSU(1,1)$ we describe conjugacy classes in the following definition.

Definition 5.12. *$\gamma \in PSU(1,1)$ is called*

1. *Elliptic if it has a fixed point in $H^1_{\mathbf{C}}$.*

2. *Parabolic if it has a unique fixed point in $\partial H^1_{\mathbf{C}}$.*

3. *Hyperbolic if it has two fixed points in $\partial H^1_{\mathbf{C}}$.*

There exists a convenient description of the conjugacy classes using trace computations on matrices:

Proposition 5.13. *Let $\gamma \in PSU(1,1)$ and consider a lift $\tilde{\gamma} \in SU(1,1)$. Then γ is*

1. *elliptic if and only if $tr^2\tilde{\gamma} < 4$,*

2. *parabolic if and only if $tr^2\tilde{\gamma} = 4$ and γ is not the identity,*

3. *hyperbolic if and only if $tr^2\tilde{\gamma} > 4$.*

Observe, however, that conjugation in $PSU(1,1)$ splits certain conjugacy classes in $PSL(2,\mathbf{C})$ (also, some disappear because they don't correspond to elements in $PSU(1,1)$). For instance, the parabolic class is split in two: $z \to z+1$ and $z \to z-1$. Analogously, the elliptic class $z \to e^{i\theta}$, $0 \le \theta \le \pi$ splits in two, so that $0 \le \theta < 2\pi$ is the parameterization of the classes. On the other hand, the only loxodromic classes which appear in $PSL(2,\mathbf{C})$ are those with $\lambda > 1$ and they don't split.

Remark 5.14. *Let $\widehat{PSU}(1,1) = \langle PSU(1,1), z \to \bar{z}\rangle$. Using conjugation on that group we can collapse again the splitting. In particular $z \to z+1$ and $z \to z-1$ are conjugate in the corresponding group $\widehat{PSL}(2,\mathbf{R})$.*

5.2 The complex plane C and its quotients

Theorem 5.15. *A Riemann surface is covered by $\mathbf{C}$ if and only if it is biholomorphic to $\mathbf{C}$, $\mathbf{C} \setminus \{0\}$ or a torus.*

Proof. We prove first the only if part. The other implication is a consequence of the next proposition. Let $\Gamma \subset Aut(\mathbf{C})$ be the covering group. If $\gamma(z) = az + b$ is an element of Γ then $a = 1$, otherwise γ would have a fixed point. So Γ is generated by translations. We saw in theorem 3.28 that a discrete subgroup of $Aut(\mathbf{C})$ generated by translations is one of the following:

1. $\{id\}$

2. $<\gamma>=\mathbf{Z}$, a group generated by one translation $\gamma(z)=z+\omega$

3. $<\gamma_1,\gamma_2>=\mathbf{Z}\oplus\mathbf{Z}$, a group generated by two translation $\gamma_1(z)=z+\omega_1$ and $\gamma_2(z)=z+\omega_2$ with ω_1 and ω_2 linearly independent over $\mathbf{R}$.

The first case corresponds to $\mathbf{C}$. For the second case the function $z\to e^{2\pi z/\omega}$ establishes a biholomorphism between $\mathbf{C}/<\gamma>$ and $\mathbf{C}\setminus\{0\}$. In the third case the quotient manifold is diffeomorphic to a torus. □

To complete the theorem we need to show that any torus is covered by $\mathbf{C}$. That is, using the uniformization theorem, the complex disc (or the half plane) cannot cover a torus. This follows from the following proposition.

Proposition 5.16. *Let $\Gamma\subset Aut(H_{\mathbf{R}})$ be a discrete group without fixed points. If Γ is abelian, then it is cyclic.*

Proof. There are two cases. If $Id\neq\gamma\in\Gamma$ is parabolic we can, without loss of generality, suppose that $\gamma(z)=z+x$, where $x=\pm1$. A computation then shows that any commuting element is parabolic. Indeed,

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

implies

$$\begin{pmatrix}a+xc & b+xd\\ c & d\end{pmatrix}\begin{pmatrix}a & ax+b\\ c & cx+d\end{pmatrix}$$

So $xc=0$ and $x(a-d)=0$ which implies $c=0$ and $a=d$. That is, the commuting element is parabolic. By discreteness we obtain that the group generated by the two elements is cyclic. Analogously, if γ is hyperbolic, without loss of generality, suppose that $\gamma(z)=\lambda z$. We easily conclude (by the lemma bellow) that an element commuting with it is of the same form and using discreteness we conclude that the subgroup is cyclic. □

Lemma 5.17. *Two hyperbolic elements commute if and only if they have the same fixed points.*

Proof. We write one element as $z\to\lambda z$ and the other by a general Möbius transformation. Then, by commutativity

$$\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

A computation shows that $b=c=0$. □

Remark 5.18. *Observe that if G (any group) acts on M (any space) and g_1 commutes with g_2 the fixed points of g_1 are preserved by g_2 and the fixed points of g_2 are preserved by g_1, indeed,*

$$g_1(x) = x \Rightarrow g_2 g_1(x) = g_2(x) \Rightarrow g_1(g_2(x)) = g_2(x).$$

If γ has only one fixed point any commuting element will have precisely the same fixed point (so if γ is parabolic the commuting element is also parabolic).

5.3 Fuchsian groups

Definition 5.19. *A Fuchsian group is a discrete subgroup of $PSU(1,1)$.*

In order to define a quotient of the disc by a discrete group as a Riemann surface we need to verify that the action is free and properly discontinuous. The action is free if there are no elliptic elements, also called torsion elements. On the other hand, the action is always properly discontinuous as is shown by the next theorem.

Theorem 5.20. *A subgroup $\Gamma \subset Aut(H^1_{\mathbf{C}})$ is Fuchsian if and only if it acts properly discontinuously.*

Proof. Clearly if Γ acts properly discontinuously then it is discrete. Now suppose it is discrete and it does not act properly discontinuously. Recall the normal family theorem:

Theorem 5.21 (Normal family theorem)**.** *Suppose $f_n : \Omega \to \mathbf{C}$ is a family of holomorphic functions defined on a region of $\mathbf{C}$. If f_n is uniformly bounded on each compact subset of Ω (a normal family) then there exists a subsequence which converges uniformly on compact subsets (the limit function will then be holomorphic).*

We need the following lemma

Lemma 5.22. *If a sequence $\gamma_n \in Aut(H^1_{\mathbf{C}})$ converges uniformly on compact subsets to γ then*

1. *$\gamma \in Aut(H^1_{\mathbf{C}})$ or*
2. *γ is a constant function with value some $e^{i\theta}$ in the boundary of $H^1_{\mathbf{C}}$.*

Proof. If there exists $x_0 \in H^1_{\mathbf{C}}$ such that $\gamma_n(x_0) \to b$ with $|b| = 1$ then by the maximum modulus principle $\gamma(z) = \gamma(x_0) = b$, for all $z \in H^1_{\mathbf{C}}$. Otherwise we have $\gamma : H^1_{\mathbf{C}} \to H^1_{\mathbf{C}}$ and taking a subsequence if necessary γ_n^{-1} converges uniformly on compact subsets to $\delta : H^1_{\mathbf{C}} \to H^1_{\mathbf{C}}$ such that $\delta \circ \gamma = Id$. Therefore $\gamma \in H^1_{\mathbf{C}}$. □

Back to the proof: if the action is not properly discontinuous there exists a compact $K \subset H^1_{\mathbf{C}}$ and a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\gamma_n(K) \cap K \neq \emptyset$. Clearly the sequence γ_n is a normal family. Therefore, taking perhaps a subsequence, it converges uniformly on compact subsets to a holomorphic function. Taking a subsequence if necessary we have $\gamma_n(x_n) = y_n$ for two sequences (x_n) and (y_n) in K with $\lim x_n = x$ and $\lim y_n = y$, therefore $\lim \gamma_n(x) = y$. We conclude, using the lemma, that γ_n converges to an element of $Aut(H^1_{\mathbf{C}})$, therefore the group is not discrete. □

The following lemma is an important technical component of the next theorem.

Lemma 5.23 (Shimizu)**.** *If $z \to z+1$ belongs to a Fuchsian group in $PSL(2,\mathbf{R})$, then every other element γ of the form*

$$\frac{az+b}{cz+d}$$

satisfies $|c| \geq 1$, provided $c \neq 0$.

Proof. We set

$$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and define by induction for $n \geq 1$,

$$A_{n+1} = A_n A_0 A_n^{-1}.$$

We compute the coeficients of A_{n+1} obtaining

$$\begin{aligned} a_{n+1} &= 1 - c_n a_n \\ b_{n+1} &= a_n^2 \\ c_{n+1} &= -c_n^2 \\ d_{n+1} &= 1 + a_n c_n \end{aligned}$$

If $c < 1$ then c_n converges, in fact $|c_n| = |c|^{2^{n-1}}$. We claim that $\lim a_n = 1$. Observe that $|a_{n+1}| \leq 1 + |a_n c_n| \leq 1 + |a_n|$. By induction then $|a_{n+1}| \leq n + |a|$. We obtain then $|a_{n+1}| \leq 1 + |a_n c_n| \leq 1 + |c_n|(n+|a|) \leq 1 + |c|^{2^{n-1}}(n+|a|)$ and the result follows. □

A Fuchsian group $\Gamma \subset PSL(2,\mathbf{R})$ is said to be co-compact if the quotient $H^1_{\mathbf{C}}/\Gamma$ is compact. From Shimizu lemma we conclude the following theorem which says that if a Riemann surface is compact and not the sphere or a quotient of the complex plane then its fundamental group does not have parabolics.

Theorem 5.24. *If $\Gamma \subset PSL(2,\mathbf{R})$ is co-compact without torsion then any non-trivial element is hyperbolic.*

Proof. If there were a parabolic element, by conjugation we may suppose it $z \to z+1$ and generator of the parabolic group Γ_∞ fixing ∞. As

$$Im(\gamma(z)) = \frac{Im(z)}{|cz+d|^2}$$

for any $\gamma(z) = \frac{az+b}{cz+d}$ in Γ we estimate using Shimizu's lemma that if $Im(z) > 1$ then

$$Im(\gamma(z)) \leq \frac{1}{|c|^2 Im(z)} < 1$$

for γ not in Γ_∞. Therefore the set $\{ z \mid -\frac{1}{2} < \mathrm{Re}\, z < \frac{1}{2}\ ,\ Im(z) > 1 \}$ passes to the quotient, but it is not compact, a contradiction. □

5.4 Fundamental domains

Definition 5.25. *A fundamental domain of a properly discontinous action on a topological manifold,* $\Gamma \times X \to X$ *is an open set* $F \subset X$ *such that*

1. $\bigcup_{\gamma\in\Gamma} \gamma\overline{F} = X$*, where* $\overline{F}$ *is the closure of* F
2. *If* $x, y \in F$ *they are not in the same orbit.*

We do not suppose that the action is free but observe that a fixed point of an element in Γ is never contained in F. It might be contained in the closure of F.

Example 5.26. *A fundamental domain for the action of the additive group generated by the translations* $z \to z+1$ *and* $z \to z+\tau$ *is the parallelogram defined by the sides* $1, \tau$.

5.4.1 $PSL(2,\mathbf{Z})$

Theorem 5.27. $D = \{ z \in H^1_{\mathbf{C}} \mid |z| > 1, -1/2 < Re(z) < 1/2 \}$ *is a fundamental domain for* $PSL(2,\mathbf{Z})$.

Proof. Again we use

$$Im(\gamma(z)) = \frac{Im(z)}{|cz+d|^2}$$

to observe that fixing $\tau \in \mathbf{C}$, there is only a finite number of elements $\gamma \in PSL(2,\mathbf{Z})$ with $|c\tau+d|^2 < M$ for a fixed bound M. This follows because $\mathbf{Z}\tau + \mathbf{Z}$ is a discrete group. Take γ such that $Im(\gamma(\tau))$ is maximum. Using the translation we can suppose without loss of generality that $-1/2 \leq Re(\tau) \leq 1/2$. We claim that $|\gamma(\tau)| \geq 1$, otherwise using the inversion $s(z) = -1/z$ we would get $Im(s\gamma(\tau)) = \frac{Im(\gamma(\tau))}{|\gamma(\tau)|^2} > Im(\gamma(\tau))$. A contradiction.

Figure 4: A fundamental domain for a triangle group containing $PSL(2,\mathbf{Z})$ as an index two subgroup. The fundamental domain for $PSL(2,\mathbf{Z})$ is the symmetric double of the grey region.

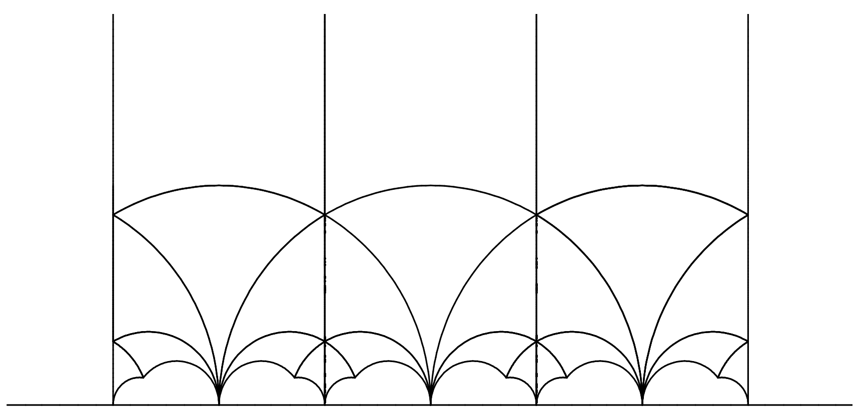


Figure 5: A fundamental domain for $PSL(2,\mathbf{Z})$ and some of its translates.

Suppose now that τ and $\gamma(\tau)$ belong to $\bar{D}$. Without loss of generality we may assume that $Im(\gamma(\tau)) \geq Im(\tau)$. Therefore

$$|c\tau + d| \leq 1.$$

Just looking at the imaginary part, that is, $\mathrm{Im}\,(c\tau + d) = c\,\mathrm{Im}\,\tau \geq c\frac{\sqrt{3}}{2}$, we obtain that the only possibilities are $c = 0, 1, -1$. If $c = 0$ it follows easily that γ is either the translation or the identity. If $c = 1$, we must have $|z + d| \leq 1$. We claim that that is only possible if $z = \omega$ or $z = -\bar{\omega}$ or $z = i$. That can be seen easily in the picture. Analogously we obtain those two points if $c = -1$. □

5.4.2 $\Gamma(2)$

Let $\pi_N : SL(2,\mathbf{Z}) \to SL(2,\mathbf{Z}_N)$ be the homomorphism obtained by reducing modulo N. It passes to the quotients

$$\phi_N : SL(2,\mathbf{Z})/\{I,-I\} \to SL(2,\mathbf{Z}_N)/\{I,-I\}.$$

The kernel of this homomorphism is called the principal congruence group of level N, $\Gamma(N) \subset PSL(2,\mathbf{Z})$.

The simplest case, $\Gamma(2)$, acts freely on the complex disc so that $H^1_{\mathbf{C}}/\Gamma(2)$ is a sphere with three points deleted.

To understand the action, observe first that the homomorphism ϕ_N is clearly surjective and, as $SL(2,\mathbf{Z}_2) = PSL(2,\mathbf{Z}_2)$ has 6 elements which can easily be enumerated:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

we have, therefore, that $\Gamma(2) \subset PSL(2,\mathbf{Z})$ is of index 6.

The fundamental domain of subgroups of finite index can be computed using the following lemma.

Lemma 5.28. *Suppose D is a fundamental domain for a group G acting on a space M. Let $H \subset G$ be a subgroup of index k and $Hg_1, \cdots, Hg_k$ be its left cosets. Then $D_H = g_1 D \cup \cdots \cup g_k D$ is a fundamental domain for H.*

Proof. If $x, y \in D_H$ and there exists $h \in H$ such that $y = hx$ then, as $x \in g_i D$ and $y \in g_j D$, we might suppose that $g_j \bar{y} = h g_i \bar{x}$ for $\bar{x}, \bar{y} \in D$. That is, $\bar{y} = g_j^{-1} h g_i \bar{x}$ which contradicts the fact that D is a fundamental domain for G. On the other hand, $H\overline{D_H} = M$ follows because $G = \bigcup H g_i$. □

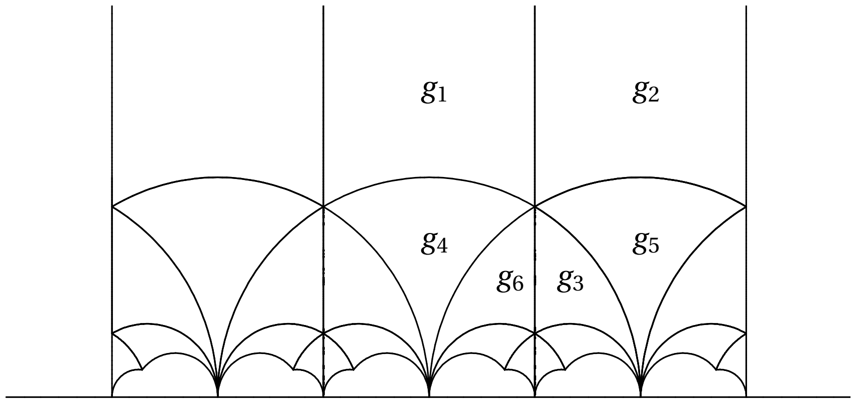


Figure 6: A fundamental domain for $\Gamma(2)$ showing the six translates of the fundamental region of $PSL(2,\mathbf{Z})$ corresponding to each coset.

Left coset representatives of $\Gamma(2)$ are obtained by chosing an inverse image for each element of $SL(2,\mathbf{Z}_2)$:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g_5 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} g_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

The boundary of the fundamental domain consists of 2 vertical half lines paired by the parabolic element

$$\gamma_1 = z \to z+2$$

and two pairs of arcs paired by parabolic elements in the group:

$$\gamma_2 = g_4\gamma_1 g_4^{-1} = z \to \frac{z}{2z+1}$$

for the sides of the region $g_4D \cup g_6D$ (where D is the fundamental domain for $PSL(2,\mathbf{Z})$ found before),

$$z \to \frac{3z-2}{2z-1}$$

for the sides of the region $g_3D \cup g_5D$. One should observe that the three points of $H^1_{\mathbf{C}}$ in the boundary of the region are identified by those pairings and, around that point, the regions match together to form a complex disc. The quotient is the sphere where 3 points are deleted.

6 Riemann surfaces as algebraic curves

The principal source of examples of Riemann surfaces comes from subsets of $\mathbf{C}^n$ or complex projective spaces $\mathbf{CP}^n$ defined by zeros of polynomials. They are called algebraic curves. It turns out that every compact Riemann surface can be embedded as an algebraic curve in $\mathbf{CP}^3$. Indeed, a deep theorem proves that any compact Riemann surface can be embedded as a projective algebraic curve in some $\mathbf{CP}^n$. A simple argument shows then that any complex algebraic curve $\mathbf{CP}^n$, $n > 3$, can be projected as an embedding into a $\mathbf{CP}^3$.

6.1 Affine plane curves

Let

$$F(x, y) = \sum_{r,s} c_{r,s} x^r y^s$$

be a polynomial in two variables with complex coefficients. That is, $F \in \mathbf{C}[x, y]$.

Definition 6.1. *The affine complex plane curve defined by a non-constant polynomial F is the set*

$$C_F = \{(x, y) \in \mathbf{C}^2 \mid F(x, y) = 0\}$$

Examples:

1. A complex line is given by the equation $ax + by + c = 0$.
2. A conic is given by the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$.
3. (Exercise) A homogeneous polynomial in two variables can be factored as a product of linear polynomials.

The definition has some obvious problems. Namely, two different polynomials might define the same curve (think of $F(x, y)$ and $F(x, y)^2$) and the set C_F might not be connected ($F(x, y) = x(x + 1)$). Another problem is that the set C_F might not be a smooth subvariety of $\mathbf{C}^2$.

The important notion to address the first problem is that of irreducible polynomial. F (non-constant polynomial) is irreducible if it cannot be written as $F = Q.R$ where Q and R are non-constant polynomials. Any polynomial can be written in a unique way (up to multiplicative constants and permutation of factors) as a product of irreducible factors. The following theorem shows that C_F is determined by the irreducible factors of F. One can also show that if F is irreducible C_F is connected (this is not trivial, see Milne, Algebraic Geometry, prop 15.1 https://www.jmilne.org/math/CourseNotes/ AG15.pdf). We say that a curve C_F is irreducible if F is irreducible. We will admit the following fundamental theorem which we state in the special case of polynomials in two variables:

Theorem 6.2 (Hilbert Nullstellensatz)**.** *If F and Q are two polynomials, then Q vanishes on C_F if and only if there exists $n \in \mathbf{N}^*$ and a polynomial $H \in \mathbf{C}[x, y]$ such that $Q^n = FH$. That is, Q^n is in the ideal $(F) \subset \mathbf{C}[x, y]$ generated by F.*

Therefore, if a polynomial is factored into its prime factors as

$$F = f_1^{n_1} \cdots f_k^{n_k},$$

where $n_i \geq 1$, then

$$C_F = C_{f_1 \cdots f_k}.$$

We will say that $f_1 \cdots f_k$ is a minimal polynomial. The curves C_{f_i} defined by the irreducible factors of F are the irreducible components of C_F.

Definition 6.3. *The degree of a curve C_F defined by a minimal polynomial F is the degree of F, that is*

$$d = max\{\, r + s \mid c_{r,s} \neq 0 \,\}.$$

Definition 6.4. *A point $(x_0, y_0) \in C_F$ is singular if*

$$\frac{\partial F}{\partial x}(x_0, y_0) = \frac{\partial F}{\partial y}(x_0, y_0) = 0.$$

Otherwise, it is called a non-singular point. We say a curve is non-singular if it does not have singular points.

By the implicit function theorem, the curve $C_F - \{\, singular\ points \,\}$ is a complex submanifold. At a singular point (x_0, y_0), we can further analyse the curve by computing the Taylor polynomial

$$F(x, y) = \sum_{m \geq 1} \sum_{i+j=m} \frac{1}{i!j!} \frac{\partial^m F}{\partial x^i \partial y^j}(x_0, y_0)(x - x_0)^i (y - y_0)^j.$$

The smallest m with $\frac{\partial^m F}{\partial x^i \partial y^j}(x_0, y_0) \neq 0$ is the order of the singular point. Then, the homogeneous polynomial

$$\sum_{i+j=m} \frac{1}{i!j!} \frac{\partial^m F}{\partial x^i \partial y^j}(x_0, y_0)(x - x_0)^i (y - y_0)^j$$

has linear irreducible components. Each irreducible component defines a line which is tangent to the curve at the singular point. We say that the singular point is ordinary if the number of lines equals the order of the singular point.

Example 6.5. *Let $h(y)$ be a polynomial with no multiple roots (there are no common roots of h and h'). Set $F(x,y) = x^2 - h(y)$ and*

$$C_F = \{(x,y) \in \mathbf{C}^2 \mid x^2 - h(y) = 0\}.$$

Note that $x^2 - h(y)$ is irreducible (prove it). Singular points in the curve satisfy

$$\frac{\partial F}{\partial x} = 2x = 0 \quad \frac{\partial F}{\partial y} = h'(y) = 0.$$

There are none. Therefore C_F is a conected Riemann surface. What happens if h is not a perfect square? Is C_F connected? Is it non-singular?

6.2 Projective plane curves

Affine curves are never compact. Indeed, supposing that $F(x,y)$ depends on x, the function $(x,y) \to x$ is a non-constant holomorphic function on C_F. In order to consider compact surfaces we define projective curves in $\mathbf{CP}^2$. We start with a homogeneous polynomial $F(x,y,z)$ defined on $\mathbf{C}^3$.

Definition 6.6. *The projective complex curve defined by F is the set*

$$C_F = \{[x,y,z] \in \mathbf{CP}^2 \mid F(x,y,z) = 0\}.$$

We factor homogeneous polynomials by irreducible homogeneous polynomial and Hilbert's Nullstellensatz is valid for homogeneous polynomials. So a projective curve is defined by a minimal polynomial whose irreducible factors have multiplicity one. We define, as for affine curves, the irreducible components of C_F to be the projective curves defined by the irreducible factors of F.

Definition 6.7. *The degree of a curve C_F defined by a minimal polynomial F is the degree of F.*

In order to interpret geometrically the degree we define first the intersection multiplicity of a line and a projective curve. Suppose $L \subset \mathbf{CP}^2$ is a complex line which is not an irreducible component of a projective curve C_F given by a polynomial F. By changing coordinates we may suppose that $L = \{[x,y,0]\}$. To find the intersections we solve the equation

$$F(x,y,0) = 0.$$

As L is not a component, $F(x,y,0) \neq 0$. Also, remark that $F(x,y,0)$ is homogeneous and therefore it can be factored into $degF$ linear factors which might be repeated. Each factor is of the form $(b_i x - a_i y)$ and the point $[a_i, b_i, 0]$ is an intersection point with a multiplicity defined by the number of times the same factor appears.

Definition 6.8. *A point* $[x_0, y_0, z_0] \in C_F$ *is singular if*

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0) = \frac{\partial F}{\partial y}(x_0, y_0, z_0) = \frac{\partial F}{\partial z}(x_0, y_0, z_0) = 0.$$

Otherwise, it is called a non-singular point.

Example: A projective line in $\mathbf{CP}^2$ is defined by the equation $ax + by + cz = 0$.

The relation between affine curves and projective curves is made explicit by writing $\mathbf{CP}^2 = \mathbf{C}^2 \cup \mathbf{CP}^1 = \{\, [x, y, z] \mid z \neq 0 \,\} \cup \{\, [x, y, 0] \,\}$. A homogeneous polynomial of degree d, $F(x, y, z)$, which does not have z as a factor, defines a polynomial $F(x, y, 1)$ on $\mathbf{C}^2$ of degree d. And reciprocaly, if $F(x, y) = \sum_{r,s} c_{r,s} x^r y^s$ is a polynomial of degree d on $\mathbf{C}^2$ we define a degree d homogeneous polynomial on three variables

$$\tilde{F}(x, y, z) = \sum_{r,s} c_{r,s} x^r y^s z^{d-r-s}.$$

One can interpret the projective curve $C_{\tilde{F}}$ as the compactification of the affine curve C_F. The points at infinity are

$$\{\, [x, y, 0] \mid \sum_{0 \le r \le d} c_{r,d-r} x^r y^{d-r} = 0 \,\}.$$

To each infinity point (a_i, b_i) corresponds an asymptote line in $\mathbf{C}^2$ given by

$$a_i x - b_i y = 0.$$

It is also clear that $F(x, y, z)$ is irreducible if and only if $F(x, y, 1)$ is irreducible.

The tangent line at a non-singular point is the projective line defined by the equation

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)\, x + \frac{\partial F}{\partial y}(x_0, y_0, z_0)\, y + \frac{\partial F}{\partial z}(x_0, y_0, z_0)\, z = 0.$$

Exercise : Prove Euler's relation: If F is homogeneous of degree d then

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)\, x_0 + \frac{\partial F}{\partial y}(x_0, y_0, z_0)\, y_0 + \frac{\partial F}{\partial z}(x_0, y_0, z_0)\, z_0 = dF(x_0, y_0, z_0).$$

The following Lemma relates non-singular points of a projective curve and its affine curve. It follows immediately from Euler's relation.

Lemma 6.9. $[x_0, y_0, z_0]$, *with* $z_0 \neq 0$ *is a non-singular point of a projective curve defined by* $F(x, y, z)$ *if and only if* $(x_0/z_0, y_0/z_0)$ *is a non-singular point of the affine curve defined by* $F(x, y, 1)$. *The tangent line of* $C_{F(x,y,z)}$ *at* $[x_0, y_0, z_0]$ *(restricted to* $\mathbf{C}^2 \subset \mathbf{CP}^2$*) coincides with the tangent line of* $C_{F(x,y,1)}$ *at* $(x_0/z_0, y_0/z_0)$.

Using the previous lemma for each affine coordinate chart of $\mathbf{CP}^2$ we conclude that a projective curve whose points are non-singular is a Riemann surface (one can show that if all points are non-singular then the homogeneous polynomial is irreducible and this implies that the curve is connected, this is not true for affine curves as the following example shows). It is called a smooth projective plane curve.

Example 6.10. *Let* $f(x, y) = x(x-1)$. *The affine curve* C_f *is smooth and reducible (the union of two parallel lines). On the other hand its compactification* C_F *is given by the algebraic curve in* $\mathbf{CP}^2$ *defined by* $F(x, y, z) = x(x-z)$, *a reducible polynomial. Note that now,* C_F *is not smooth. Indeed, the point* $(0, 1, 0)$ *is a singular point.*

Example 6.11. *Consider the curve defined for* $g \geq 1$ *and pairwise distinct* $a_i \in \mathbf{C}$, $1 \leq i \leq 2g$*:*

$$C = \{[x, y, z] \in \mathbf{CP}^2 \mid F(x, y, z) = y^2 z^{2g-2} - (x - a_1 z) \cdots (x - a_{2g} z) = 0\}.$$

We compute the partial derivatives:

$$\frac{\partial F}{\partial x} = -\sum_i (x - a_1 z) \cdots (x - \hat{a_i} z) \cdots (x - a_{2g} z)$$

$$\frac{\partial F}{\partial y} = 2yz^{2g-2}$$

$$\frac{\partial F}{\partial z} = (2g-2) y^2 z^{2g-1} + \sum_i a_i (x - a_1 z) \cdots (x - \hat{a_i} z) \cdots (x - a_{2g} z)$$

To compute the singular points, observe that from the second equation $z = 0$ *or* $y = 0$. *If* $y = 0$ *then* $z \neq 0$ *(otherwise we also have* $x = 0$*). We may suppose that* $z = 1$ *in that case and as the* a_i *are pairwise distinct there are no solutions to the first equation in* C. *If* $z = 0$ *analogously we have* $y \neq 0$. *Making* $x = 0$ *we see that* $[0, 1, 0]$ *is the unique solution of the equations and therefore is the unique singular point of the curve.*

Exercise: Any projective line is biholomorphic to $\mathbf{CP}^1$.

Exercise: A conic in $\mathbf{CP}^2$ is defined by a degree two homogeneous polynomial

$$F(x,y,z) = ax^2 + dy^2 + fz^2 + 2bxy + 2cxz + 2eyz$$

which can be written as $X^T A_F X$ where

$$A_F = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

and

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

1. Prove that C_F is non-singular if and only if $det A_F \neq 0$.

2. Prove that any smooth projective conic is isomorphic to $\mathbf{CP}^1$.

6.3 Algebraic sets and algebraic curves

In order to give some perspective we give in this section a very short introduction to algebraic geometry. Indeed, algebraic sets in $\mathbf{C}^n$ of any dimension are defined as follows.

Consider $A = \mathbf{C}[x_1, \cdots, x_n]$ the polynomial ring in n-variables over $\mathbf{C}$.

Definition 6.12. *An affine algebraic set defined by a subset $T \subset A$ is*

$$Z(T) = \{x \in \mathbf{C}^n \mid F(x) = 0 \textit{ for all } F \in T\ \}.$$

So the empty set, any finite subset of $\mathbf{C}^n$, the whole $\mathbf{C}^n$ and affine algebraic curves are examples of algebraic sets. An hypersurface, is an algebraic set defined by one polynomial. In particular, if the polynomial is linear, the algebraic set is called an hyperplane. Again, the fact that $Z(T)$ might have different defining sets is an obvious problem. One can show that any algebraic set is a finite union of *irreducible* algebraic sets which are themselves related to prime ideals of A.

Definition 6.13. *An irreducible affine algebraic set (or algebraic variety) X is an algebraic set whose ideal*

$$I(X) = \{F \in A \mid F(x) = 0 \textit{ for all } x \in X\ \}$$

is prime.

Recall that a prime ideal $I \subset A$ is a proper ideal such that if $ab \in I$, either $a \in I$ or $b \in I$. As an example, if $F \in \mathbf{C}[x, y]$ is an irreducible polynomial, then the ideal generated bt F is prime and the complex algebraic curve is therefore an algebraic variety. We define projective algebraic varieties analogously by considering homogeneous polynomials. In principle, in $\mathbf{CP}^n$ we need $n-1$ equations but one sometimes need more equations. The best possible situation is given in the following Definition.

Definition 6.14. *A smooth complete intersection curve is the set*

$$C = \{[x] \in \mathbf{CP}^n \mid F_1(x) = \cdots = F_{n-1}(x) = 0\}.$$

where F_i are homogeneous polynomials in $\mathbf{C}^{n+1}$ such that the $(n-1)\times(n+1)$ matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)$$

has maximal rank at each point in C.

As for plane curves we can prove, using the implicit function theorem, that a complete intersection is a complex submanifold. It defines therefore a compact Riemann surface.

Not all projective curves are complete intersections. But one can show that every embedding of a Riemann surface in projective space $\mathbf{CP}^n$ is a local complete intersection, meaning that it is a projective curve defined by a finite number of homogeneous polynomials which is locally defined by only $(n-1)$ polynomials satisfying the rank condition above.

Example The classic example of a local complete intersection is the twisted cubic:

$$t : \mathbf{CP}^1 \to \mathbf{CP}^3$$

defined by $t([x, y]) = [x^3, x^2y, xy^2, y^3]$. Observe that if $x \neq 0$, one can write in a local chart $t([1, y]) = [1, y, y^2, y^3]$. Otherwise, $t([0, 1]) = [0, 0, 0, 1]$. The curve is defined by three equations: $x_0x_3 = x_1x_2$, $x_0x_2 = x_1^2$ and $x_1x_3 = x_2^2$. On each chart $x_i \neq 0$ one can use two of them. But one cannot define the curve using only two equations.

6.4 All projective curves can be embedded in $\mathbf{CP}^3$

Given a point $v \in \mathbf{CP}^n$ and a hyperplane L not containing it we may define the projection from v to L, $\pi : \mathbf{CP}^n \setminus \{v\} \to L$; choose lifts $\tilde{L}$ and $\tilde{p}$ to $\mathbf{C}^{n+1}$. Given $z \in \mathbf{CP}^n \setminus \{v\}$, choose a lift $\tilde{z}$ in $\mathbf{C}^{n+1}$. Define $\pi(z) = [span(\tilde{z}, \tilde{v}) \cap \tilde{L}] \in L$. Here $span(\tilde{z}, \tilde{v})$ is the vector space generated by $\tilde{z}, \tilde{v}$ and the intersection is not empty as $dim(\tilde{L}) = n$ and $dim(\tilde{z}, \tilde{v}) = 2$. We could also consider, more intrinsically, the projective space defined by all lines passing through v and the projection π to be given by $z \to [span(\tilde{z}, \tilde{v})]$ in this space. In the case $v = [0, 0, \cdots, 1]$ the projection is given by

$$[x_0, \cdots, x_n] \to [x_0, \cdots, x_{n-1}, 0].$$

Proposition 6.15. *Any smooth projective curve can be embedded in* $\mathbf{CP}^3$.

Proof. The proof is obtained by projecting a curve embedded in $\mathbf{CP}^n$ from a linear space into a convenient $\mathbf{CP}^3$. If we want that the projection be an embedding we need to be careful. The linear space from where we should project should avoid secants and tangents.

Definition 6.16. *A complex line passing through two points of a projective curve is called a secant.*

Suppose that $v \in \mathbf{CP}^n$ and X a projective curve disjoint from v. Clearly, the projection from p is injective restricted to X if and only if v is not contained in any of the secants to X.

Lemma 6.17. *Let $p \in X$ be a point in a smooth projective curve and $v \in \mathbf{CP}^n$ disjoint from all the secants of X. The projection from p restricted to X is an embedding at p if and only if v is disjoint from the tangent line to X at p.*

Proof. We may suppose that $p = [1, 0, \cdots, 0]$ and $v = \{[0, \cdots, 0, 1]\}$. The projection from v is given by $[x_0, \cdots, x_n] \to [x_0, \cdots, x_{n-1}, 0]$. On a neighborhood of p, the smooth projective curve is given by $[1, g_1(z), \cdots, g_n(z)]$ with $g_i'(z) \neq 0$ for some $1 \leq i \leq n-1$ if we impose that the tangent line does not contain v. This completes the proof. □

To prove the proposition, we start with a projective curve. Define the complex manifold defined by triples of points (x, y, z) such that $x \neq y$ are points in X and z a point in the secant between x and y. It is of dimension 3 and therefore, its image by the projection $(x, y, z) \to z$ is of maximal dimension 3. We conclude that there are points in $\mathbf{CP}^n$ which are not contained in any secant. Analogously, we may conclude that the set of points contained in a tangent line is of dimension at most 2. If the projective curve is embedded into a projective space of dimension greater than or equal to 4 we obtain a point not contained in any secant or tangent line and the projection from that point embeds X in a projective space of one dimension smaller. We may proceed with projections until an embedding into $\mathbf{CP}^3$. □

Remark 6.18. *An embedding into* $\mathbf{CP}^2$ *is not always possible but one can project any projective algebraic curve onto a singular curve whose singular points are all ordinary double singularities.*

6.5 Intersections of projective curves: Bézout's theorem

In this section we prove a formula which counts the intersection number of two projective curves. The formula involves a definition of multiplicity and is best described using the notion of a divisor. Meromorphic functions on projective curves are obtained by taking quotients of homogeneous polynomials of the same degree.

Consider a smooth projective curve X and a non-zero homogeneous polynomial F of degree d.

Definition 6.19. *The intersection divisor of F on X,* $div(F) = \sum n_p p$*, is the formal sum of points* $p \in X$ *where* $F(p) = 0$ *with* n_p *being the order of the meromorphic function obtained from F by dividing it by a homogeneous polynomial G of the same degree which is non-vanishing at* p.

Observe that the order of the meromorphic function does not depend on the choice of the non-vanishing homogeneous polynomial G because $G(p) \neq 0$. If F is linear, we call $div(F)$ a hyperplane divisor.

In general, the degree of a divisor $D = \sum n_p p$ is $deg(D) = \sum n_p$. If F_1 and F_2 are homogeneous polynomials of the same degree than $div(F_1) - div(F_2) = div(F_1/F_2)$ which is the divisor of a meromorphic function. But the degree of a principal divisor is 0 so $deg(div(F_1)) = deg(div(F_2))$. In particular all hyperplane divisors have the same degree.

Definition 6.20. *The degree of a smooth projective curve,* $deg(X)$ *is the degree of a hyperplane divisor.*

Exercise: The degree of a smooth plane projective curve coincides with the degree of the irreducible polynomial defining it.

Bézout's theorem computes the degree of an intersection divisor:

Theorem 6.21 (Bézout's theorem)**.** *Let X be a smooth curve and F a non-zero homogeneous polynomial . Then*

$$deg(div(F)) = deg(X)deg(F).$$

Proof. Let H a homogeneous polynomial of degree 1. Then $deg(div(H^{degF})) = deg(div(F))$. Now $deg(div(H^{degF})) = deg(F)deg(div(H)) = deg(F)deg(X)$. □

6.6 Algebraic curves and ramified covers: Plücker's formula

Given a smooth projective plane curve $X \subset \mathbf{CP}^2$, not containing the point $[0,1,0]$, defined by a homogeneous polynomial F we can define a ramified cover $\pi : X \to \mathbf{CP}^1$ by taking the projection from the point $[0,1,0]$, that is $\pi : [x, y, z] \to [x, z]$. We obtain the Riemann surface which as ramified cover of $\mathbf{CP}^1$.

Proposition 6.22. *Let X be a smooth algebraic curve defined by the homogeneous polynomial F in $\mathbf{CP}^2$ not containing the point $[0,1,0]$ and $\pi : X \to \mathbf{CP}^1$ the projection as above. Then, the ramification divisor $R_\pi \subset X$ is equal to $div(\frac{\partial F}{\partial y})$ (which is a homogeneous polynomial).*

Proof. Without loss of generality we will work on a chart with $z \neq 0$. We suppose therefore $z = 1$. By the implicit function theorem, if $\frac{\partial F}{\partial y}(x_0, y_0, 1) \neq 0$ the projection has multiplicity one in $[x_0, y_0, 1]$. On the other hand, if $\frac{\partial F}{\partial y}(x_0, y_0, 1) = 0$ we have $\frac{\partial F}{\partial x}(x_0, y_0, 1) \neq 0$ and therefore there exists a holomorphic function g defined on a neighborhood of y_0 such that $F(g(y), y, 1) = 0$. In that case

$$\frac{dF(g(y), y, 1)}{dy} = \frac{\partial F}{\partial x}(g(y), y, 1)g'(y) + \frac{\partial F}{\partial y}(g(y), y, 1) = 0,$$

so that $g'(y_0) = 0$. In fact, differentiating again and again we observe that the order of g' at y_0 is the same as the order of the function $\frac{\partial F}{\partial y}$ at y_0 (where x_0 is fixed). Therefore $\pi([g(y), y, 1]) = [g(y), 1]$ which in charts is writen $y \to g(y)$ has multiplicity at (x_0, y_0) given by $ord_{y_0} \frac{\partial F}{\partial y}(x_0, y) + 1$.

□

Example 6.23. *Consider the Fermat curve for $d \geq 1$:*

$$C = \{[x, y, z] \in \mathbf{CP}^2 \mid x^d + y^d + z^d = 0\}.$$

It is a smooth curve. Let $\pi : [x, y, z] \to [x, z]$ be the projection as above. Observe that the point $[0,1,0]$ does not belong to C_F. We have $\frac{\partial F}{\partial y}(x, y, z) = dy^{d-1}$. The ramification points correspond to $y = 0$ and are given by solutions to the equation $x^d + z^d = 0$. That gives d solutions. The multiplicity at each solution is $ord_{y=0} \frac{\partial F}{\partial y} + 1 = d$.

By Riemann-Hurwitz, we obtain that

$$\chi(C) = d\chi(\mathbf{CP}^1) - d(d-1)$$

which gives its genus $g = \frac{(d-1)(d-2)}{2}$.

This computation can be carried on for a any smooth projective plane curve:

Theorem 6.24 (Plücker's formula)**.** *Let $X \in \mathbf{CP}^2$ be a smooth plane projective curve of degree d. Then, the genus of X is*

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. Suppose $X = \{ [x,y,z] \in \mathbf{CP}^2 \mid p(x,y,z) = 0 \}$ and consider the projection $\pi : [x,y,z] \to [x,z]$ as above (suppose without loss of generality that $[0,1,0]$ does not belong to X) and therefore $R_\pi = div(\frac{\partial p}{\partial y})$.

Now, by Bézout, as $\deg p = d$ and $\deg \frac{\partial p}{\partial y} = d-1$, we obtain that

$$degR_\pi = \deg(div(\frac{\partial p}{\partial y})) = \deg p . \deg \frac{\partial p}{\partial y} = d(d-1).$$

Therefore

$$\chi(X) = d\chi(\mathbf{CP}^1) - d(d-1).$$

□

The relation between the compact Riemann surface constructed from an irreducible polynomial in two variables and the complex algebraic curve obtained through the associated homogeneous polynomial is given in the following discussion.

Let $P(x,y)$ be an irreducible polynomial of degree d in y. Set $V_P = \{ (x,y) \in \mathbf{C}^2 \mid P(x,y) = 0 \}$ and $Y \to \mathbf{CP}^1$ be the compact Riemann surface constructed in theorem 4.8 . In particular, Y contains, as a dense subset, the set $V_P \setminus \Sigma$ where

$$\Sigma = \{ (x,y) \in \mathbf{C}^2 \mid \deg P(x,\cdot) < d \text{ or } \frac{\partial P}{\partial y}(x,y) = \frac{\partial P}{\partial y}(x,y) = 0 \}.$$

One can homogenize P to obtain the (irreducible) homogeneous polynomial $\tilde{P}(x,y,z)$. Note that the complex curve $V_{\tilde{P}} \subset \mathbf{CP}^2$ might have singularities. On the other hand Y, by construction, is smooth. The relation between the two constructions is given in the following:

Proposition 6.25. *Let $P(x,y)$ be an irreducible polynomial and consider the dense subset $V_P \setminus \Sigma \subset Y$ as above. Then, the inclusion $V_P \setminus \Sigma \subset V_{\tilde{P}} \subset \mathbf{CP}^2$ extends to a holomorphic surjection*

$$Y \to V_{\tilde{P}}.$$

7 Riemann surfaces and hyperbolic geometry

An important development was the discovery by Poincaré was that Möbius transformations preserving the disc were, in fact, isometries of the disc equipped with a metric of constant negative curvature.

7.1 Riemannian manifolds

A Riemannian manifold is a manifold equipped with a positive definite scalar product $\langle\ ,\ \rangle$ defined on the tangent space at each point. Using the Riemannian metric one defines the length of curves and a metric on the manifold so that the distance between two points is the infimum of all lengths of curves joining them:

$$d(p,q) = \inf_{\gamma(0)=p,\gamma(1)=q} L(\gamma)$$

where

$$L(\gamma) = \int_0^1 \sqrt{\langle\dot{\gamma},\dot{\gamma}\rangle}\,dt$$

The group of isometries, that is, distance preserving diffeomorphisms of a metric space M, will be denoted by $Isom(M)$. Isometries are determined by their derivative at one point:

Proposition 7.1. *Let M be a connected Riemannian manifold, $\phi : M \to M$ be an isometry with $\phi(p) = p$. Then $\phi_* : T_pM \to T_pM$ determines ϕ.*

Exercise 7.2. *Let E^n be the n-dimensional Euclidean space. Show that $Isom(E^n)$ is the group $\{x \to Ax + B\}$ where A is orthogonal.*

Exercise 7.3. *Prove the following exact sequence*

$$0 \to \mathbf{R}^n \to Isom(E^n) \to O(n) \to 1$$

Exercise 7.4. *The finite subgroups of O(2) are the cyclic group generated by a rotation and the dihedral group generated by two reflections.*

The discrete subgroups of $Isom(E^2)$ were classified in the 19th century. The classification starts writing the discrete group Γ inside the exact sequence

$$0 \to T \to \Gamma \to H \to 1$$

where T is the subgroup of translations of Γ and H is a subgroup of $O(2)$. As Γ is discrete, T is also discrete. Therefore it is either trivial or $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$. If T is trivial Γ is either finite cyclic or dihedral. If T has one generator it is one of the seven strip patterns. If T is a lattice it is one of the 17 crystallographic groups.

Example 7.5. *The triangle groups are those groups generated by reflections in three lines. If the angles are π/p, π/q and π/r for positive integers p,q,r we should have $\pi/p+\pi/q+\pi/r=\pi$ and in this case the group is discrete. That gives three possibilities for (p,q,r), that is, $(3,3,3)$, $(2,3,6)$ and $(2,4,4)$. The region inside the triangle is a fundamental domain for the triangle group. (reflections on the sides of the triangle of angles $2\pi/3$, $\pi/6$ and $\pi/6$ also defines a discrete group, this is the only non-obtuse triangle leading to a discrete group)*

Example 7.6. *The index two subgroup of orientation preserving isometries of a triangle group has two generators. If we denote r_1, r_2 and r_3 the reflections on the sides of the triangles, the subgroup of orientation preserving isometries is generated by $r_1\circ r_2$ and $r_1\circ r_3$. A fundamental domain consists of any two adjacent triangles.*

Exercise 7.7.

1. *Consider the Riemannian manifold obtained by identifying the two vertical lines $\{\mathrm{Re}\, z=1\}$ and $\{\mathrm{Re}\, z=2\}$ on the upper half-plane via the isometry $z\to z+1$. Prove that this manifold is complete. Hint: show that any geodesic is defined on $\mathbf{R}$ by glueing copies of the vertical band to form the complete Poincaré half-plane.*

2. *Consider the Riemannian manifold obtained by identifying the vertical lines $\{\mathrm{Re}\, z=1\}$ and $\{\mathrm{Re}\, z=2\}$ on the upper half-plane via the isometry $z\to 2z$. That manifold is not complete. Prove that the sequence $(1,2^i)$ is a Cauchy sequence but it is not convergent.*

Local isometries between Riemannian spaces are very special:

Proposition 7.8. *Let $d: M\to N$ be a surjective local isometry between Riemannian manifolds. If M is complete and connected then d is a covering.*

7.2 Hyperbolic surfaces

We will start with the half-plane model and define the metric

$$ds^2=\frac{dx^2+dy^2}{y^2}=\frac{|dz|^2}{Im(z)^2}$$

Given a metric g on a Riemannian manifold we can define a volume form dv by imposing $dv(X_1,\cdots,X_n)=1$ for an orthonormal basis. In local coordinates we have $dv=\sqrt{det(g_{ij})}dx^1\cdots dx^n$. For hyperbolic geometry we get

$$dv=\frac{1}{y^2}dxdy.$$

Proposition 7.9. $PSL(2,\mathbf{R}) \subset Isom(H^1_{\mathbf{C}})$.

Proof. We need to show that

$$\frac{|d\gamma(z)|^2}{(Im\gamma(z))^2} = \frac{|dz|^2}{(Imz)^2}.$$

This follows from a simple computation. $\square$

Proposition 7.10. *The geodesics of $H^1_{\mathbf{C}}$ are vertical lines or circles perpendicular to the $\mathbf{R}$-axis.*

Proof. We first observe that given two points with the same x-coordinate, $p = (x, y_1)$ and $q = (x, y_2)$ (without loss of generality we suppose $y_2 > y_1$), then

$$d(p,q) = \inf \int \frac{\sqrt{dx^2 + dy^2}}{y}.$$

But $\int \frac{\sqrt{dy^2}}{y} \leq \int \frac{\sqrt{dx^2+dy^2}}{y}$. As $\int \frac{\sqrt{dy^2}}{y} \geq \ln(y_2/y_1)$ we conclude that

$$d(p,q) = \ln(y_2/y_1).$$

We use now the fact that geodesics are preserved by isometries and that vertical lines are transformed to circles orthogonal to the real axis or to vertical lines by $PSL(2,\mathbf{R})$. $\square$

In the following we will call a hyperbolic triangle a simplex in $H^1_{\mathbf{C}}$ whose boundary is formed by three geodesic segments.

Proposition 7.11. *Let Δ be an hyperbolic triangle with angles α, β and γ. Then*

$$Area(\Delta) = \pi - \alpha - \beta - \gamma$$

Proof. Suppose first that the triangle has an ideal point, that is, one of the angles is null, or, equivalently, one of its vertices is in the boundary of $H^1_{\mathbf{C}}$. Without loss of generality we might suppose that the vertex is ∞ and one of the geodesics is the half circle of radius one centered at the origin. The other two are vertical lines which form angle α and β with the circle. then

$$Area(\Delta) = \int\int \frac{dxdy}{y^2} = \int_a^b dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{1-x^2}}.$$

By a change of coordinate $x = \cos\theta$ we get

$$\int_{\pi-\alpha}^{\beta} \frac{-\sin\theta}{\sin\theta} d\theta = \pi - \alpha - \beta$$

.

If the triangle Δ_1 is compact we choose one of the vertices (say the one with angle γ) and prolong one of the sides containing it up to the boundary of $H^1_{\mathbf{C}}$. We have three triangles one (containing an ideal point) being the union of the other two. Comparing their areas:

$$\Delta_1 = \Delta_1 + \Delta_2 - \Delta_2$$

$$A(\Delta_1) = \pi - \alpha - (\beta - \theta) - (\pi - (\pi - \gamma) - \theta) = \pi - \alpha - \beta - \gamma.$$

□

Decomposing a polygon in triangles we obtain the following

Corollary 7.12. *For a geodesic polygon with n sides denote by α the sum of the internal angles. Then*

$$A = n\pi - 2\pi - \alpha.$$

Using a geodesic triangulation one can prove Gauss-Bonnet theorem:

Theorem 7.13. *If S_g is a hyperbolic surface, then*

$$A = -2\pi\chi.$$

Proof. We have $A = \sum(\pi - \alpha_i - \beta_i - \gamma_i)$ summing over all triangles, say F of them. The angles sum to 2π times the number of vertices, say V. Therefore $A = \pi(F - 2V)$. On the other hand the number of edges is precisely $E = 3F/2$. We conclude that $\chi = F - E + V = F - 3F/2 + V = -F/2 + V = A/(-2\pi)$ □

The Poincaré metric on the disc is given by

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

The geodesics of the hyperbolic disc are described in the following

Proposition 7.14. *The geodesics of the hyperbolic disc are sub-arcs of circles orthogonal to the boundary of the disc.*

Proposition 7.15. *Let S be a Riemann surface with a Fuchsian model $H^1_{\mathbf{C}}/\Gamma$. If γ is a hyperbolic element and L_γ the geodesic obtained by projection of its axis then*

$$|tr(\gamma)| = 2\cosh\left(\frac{l(L_\gamma))}{2}\right)$$

where $l(L_\gamma)$ is the length of the geodesic.

Proof. We may assume that $\gamma(z) = \lambda^2 z$. Then $l(L_\gamma) = \int_1^{\lambda^2} \frac{dy}{y} = \ln\lambda^2$. □

7.3 Poincaré's polyhedron theorem

Poincaré's theorem is an efficient method to prove that a given set of transformations of $H^1_{\mathbf{C}}$ generates a discrete group and to determine the topology of the quotient.

Consider a domain $P \subset H^1_{\mathbf{C}}$ whose boundary is a finite union of geodesics segments c_i (called sides). Suppose that the sides are paired. That is, for each c_i there exists another side c'_i and an isometry (called a side-pairing) γ_i such that $c'_i = \gamma_i c_i$ and $c_i = \gamma_i^{-1} c'_i$ (the side might be paired to itself). We will supoose the side pairings reverse the orientation of the segments. For simplicity we may orient the boundary in the direct sense and define for each vertex of v_0 the image $v_1 = \gamma v_0$ where γ is the side-pairing associated to the side starting at v_0. The vertices of the polygonal boundary are then partioned into cycles. We define the dyhedral angle θ_v at a vertex v to be the positive internal angle betwwen the sides meeting at v.

Theorem 7.16. *Suppose P is a domain with geodesic sides $\{c_i\}$ and side pairings γ_i. Suppose that for each cycle $\mathscr{C}$,*

$$\sum_{v_i \in \mathscr{C}} \theta_{v_i} = 2\pi$$

where the sum is over all vertices of the cycle. Then, the group Γ generated by the side-pairings is discrete and the quotient $H^1_{\mathbf{C}}/\Gamma$ is a Riemann surface. For each cycle $\mathscr{C}_k$, $1 \leq k \leq N$, let $\gamma^k_1, \cdots, \gamma^k_{n_k}$ be the sequence of side pairings such that $\gamma^k_1 v^k_0 = v^k_1, \cdots, v^k_0 = \gamma^k_{n_k} v^k_n$. A presentation of Γ is then given by

$$\langle \gamma^k_i \ (1 \leq k \leq N) \mid \gamma^k_1 \cdots \gamma^k_{n_k} = 1 \ \ for\, 1 \leq k \leq N \,\rangle.$$

Example: Consider the n-roots of unity in S^1. Take the geodesics (circle segments perpendicular to the boundary) centred at each of these roots with the same radii. If the radius is near 0 we get n disjoint circle segments. On the opposite case, if the radius approaches 1, then ,near the origin, we obtain a region which is nearly a regular euclidean polygon. The angle at a vertex, therefore, varies from $\pi - 2\pi/n$ (the almost euclidean regular polygon) to 0 (the ideal regular polygon). Clearly, the angle is a continuous function and there exists a radius such that the angle between the circles will be

$$\theta = \frac{2\pi}{n}.$$

In that case we can apply Poincaré's theorem to side pairings as in the canonical polygon defining a surface of genus $g \geq 2$. We obtain a Riemann surface of genus g as a quotient of the disc by the discrete subgroup generated by the side-pairings.

Remarks:

1. To obtain non-compact Riemann surfaces with finite volume we may allow certain vertices in the boundary. The angles at these vertices are 0. We need a further hypothesis: the cycle map $\gamma_1 \cdots \gamma_n$ defined as before starting with an ideal vertex should be parabolic. One can prove that this is equivalent to suppose that the space $P/\equiv$, obtained by identifying the sides with the pairings, is a complete space. In the previous example, if the geodesic segments touch at infinity we obtain a Riemann surfce of genus g with one puncture.

2. To obtain subgroups with torsion elements we impose that the cycle satisfies

$$\sum_{v_i \in \mathscr{C}} \theta_{v_i} = 2\pi/r$$

Then, the group Γ generated by the side-pairings (in that case we allow a side to be paired to itself) is discrete with a presentation given by

$$\langle \gamma_i^k \ (1 \le k \le N) \mid (\gamma_1^k \cdots \gamma_{n_k}^k)^{r_k} = 1 \ \text{ for } 1 \le k \le N \rangle.$$

We might also suppose that the side-pairings are not holomorphic (isometries which don't preserve the orientation). In that case the quotient is not a Riemann surface but there will exist a subgroup of finite index which does not have torsion elements whose quotient is a Riemann surface. The simplest examples of discrete groups obtained using that version of Poincaré's theorem are the triangle groups. We consider a geodesic triangle with angles $\pi/p, \pi/q, \pi/r$, with positive integers p, q, r, at the three vertices. The necessary and sufficient condition for the existence of the triangle is that $\pi/p + \pi/q + \pi/r < \pi$. Granted that condition, the subgroup generated by reflections on each side is discrete and has a presentation of the form

$$\langle r_1, r_2, r_3 \mid r_i^2 = (r_1 \circ r_2)^p = (r_2 \circ r_3)^q = (r_3 \circ r_1)^r = 1 \rangle.$$

3. To obtain surfaces which are not of finite volume we allow the polygon to have sides on the boundary. There is no side-pairing between them. A vertex which is in a boundary side is paired to another vertex of the same type by a loxodromic element (it is a side pairing of the corresponding sides in the interior of hyperbolic space). The simplest case is that of Schottky groups. The interior sides of the polygon are given by an even number of non-intersecting geodesics.

8 Calculus on a Riemann surface: Hodge theorems

8.1 Forms

We first recall the definitions and introduce notations describing forms on a real two dimensional manifold X. A 0-form defined on an open subset U of a Riemann surface is simply a function (complex) defined on an open subset $U \subset X$ and we write $\mathscr{E}^0(U)$ for the space of smooth functions defined over U. A smooth differential 1-form α is written in local coordinates $\phi : U \to \mathbf{R}^2$ as

$$\phi^* \alpha = \phi_1 dx^1 + \phi_2 dx^2.$$

Here, the coefficients ϕ_i are complex functions. For a change of coordinates $\tilde{x}^i = \tilde{x}^i(x^1, x^2)$, it satisfies the relation

$$\tilde{\phi}_i = \sum \frac{\partial x^j}{\partial \tilde{x}^i} \phi_j.$$

The space of smooth 1-forms over U will be denoted $\mathscr{E}^1(U)$.

The space of 2-forms over U will be denoted $\mathscr{E}^2(U)$. In local coordinates one writes

$$f dx^1 \wedge dx^2,$$

where f is a (complex) function. For a change of coordinates, we obtain

$$\tilde{f} = \frac{\partial(x^1, x^2)}{\partial(\tilde{x}^1, \tilde{x}^2)},$$

where $\frac{\partial(x^1,x^2)}{\partial(\tilde{x}^1,\tilde{x}^2)}$ is the Jacobian determinant.

On a Riemann surface we may use complex charts $z = x + iy$ and then write $dz = dx + i dy$ and $d\bar{z} = dx - i dy$. In terms of dz and $d\bar{z}$ a 1-form is written locally as

$$a dz + b d\bar{z}.$$

The space of 1-forms which can be written for every point as $a dz$ in one chart centred at the point (and therefore in all charts of the Riemann surface) are called forms of type $(1,0)$. We write $\mathscr{E}^{1,0}(U)$ the space of 1-forms on $U \subset X$ of type $(1,0)$. Analogously the forms of type $(0,1)$ are written locally as $a d\bar{z}$ and the space of these forms defined on U is denoted $\mathscr{E}^{0,1}(U)$. We have the decomposition

$$\mathscr{E}^1(U) = \mathscr{E}^{1,0}(U) \oplus \mathscr{E}^{0,1}(U).$$

Using local coordinates $z = x + iy$ we may write a differential 2-form as

$$f\,dx \wedge dy = \frac{i}{2} f\,dz \wedge d\bar{z}.$$

We also denote then by $\mathscr{E}^{1,1}(U) = \mathscr{E}^2(U)$ the space of all 2-forms over U.

One usually considers 1-forms as smooth sections of the cotangent bundle T^*U and 1-forms of type $(1,0)$ as sections of $T^{*\,1,0}$, a holomorphic line bundle over X (see the next chapter).

Recall the exterior differentiation of a 0-form f defined on a surface is, in local coordinates (x^1, x^2), given

$$df = \frac{\partial f}{\partial \tilde{x}^1} dx^1 + \frac{\partial f}{\partial \tilde{x}^2} dx^2.$$

For a 1-form $\alpha = \phi_1 dx^1 + \phi_2 dx^2$ it is

$$d\alpha = (\frac{\partial \phi_2}{\partial x^1} - \frac{\partial \phi_1}{\partial x^2}) dx^1 \wedge dx^2.$$

On a Riemann surface one introduces operators ∂ and $\bar{\partial}$ as projections of the exterior differentiation into the spaces $\mathscr{E}^{1,0}(U)$ and $\mathscr{E}^{0,1}(U)$ respectively. In coordinates,

$$\partial f = \frac{\partial f}{\partial z} dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

and

$$\partial(a\,dz + b\,d\bar{z}) = \frac{\partial b}{\partial z} dz \wedge d\bar{z}, \quad \bar{\partial}(a\,dz + b\,d\bar{z}) = \frac{\partial a}{\partial \bar{z}} d\bar{z} \wedge dz.$$

Definition 8.1. *A 1-form $\alpha \in \mathscr{E}^1(U)$ is holomorphic on $U \subset X$ if locally it is written as $g(z)dz$ with g holomorphic. A 1-form defined on the complement of a discrete and closed subset of U is meromorphic if locally it is written as $g(z)dz$ with g meromorphic.*

Remark that α is holomorphic if and only if $\bar{\partial}\alpha = 0$.

8.2 Integration

Given a differential form α on a surface X and a piece-wise smooth curve $c : [0,1] \to X$ we define the integral

$$\int_c \alpha$$

using local charts $\phi : U \to \mathbf{C}$ with coordinates (x, y). That is, suppose $Im(c) \subset U$ and $\alpha = \phi_1 dx + \phi_2 dy$ then

$$\int_c \alpha = \int \left(\phi_1 \dot{x} + \phi_2 \dot{y}\right) dt.$$

If $Im(c)$ is not contained in a single coordinate chart we use a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ so that each $c([t_i, t_{i+1}])$ is contained in a coordinate chart. Clearly this definition does not depend on the chart because if $(\tilde{x}, \tilde{y})$ are different coordinates then $\alpha = \phi_1 dx + \phi_2 dy = \left(\phi_1 \frac{\partial x}{\partial \tilde{x}} + \phi_2 \frac{\partial y}{\partial \tilde{x}}\right) d\tilde{x} + \left(\phi_1 \frac{\partial x}{\partial \tilde{y}} + \phi_2 \frac{\partial y}{\partial \tilde{y}}\right) d\tilde{y}$ and therefore by the chain rule

$$\phi_1 \dot{x} + \phi_2 \dot{y} = \phi_1 dx + \phi_2 dy = \left(\phi_1 \frac{\partial x}{\partial \tilde{x}} + \phi_2 \frac{\partial y}{\partial \tilde{x}}\right) \dot{d\tilde{x}} + \left(\phi_1 \frac{\partial x}{\partial \tilde{y}} + \phi_2 \frac{\partial y}{\partial \tilde{y}}\right) \dot{d\tilde{y}}$$

and the integrals are the same.

Proposition 8.2. *Let α be a closed form and c, c' be homotopic curves between two points x_0, x_1 on a surface. Then $\int_c \alpha = \int_{c'} \alpha$.*

Proof. By Stokes theorem (see next section). □

Theorem 8.3. *On a simply connected surface every closed 1-form α is exact. That is, there exists a function F (called a primitive of α) such that $\alpha = dF$. Two primitives differ by a constant.*

Proof. It follows from the previous proposition by defining $F(x) = \int_{x_0}^{x} \alpha$ as the integral does not depend on the path of integration. □

In general, if X is a Riemann surface and $\pi : \tilde{X} \to X$ is its universal cover, then $\int_{\tilde{c}} \pi^* \alpha = \int_{\pi \tilde{c}} \alpha$. So if α is a 1-form on a Riemann surface X we can compute its integral

$$\int_c \alpha = F(\tilde{c}(1)) - F(\tilde{c}(0))$$

where $\tilde{c}$ is a lift of c to the universal cover of X and F is a primitive of $\pi^* \alpha$.

Remark: Let Γ be the group of Deck transformations of the cover $\pi : \tilde{X} \to X$. If F is a primitive of the form $\pi^* \alpha$ then $F \circ \gamma$ is also a primitive because $d(F \circ \gamma) = d\gamma^* F = \gamma^* dF = \gamma^* \pi^* \alpha = (\pi\gamma)^* \alpha = \pi^* \alpha$. As two primitives differ by a constant we obtain that $F \circ \gamma = F + a_\gamma$.

Definition 8.4. *Let α be a closed one-form defined on a surface X. The period map associated to α is the homomorphism*

$$\pi_1(X, x_0) \to \mathbf{C} \quad \textit{given by} \quad c \to \int_c \alpha.$$

Let Γ be the group of Deck transformations of the cover $\pi : \tilde{X} \to X$ and F a primitive of α defined on $\tilde{X}$, then the image of the period map is given by the set $\{ a_\gamma \mid \gamma \in \Gamma \}$ where a_γ are defined in the remark above. This can be seen easily if we interpret an element of Γ as a closed curve c with lift $\tilde{c}$. Then

$$\int_c \omega = F(\tilde{c}(1)) - F(\tilde{c}(0)) = F(\gamma \tilde{c}(0)) - F(\tilde{c}(0)) = a_\gamma.$$

Theorem 8.5. *Suppose a closed differential form has all periods zero. Then it has a primitive.*

Proof. Construct explicitly the primitive as $F(z) = \int_{z_0}^{z} \alpha$ where z_0 is a point in X. This function is well defined as the periods are null. □

Corollary 8.6. *If ω is a closed holomorphic form on a compact Riemann surface such that the associated period map is zero then $\omega = 0$.*

Proof. By the previous theorem the form ω has a primitive. It is holomorphic on a compact Riemann surface therefore constant. □

If $\phi : V \to U$ is a diffeomorphism, recall the change of variable formula

$$\int\int_U f dx dy = \int\int_V \phi^* f du dv$$

which can be written more explicitly as

$$\int\int_U f dx dy = \int\int_V f \circ \phi |\frac{\partial(x,y)}{\partial(u,v)}| du dv$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian determinant. In the case ϕ is a biholomorphism we have

$$\int\int_U f dz \wedge d\bar{z} = \int\int_V f \circ \phi |\frac{dz}{dw}|^2 dw \wedge d\bar{w}$$

To define the integral of a 2-form on a Riemann surface we use a partition of unit subordinated to a cover by charts. The fundamental theorem we will use is the following version of Stokes theorem.

Theorem 8.7 (Stokes Theorem)**.** *Let α be a smooth 1-form defined on a neighborhood of a domain Ω with piecewise smooth boundary $\partial\Omega$ contained in a surface.*

$$\int_{\partial\Omega} \alpha = \int_\Omega d\alpha.$$

8.2.1 The residue theorem

We will admit the following integral formula (for a proof see [Hörmander]).

Theorem 8.8. *Let $\Omega \subset \mathbf{C}$ be a connected open domain whose boundary is a union of finitely many C^1 Jordan curves. Let $f \in C^1(\bar{\Omega})$. Then, for $z \in \Omega$,*

$$2\pi i f(z) = \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\Omega} \frac{\partial f(\zeta)/\partial\bar{\zeta}}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

Let ω be a meromorphic 1-form which is not identically null. Let $p \in X$ and $z : U \to \mathbf{C}$ be a chart such that ω is holomorphic on $U \setminus \{p\}$. We define the residue of ω at p as

$$res_p(\omega) = \frac{1}{2\pi i} \int_{\gamma} \omega$$

where γ is a curve with winding number 1 around p contained in U. It is easy to see that this integral is well defined. It can be computed using a Taylor expansion; write, using local coordinates, $\omega = f(z)dz$ where $f(z)$ has a pole at p and the residue is simply the coeficient of the term $\frac{1}{z}$ in the Taylor expansion. If we change the local chart then $\omega = g(w)dw = f(z)\frac{dw}{dz}dz$ and the residue is the same.

Proposition 8.9. *If X is compact then*

$$\sum_{p \in X} res_p(\omega) = 0$$

Proof. Stokes theorem. Suppose $D = \{p_i\}_{1 \leq i \leq n}$ are the poles of ω. Choose non-intersecting neighborhoods U_i containing each p_i with boundary γ_i and compute

$$\sum_i \int_{\gamma_i} \omega = -\int_{X - \bigcup U_i} d\omega = 0$$

because $d\omega = \bar{\partial}\omega + \partial\omega = 0$. □

Proposition 8.10. *If X is compact and f is a meromorphic function, then the degree of the divisor $div(f)$ is zero.*

Proof. This follows from the proposition above and the fact that $deg(f) = \sum_{p \in X} res_p(\omega)$ for $\omega = df/f$. □

8.3 Homology and Cohomology

8.3.1 The de Rham complex

The de Rham complex over a surface X is

$$0 \to \mathbf{R} \to \mathscr{E}^0(X) \xrightarrow{d} \mathscr{E}^1(X) \xrightarrow{d} \mathscr{E}^2(X) \xrightarrow{d} 0.$$

where $\mathscr{E}^0(X) = \mathscr{C}^\infty(X)$ is the space of C^∞ functions on X, $\mathscr{E}^i(X)$ is the space of i-forms on X and d is the exterior differentiation. Poincaré's lemma says that the sequence is locally exact. The cohomology groups measure how much the sequence is far from being exact. Observe that the space of closed or exact forms (respectively forms α such that $d\alpha = 0$ or $\alpha = d\beta$ for a form β) are vector spaces.

Definition 8.11. *The i-th cohomology group $H^i(X, \mathbf{R})$, of the surface X is the quotient of the space of closed i-forms by the space of exact i-forms.*

Observe that $dimH^0(X, \mathbf{R})$ is the number connected components of X. In fact the space of exact 0-forms is formed by the trivial vector space of null functions.

In order to compute $H^1(X, \mathbf{R})$ we will introduce the singular homology. A singular p-simplex is a differential map from a p-simplex to X. We will write sometimes (P) for a singular 0-simplex, (P_1, P_2) for a singular 1-simplex and (P_1, P_2, P_3) for a singular 2-simplex. Fix now an abelian group G (we will mostly use $\mathbf{Z}, \mathbf{R}$ or $\mathbf{C}$). A p-chain is a finite linear combination of singular p-simplices with coefficients in G. The space of p-chains will be noted C_p (with a convention that $C_{-1} = \{0\}$). There exists a boundary operator $\partial : C_p \to C_{p-1}$ satifying $\partial^2 c = 0$ for any chain c. It is defined on singular simplices by the formulas (using the obvious notation for the restriction of maps to the boundary of a simplex)

$$\partial(P) = 0 \quad \partial(P_1, P_2) = (P_2) - (P_1) \quad \partial(P_1, P_2, P_3) = (P_2, P_3) - (P_1, P_3) + (P_1, P_2)$$

and extended by linearity to all chains.

A chain c is called a cycle if $\partial c = 0$ and a boundary if there exists a chain $\tilde{c}$ such that $\partial \tilde{c} = c$. We define

Definition 8.12. *The p-th homology group, $H_p(X, G)$, is the quotient of the space of cycles, Z_n, by the space of boundaries, B_n.*

If the surface X is connected $dim\, H_0(X, \mathbf{R}) = 1$. If X is compact, orientable and connected then $dim\, H_2(X, \mathbf{R}) = 1$. To compute $H_1(X, \mathbf{Z})$, we will invoke van Kampen theorem, that describes the first homology as the abelianization of the fundamental group:

$$H_1(X,\mathbf{Z}) = \frac{\pi_1(X,z)}{\{\langle [a,b]\rangle | a,b \in \pi_1(X,z)\}}.$$

Using the generators a_i, b_i, $1 \le i \le g$ for a compact surface of genus g we obtain that $H_1(X,\mathbf{Z}) = \mathbf{Z}^{2g}$. The generators a_i, b_i, viewed as a basis of $H_1(X,\mathbf{Z})$ are also called a canonical basis for the homology. It follows from general theorems on the homology that we also have $H_1(X,\mathbf{R}) = \mathbf{R}^{2g}$.

The relation between homology and cohomology is essentially given by Stokes theorem on a chain c:

$$\int_{\partial c} \omega = \int_c d\omega.$$

Lemma 8.13. *If ω is closed and c_1 and c_2 are two homologous chains then*

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

Proof. By hypothesis $c_2 - c_1 = \partial C$. Apply Stokes theorem. □

This lemma shows that the bilinear map in the following theorem is well defined.

Theorem 8.14. *Let X be a compact orientable surface of genus p. The bilinear map $H_1 \times H^1 \to \mathbf{R}$ defined by*

$$(c,\omega) \to \int_a \omega$$

is non-degenerate.

Proof. The fact that $(\cdot,\omega)$ is non-zero follows from the fact that if all periods are null, the form ω is null. On the other hand, given an element $c \in H_1$ we construct a form such that $(c,\omega) \neq 0$ in the following two lemmas. □

Suppose X is orientable. Let γ be simple closed curve in X. We consider an annulus A containing γ and let A^- be the left side and A^+ the right side. Let f be a function with compact support on A^- which is one on A^- intersected with a neighborhood of γ. Define then $\eta_\gamma = df$. Even if f is not continuous, η_γ is C^∞ 1-form. On the other hand η_γ is not exact in general. The form η_γ is dual to γ in the sense of the following lemma.

Lemma 8.15. *Let ω be a closed 1-form. Then*

$$\int_\gamma \omega = \int_X \eta_\gamma \wedge \omega.$$

Proof. We compute

$$\int_X \eta_\gamma \wedge \omega = \int_{A^-} df \wedge \omega = \int_{A^-} d(f\omega) - \int_{A^-} f\,d\omega = \int_\gamma f\omega = \int_\gamma \omega.$$

□

Remark: Using notation of the next section we write $\int_\gamma \omega = (\omega, *\eta_\gamma)$.

Lemma 8.16. *Let a_i, b_i be an homology basis. Then*

$$\int_{a_i} \eta_{a_j} = \int_{b_i} \eta_{b_j} = 0 \quad \int_{a_i} \eta_{b_j} = -\int_{b_i} \eta_{a_i} = \delta_{ij}.$$

Proof. The first equality follows from the previous lemma. For the second one, we compute in the case that a, b are two loops intersecting once at a point with orientation given by the tangent vectors to a and b at the point of intersection in that order. we denote by f_b a function associated to the loop b with support in A_b^- as before. We obtain

$$= \int_a \eta_b = \int_a df_b = 1.$$

The last equality follows from the explicit form of the function f_b at the intersection point; it corresponds to the integration on a closed interval $[0,1]$ of the derivative of a function such that $f(0) = 0$ and $f(1) = 1$. □

8.4 The Dolbeault complex

Recall the Cauchy-Riemann operator $\bar\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ defined on functions on an open subset $U \subset \mathbf{C}$. It is better understood in the guise of an operator:

$$\bar\partial : \mathscr{C}^\infty(U) \to \mathscr{E}^{0,1}(U)$$

given by $f \to \frac{\partial f}{\partial \bar z} d\bar z$.

Local solvability of the Cauchy-Riemann equation: for each $g \in C^\infty(U)$ there exists $V \subset U$ and $f \in C^\infty(V)$ such that

$$\frac{\partial f}{\partial \bar z} = g.$$

on V. A stronger result is true:

Proposition 8.17 (Dolbeault's lemma)**.** *Let $\Omega \subset \mathbf{C}$ be an open subset and $g \in C^\infty(\Omega)$. Then there exists a function $f \in C^\infty(\Omega)$ such that*

$$\frac{\partial f}{\partial \bar z} = g.$$

Proof. There are two cases:

1. In the first case we suppose g of compact support. An explicit solution is given in terms of the integral formula
$$f(z) = \frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{g(w)}{w-z} dw \wedge d\bar{w}.$$
The integral is well defined as can be seen by using polar coordinates $w - z = re^{i\theta}$ so that $\frac{1}{w-z} dw \wedge d\bar{w} = \frac{-2ir}{re^{i\theta}} dr \wedge d\theta$. Because g is of compact support, the integration is made in a sufficiently large rectangle and therefore we may differentiate under the integral sign. We obtain making the change w for $w - z$
$$\frac{\partial f(z)}{\partial \bar{z}} = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int\int_{|w|>\varepsilon} \frac{\partial g(z+w)}{\partial \bar{z}} \frac{1}{w} dw \wedge d\bar{w}.$$
So
$$\frac{\partial f(z)}{\partial \bar{z}} = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int\int_{|w|>\varepsilon} \frac{\partial}{\partial \bar{w}}\left(\frac{g(z+w)}{w}\right) dw \wedge d\bar{w} = -\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int\int_{|w|>\varepsilon} d\left(\frac{g(z+w)}{w} dw\right)$$
$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{g(z+w)}{w} dw = g(z).$$
2. If $suppg \subset \Omega$ is not compact we construct an exhaustion sequence of compact sets K_n ($K_n \subset Int(K_{n+1})$ with $\Omega \setminus K_n$ having no relatively compact component) and cut-off functions ϕ_n with $\phi_{n|K_n} = 1$ and $\phi_{n|K_{n+1}} = 0$. We solve
$$\frac{\partial f_n}{\partial \bar{z}} = \phi_n g.$$
We would like to make sense of
$$f = f_n + (f_{n+1} - f_n) + (f_{n+2} - f_{n+1}) + \cdots$$
As this sum might not converge we modify each term by a holomorphic function using Runge's theorem: As $f_{m+1} - f_m$, $m \geq 1$, is holomorphic on a neighborhood of K_n there exists a holomorphic function h_m on Ω such that
$$|f_{m+1} - f_m - h_m| < \frac{1}{2^m}$$
on K_m. We redefine the sum to be
$$f = f_n + (f_{n+1} - f_n - h_n) + (f_{n+2} - f_{n+1} - h_{n+1}) + \cdots$$
Now the sum is uniformly convergent on K_m for each $m \geq n$ so f is well defined on Ω. Moreover we imediately see that on each K_m f solves the equation.

□

Remark 8.18. *On an n-dimensional complex manifold we have the following exact sequence*

$$0 \to \mathcal{O} \longrightarrow \mathcal{C}^{\infty} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,2} \cdots,$$

and more generally

$$0 \to \Omega^{p,q} \longrightarrow \mathcal{E}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+2} \cdots.$$

where the vector spaces in the exact sequence are germs of of differential forms. A better formulation is obtained using sheaf theory.

8.5 Poisson equation and functional analysis

8.5.1 The Laplacian on a Riemann surface

Given a Riemannian manifold the Laplacian operator can be defined. In the case of real two dimensional manifolds one does not need a metric to define a Laplacian, but instead a conformal Riemannian structure is enogh.

We write $\mathcal{E}^1$ as the space of $\mathbf{C}$-valued 1-forms. If X is a Riemann surface we define the space $\mathcal{E}^{1,0}$ of forms of type $(1,0)$ and the space $\mathcal{E}^{0,1}$ of forms of type $(0,1)$. One has the decomposition $\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$. Note that complex conjugation interchanges $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$.

The Hodge star operator on 1-forms on a Riemann surface is the following:

Definition 8.19 (Hodge star operator)**.** *Let $\alpha \in \mathcal{E}^1$ and write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in \mathcal{E}^{1,0}$, $\alpha_2 \in \mathcal{E}^{0,1}$. Define*

$$*\alpha = -i\alpha_1 + i\alpha_2.$$

In complex coordinates, for $\alpha = adz + bd\bar{z}$, we obtain $*\alpha = -iadz + ibd\bar{z}$. On real coordinates such that $z = x + iy$, we have $*dx = dy$ and $*dy = -dx$. The geometric interpretation of the $*$-operator acting on exact 1-forms is given by the formula

$$(*df)(v) = df(Jv).$$

That is the dual of the J-operator acting on vectors.

Remark 8.20. *Note that we don't need a metric to define the star operator on $\mathcal{E}^1$ on Riemann surface.*

By a straight computation one can verify that the Hodge star operator defined above satisfies the following properties:

Proposition 8.21. *Let $\alpha \in \mathscr{E}^1$. Then*

1. $* * \alpha = -\alpha$

2. $*\bar{\alpha} = \overline{*\alpha}$

Proposition 8.22. *Let $\alpha_1 \in \mathscr{E}^{1,0}$, $\alpha_2 \in \mathscr{E}^{0,1}$ and $f \in \mathscr{E}^0$. Then*

1. $d * \alpha_1 = -i\bar{\partial}\alpha_1$

2. $d * \alpha_2 = i\partial\alpha_2$

3. $*\partial f = -i\partial f$

4. $*\bar{\partial} f = i\bar{\partial} f$

5. $d * df = 2i\partial\bar{\partial} f = -2i\bar{\partial}\partial f$

Definition 8.23. *Let $f \in \mathscr{E}^0$. Define the Laplacian $\Delta : \mathscr{E}^0 \to \mathscr{E}^2$ by the formula*

$$\Delta f = d * df.$$

We say f is harmonic if $\Delta f = 0$.

In local coordinates $z = x + iy$ we obtain the formula

$$\Delta f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy.$$

Using the star operator we define a hermitian product on 1-forms over a compact Riemann surface:

Definition 8.24. *Let X be a compact Riemann surface and α_1, α_2 1-forms in $\mathscr{E}^1(X)$. Define*

$$\langle \alpha_1, \alpha_2 \rangle = \int_X \alpha_1 \wedge *\overline{\alpha}_2.$$

Clearly $\langle \alpha_1, \alpha_2 \rangle = \overline{\langle \alpha_2, \alpha_1 \rangle}$. To show that $\langle \alpha, \alpha \rangle > 0$ for non-vanishing α, write in local coordinates $\alpha = adz + bd\bar{z}$. Then $\alpha \wedge *\alpha = i(|a|^2 + |b|^2)dz \wedge d\bar{z} = 2(|a|^2 + |b|^2)dx \wedge dy$. Therefore the integrand is a positive form and the product is 0 if and only if $\alpha = 0$.

Remark 8.25. *Note that this Hermitian metric on the space of 1-forms does not come from a pointwise Hermitian metric. On the other hand, once a volume form v is fixed, we can define a pointwise Hermitian metric $(\cdot,\cdot)$ by the formula*

$$\alpha_1 \wedge *\overline{\alpha}_2 = (\alpha_1, \alpha_2)\, v.$$

Definition 8.26. *Let $\alpha \in \mathscr{E}^1$.*

- *α is closed if $d\alpha = 0$.*
- *α is co-closed if $d * \alpha = 0$.*
- *α is harmonic if $d\alpha = 0$ and $d * \alpha = 0$.*

Observe that if α is harmonic then, from $d\alpha = 0$ and Poincaré's lemma, we may write, locally, $\alpha = df$, where $f \in \mathscr{E}^0$. Therefore α is harmonic if and only if, locally, one can write $\alpha = df$ where f is harmonic.

Exercise 8.27.

Prove that $\alpha \in \Omega^{1,0}$ (that is, α is holomorphic) if and only if, locally, $\alpha = df$ with f holomorphic.

The following proposition is left as an exercise.

Proposition 8.28. *The following are equivalent:*

1. *α is harmonic*
2. *$\partial\alpha = \bar{\partial}\alpha = 0$*
3. *$\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in \Omega^{1,0}$ and $\alpha_2 \in \overline{\Omega^{1,0}}$*

8.5.2 Riez representation theorem: weak solutions

A fundamental theorem in functional analysis is the following:

Theorem 8.29 (Riez representation theorem)**.** *Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathbf{R}$ be a bounded linear map. Then there exists $x_T \in \mathscr{H}$ such that for every $x \in \mathscr{H}$, $T(x) = \langle x_T, x\rangle$.*

If $\mathscr{H}$ is defined to be the completion of a vector space H equipped with a scalar product $\langle\cdot,\cdot\rangle$, Riez representation theorem says that if $T : H \to \mathbf{R}$ is a bounded linear map, then there exists a Cauchy sequence (h_n) in H (called a weak solution) such that, for each v in H

$$T(v) \to \lim_{n\to\infty} \langle h_n, v\rangle.$$

Therefore, in the case of Hilbert spaces obtained by completions of spaces of smooth functions, one can usually make arguments which only involve estimates on smooth functions. In particular, if one can prove that the Cauchy sequence (h_n) converges to an element of H one obtains a smooth representation of T (in the context of the Laplacian this result is called Weyl's lemma).

We use Riez representation theorem and Weyl's lemma to find solutions to an equation

$$P\phi = \rho,$$

where P is a differential operator and ρ is a given C^∞ function (or section of a bundle). To find a C^∞ solution directly is most of the times hard. The idea therefore is first to identify a convenient Hilbert space and a linear operator T_ρ on this Hilbert space related to the equation. Then find a 'weak' solution as an element in the Hilbert space using Riez theorem. Finally, prove that the 'weak' solution is in fact regular using Weyl's lemma. It turns out that for an important class of operators, one can find solutions outside a finite dimensional subset of the space of functions (or sections).

8.5.3 The Poisson equation

Consider the equation

$$\Delta\phi = \rho,$$

where ρ is a smooth 2-form on a Riemann surface X. Observe that if ϕ exists, for any smooth function ψ we have that

$$\int_X \psi\Delta\phi = \int_X \psi\rho.$$

In particular,

$$\int_X \Delta\phi = \int_X \rho,$$

which, from Stokes theorem, implies

$$\int_X \rho = 0.$$

This is a necessary condition which turns out to be sufficient:

Theorem 8.30. *On a compact Riemann surface, for any smooth 2-form ρ satisfying $\int_X \rho = 0$, there exists a smooth function ϕ such that $\Delta\phi = \rho$.*

The identity

$$\int_X \psi \Delta\phi = -\int_X d\psi \wedge * d\phi$$

suggests the Hilbert space we will work with. Consider $C^\infty(X)$ and the bilinear form

$$\langle \psi, \phi \rangle = \int_X d\psi \wedge * d\phi.$$

This is clearly a metric on the space of smooth functions modulo an additive constant, $C^\infty(X)/\mathbf{R}$. To avoid considering the quotient modulo constants we fix a volume form v on the surface. Define the space of smooth functions satisfying

$$\int_X \psi v = 0.$$

The completion W of this metric will be our Hilbert space and the operator

$$T_\rho(\psi) = \int_X \psi \rho$$

will be the linear form associated to the differential operator Δ. The first step then is to show that this operator is bounded in order to apply Riez representation theorem. That is, there exists a constant C such that, for all ψ in the Hilbert space,

$$\left| \int_X \psi \rho \right|^2 \leq C \int_X d\psi \wedge * d\psi.$$

Clearly, it suffices to prove this bound for smooth functions with null average as this space is dense in the Hilbert space.

Theorem 8.31. *Let X be a compact Riemannn surface with a fixed volume form v and ρ a smooth 2-form on X satisfying $\int_X \rho = 0$. Then, there exists a constant C, such that for any smooth function ψ on X with $\int_X \psi v = 0$,*

$$\left| \int_X \psi \rho \right| \leq C \left(\int_X d\psi \wedge * d\psi \right)^{1/2}.$$

Proof. Recall first that $\int_X \alpha \wedge * \beta$ defines a metric on the space of real 1-forms. Cauchy-Schwartz inequality states then that

$$\left| \int_X \alpha \wedge * \beta \right| \leq ||\alpha||.||\beta||,$$

where $||\alpha|| = \int_X \alpha \wedge *\alpha$ As $\int_X \rho = 0$ and v is a generator of $H^2(X, \mathbf{R})$, we obtain that $\rho = d\beta$ where β is a smooth one form. Now, by Stokes and then Cauchy-Schwartz:

$$\left|\int_X \psi\rho\right| = \left|\int_X \psi d\beta\right| = \left|\int_X d\psi \wedge \beta\right| \leq ||d\psi||.||\beta|| = C\left(\int_X d\psi \wedge *d\psi\right)^{1/2}$$

for a constant $C = \left(\int_X \beta \wedge *\beta\right)^{1/2}$.

$\square$

8.5.4 Weyl's lemma

The last part of the proof is the regularity proof. It is a special case of more general results for elliptic operators. We want to show that a weak solution ϕ (that is a convergent sequence $\phi_n \to \phi$) to the Poisson equation $\Delta\phi = \rho$ is smooth:

Theorem 8.32. *Let ϕ be a weak solution of the equation $\Delta\phi = \rho$ where ρ is a smooth 2-form on a closed Riemann surface. Then ϕ may be represented as a smooth function.*

The first observation is that the weak solution can be thought as an element in $L^2(X)$. Indeed, from Poincaré's inequality one has $\int_X |\phi_i - \phi_j| v \leq C \int d(\phi_i - \phi_j) \wedge *d(\phi_i - \phi_j)$ (Here, as before, we used a fixed volume form v on X in order to define L^2). As ϕ_i is a Cauchy sequence in the Hilbert space defined by the metric on the space of differentials, it implies that it is also a Cauchy sequence in the L^2 norm.

The second observation is that it suffices to prove the result on a local chart. Indeed, the solution $\phi \in L^2(X)$ (where $\phi_n \to \phi$ in L^2) to $\Delta\phi = \rho$ satisfies

$$\int_X \psi\rho = lim_{n\to\infty} \int_X d\psi \wedge *d\phi_n = lim_{n\to\infty} \int_X \phi_n d * d\psi = \int_X \phi d * d\psi.$$

Consider only test functions ψ with compact support inside an open subset $U' \subset U \subset X$, where $U' \subset U$ is of compact closure and U carries a chart. We must have then

$$\int_U \psi\rho = \int_U \phi d * d\psi.$$

In the coordinates of the chart we have

$$\int_\Omega \psi f dx = \int_\Omega \phi\Delta\psi,$$

with f smooth of compact support in Ω and $\psi \in C_0^\infty(\Omega')$.

We have to show that ϕ is smooth. The third observation is that it is enough to deal with $\rho = 0$. Indeed, first prove that the Poisson equation $\Delta\phi = f$, for $f \in C_0^\infty(\Omega)$, on an open subset $\Omega \subset \mathbf{R}^2$ has a particular explicit solution $\phi_0 \in C^\infty(\Omega)$. We will have to show that all of them are in $C^\infty(\Omega)$. Now, if ϕ is any solution, $\phi - \phi_0$ is a solution of the Poisson equation with $f = 0$. To obtain regularity of solutions it suffices then to show that all solutions of $\Delta\phi = 0$ are smooth.

The particular solution is given by a convolution with the function $K(x) = \frac{1}{2\pi} \ln|x|$ thought as a locally L^1 function on $\mathbf{R}^2$. Indeed, in polar coordinates,

$$\frac{1}{2\pi} \int_{|x|<\varepsilon} \ln|x| = \frac{1}{2\pi} \int_{0\le r<\varepsilon} (\ln r) r\, dr\, d\theta = \int_{0\le r<\varepsilon} r \ln r\, dr < \infty.$$

Proposition 8.33. *Let $f \in C_0^\infty(\mathbf{R}^2)$. Then $K * f \in C^\infty(\mathbf{R}^2)$ and*

$$\Delta(K * f) = f.$$

That is $K * f$ is a smooth solution to the Poisson equation.

Proof. From the definition, $K * f(x) = \int_{\mathbf{R}^2} K(y) f(x-y)\, dy$ which implies that it is in $C^\infty(\mathbf{R}^2)$ as K is locally L^1 and f is smooth. Compute

$$\Delta \int_{\mathbf{R}^2} K(y) f(x-y)\, dy = \int_{\mathbf{R}^2} K(y) \Delta f(x-y)\, dy = \lim_{\varepsilon\to 0} \int_{|y|\ge\varepsilon} K(y) \Delta f(x-y)\, dy.$$

The limit can be shown to be 0 (exercise). □

We finally prove the necessary local regularity result for the Laplace equation (here we use the notation Ω for Ω'):

Proposition 8.34 (Weyl's lemma)**.** *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain and $\phi \in L^2(\Omega)$ be a weak solution of $\Delta\phi = 0$ on Ω. Then $\phi \in C^\infty(\Omega)$.*

Proof. We consider $C^\infty(\Omega)$ smoothing deformations of ϕ: Define a function χ with compact support on the interval $[0,1)$with value one at a neighborhood of 0 and such that $\int_{[0,1)} r\chi(r)\, dr = 1$. Define also the family of functions $\chi_\varepsilon : D \to \mathbf{R}$ on the unit disc by the formula

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^2} \chi\left(\frac{|x|}{\varepsilon}\right).$$

Observe that $\int_{\mathbf{R}^2} \chi_\varepsilon(x)\, dx = 2\pi \int_0^\varepsilon \frac{1}{\varepsilon^2} \chi(\frac{r}{\varepsilon}) r\, dr = 1$. The smoothing is a convolution with χ_ε. That is, define, for $\phi \in L^2(\Omega)$, $\chi_\varepsilon * \phi(x) = \int_\Omega \chi_\varepsilon(x-y)\phi(y)\, dy$, which is smooth and such that $\chi_\varepsilon(x-y)$ has compact support in Ω if $x \in \Omega_\varepsilon = \{\, x \in \Omega \mid x + B(x,\varepsilon) \subset \Omega \,\}$.

The proof is completed by computing $\Delta(\chi_\varepsilon * \phi) = 0$ and showing that $\chi_\varepsilon * \phi \to \phi$ in the C^∞ norm:

First,

$$\Delta(\chi_\varepsilon * \phi) = \Delta(\int_\Omega \chi_\varepsilon(x-y)\phi(y)dy) = \int_\Omega \Delta_x \chi_\varepsilon(x-y)\phi(y)dy = \int_\Omega \Delta_y \chi_\varepsilon(x-y)\phi(y)dy$$

Observe that for $x \in \Omega_\varepsilon$, $\chi_\varepsilon(x-y)$ has compact support and as ϕ is a weak solution we conclude that $\Delta(\chi_\varepsilon * \phi) = 0$.

Secondly, in order to show convergence to a smooth function, we use the mean value property of harmonic functions. That is, for a smooth harmonic function g defined on Ω, $y \in \Omega$ and a relatively compact disc $B(y,r) \subset \Omega$, we have

$$g(y) = \frac{1}{\pi r^2} \int_{B(y,r)} g(x)dx.$$

This equality implies uniform bounds in all derivatives of the harmonic function by L^1 norms of g (exercise).

The family $\chi_\varepsilon * \phi$, using Arzela-Ascoli theorem contains a sequence $\chi_{\varepsilon_n} * \phi$ converging together to all its derivatives to a smooth function $\tilde{\phi}$. But this family also converges in L^2 to ϕ. This concludes the proof. □

8.6 Hodge theory

8.6.1 Hodge theorem

In this section we establish the relations between cohomology groups on a Riemann surface X (which we suppose connected) associated to the following sequences. We will see later a formulation using sheaf cohomology. The following sequences of homomorphisms are not exact and give origin to cohomology groups. The first arrow in each sequence is the embedding as a subset.

$$0 \to \mathbf{C} \longrightarrow \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to 0,$$

$$0 \to \mathscr{O} \longrightarrow \mathscr{E}^0 \xrightarrow{\bar{\partial}} \mathscr{E}^{0,1} \to 0$$

and

$$0 \to \Omega^{1,0} \longrightarrow \mathscr{E}^{1,0} \xrightarrow{\bar{\partial}} \mathscr{E}^{1,1} \to 0.$$

Here $\mathscr{O}$ is the set of holomorphic functions over the Riemann surface, $\Omega^{1,0}$ is the set of holomorphic differentials, $\mathscr{E}^0 = \mathscr{C}^\infty$ is the set of C^∞ functions and $\mathscr{E}^{i,j}$ is the set of smooth forms of type (i,j). Note that $\mathscr{E}^{1,1} = \mathscr{E}^2$. In the case of compact Riemann surfaces $\mathscr{O} = \mathbf{C}$.

The cohomology groups are, for the first sequence, the usual de Rham cohomology groups

$$H^0(X,\mathbf{C}) = \ker d = \mathbf{C},\ \ H^1(X,\mathbf{C}) = \frac{\ker d}{d\mathscr{E}^0},\ \ H^2(X,\mathbf{C}) = \frac{\mathscr{E}^2}{d\mathscr{E}^1}.$$

For the second sequence:

$$H^{0,0}(X,\mathbf{C}) = \ker \bar{\partial}_{|\mathscr{E}^0} = \mathscr{O},\ \ H^{0,1}(X,\mathbf{C}) = \frac{\mathscr{E}^{0,1}}{\bar{\partial}(\mathscr{E}^0)}.$$

For the third:

$$H^{1,0}(X,\mathbf{C}) = \ker \bar{\partial}_{|\mathscr{E}^{1,0}} = \Omega^{1,0},\quad H^{1,1}(X,\mathbf{C}) = \frac{\mathscr{E}^{1,1}}{\bar{\partial}(\mathscr{E}^{1,0})}.$$

On a compact Riemann surface we clearly have $H^{0,0}(X,\mathbf{C}) = H^0(X,\mathbf{C}) = \mathbf{C}$. We used in the proof of the existence theorem the result $\dim_{\mathbf{C}} H^2(X,\mathbf{C}) = 1$. On the other hand, if theorem 8.30 is known, one can use the solution to Poisson equation to compute that $\dim_{\mathbf{C}} H^2(X,\mathbf{C}) = 1$. Indeed, consider $\rho \in \mathscr{E}^2$. Let v be a volume form for X which we normalize so that $\int_X v = 1$. If $\int_X \rho = \lambda \neq 0$, then $\int_X(\rho - \lambda v) = 0$ and therefore there exists a smooth function $f \in \mathscr{E}^0$ such that $\Delta f = \rho - \lambda v$. This implies that $[\rho] = \lambda[v]$ in $H^2(X,\mathbf{C})$.

The main theorem which describes the relations between the cohomology groups is the following decomposition theorem:

Theorem 8.35. *On a compact Riemann surface X we have*

$$H^{1,1}(X,\mathbf{C}) \cong H^2(X,\mathbf{C}),\quad H^{1,0}(X,\mathbf{C}) \cong H^{0,1}(X,\mathbf{C}).$$

Moreover,

$$H^1(X,\mathbf{C}) \cong H^{1,0}(X,\mathbf{C}) \oplus H^{0,1}(X,\mathbf{C}).$$

Remark 8.36. *Note that the theorem implies that if X is a surface of topological genus g then* $\dim H^{1,0}(X,\mathbf{C}) = g$ *and we computed the dimensions of all cohomology groups.*

Proof. The proof consists in defining explicit isomorphisms and using the solution of the Poisson equation.

For the first isomorphism: note that $\mathscr{E}^{1,1} \to \mathscr{E}^2$ is an isomorphism of real vector spaces which induces a (surjective) homomorphism $H^{1,1}(X,\mathbf{C}) \to H^2(X,\mathbf{C})$ because, clearly, $\bar{\partial}(\mathscr{E}^{1,0}) = (\bar{\partial} + \partial)(\mathscr{E}^{1,0}) \subset d\mathscr{E}^1$. We need to show it is injective. Suppose $\rho \in \mathscr{E}^{1,1}$ is such that there exists $\beta \in \mathscr{E}^1$ satisfying $d\beta = \rho$ (that is ρ is trivial in the cohomology $H^2(X,\mathbf{C})$). This implies that $\int_X \rho = 0$ and therefore by the solution of the Poisson equation there exists a smooth function $f \in \mathscr{E}^0$ such that $\Delta f = -2i\bar{\partial}\partial f = \rho$. This shows that $\rho \in \bar{\partial}(\mathscr{E}^{1,0})$.

For the second isomorphism, consider the sequence

$$\Omega^{1,0} \to \mathscr{E}^{1,0} \longrightarrow \mathscr{E}^{0,1} \longrightarrow \frac{\mathscr{E}^{0,1}}{\bar{\partial}(\mathscr{E}^0)} = H^{0,1}(X,\mathbf{C}),$$

where the second arrow is complex conjugation and the third is the quotient map. We want to show that the composition

$$\Omega^{1,0} \longrightarrow \frac{\mathscr{E}^{0,1}}{\bar{\partial}(\mathscr{E}^0)}$$

is an isomorphism of real vector spaces. First we show surjectivity: suppose $\alpha \in \mathscr{E}^{0,1}$ we need to find $\beta \in \Omega^{1,0}$ and $f \in \mathscr{E}^0$ such that $\alpha = \bar{\beta} + \bar{\partial} f$. That is, we need to find f such that $\bar{\partial}\overline{(\alpha - \bar{\partial} f)} = 0$. As $\Delta = -2i\bar{\partial}\partial$, this equation is

$$\Delta \bar{f} = \bar{\partial}\bar{\alpha} = d\bar{\alpha}.$$

This is a Poisson equation which admits a solution. To prove injectivity observe that if there exists $\beta \in \Omega^{1,0}$ such that $\bar{\beta} = \bar{\partial} f$ (that is, its image into $H^{0,1}(X,\mathbf{C})$ is null), then

$$\int_X \beta \wedge \bar{\beta} = \int_X \beta \wedge \bar{\partial} f = -\int_X \bar{\partial}(f\beta) + \int_X f\bar{\partial}\beta = 0.$$

This forces $\beta = 0$.

In order to prove the last isomorphism, define for $\beta_1 \in H^{1,0}(X,\mathbf{C}) = \ker(\bar{\partial})$ and $[\beta_2] \in H^{0,1}(X,\mathbf{C})$ (with $\beta_1 \in \mathscr{E}^{1,0}$ a $\bar{\partial}$ closed form and we choose $\beta_2 \in \mathscr{E}^{0,1}$ which is ∂ closed by adding a convenient $\bar{\partial} f$ found by solving Poisson equation), $\beta = \beta_1 + \beta_2$. We see then that $d\beta = (\bar{\partial} + \partial)(\beta_1 + \beta_2) = \bar{\partial}\beta_1 + \partial\beta_2 = 0$. First surjectivity: For an element $\beta \in \mathscr{E}^1$ (with decomposition $\beta = \beta_1 + \beta_2$) such that $d\beta = 0$ we show that there exists $f \in \mathscr{E}^0$ such that

$$\beta + df = (\beta_1 + \partial f) + (\beta_2 + \bar{\partial} f),$$

with $\bar{\partial}(\beta_1 + \partial f) = 0$ and $\partial(\beta_2 + \bar{\partial} f) = 0$. This follows from the solution of the Poisson equation as $d\beta = 0$ implies $\bar{\partial}\beta_1 = -\partial\beta_2$. For the injectivity, observe that, if $\beta = df$ then $\beta_1 = \partial f$ and $\beta_2 = \bar{\partial} f$ but then, $\bar{\partial}\beta_1 = \bar{\partial}\partial f = 0$ and therefore f is harmonic so constant by the maximal principle. □

8.6.2 Duality

There exist a duality between $H^{1,0}(X,\mathbf{C})$ and $H^{0,1}(X,\mathbf{C})$ which was implicit in the proof of the isomorphism $H^{1,0}(X,\mathbf{C}) \cong H^{0,1}(X,\mathbf{C})$. Define first the bilinear map $b : H^{1,0}(X,\mathbf{C}) \times H^{0,1}(X,\mathbf{C}) \to \mathbf{C}$ by taking representatives $\beta^1 \in \Omega^{1,0}$ and $\beta^2 \in \mathscr{E}^{0,1}$

$$B(\beta^1, [\beta^2]) = \int_X \beta^1 \wedge \beta^2.$$

Stokes theorem implies that the bilinear map is well defined and does not depend on the choice of the representatives.

Proposition 8.37 (Duality)**.** *On a compact Riemann surface*

$$H^{1,0}(X,\mathbf{C})^* \cong H^{0,1}(X,\mathbf{C}).$$

Proof. From the definition of the bilinear map, each element in $H^{0,1}(X,\mathbf{C})$ defines an element of the dual $H^{1,0}(X,\mathbf{C})^*$. It remains to show that the bilinear map is non-degenerate. Suppose $\alpha \in \mathscr{E}^{0,1}$ is such that $\int_X \beta\wedge\alpha = 0$ for all $\beta \in \Omega^{1,0}$. But in the previous theorem we showed that there exists an element $\beta \in \Omega^{1,0}$ such that $\alpha = \bar{\beta} + \bar{\partial} f$. For this element

$$\int_X \beta\wedge\alpha = \int_X \beta\wedge\bar{\beta} > 0.$$

□

8.6.3 Orthogonality and Harmonic forms: Hodge theorem

Using the star operator, recall that we defined a hermitian product on 1-forms over a compact Riemann surface X: For α_1, α_2 1-forms in $\mathscr{E}^1(X)$,

$$\langle\alpha_1,\alpha_2\rangle = \int_X \alpha_1\wedge *\overline{\alpha}_2.$$

Proposition 8.38. *Let X be a compact Riemann surface. Then*

1. *$\partial\mathscr{E}^0(X)$, $\bar{\partial}\mathscr{E}^0(X)$, $\Omega^{1,0}(X)$ and $\overline{\Omega^{1,0}(X)}$ are pairwise orthogonal.*

2. *$d\mathscr{E}^0(X)$ and $*d\mathscr{E}^0(X)$ are orthogonal*

3. *$d\mathscr{E}^0(X)\oplus *d\mathscr{E}^0(X) = \partial\mathscr{E}^0(X)\oplus\bar{\partial}\mathscr{E}^0(X)$.*

Proof. The proof of the first two items is an application of Stokes theorem.

1. Clearly, $\mathscr{E}^{1,0}(X)$ and $\overline{\mathscr{E}^{1,0}(X)} = \mathscr{E}^{0,1}(X)$ are orthogonal and therefore $\Omega^{1,0}(X)$ and $\overline{\Omega^{1,0}(X)}$ are also orthogonal. For the same reason $\partial\mathscr{E}^0(X)$, $\bar{\partial}\mathscr{E}^0(X)$ are orthogonal. In order to prove that $\partial\mathscr{E}^0(X)$ is orthogonal to $\Omega^{1,0}(X)$ compute by Stokes, for $f\in\mathscr{E}^0$ and $\beta\in\Omega^{1,0}(X)$:

$$\int_X \partial f\wedge\overline{*\beta} = \int_X \partial(f\overline{*\beta}) - \int_X f\partial\overline{*\beta} = 0,$$

The other cases are similar.

2. Compute, by Stokes theorem,

$$\langle df, *dg\rangle = \int_X df \wedge \overline{**dg} = -\int_X df \wedge d\overline{g} = 0.$$

3. Observe that $d\alpha = \partial\alpha + \bar{\partial}\alpha$ and $*d\alpha = -i\partial\alpha + i\bar{\partial}\alpha$ and therefore we have the equality.

□

The space $\Omega^{1,0}(X) \oplus \overline{\Omega^{1,0}}(X)$ is the space of harmonic forms. Indeed, by the formulae $d\alpha = \partial\alpha + \bar{\partial}\alpha$ and $d*\alpha = -i\partial\alpha + i\bar{\partial}\alpha$ we obtain $d\alpha = d*\alpha = 0$ if and only if $\alpha \in \Omega^{1,0}(X) \oplus \overline{\Omega^{1,0}}(X)$. The following decomposition theorem is a consequence of Theorem 8.35.

Theorem 8.39 (Smooth Hodge decomposition)**.** *For a compact Riemann surface X,*

$$\mathscr{E}^1(X) = d\mathscr{E}^0(X) \oplus *d\mathscr{E}^0(X) \oplus \Omega^{1,0}(X) \oplus \overline{\Omega^{1,0}(X)}.$$

Proof. If $\alpha \in \mathscr{E}^1(X)$ then one can solve $d*df = d\alpha$. Moreover if $d\alpha \neq 0$ then one obtains a non-trivial $\alpha = *df$. By theorem 8.35 any form α such that $d\alpha = 0$ satisfies $\alpha \in d\mathscr{E}^0(X) \oplus \Omega^{1,0}(X) \oplus \overline{\Omega^{1,0}(X)}$. This proves the decomposition.

□

8.7 Existence of meromorphic functions

The existence of meromorphic functions with prescribed singularities follows from Theorem 8.30 or, more precisely, its consequence Theorem 8.35.

Theorem 8.40. *Let X be a compact Riemann surface of genus g and $z_0 \in X$. There exists a holomorphic function on $X \setminus \{z_0\}$, meromorphic on X with a pole of order at most $g+1$.*

Proof. Let $U \subset X$ be a neighborhood in a Riemann surface defined in local coordinates by $|z| < 1$. With a slight abuse of notation, we write a function on U using the local coordinate. For instance, we say that $1/z^n$ is a function defined on U with a singularity at $z_0 = 0$.

Let $\chi \in C_0^\infty(X)$ with support in U and such that it is the identity on $|z| < 1/2$. We define the differential $\alpha_n = \bar{\partial}(\chi/z^n) \in X \setminus \{z_0\}$. Observe that α_n is null on $|z| < 1/2$ and therefore α_n is smooth on X.

We want to solve

$$\bar{\partial}u = -\alpha_n.$$

If there exists a solution, then $f = u + \chi/z^n$ is a holomorphic function on $X \setminus \{z_0\}$, meromorphic on X with a pole of order n at z_0. The problem is that this equation does not always have solutions. Indeed, we have $\alpha_n \in \ker\bar\partial$ but

$$H^{0,1} = \frac{\ker\bar\partial}{\bar\partial\mathscr{E}^0}$$

might not be trivial.

We know by now that $\dim H^{0,1} = g$, the genus of X. In particular, if $g = 0$, one can always solve the equation for any $n \in \mathbf{N}$. The argument to show existence is to consider the set of forms $\{\alpha_n\}_{1\le n\le g+1}$. It gives rise to a set $\{[\alpha_n]\}_{1\le n\le g+1}$ of classes in $H^{0,1}$ which therefore satisfies a linear relation

$$\left[\bar\partial\left(\sum_1^{g+1} c_i \frac{\chi}{z^i}\right)\right] = 0.$$

One can solve now equation $\bar\partial u = -\alpha$ with $\alpha = \bar\partial\left(\sum_1^{g+1} c_i \frac{\chi}{z^i}\right)$ to obtain a meromorphic function f with one single pole at z_0 of order at most $g+1$.

□

Remark 8.41. *1. The same argument proves the existence of a meromorphic function with only possible poles at points $z_1, \cdots, z_k$ of order $n_1, \cdots, n_k$ such that $\sum_1^k n_k \le g+1$.*

2. The proof depends only on the fact that $H^{0,1}$ has a finite dimension.

9 Riemann-Roch theorem and applications

A quantitative version to the existence of meromorphic functions is given by the Riemann-Roch theorem. Here we give the classical formulation of the result and in a later section we will give a formulation using cohomology theory. Let $D = \sum n_i z_i$ be a divisor on a Riemann surface X. The relevant spaces are the following:

Definition 9.1. 1. *$L(D)$ is the vector space whose non-null elements are meromorphic functions f satisfying $(f) \geq -D$.*

2. *$L(K-D)$ is the vector space whose non-null elements are meromorphic functions f satisfying $(f) \geq D-K$, where K is the divisor associated to any holomorphic 1-form.*

Observe that if ω is any meromorphic form, $div(\omega)$ (called a canonical divisor) is computed using the expression of ω in coordinates, $f(z)dz$, and computing $ord_p f(z)$. As any two meromorphic forms ω_1 and ω_2 are related by a meromorphic function f, that is, $\omega_2 = f.\omega_1$ and therefore $div(\omega_2) = div(\omega_1) + div(f)$. We proved:

Lemma 9.2. *Any two canonical divisors are linearly equivalent.*

Proposition 9.3. *Let K be a canonical divisor of a surface X of genus g. Then*

$$deg\, K = 2g-2.$$

Proof. Consider a meromorphic function $f : S \to \mathbf{C}P^1$. Suppose there are n poles (counted with multiplicity). The map f is a ramified cover of degree n. We have that, by Riemann-Hurwitz,

$$2-2g = n.2 - \sum(ord_z f - 1).$$

But $deg\, df = \sum(ord_{z_0} f - 1) - \sum(ord_{z_\infty} f + 1)$, where z_0 and z_∞ are, respectively, the zeros and poles of f. Therefore $deg\, df = 2g-2$. □

One can think of $L(D)$ as the space of meromorphic functions having at worst singularities at z_i of order n_i. On the other hand, $L(K-D)$ is identified to the space of meromorphic 1-forms vanishing at least in order n_i at z_i. Indeed

$$L(K-D) = \{f \in \mathcal{M}(X) \mid div(f) \geq -K + D\} = \{f \in \mathcal{M}(X) \mid div(f) + K \geq D\}$$

$$= \{f \in \mathcal{M}(X) \mid div(f\omega) \geq D\}$$

where ω is a meromorphic form.

Theorem 9.4 (Riemann-Roch)**.** *Given a divisor D of degree d on a compact Riemann surface of genus g, we have*

$$\dim L(D) - \dim L(K-D) = d - g + 1.$$

We postpone the proof of the theorem and give only a heuristic argument. Indeed, $\dim L(D)$ (we suppose D positive) should be at most the $d+1$ the number of slots in the Laurent development of a meromorphic function at the possible poles plus one for the constant functions. Now each holomorphic 1-form ω_i, $1 \le i \le g$ gives a constraint

$$\sum Res_{z_i}(f\omega_i) = 0.$$

Therefore, we obtain Riemann's inequality.

$$\dim L(D) \ge d - g + 1.$$

In order to obtain an equality we take into account the holomorphic 1-forms whose products with a meromorphic function do not have residues. That is precisely $L(K-D)$. Those 1-forms do not pose constraints in the account. We obtain then $\dim L(D) = d - g + 1 + \dim L(K-D)$. In the following, we will use the notation $l(D) = \dim L(D)$.

9.1 Applications

We write, for a divisor D on a Riemann surface X,

$$|D| = \{\, D' \in Div(S) \mid \text{effective and linearly equivalent to } D \,\}.$$

This is not a vector space but it is related to $L(D)$. Indeed, observe that if $D' = D + (g)$, then (g) is determined up to a multiplicative constant. From the definitions, $l(D) = 0$ if and only if $|D|$ is empty. On the other hand if $l(D) \ge 1$, the space $|D|$ is the projective space defined by $L(D) \setminus \{0\}$, so that

$$l(D) = dim_{\mathbf{C}}|D| + 1.$$

Lemma 9.5. *If $degD < 0$ then $l(D) = 0$.*

Proof. If $f \in L(D)$ is non-zero we have $f \ge -D$ and so $deg(f) \ge -deg(D) > 0$. But the degree of any non-vanishing principal divisor is zero. □

Definition 9.6. *Define the vector space*

$$\Omega(D) = \{\textit{abelian differentials which are multiples of } -D\,\}$$

and its dimension $i(D) = dim_{\mathbf{C}}\Omega(D)$ (the index of speciality).

In this section we prove that any compact Riemann surface is embedded in a projective space. We start with some simple consequences of Riemann-Roch. Even if they have been obtained before, it is worth to see how one can obtain them directly from the formula.

1. $l(K) = g$ follows by taking $D = 0$ in the formula.
2. $deg\,K = 2g-2$ follows by taking $D = K$ and the previous result.
3. If $deg\,D > 2g-2$ then $l(D) = deg(D) - g + 1$ because $l(K-D) = 0$ $(deg(K-D) < 0)$.
4. If $l(p) \geq 2$ then the surface is $\mathbf{C}P^1$. Indeed, in that case, there is a nontrivial meromorphic function $f : S \to \mathbf{C}P^1$ with one simple pole at p. f is a biholomorphism. In fact $g = 0$ and we obtain $l(p) = 2$.
5. Elliptic curves. Suppose $g = 1$. We have $deg\,K = 2g-2 = 0$ and therefore $l(p) = 1 - 1 + 1 + l(K-p) = 1$. That means that there are only constant functions on $L(p)$. On the other hand $l(2p) = 2$. So there exists a non-constant meromorphic function, say x, with a double pole at p. Also, $l(3p) = 3$ so there exists a meromorphic function, y, with a triple pole at p. As $l(6p) = 6$ and $1, x, x^2, x^3, y, y^2, xy$ are all in $L(6p)$, there exists a linear relation between them of the form

$$a + bx + cx^2 + dx^3 + dy + ey^2 + fxy = 0,$$

with $e \neq 0$ (otherwise y would have an even order pole). By a linear change of coordiantes w can write $y^2 = x^3 + g_2 x + g_3$.

Proposition 9.7. *If S is a compact surface which is not biholomorphic to $\mathbf{C}P^1$ then at each point $z \in S$ there exists a holomorphic 1-form ω with $\omega(p) \neq 0$.*

Proof. If that is not the case, as all meromorphic one forms are given by $g\omega_0$ (where g is meromorphic and ω_0 a fixed holomorphic 1-form) and so holomorphic one forms are $L(K) = \{\ g\ ;\ (g) + (\omega_0) \geq 0\ \}$, we obtain that $L(K-z) = L(K)$. By Riemann-Roch, $l(z) = 1 - g + 1 - g = 2$. This means that there exists a meromorphic function with only one pole at z. This is impossible if the Riemann surface is not $\mathbf{C}P^1$. □

From the proposition it is easy to see that the following map is well defined.

Definition 9.8. *Let ω_i be a basis of holomorphic differentials of a surface X of genus $g \geq 1$. Write $\omega_i(z) = f_i(z)dz$ in local coordinates. The canonical map is the map*

$$\phi_K : X \to \mathbf{C}P^{g-1}$$

given by $\phi_K(z) = [f_1(z), \cdots, f_g(z)]$.

Lemma 9.9. *If $\phi_K(p) = \phi_K(q)$ for two distinct points $p, q \in X$ then there exists a ramified cover of $f : X \to \mathbf{CP}^1$ of degree 2.*

Proof. If $\phi_K(p) = \phi_K(q)$ then $\omega_i(p) = \lambda\omega_i(q)$ for all $1 \leq i \leq g$ and a constant λ. Therefore, for any holomorphic form ω, $div\omega \geq p + q$ if and only if $div\omega \geq p$. That means, for $D = p + q$ that $L(K - D) = L(K - p)$. By Riemann-Roch

$$l(p+q) - l(K-p) = 2 - g + 1 \text{ and } l(p) - l(K-p) = 1 - g + 1.$$

From the second equation we obtain (as X is not of genus 0, $l(p) = 1$) we obtain $l(K-p) = g-1$. Substituting in the first equation one obtains $l(p+q) = 2-g+1+g-1 = 2$. That is, there exists a meromorphic function with only simple poles at p and q. Therefore it defines a ramified cover of CP^1 of degree 2. □

Proposition 9.10. *If X of genus $g \geq 2$ is not hyperelliptic than ϕ_K is an embedding into $\mathbf{CP}^{g-1}$.*

Proof. ϕ_K is of rank one if for any $p \in X$ there exists $\omega \in \Omega^{1,0}$ with a zero of order one at p. Analogously, as in the previous lemma if this does not happen at p then

$$l(K - 2p) = l(K - p).$$

Then one concludes by Riemann-Roch that $l(2p) = 2$. We conclude that there exists a ramified cover of degree 2 $X \to \mathbf{CP}^1$. □

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