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MACDONALD INTEGRALS AND MONODROMY

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We prove several results on monodromies associated to Macdonald integrals, that were used in our previous work on the finite field analogue of a conjecture of Macdonald. We also give a new proof of our formula expressing recursively the zeta function of the local monodromy at the origin of the discriminant of a finite Coxeter group in terms of the degrees of the group.

1. Introduction

1.1. Let $f : \mathbf{C}^n \to \mathbf{C}$ be a polynomial map having a singularity at the origin 0 in $\mathbf{C}^n$. For $0 < \eta \ll \varepsilon \ll 1$, the restriction of f to $B(0,\varepsilon) \cap f^{-1}(D_\eta \setminus \{0\})$, with $B(0,\varepsilon)$ the open ball of radius ε centered at 0 and D_η the open disk of radius η centered at 0, is a locally trivial fibration, the Milnor fibration, onto $D_\eta \setminus \{0\}$ with fiber F_0, the Milnor fiber at 0. The action of a characteristic homeomorphism of this fibration on cohomology gives rise to the monodromy operator $M : H^*(F_0, \mathbf{Q}) \to H^*(F_0, \mathbf{Q})$. Define the monodromy zeta function as

$$Z_{f,0}(T) := \prod_{i \geq 0} [\det(\mathrm{Id} - TM, H^i(F_0, \mathbf{Q}))]^{(-1)^{i+1}} .$$

In case the hypersurface $f = 0$ has an explicit embedded resolution, a formula due to A'Campo [1] may be used to compute $Z_{f,0}$. However, in general, calculating explicitely $Z_{f,0}$ happens to be a quite dificult task.

1.2. Let V be a complex vector space of finite dimension n and let G be a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo-reflections, i.e. endomorphisms of finite order fixing pointwise an hyperplane. Such a group is called a finite complex reflection group. Pseudo-reflections of order 2 will be called reflections. Denote by $\mathbf{C}[V]$

the algebra of polynomial functions on V. By Chevalley's Theorem the ring of invariants $\mathbf{C}[V]^G$ is a free polynomial algebra on n homogeneous invariant polynomials, whose degrees $d_1, \ldots, d_n$ only depend on G and are called the degrees of the group G. For every pseudo-reflection in G with corresponding hyperplane H, choose a linear form ℓ_H defining H and denote by $e(H)$ the order of the subgroup of elements of G fixing H pointwise. Set $\Delta := \prod_H \ell_H^{e(H)}$, the product being over all pseudo-reflection hyperplanes in G. The induced function $\tilde{\Delta} : V/G \to \mathbf{C}$ is the discriminant of G. When $V = \mathbf{C}^n$, and G is furthermore a subgroup of $\mathrm{GL}(\mathbf{R}^n)$, G is called a finite Coxeter group. In this case the integers e_H are all equal to 2. Now consider $Z(T, G) := Z_{\tilde{\Delta},0}(T)$ the zeta function of the local monodromy of the discriminant at the origin.

In the paper [7], we proved the following remarkable recursion formula for $Z(T, G)$:

Theorem 1.3. *For G a finite Coxeter group we have*

$$\prod_{\mathcal{E} \text{ connected subgraph}} Z(-T, G(\mathcal{E}))^{(-1)^{|\mathcal{E}|}} = \prod_{i=1}^{n} \frac{1 - T^{d_i}}{1 - T},$$

where the product on the left-hand side runs over all connected subgraphs $\mathcal{E}$ of the Coxeter diagram of G, $G(\mathcal{E})$ denotes the Coxeter group with diagram $\mathcal{E}$, and $|\mathcal{E}|$ the number of vertices of $\mathcal{E}$.

The proof of Theorem 1.3 given in [7] was based on some new properties of Springer's regular elements [19] in finite complex reflection groups, which have been since further investigated by Lehrer and Springer in [12, 13]. In fact, we computed in [7] the zeta function of the local monodromy of the discriminant for all irreducible finite complex reflection groups. This involved a case by case analysis, already for finite Coxeter groups.

1.4. Though the techniques in the paper [7] are mostly group theoretic, that work arouse from our study of a finite field analogue of Macdonald's conjecture. Let us recall the statement of Macdonald's conjecture as formulated in [8]. Let G be a finite subgroup of $\mathrm{GL}(\mathbf{R}^n)$ which is generated by reflections and let q be a positive definite quadratic form which is invariant under G. We denote by $d_1, \ldots, d_n$ the degrees of G. Let $\ell_1, \ldots, \ell_N$ be equations for the N distinct reflection hyperplanes and set $\Delta := (\prod_{i=1}^{N} \ell_i)^2$. Macdonald's conjecture [15], proved by Opdam [17], is the following result.

Theorem 1.5. *The integral*

$$I(s) := \int_{\mathbf{R}^n} \Delta(x)^s e^{-q(x)} dx \tag{1.5.1}$$

may be expressed as

$$I(s) = \pi^{n/2} \kappa^s \left(\prod_{i=1}^{n} \frac{\Gamma(d_i s + 1)}{\Gamma(s + 1)} \right) (\mathrm{discr}\, q)^{-1/2}, \tag{1.5.2}$$

with

$$\kappa = \prod_{i=1}^{N} \frac{q(\ell_i)}{4}, \tag{1.5.3}$$

where we consider ℓ_i *in* $q(\ell_i)$ *as a vector in* $\mathbf{R}^n$*, identifying* $\mathbf{R}^n$ *with its dual, by means of the quadratic form* q.

1.6. Let us now state the finite field analogue. Let $\mathbf{F}$ be a finite field of characteristic p different from 2. We consider a finite-dimensional $\mathbf{F}$-vector space V, a finite subgroup G of $\mathrm{GL}(V)$ generated by reflections, and q a G-invariant, non degenerate, symmetric bilinear form on V. If p does not divide $|G|$, one may define the degrees of G, $d_1, \ldots, d_n$ as in the complex case. One also defines Δ similarly. Because $p \neq 2$, we may define an element κ of $\mathbf{F}$ by (1.5.3). Fix a non trivial additive character $\psi : \mathbf{F} \to \mathbf{C}$. The analogue of the integral in (1.5.1) will be the character sum

$$S_G(\chi) := \sum_{x \in (U/G)(\mathbf{F})} \chi(\Delta(x))\psi(q(x)) \tag{1.6.1}$$

where $\chi : \mathbf{F}^\times \to \mathbf{C}^\times$ is a multiplicative character and U denotes the complement of the hypersurface $\Delta(x) = 0$ in V. Here we write $q(x)$ instead of $q(x, x)$. The analogue of the Gamma function will be the Gauss sum $-g(\chi)$, where $g(\chi) := -\sum_{x \in \mathbf{F}^\times} \chi(x)\psi(x)$. Our main result in the paper [8] is the following finite field analogue of Macdonald's conjecture.

Theorem 1.7. *Assume that* p *does not divide* $|G|$*. Then* $\kappa \neq 0$ *and, for every multiplicative character* $\chi : \mathbf{F}^\times \to \mathbf{C}^\times$,

$$S_G(\chi) = (-1)^n \phi(\operatorname{discr} q) g(\phi)^n \phi(\kappa)\chi(\kappa) \prod_{i=1}^{n} \frac{g((\phi\chi)^{d_i})}{g(\phi\chi)},$$

where ϕ *denotes the unique multiplicative character of order* 2.

In the special case when G is the symmetric group S_n, this identity was conjectured by Evans [9] and proved ten years later by him [10] by using important work of Anderson [2].

1.8. The aim of the present paper is doublefold. Firstly, the proof of Theorem 1.7 in [8], not only used Theorem 1.3, but also some other results on the monodromy of functions related to Macdonald's integrals. These monodromy calculations were required in order to use Laumon's product formula [11]. We present direct, self-contained proofs of these results in Secs. 4, 5, and 6. More precisely, in Theorem 4.2 we express the global monodromy at the origin of the restriction of the discriminant to the quotient of the quadric $q = 1$ by G in terms of local monodromies along strata of the discriminant. In Theorem 5.4, we prove a strange "Complement formula" expressing the local monodromy of the discriminant at the origin as the sum of the global monodromy at infinity of the restriction of the discriminant to the quotient of the quadric $q = 1$ by G and the global monodromy of the morphism induced by q^N at the origin in the quotient space U/G, where U denotes the complement of the hypersurface $\Delta(x) = 0$ in V. In Theorem 6.3, we explicitely compute the monodromy at the origin of the morphism induced by q in the quotient space U/G and more generally for the constant sheaf replaced by the pullback of a Kummer sheaf by the function defining the discriminant. Secondly, we derive in Sec. 7 two consequences of Macdonald's formula (1.5.2). The first one, Theorem 7.2, whose proof is simple and elementary, is the calculation of the maximum of the function Δ on the real points of the quadric defined by $q = 1$. This formula is used in our proof of Theorem 1.7 and is the only place in that proof relying on Macdonald's formula. Then, we explain how, using work of Anderson [3] and Loeser and Sabbah [14] on determinants of Aomoto complexes and determinants of integrals, one can derive Theorem 1.3 from Macdonald's formula (in fact Theorem 1.3 is equivalent to knowing the precise form of the gamma factors in Macdonald's formula) making the full circle of the story.

At the end of the paper we give a complete list of the assertions in [8] whose proofs were postponed to the present work.

2. Coxeter Arrangements

2.1. Arrangements. Let V be a finite dimensional vector space over a field k and $V \hookrightarrow \mathbf{P}$ its canonical projective space compactification. By an *hyperplane arrangement* $\mathcal{A}$ in V we mean a finite set of affine hyperplanes in V. Similarly, if $\mathbf{P}$ is a projective space over k, an hyperplane arrangement in $\mathbf{P}$ will be a finite set of projective hyperplanes in $\mathbf{P}$.

If $\mathcal{A}$ is an arrangement in V, we denote by $\bar{\mathcal{A}}$ the projective arrangement in $\mathbf{P}$ defined by taking the closure of the hyperplanes in $\mathcal{A}$. We shall denote by $V \setminus \mathcal{A}$ the complement in V of the union of the hyperplanes belonging to $\mathcal{A}$. When $k = \mathbf{R}$, we shall call a connected component of $V \setminus \mathcal{A}$ a *chamber* of $V \setminus \mathcal{A}$. If all the hyperplanes contain 0, the arrangement is said to be *central.* We call an endomorphism of V a *reflection* if it has order 2 and fixes pointwise some hyperplane.

Lemma 2.2. *Let q be a non degenerate quadratic form on $\mathbf{C}^n$ and denote by Q the quadric defined by $q = 1$ in $\mathbf{C}^n$. Let $\mathcal{A}$ be a non empty central arrangement in $\mathbf{R}^n$ such that the restriction of q to every non empty intersection of hyperplanes in $\mathcal{A}$ is also non degenerate. Then the number of chambers* $\mathrm{ch}(\mathbf{R}^n \setminus \mathcal{A})$ *of $\mathbf{R}^n \setminus \mathcal{A}$ is equal to $(-1)^{n-1}\chi(Q \setminus (\mathcal{A} \otimes \mathbf{C}))$. Here $\mathcal{A} \otimes \mathbf{C}$ denotes the central arrangement in $\mathbf{C}^n$ obtained by extension of scalars.*

Proof. When $|\mathcal{A}| = 1$, the result is clear, since the Euler characteristic of the smooth quadric Q in $\mathbf{C}^n$ is equal to $1 + (-1)^{n-1}$. So assume $|\mathcal{A}| > 1$ and choose an hyperplane H in $\mathcal{A}$. Denote by $\mathcal{A}'$ the arrangement in $\mathbf{R}^n$ obtained by deleting

H from $\mathcal{A}$ and by $\mathcal{A}''$ the arrangement in H obtained by intersecting H with the hyperplanes in $\mathcal{A}'$. Since $\mathrm{ch}(\mathbf{R}^n \setminus \mathcal{A}) = \mathrm{ch}(\mathbf{R}^n \setminus \mathcal{A}') + \mathrm{ch}(H \setminus \mathcal{A}'')$ and

$$\chi(Q \setminus (\mathcal{A} \otimes \mathbf{C})) = \chi(Q \setminus (\mathcal{A}' \otimes \mathbf{C})) - \chi((Q \cap H) \setminus (\mathcal{A}'' \otimes \mathbf{C})),$$

the result follows by induction on the number of hyperplanes. □

2.3. Coxeter arrangements. We define a *classical Coxeter arrangement* as a triple $A = (V, G, q)$, where V is a finite dimensional vector space over $\mathbf{R}$, G is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections, and q is a G-invariant *positive definite* symmetric bilinear form on V. We define a *Coxeter arrangement over* $\mathbf{C}$ as a triple $A = (V, G, q)$, where V is a finite dimensional vector space over $\mathbf{C}$, G is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections, and q is a G-invariant symmetric bilinear form on V, which arises by extension of scalars from a classical Coxeter arrangement. (In fact it follows from the argument given at the end of the proof of [8, Proposition 1.6] that the last condition is always satisfied.) We denote by $\mathcal{A}_G$ the central arrangement consisting of all reflection hyperplanes of G.

Proposition 2.4. *Let $A = (V, G, q)$ be a complex Coxeter arrangement and denote by Q the quadric defined by $q = 1$ in V. Set $B = Q \cap (V \setminus \mathcal{A}_G)$. Then*

$$\chi(B) = (-1)^{n-1}|G| \quad \textit{and} \quad \chi(B/G) = (-1)^{n-1}.$$

Proof. Since a Coxeter group acts transitively on the corresponding set of chambers (cf. [5, Sec. 3.1, Lemme 2]), this follows from Lemma 2.2. □

2.5. Canonical embedded resolution. Let V be a vector space of dimension n over a field k and let $\mathcal{A}$ be an hyperplane arrangement in V. By an intersection space of $\mathcal{A}$ we shall mean a non empty subset of V which is the intersection of some non empty family of hyperplanes in $\mathcal{A}$. By a stratum of $\mathcal{A}$ we shall mean an intersection space of $\mathcal{A}$ minus the union of all hyperplanes of $\mathcal{A}$ that do not contain that intersection space. Similarly one defines intersection spaces and strata of projective arrangements, and intersection spaces of $\bar{\mathcal{A}}$ are just the closure of intersection spaces of $\mathcal{A}$. Let $\mathcal{A}$ be an hyperplane arrangement in V or in the canonical projective space compactification $\mathbf{P}$ of V respectively. We set $X_0 = V$ respectively $\mathbf{P}$. We define $h_1 : X_1 \to X_0$ to be the blowing up of all dimension zero intersection spaces of $\mathcal{A}$ and then, by induction, for $2 \leq i \leq n-1$, $h_i : X_i \to X_{i-1}$ to be the blowing up of the union Y_{i-1} of the strict transforms of all $i-1$-dimensional intersection spaces of $\mathcal{A}$ in X_{i-1}. Note that Y_{i-1} is the disjoint union of these strict transforms, hence is in particular smooth. We set $X_{\mathcal{A}} := X_{n-1}$ and denote by $h_{\mathcal{A}} : X \to X_0$ the composition of the morphisms h_i. Remark that $X_{\mathcal{A}}$ is smooth and that the inverse image of the union of hyperplanes in $\mathcal{A}$ by $h_{\mathcal{A}}$ is a divisor with (global) normal crossings, and that its set of irreducible components is in natural bijection with the set of strata of $\mathcal{A}$.

2.6. Assume now $A = (V, G, q)$ is a complex Coxeter arrangement, fix equations $\ell_1 = 0, \ldots, \ell_N = 0$ for the N distinct reflection hyperplanes and set $\Delta := (\prod_{i=1}^{N} \ell_i)^2$. Remark in this case $\mathcal{A}_G$ and $\bar{\mathcal{A}}_G$ have only one zero-dimension stratum, namely the origin, hence h_1 is just the blow up of the origin. It follows that $h_{\bar{\mathcal{A}}_G} : X_{\bar{\mathcal{A}}_G} \to \mathbf{P}$ is an embedded resolution of the divisor $(\Delta = 0) \cup \bar{Q} \cup H_\infty$ and the divisor $(\Delta = 0) \cup \bar{Q}_0 \cup H_\infty$ with $\bar{Q}$, respectively $\bar{Q}_0$, the closure of the locus of $q = 1$, respectively $q = 0$, in $\mathbf{P}$ and H_∞ the hyperplane at infinity. One should also remark that the polar divisors of Δ and q in $X_{\bar{\mathcal{A}}_G}$ have only one irreducible component, namely the strict transform of hyperplane at infinity H_∞. Furthermore, if we denote by $X_{\bar{\mathcal{A}}_G, \bar{Q}}$ the strict transform of Q in $X_{\bar{\mathcal{A}}_G}$, the morphism $X_{\bar{\mathcal{A}}_G, \bar{Q}} \to \bar{Q}$ yields an embedded resolution of the divisor $((\Delta = 0) \cup H_\infty) \cap \bar{Q}$ in $\bar{Q}$.

3. Monodromy Computations for Group Actions

3.1. Monodromic Grothendieck group. To agree with notation used in [8], we shall use the terminology from [18] Exp. XIII–XIV concerning vanishing cycles. We shall explain in 3.2 how statements about elements of the monodromic Grothendieck group can be equivalently expressed in terms of the more classical zeta functions of the monodromy used in [1, 16].

We denote by $\bar{\eta}_0$ the generic geometric point of the henselization of the complex affine line $\mathbf{A}^1$ at 0, by I_0 its inertia group (i.e. the fundamental group of the complement of 0 in a small disk around 0) and by K_{I_0} the Grothendieck group of finite dimensional $\mathbf{C}$-vector spaces with I_0-action. If $\mathcal{L}$ is an object in $D_c^b(\mathbf{G}_m, \mathbf{C})$, the derived category of bounded complexes of $\mathbf{C}$-sheaves with constructible cohomology on the multiplicative group $\mathbf{G}_m := \mathbf{A}^1 \setminus \{0\}$, we denote by $[\mathcal{L}_{\bar{\eta}_0}]$ the class of $\sum (-1)^i [\mathcal{H}^i(\mathcal{L})_{\bar{\eta}_0}]$ in K_{I_0} and we set $[\mathcal{L}_{\bar{\eta}_\infty}] = [\mathrm{inv}^*(\mathcal{L})_{\bar{\eta}_0}]$, where inv is the morphism $x \mapsto x^{-1}$. If a finite group G acts on $\mathcal{L}$ we denote by $[\mathcal{L}^G_{\bar{\eta}_0}]$ the class of $\sum (-1)^i [\mathcal{H}^i(\mathcal{L})^G_{\bar{\eta}_0}]$ and we define similarly $[\mathcal{L}^G_{\bar{\eta}_\infty}]$. For any character $\chi : I_0 \to \mathbf{C}^\times$ we denote by V_χ the class in K_{I_0} of the rank one object with action given by χ, hence $V_\chi = [(\mathcal{L}_\chi)_{\bar{\eta}_0}]$, with $\mathcal{L}_\chi$ the local system of rank one on $\mathbf{G}_m$ and monodromy χ around the origin. For any natural number $m \geq 1$, we set $V_m = [(\pi_{m*}\mathbf{C})_{\bar{\eta}_0}]$, where $\pi_m : \mathbf{G}_m \to \mathbf{G}_m$ is given by $x \mapsto x^m$, so we have $V_m = \sum_{\chi^m = 1} V_\chi$.

3.2. Let ϱ be the standard topological generator of I_0, corresponding to counterclockwise rotation around 0 in $\mathbf{C}$. Since as an abelian group K_{I_0} is generated by the elements V_χ, there is a unique morphism of abelian groups

$$Z : K_{I_0} \longrightarrow \mathbf{C}(T)^\times$$

sending V_χ to $Z(V_\chi) := (1 - T\chi(\varrho))^{-1}$. In particular we have $Z(V_m) = (1 - T^m)^{-1}$. If $\mathcal{L}$ is an object in $D_c^b(\mathbf{G}_m, \mathbf{C})$, $Z([\mathcal{L}_{\bar{\eta}_0}])$, respectively $Z([\mathcal{L}_{\bar{\eta}_\infty}])$ is nothing else than the monodromy zeta function of $\mathcal{L}$ at the origin, respectively at infinity, of $\mathcal{L}$. Remark that Z is clearly an isomorphism onto its image. Hence, depending on convenience, we shall either formulate results in K_{I_0} or in terms of monodromy zeta functions.

3.3. We consider the following geometric situation. Let $j : U \to X$ be the immersion of a dense open subset U in a smooth complex projective variety X. We also assume to be given a morphism $f : U \to \mathbf{G}_m$ and a proper morphism $g : X \to \mathbf{P}^1$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\;j\;} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{G}_m & \xrightarrow{\;j_0\;} & \mathbf{P}^1 \end{array}$$

is commutative, with $j_0 : \mathbf{G}_m \to \mathbf{P}^1$ the standard immersion. We assume $X - U$ is a divisor with normal crossings. We denote by E_i, $i \in J$, the irreducible components of $X - U$. In particular the E_i's are smooth. We set $E = (g^{-1}(0))_{\mathrm{red}} = \cup_{i \in J_0} E_i$, $E^0 = E - \cup_{i \notin J_0} E_i$, and $g^{-1}(0) = \sum_{i \in J_0} N_i E_i$.

3.4. We assume a finite group G acts on X and that U is stable. We also assume the action of G on U is free and that f is G-invariant. For $s \in E$ we denote by G_s the stabilizer at s and by $[(R\psi_g \mathbf{C})_s^{G_s}]$ the class of $\Sigma(-1)^j [(R^j \psi_g \mathbf{C})_s^{G_s}]$ in K_{I_0}, with $R\psi_g \mathbf{C}$ the complex of nearby cycles of g. For $s \in E^0$, we set $J^s = \{i \in J | s \in E_i\} \subset J_0$ and we denote by C_s the set of connected components of the Milnor fiber of g at s. There is a natural action of G_s on C_s.

The following result is proved in [6, Sec. 3]:

Proposition 3.5. *Assume the previous assumptions hold.*

(1) *Let $\mathcal{S}$ be a partition of E^0 into constructible subsets S such that $|G_s|$ and $[(R\psi_g \mathbf{C})_s^{G_s}]$ are constant on every S. Then the following holds in K_{I_0}:*

$$[(Rf_! \mathbf{C})_{\bar{\eta}_0}^G] = |G|^{-1} \sum_{S \in \mathcal{S}} \chi(S, \mathbf{C}) |G_s| [(R\psi_g \mathbf{C})_s^{G_s}] ,$$

where $s \in S$.

(2) *For every closed point s of E, the stabilizer G_s acts freely on C_s.*

(3) *Assume for every $i \in J_0$ and every $\sigma \in G$, $\sigma(E_i) = E_i$ or $\sigma(E_i) \cap E_i = \emptyset$. Then, for every $s \in E^0$,*

$$[(R\psi_g \mathbf{C})_s^{G_s}] = \begin{cases} 0 & \text{if } |J^s| > 1 , \\ (\mathbf{C}^{C_s/G_s})^\vee & \text{if } |J^s| = 1 , \end{cases}$$

with $(\mathbf{C}^{C_s/G_s})^\vee$ the dual of the module $\mathbf{C}^{C_s/G_s}$ endowed with its natural I_0-action.

3.6. In fact, we shall also need to consider the case where the morphism f, instead of having an extension to an actual morphism $g : X \to \mathbf{P}^1$, only extends to a rational map $g : X \to \mathbf{P}^1$. Denote by $g^{-1}(0)$ and $g^{-1}(\infty)$ the divisor of zeroes and poles respectively. Remark that g is a morphism outside $g^{-1}(0) \cap g^{-1}(\infty)$ and write $g^{-1}(\infty) = \sum_{i \in J_\infty} N_i E_i$. We replace the condition "$X \setminus U$ is a divisor with normal crossings" by "$X \setminus U$ is a divisor with normal crossings on a Zariski neighbourhood of $g^{-1}(0)$".

The following generalization of Proposition 3.5 is also proved in [6, Sec. 3] and shows that the exceptional divisors of the additional blowing ups needed to make the extensions of f and g regular do not contribute to our monodromy calculations:

Proposition 3.7. *Assume for every i in $J_0 \cup J_\infty$ and every σ in G, $\sigma(E_i) = E_i$ or $\sigma(E_i) \cap E_i = \emptyset$. Then the statements in Proposition* 3.5 *still hold in the setting* 3.6.

4. Expressing the Global Monodromy at the Origin in Term of Local Monodromies

4.1. Let $A = (V, G, q)$ be a complex Coxeter arrangement. As in 2.3, we fix equations for the hyperplanes and consider the corresponding function Δ. Let $(R\psi_\Delta(\mathbf{C}))_0$ be the stalk at zero of the complex of nearby cycles with respect to Δ. We set $M_G := [(R\psi_\Delta(\mathbf{C}))_0^G]$ and $\bar{M}_G := (-1)^{n-1}[(R\psi_\Delta(\mathbf{C}))_0^G]$, with n the dimension of V. We also set $B := Q \setminus \mathcal{A}_G$.

Theorem 4.2. *The relation*

$$[(R\Delta_{|B!}\mathbf{C})_{\bar{\eta}_0}^G] = (-1)^n \sum_{\substack{\mathcal{E} \text{ connected subgraph} \\ G(\mathcal{E}) \neq G}} \bar{M}_{G(\mathcal{E})}$$

holds in K_{I_0}.

Proof. The polynomial Δ being homogeneous polynomial of degree $2N$, its local Milnor fiber at the origin is diffeomorphic to any of the global fibers $F_\lambda = \{x \in V | f(x) = \lambda\}$ with $\lambda \neq 0$, and the transformation $x \mapsto e^{2i\pi/2N}x$ on F_λ corresponds to a characteristic homeomorphism of the local Milnor fibration via this diffeomorphism (cf. [16, Lemma 9.4]). Since, by Poincaré duality, $[(R\Delta_*\mathbf{C})_{\bar{\eta}_0}^G] = [(R\Delta_!\mathbf{C})_{\bar{\eta}_0}^G]$, it follows we may write $M_G = [(R\Delta_!\mathbf{C})_{\bar{\eta}_0}^G]$, which allows us to calculate M_G by applying Proposition 3.7 to the resolution $X_{\bar{\mathcal{A}}_G} \to \mathbf{P}$ defined in 2.6. Since the Euler characteristic of a complex algebraic variety with a free $\mathbf{G}_m$-action is zero (see, e.g. [4]), we have to sum in the formula of Proposition 3.5(1) for M_G only over strata lying inside the strict transform in $X_{\bar{\mathcal{A}}_G}$ of the exceptional divisor H of the blow up $h_1 : X_1 \to V$ of 0 in V. Let Z be the complement in H of the strict transform in X_1 of the locus of $\Delta = 0$, and let $\mathcal{W}$ be a partition of Z into constructible subsets W on which G_s and $[(R\psi_{\Delta \circ h_1}(\mathbf{C}))_s^{G_s}]$ are constant. Then the above discussion yields

$$M_G = |G|^{-1} \sum_{W \in \mathcal{W}} \chi(W)|G_s|[(R\psi_{\Delta \circ h_1}(\mathbf{C}))_s^{G_s}]\,,$$

where s is any point in W. A similar formula holds also for M_{G_S}, where G_S is defined below. Applying now Proposition 3.7 to the resolution $X_{\bar{\mathcal{A}}_G, \bar{Q}} \to \bar{Q}$ defined in 2.6, and using [5, Sec. 3.3, Proposition 1], we obtain the relation

$$[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_0}] = \frac{1}{|G|} \sum_{S \text{ stratum}} \chi(S \cap Q) |G_S| M_{G_S} \,,$$

where, for any subset S of V, G_S is the group generated by the reflections with respect to the walls containing S. Here by stratum, we mean non dense stratum of the arrangement associated to G. Using Lemma 2.2, we obtain

$$\begin{aligned}[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_0}] &= \frac{1}{|G|} \sum_{S \text{ stratum}, S \neq \{0\}} (-1)^{\dim S - 1} \left(\sum_{\substack{F \text{ face} \\ F \subset S}} 1 \right) |G_S| M_{G_S} \\ &= \frac{1}{|G|} \sum_{F \text{ face}, F \neq \{0\}} (-1)^{\dim F - 1} |G_F| M_{G_F} \,,\end{aligned}$$

where by "face" we mean any face of codimension ≥ 1 of some chamber of the arrangement associated to G. The faces which are contained in a stratum S are indeed exactly the chambers of the set of real points of S. Let C_0 be a fixed chamber. For each face F, there exists a unique face F_0 of C_0 such that there exists w in G with $F_0 = w(F)$ (see [5, Sec. 3.3, Remarque 1]). We say that the faces F and F_0 are *related*. We have

$$\begin{aligned}[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_0}] &= \frac{1}{|G|} \sum_{\substack{F_0 \text{ face of } C_0 \\ F_0 \neq \{0\}}} \sum_{\substack{F \text{ face} \\ F \text{ related to } F_0}} (-1)^{\dim F - 1} |G_F| M_{G_F} \\ &= \frac{1}{|G|} \sum_{\substack{F_0 \text{ face of } C_0 \\ F_0 \neq \{0\}}} (-1)^{\dim F_0 - 1} |G_{F_0}| M_{G_{F_0}} N(F_0) \,,\end{aligned}$$

with $N(F_0)$ the number of faces related to F_0. Note that $N(F_0)$ is equal to the total number of chambers divided by the number of chambers in the arrangement associated to G_{F_0}, since every chamber has exactly one face related to F_0, and the number of chambers containing the same face F is equal to the number of chambers in the arrangement associated to G_F. Hence, since the number of chambers of a Coxeter arrangement is equal to the order of the group, we obtain

$$[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_0}] = \sum_{\substack{F_0 \text{ face of } C_0 \\ F_0 \neq \{0\}}} (-1)^{\dim F_0 - 1} M_{G_{F_0}} \,.$$

Associating to F_0 the subgraph of the Coxeter diagram $\mathcal{G}$ of G whose vertices correspond to the walls of C_0 that contain F_0, we get

$$[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_0}] = (-1)^n \sum_{\mathcal{E} \text{ proper subgraph of } \mathcal{G}} \bar{M}_{G(\mathcal{E})} \,.$$

Since, by [7, Corollary 3.3], which is given in [7] a direct self contained proof, $\bar{M}_{G(\mathcal{E})} = 0$ when $\mathcal{E}$ is not connected, the result follows. □

5. The Formula of the Complement

As before, U denotes the complement in V of the hypersurface defined by $\Delta = 0$ and B denotes the intersection of U with the quadric $q = 1$. We also denote by B_0 the intersection of U with the quadric $q = 0$.

Lemma 5.1. *The following holds in* K_{I_0}:

$$M_G = [(R\Delta_{|B_0!}\mathbf{C})^G_{\bar\eta_0}] + [(Rq^N_{|U!}\mathbf{C})^G_{\bar\eta_0}]\,.$$

Proof. For any element W of K_{I_0}, the monodromy zeta function $Z(W)$ of W is equal to $\exp\sum_{j=1}^{\infty} \mathrm{Tr}(\varrho^j, W)T^j/j$, where ϱ is as in 3.2 (see, e.g. [16, p. 77]). Hence, we have to prove that

$$\mathrm{Tr}(\varrho^j, M_G) = \mathrm{Tr}(\varrho^j, (R\Delta_{|B_0!}\mathbf{C})^G_{\bar\eta_0}) + \mathrm{Tr}(\varrho^j, (Rq^N_{|U!}\mathbf{C})^G_{\bar\eta_0})\,, \tag{5.1.1}$$

for every j in $\mathbf{N}\setminus\{0\}$. Since both functions Δ and q^N are homogeneous of degree $2N$, the map $h : \mathbf{C}^n \to \mathbf{C}^n$, $x \mapsto e^{2i\pi/2N}x$ induces the monodromy action ϱ on the elements of K_{I_0} appearing in (5.1.1). Hence the traces in (5.1.1) are equal to the Euler characteristic of the fixed point manifold of h^j restricted to the generic fiber of respectively $\Delta_{|U/G}$, $\Delta_{|B_0/G}$ and $q^N_{|U/G}$ (see, e.g. [16, Lemma 9.5]). Thus, we may write

$$\begin{aligned}\mathrm{Tr}(\varrho^j, M_G) &= \chi(\{x\in U/G | \Delta(x) = 1, h^j(x) = x \bmod G\})\\ &= \chi(\{x\in U/G | q(x) = 0, \Delta(x) = 1, h^j(x) = x \bmod G\})\\ &\quad + \chi(\{x\in U/G | q(x) \neq 0, \Delta(x) = 1, h^j(x) = x \bmod G\})\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Tr}(\varrho^j, (R\Delta_{|B_0!}\mathbf{C})^G_{\bar\eta_0}) + \mathrm{Tr}(\varrho^j, (Rq^N_{|U!}\mathbf{C})^G_{\bar\eta_0})\\ &\quad= \mathrm{Tr}(\varrho^j, (R\Delta_{|B_0!}\mathbf{C})^G_{\bar\eta_0})\\ &\qquad + \chi(\{x\in U/G | q^N(x) = 1, \Delta(x) \neq 0, h^j(x) = x \bmod G\})\,.\end{aligned}$$

Hence it is enough to prove that

$$\begin{aligned}&\chi(\{x\in U/G | q(x) \neq 0, \Delta(x) = 1, h^j(x) = x \bmod G\})\\ &\quad= \chi(\{x\in U/G | q^N(x) = 1, \Delta(x) \neq 0, h^j(x) = x \bmod G\})\,.\end{aligned} \tag{5.1.2}$$

This follows from the fact that the variety

$$\{(x,z)\in U/G\times\mathbf{C}^\times | q(x)\neq 0, \Delta(x) = 1, h^j(x) = x \bmod G, z^{2N} = q^N(x)\}$$

is isomorphic to the variety

$$\{(x,z)\in U/G\times\mathbf{C}^\times | q^N(x) = 1, \Delta(x)\neq 0, h^j(x) = x \bmod G, z^{2N} = \Delta(x)\}$$

through the morphism $(x, z) \mapsto (xz^{-1}, z^{-1})$, whose inverse is given by the same formula, since the two varieties are tale covers of degree $2N$ of the two varieties occuring in (5.1.2). □

Proposition 5.2. *We have*

$$[(R\Delta_{|B_0!}\mathbf{C})^G_{\bar{\eta}_0}] = [(R\Delta_{|B_0!}\mathbf{C})^G_{\bar{\eta}_\infty}] = [(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_\infty}]$$

in K_{I_0}.

Proof. Let us prove the equality $[(R\Delta_{|B_0!}\mathbf{C})^G_{\bar{\eta}_\infty}] = [(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_\infty}]$. Recall that the polar divisor of Δ in $X_{\bar{\mathcal{A}}_G}$ has only one irreducible component, the strict transform $\tilde{H}_\infty$ of the hyperplane at infinity H_∞. On the other hand, the strict transforms of the divisors $\bar{Q}$ and $\bar{Q}_0$ (respectively the closure of the locus of $q = 1$ and $q = 0$ in $\mathbf{P}$) have the same intersection with $\tilde{H}_\infty$. Hence, if one computes $[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_\infty}]$ applying Proposition 3.7 to the resolution $X_{\bar{\mathcal{A}}_G,\bar{Q}} \to \bar{Q}$, defined in 2.6, one gets the same result as if one does the similar calculation for $[(R\Delta_{|B_0!}\mathbf{C})^G_{\bar{\eta}_\infty}]$. To prove the equality $[(R\Delta_{|B_0!}\mathbf{C})^G_{\bar{\eta}_0}] = [(R\Delta_{|B_0!}\mathbf{C})^G_{\bar{\eta}_\infty}]$, one remarks that Δ induces a G-equivariant fibration $B_0(\mathbf{C}) \to \mathbf{C}^\times$, hence $(R\Delta_{|B_0!}\mathbf{C})_{\bar{\eta}_0}$ and $(R\Delta_{|B_0!}\mathbf{C})_{\bar{\eta}_\infty}$ are already isomorphic as complexes of sheaves with G-action. □

Remark 5.3. It seems quite likely that $(R\Delta_{|B_0!}\mathbf{C})_{\bar{\eta}_\infty}$ and $(R\Delta_{|B!}\mathbf{C})_{\bar{\eta}_\infty}$ are already isomorphic as complexes of sheaves with G-action.

Now we are able to deduce the following result:

Theorem 5.4 (Formula of the Complement). *The equality*

$$[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_\infty}] + [(Rq^N_{|U!}\mathbf{C})^G_{\bar{\eta}_0}] = M_G$$

holds in K_{I_0}.

Proof. Follows directly from Lemma 5.1 and Proposition 5.2. □

6. Calculation of $[(Rq_{|U!}\Delta^*\mathcal{L}_\chi)^G_{\bar{\eta}_0}]$

We first begin by proving the following general result:

Proposition 6.1. *Let f in $\mathbf{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree N. Let G be a finite subgroup of* $\mathrm{GL}(\mathbf{R}^n)$, *and let q be a positive definite quadratic form on $\mathbf{R}^n$ which is G-invariant. We assume that* $f \circ \sigma = \det(\sigma) f$ *for every σ in G. We set $\Delta = f^2$, we denote by U the complement in $\mathbf{C}^n$ of the hypersurface* $\Delta = 0$, *and by Q the quadric $q = 1$ in $\mathbf{C}^n$. We set $B := Q \cap U$ and we denote by ϕ the unique character of order 2 of I_0. Assume that*

$$H^i_c(B, \Delta^*\mathcal{L}_\chi) = 0 \quad \textit{for } i \neq n - 1 \qquad (6.1.1)$$

and

$$\dim H_c^{n-1}(B, \Delta^* \mathcal{L}_\chi)^G = 1\,, \tag{6.1.2}$$

for almost all characters χ of finite order of I_0. Then, for almost all such χ, we have

$$[(Rq_{|U!}\Delta^* \mathcal{L}_\chi)^G_{\bar\eta_0}] = (-1)^{n-1} V_{\phi^{n+N}\chi^N} \tag{6.1.3}$$

in the Grothendieck group K_{I_0}. Here "almost all" means "outside a finite set".

Proof. Denote by $(Rq_{|U!}\Delta^* \mathcal{L}_{\phi\chi})^{\det}$ the part of $(Rq_{|U!}\Delta^* \mathcal{L}_{\phi\chi})$ on which the G-action is given by multiplication by the determinant. By a direct adaptation of the proof of [8, Lemma 2.3.1(1)], we get an isomorphism

$$(Rq_{|U!}\Delta^* \mathcal{L}_\chi)^G \simeq (Rq_{|U!}\Delta^* \mathcal{L}_{\phi\chi})^{\det}\,. \tag{6.1.4}$$

Thus, it suffices to show that

$$[(Rq_{|U!}\Delta^* \mathcal{L}_\chi)^{\det}_{\bar\eta_0}] = (-1)^{n-1} V_{\phi^n\chi^N}\,.$$

Now consider the compact set $Q(\mathbf{R}) = Q \cap \mathbf{R}^n$ and set $C := Q(\mathbf{R}) \cap U$. Since $Q(\mathbf{R})$ is compact and $\mathcal{L}_\chi$ is constant on $\Delta(C)$, the set C determines a cycle class $[C]_\chi$ in

$$H_{n-1}^{\Delta-\text{proper}}(B, \Delta^* \mathcal{L}_\chi) \simeq H^{n-1}_{\Delta-\text{proper}}(B, \Delta^* \mathcal{L}_\chi)$$

which is non zero since

$$\int_C \Delta^s \frac{dx}{dq} \neq 0$$

for every $s > 0$. We remark that, for σ in G, we have

$$\sigma([C]_\chi) = \det(\sigma)[C]_\chi\,,$$

because $[C]_\chi$ is induced by an element of $H_{n-1}(Q(\mathbf{R}), \mathbf{R})$ on which G acts by multiplication with the determinant. For almost all χ, the canonical morphism

$$H_c^{n-1}(B, \Delta^* \mathcal{L}_\chi)^{\det} \longrightarrow H^{n-1}_{\Delta-\text{proper}}(B, \Delta^* \mathcal{L}_\chi)^{\det}$$

is an isomorphism by [14, Proposition 4.2.7]. Note that $H_c^{n-1}(B, \Delta^* \mathcal{L}_\chi)^{\det}$ has rank one, for almost all χ, because of (6.1.2) and (6.1.4). Thus, when χ is general enough, the cycle class $[C]_\chi$ is a generator of $H_c^{n-1}(B, \Delta^* \mathcal{L}_\chi)^{\det}$. Choose a topological generator ϱ of I_0. It is enough to prove that

$$\varrho([C]_\chi) = (-1)^n \chi^N(\varrho)[C]_\chi\,.$$

Now remark that the map $x \mapsto \exp(2\pi i\theta/2)x$, with $\theta \in [0, 1]$, is a realization of the monodromy of q which is induced by $-\mathrm{Id}_{\mathbf{C}^n}$. Let χ be the character sending ϱ to $\exp(2\pi i a/k)$. Since $\Delta(\exp(2\pi i\theta/2)x) = \exp(2\pi i N\theta)(\Delta(x))$, we obtain that

$$\varrho([C]_\chi) = \exp(2\pi i a N/k) \det(-\mathrm{Id}_{\mathbf{C}^n})[C]_\chi = \chi^N(\varrho)(-1)^n[C]_\chi\,,$$

and the result follows. □

Proposition 6.2. *Assume we are in the Coxeter setting* 4.1. *There exists integers* $\bar{a}$ *and* $\bar{b}$ *satisfying* $\bar{a}+\bar{b}=(-1)^{n-1}$, *such that the following relations hold*:

(1) $[(Rq_{|U!}\mathbf{C})^G_{\bar{\eta}_0}] = \bar{a}V_1 + \bar{b}V_\phi = (\bar{a}-\bar{b})V_1 + \bar{b}V_2$.
(2) *For every character* χ *of* I_0, *we have* $[(Rq_{|U!}\Delta^*\mathcal{L}_\chi)^G_{\bar{\eta}_0}] = \bar{a}V_{\chi^N} + \bar{b}V_{\phi\chi^N}$.
(3) $[(Rq^N_{|U!}\mathbf{C})^G_{\bar{\eta}_0}] = (\bar{a}-\bar{b})V_N + \bar{b}V_{2N}$.

Proof. Since q is homogeneous of degree 2 and Δ is homogeneous, the map $-\mathrm{Id}_{\mathbf{C}^n}$ induces the monodromy action on $(Rq_{|U!}\mathbf{C})^G_{\bar{\eta}_0}$. This proves the existence of $\bar{a}$ and $\bar{b}$ in (1). That $\bar{a}+\bar{b}=(-1)^{n-1}$ follows directly form Proposition 2.4, since the virtual rank of $[(Rq_{|U!}\mathbf{C})^G_{\bar{\eta}_0}]$ equals $\chi(B)/|G|$. To prove (2), we first show that

$$[(Rq_{|U!}\Delta^*\mathcal{L}_\chi)^G_{\bar{\eta}_0}] = [(Rq_{|U!}q^{N*}\mathcal{L}_\chi)^G_{\bar{\eta}_0}]\,. \tag{6.2.1}$$

To show (6.2.1), we use the fact that 3.5(1) and 3.5(3) for $|J_s|>1$ remain valid when the constant sheaf is replaced by a local system on U with G-action (see [6, Remarque 3.4.2]). Moreover, this remains valid in the more general situation of Proposition 3.7. We apply this to the rational map $g: X_{\bar{\mathcal{A}}_G} \to \mathbf{P}^1$ induced by q. In this case, using the notation of 3.3, we have $E=\widetilde{H}\cup\widetilde{Q}$ where $\widetilde{H}$ is the strict transform in $X_{\bar{\mathcal{A}}_G}$ of the exceptional divisor H of the blow up $h_1: X_1 \to V$ of 0 in V, and $\widetilde{Q}$ is the strict transform in $X_{\bar{\mathcal{A}}_G}$ of the locus of $q=0$ in V. Since the Euler characteristic of a complex algebraic variety with a free $\mathbf{G}_m$-action is zero (see, e.g. [4]), we may sum in 3.5(1) only over strata lying inside $\widetilde{H}$. In fact, because of 3.5(3), we may even only sum over strata lying inside $\widetilde{H}\setminus\bigcup_{E_i\neq\widetilde{H}}E_i \simeq H\setminus S$, where S is the strict transform in X_1 of the locus of $\Delta q=0$. Equality (6.2.1) follows now directly from the above discussion and the fact that $h_1^*q^{N*}\mathcal{L}_\chi$ and $h_1^*\Delta^*\mathcal{L}_\chi$ are locally isomorphic, as local systems with G-action, on a neighbourhood of $H\setminus S$ in X_1.

But $q^{N*}\mathcal{L}_\chi \simeq q^*\mathcal{L}_{\chi^N}$, hence

$$[(Rq_{|U!}q^{N*}\mathcal{L}_\chi)^G_{\bar{\eta}_0}] = [(Rq_{|U!}(\mathbf{C})\otimes\mathcal{L}_\chi)^G_{\bar{\eta}_0}]$$

by the projection formula, and

$$[(Rq_{|U!}\Delta^*\mathcal{L}_\chi)^G_{\bar{\eta}_0}] = [(Rq_{|U!}\mathbf{C})^G_{\bar{\eta}_0}]\otimes V_\chi\,,$$

which shows that (2) follows from (1). Since $q^N=\pi_N\circ q$, (3) follows from (1). $\square$

We now determine the exact value of $\bar{a}$ and $\bar{b}$.

Theorem 6.3. *Assume we are in the Coxeter setting* 4.1. *Then, for every character* χ *of* I_0, *we have*

$$[(Rq_{|U!}\Delta^*\mathcal{L}_\chi)^G_{\bar{\eta}_0}] = (-1)^{n-1}V_{\phi^{n+N}\chi^N}$$

in the Grothendieck group K_{I_0}.

Proof. Let us check that conditions (6.1.1) and (6.1.2) are verified for almost all χ. Consider the open immersion $j : B \hookrightarrow X_{\bar{\mathcal{A}}_G,\bar{Q}}$. Since the support of the divisor of Δ on $X_{\bar{\mathcal{A}}_G,\bar{Q}}$ is exactly $X_{\bar{\mathcal{A}}_G,\bar{Q}} \setminus B$, the canonical morphism

$$Rj_!((\Delta^*\mathcal{L}_\chi)_{|B}) \longrightarrow Rj_*((\Delta^*\mathcal{L}_\chi)_{|B})$$

is an isomorphism for almost all χ, hence the canonical morphism

$$H^i_c(B, \Delta^*\mathcal{L}_\chi) \longrightarrow H^i(B, \Delta^*\mathcal{L}_\chi)$$

is an isomorphism for almost all χ. Since B is affine, $H^i(B, \Delta^*\mathcal{L}_\chi)$ is zero for $i > n-1$, hence, by Poincaré duality, it follows that (6.1.1) is verified for almost all χ. For such a χ, the rank of $H^{n-1}_c(B, \Delta^*\mathcal{L}_\chi)^G$ is equal to $(-1)^{n-1}$ times the Euler characteristic of B/G, so (6.1.2) follows from Proposition 2.4. Now the result follows by putting together Propositions 6.1 and 6.2. $\square$

7. From Macdonald Integrals to Monodromy

7.1. In this section we shall work in the framework of 1.4.

Set $S := \{x \in \mathbf{R}^n | q(x) = 1\}$ and observe that

$$\sqrt{q(\ell_i)} = \mathrm{Max}_{x\in S}\ell_i(x)\,. \tag{7.1.1}$$

Before explaining the relations with monodromy, let us first derive the following interesting consequence of Macdonald's formula (1.5.2), which is used in the paper [8]. We do not know a direct proof of this result.

Theorem 7.2. *We have*

$$\mathrm{Max}_{x\in S}\Delta(x) = \kappa\frac{\prod_{i=1}^n d_i^{d_i}}{N^N}\,.$$

Proof. Since, on a compact subset of $\mathbf{R}^n$, the L^∞-norm is the limit of the L^p-norms, $p \mapsto \infty$, we have

$$\mathrm{Max}_{x\in S}\Delta(x) = \lim_{s\to+\infty}\left(\int_S \Delta(x)^s\frac{|dx|}{|dq|}\right)^{1/s}\,. \tag{7.2.1}$$

On the other hand, by homogeneity,

$$\int_S \Delta(x)^s\frac{|dx|}{|dq|} = \frac{1}{\Gamma(Ns+\frac{n}{2})}\int_{\mathbf{R}^n}\Delta(x)^s e^{-q(x)}dx\,. \tag{7.2.2}$$

Hence we deduce from (1.5.2), that

$$\mathrm{Max}_{x\in S}\Delta(x) = \kappa\lim_{s\to+\infty}\left(\frac{1}{\Gamma(Ns+\frac{n}{2})}\prod_{i=1}^n\frac{\Gamma(d_is+1)}{\Gamma(s+1)}\right)^{1/s}\,. \tag{7.2.3}$$

The result follows now from Stirling's formula $\Gamma(x+1) \simeq \sqrt{2\pi} x^{x+1/2} e^{-x}$ and the relation $\sum_{i=1}^{n}(d_i - 1) = N$. □

Remark 7.3. Theorem 3.3 in [8] follows directly from Theorem 7.2, since the morphism $\Delta_{|B} : B \to \mathbf{G}_m$ has a compactification whose restriction to the locus at infinity is analytically trivial locally at each point of that locus, and because $\Delta_{|B}$ has only non degenerate critical points, that are all conjugate under G. As compactification one takes for instance the projection onto $\mathbf{G}_m$ from the closure in $X_{\bar{\mathcal{A}}_G, \bar{Q}} \times \mathbf{G}_m$ of the graph of $\Delta_{|B}$ (see Sec. 2.6). For this it is essential that the support of the divisor of Δ on $X_{\bar{\mathcal{A}}_G, \bar{Q}}$ is exactly $X_{\bar{\mathcal{A}}_G, \bar{Q}} \setminus B$. To prove the assertion about the critical points of $\Delta_{|B}$, it is enough to show that $\Delta_{|B}$ has only isolated critical points, because then $(-1)^{n-1}\chi(B)$ is equal to the sum of the Milnor numbers of the critical points of $\Delta_{|B}$. But $|G| = (-1)^{n-1}\chi(B)$ by Proposition 2.4, and there are at least $|G|$ critical points because the action of G on B is free. To see that $\Delta_{|B}$ has only isolated critical points, note that the zero locus Z of the section $d\Delta/\Delta$ of $\Omega^1_{X_{\bar{\mathcal{A}}_G, \bar{Q}}}(\log(X_{\bar{\mathcal{A}}_G, \bar{Q}} \setminus B))$ is closed in the proper variety $X_{\bar{\mathcal{A}}_G, \bar{Q}}$, but contained in the affine variety B, hence Z is just a finite set.

7.4. In their paper [14], Loeser and Sabbah gave a general formula for computing the determinant of a matrix whose entries are integrals of algebraic differential forms multiplied by a product of complex powers of polynomials. The gamma factors that appear in this formula are described in terms of monodromies associated with the family of polynomials. The proof involved the computation of the determinant of a complex of twisted differential forms introduced by K. Aomoto. The computation of such a determinant of a complex of twisted differential forms was carried out independently (and slightly before) by Anderson in [3]. Since stating the general formula would lead us to far away from the core of the present work, we shall only quote the result in [14] in a very special case, which will be enough for what we need in this paper.

Theorem 7.5. *Let X be a smooth connected algebraic variety over $\mathbf{R}$ of dimension n, and let $f : X \to \mathbf{G}_{m,\mathbf{R}}$ be a morphism of real algebraic varieties. Let ω be a global section of Ω^n_X and let Γ be a representative of an element in $H_n^{f-\mathrm{proper}}(X(\mathbf{R}), \mathbf{R})$, such that $f(\Gamma)$ is a bounded subset of $\mathbf{R}_+$. Assume the following assumptions hold*:

(1) *The cohomology groups $H^i(X(\mathbf{C}), f^*\mathcal{L}_\chi)$ are zero for $i \neq n$ and almost all characters χ of finite order of I_0.*
(2) *The Euler characteristic $\chi(X(\mathbf{C}))$ is equal to $(-1)^n$.*

Then

$$\int_\Gamma f^s \omega = h(s) c^s \prod_d \Gamma(ds)^{\alpha(d)} \,,$$

where c is a (non zero) positive real number[a], $h(s)$ *is a non zero rational function in* $\mathbf{C}(s)$ *and* $\alpha(d)$ *is defined by*

$$\prod_d (1 - T^d)^{\alpha(d)(-1)^n} = \frac{Z_0(T)}{Z_\infty(T)},$$

with $Z_0(T)$ *and* $Z_\infty(T)$ *respectively the zeta function of the monodromy action around* 0 *and* ∞ *on* $H_c^{\cdot}(f^{-1}(t), \mathbf{C})$ *for* t *a generic point of* $\mathbf{C}^\times$: *with notations of* 3.2, $Z_0(T) = Z([Rf_!(\mathbf{C})_{\bar{\eta}_0}])$ *and* $Z_\infty(T) = Z([Rf_!(\mathbf{C})_{\bar{\eta}_\infty}])$.

Proof. Indeed, this follows from Theorem 4.2.10 and the remark following it in [14]. □

Now we can deduce the following statement from formula (1.5.2), Theorem 7.5, and the material in Sec. 6.

Theorem 7.6. *The equality*

$$[(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_0}] - [(R\Delta_{|B!}\mathbf{C})^G_{\bar{\eta}_\infty}] - [(Rq^N_{|U!}\mathbf{C})^G_{\bar{\eta}_0}] = (-1)^n \sum_{i=1}^n (V_{\phi^{d_i}} \otimes V_{d_i} - V_\phi)$$

holds in K_{I_0}.

Proof. We first write

$$\begin{aligned} I\left(s + \frac{1}{2}\right) &= \Gamma\left(Ns + \frac{N+n}{2}\right) \int_S \Delta(x)^{s+\frac{1}{2}} \frac{|dx|}{|dq|} \\ &= |G|\Gamma\left(Ns + \frac{N+n}{2}\right) \int_{\pi(S)} \Delta(y)^s \frac{|dy|}{|dy_1|}, \end{aligned} \qquad (7.6.1)$$

considering the morphism $\pi : \mathbf{C}^n \to \mathbf{C}^n/G = \mathbf{C}^n$ sending x to y with $y_1 = q(x)$. By the proof of Theorem 6.3 and by Proposition 2.4, we may apply Theorem 7.5 to $X = B/G$, $\Gamma = \pi(S)$ and f the morphism induced by Δ. Comparing with (1.5.2), one deduces the relation (remind the duplication formula $\Gamma(x + \frac{1}{2}) = \sqrt{\pi} 2^{-2x+1} \frac{\Gamma(2x)}{\Gamma(x)}$)

$$\frac{Z_0(T)}{Z_\infty(T)} = (1 - (-1)^{\frac{N+n}{2}} T^N)^{(-1)^n} \left(\prod_{i=1}^n \frac{1 - (-T)^{d_i}}{1 - (-T)} \right)^{(-1)^{n-1}}. \qquad (7.6.2)$$

The result follows by rewriting everything in terms of virtual representations of I_0, using the fact that, by Theorem 6.3 and Proposition 6.2(3), $(1 - (-1)^{\frac{N+n}{2}} T^N)^{(-1)^n}$ is the zeta function of the monodromy action around 0 on the cohomology of the Milnor fiber of $q^N_{|U}$. □

[a]Explicitely computed in [14].

8. Conclusion

Using Theorems 4.2 and 5.4, one sees that Theorems 1.3 and 7.6 are in fact equivalent. Hence our proof of Theorem 7.6 yields a new proof of Theorem 1.3 and shows that Theorem 1.3 is in fact equivalent to knowing the precise form of the gamma factors in Macdonald's formula.

For the convenience of the reader, let us indicate for each result which has not been proved in [8] the precise place in the present paper where it is proved: assertions (1), (2), (3) and (4) in [8, Proposition 3.2.1] are proved respectively in Proposition 5.2, Theorem 5.4, Theorem 4.2 and Proposition 6.2. Proposition 3.2.3 of [8] is proved in Theorem 6.3, Theorem 3.3 of [8] in Remark 7.3, formula (0.5) of [8] in Theorem 7.2, the assertion three lines above Lemma 4.3.3 of [8] in Proposition 2.4.

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The present work was done around 1992, at about the same time as the material in [7, 8]. We apologize for the long delay we have taken to finally write it down.

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