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Motivic Integration, Quotient Singularities and the McKay Correspondence

JAN DENEF[1] and FRANÇOIS LOESER[2]
[1] *University of Leuven, Department of Mathematics, Celestijnenlaan 200B, 3001 Leuven, Belgium. e-mail: jan.denef@wis.kuleuven.ac.be*
[2] *Département de mathématiques et applications, École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France (UMR 8553 du CNRS). e-mail: François.Loeser@ens.fr*

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Abstract. The present work is devoted to the study of motivic integration on quotient singularities. We give a new proof of a form of the McKay correspondence previously proved by Batyrev. The paper contains also some general results on motivic integration on arbitrary singular spaces.

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Introduction

Let X be an algebraic variety, not necessarily smooth, over a field k of characteristic zero. We denote by $\mathcal{L}(X)$ the k-scheme of formal arcs on X: K-points of $\mathcal{L}(X)$ correspond to formal arcs $\operatorname{Spec} K[[t]] \to X$, for K any field containing k. In a recent paper [8], we developed an integration theory on the space $\mathcal{L}(X)$ with values in $\widehat{\mathcal{M}}$, a certain ring completion of the Grothendieck ring $\mathcal{M}$ of algebraic varieties over k (the definition of these rings is recalled in Section 1.9), based on ideas of Kontsevich [12]. In the most interesting cases, the integrals we consider belong to a much smaller ring $\overline{\mathcal{M}}_{\mathrm{loc}}[((\mathbf{L}-1)/(\mathbf{L}^i-1))_{i\geqslant 1}]$, on which the usual Euler characteristic and Hodge polynomial may be extended in a natural way to an Euler characteristic Eu and a Hodge polynomial H belonging respectively to $\mathbf{Q}$ and the ring

$$\mathbf{Z}[u,v][(uv)^{-1}]\left[\left(\frac{uv-1}{(uv)^i-1}\right)_{i\geqslant 1}\right].$$

When X is smooth and one considers the total measure of $\mathcal{L}(X)$, these invariants reduce to the usual Euler characteristic and Hodge polynomial, but in general one obtains interesting new invariants (see [2,3,5,8,18]).

When X is a normal variety with at most Gorenstein canonical singularities, one can use the canonical class to define a measure $\mu^{\mathrm{Gor}}(A)$ for certain subsets A of $\mathcal{L}(X)$. Now assume X is the quotient of the affine space $\mathbf{A}_k^n$ by a finite subgroup G of order d of $\mathrm{SL}_n(k)$. We make the assumption k contains all dth roots of unity. We denote by $\mathcal{L}(X)_0$ the set of arcs whose origin is the point 0 in X. One of the main results of the present paper is Theorem 3.6, which expresses $\mu^{\mathrm{Gor}}(\mathcal{L}(X)_0)$ in terms of representation theoretic weights $w(\gamma)$ of the conjugacy classes of elements γ of G, defined as

$$w(\gamma) := \sum_{1 \leqslant i \leqslant n} e_{\gamma,i}/d,$$

with $1 \leqslant e_{\gamma,i} \leqslant d$ and $\xi^{e_{\gamma,i}}$ the eigenvalues of γ for $i = 1, \ldots, n$, ξ being a fixed primitive dth root of unity in k. More precisely, the image of $\mu^{\mathrm{Gor}}(\mathcal{L}(X)_0)$ is equal to that of $\sum_{[\gamma]\in\mathrm{Conj}(G)} \mathbf{L}^{-w(\gamma)}$ in a certain quotient $\widehat{\mathcal{M}}_/$ of $\widehat{\mathcal{M}}$, with $\mathbf{L}$ the class of the affine line. The quotient $\widehat{\mathcal{M}}_/$ is defined by requiring that the class of a quotient of a vector space V by a finite group acting linearly should be that of V. This condition is mild enough to guarantee that $\mu^{\mathrm{Gor}}(\mathcal{L}(X)_0)$ and $\sum_{[\gamma]\in\mathrm{Conj}(G)} \mathbf{L}^{-w(\gamma)}$ have the same image in $\widehat{K}_0(\mathrm{CHM}_k)$, an appropriate completion of $K_0(\mathrm{CHM}_k)$, the Grothendieck group of the pseudo-Abelian category of Chow motives over k, and in particular have the same Hodge polynomial and Euler characteristic. This result – at least for the Hodge realization – is due to Batyrev [6] and implies, when X has a crepant resolution, a form of the McKay correspondence which has been conjectured by Reid [16] and proved by Batyrev [6].

The aim of the present paper is to present an alternative proof of Batyrev's result and also to develop further, some basic properties of motivic integration which were not covered in [8]. Though Batyrev also uses integration on spaces of arcs, the approach we follow here, which was inspired to us by Maxim Kontsevich, is somewhat different. One of the main differences is that we are able to work directly on the singular space X instead of going to desingularizations. This allows us to have a more local approach, in the sense that we can directly calculate the part of the motivic integral coming from each conjugacy class. More precisely, for each element γ in the group G, we consider $\mathcal{L}(X)_{0,\gamma}^g$, the set of arcs φ in $\mathcal{L}(X)_0$, which are not contained in the discriminant and may be lifted in $\mathcal{L}(\mathbf{A}_k^n)$ to a fractional arc $\tilde{\varphi}(t^{1/d})$ such that $\tilde{\varphi}(\xi t^{1/d}) = \gamma\,\tilde{\varphi}(t^{1/d})$. We prove that the image of $\mu^{\mathrm{Gor}}(\mathcal{L}(X)_{0,\gamma}^g)$ in $\widehat{\mathcal{M}}_/$ is equal to that of $\mathbf{L}^{-w(\gamma)}$.

Let us now briefly review the content of the paper. In Section 1, we recall some material on semi-algebraic geometry over $k((t))$ from [8]. In fact, we need to generalize slightly semi-algebraic geometry as developed in [8] to '$k[t]$-semi-algebraic geometry' which allows expressions involving t, since $k[t]$-morphisms naturally appear in Section 2. Fortunately, this is quite harmless, since most proofs remain the same. This material on $k[t]$-semi-algebraic geometry might be useful elsewhere. One of the main technical difficulties of the section is Theorem 1.16 were we extend the crucial change of variables formula [8] to certain maps which are not birational.

Section 2 is the heart of the paper, namely the study of the local action of the group G on arcs. We are then able to deduce the main results in Section 3. In Section 4 we explain how one can deduce statements at the level of Chow motives and then realizations, and in Section 5 we express the main results in terms of resolutions of singularities and we explain the relation with McKay's correspondence.

Let us remark that $k[t]$-semi-algebraic sets appear quite naturally in the problem, since the set $\mathcal{L}(X)^{g}_{0,\gamma}$ is $k[t]$-semi-algebraic. Nevertheless, it is possible to avoid the use of $k[t]$-semi-algebraic geometry here, by using properties of measurable sets which are developed in the appendix, in particular the fact, proved in Theorem A.8, that the image of a measurable set under a $k[t]$-morphism, for varieties of the same dimension, is again measurable.

1. Preliminaries on Semi-algebraic Geometry and Motivic Integration

1.1. In the present paper by a variety over k, or variety, we always mean a reduced separated scheme of finite type over a field k that will be assumed to be of characteristic zero throughout the paper. If X is a variety, we shall denote by X_{sing} the singular locus of X.

1.2. For X a variety over k, we will denote by $\mathcal{L}(X)$ the *scheme of germs of arcs* on X. It is a scheme over k and for any field extension $k \subset K$ there is a natural bijection

$$\mathcal{L}(X)(K) \simeq \mathrm{Mor}_{k-\text{schemes}}(K[[t]], X)$$

between the set of K-rational points of $\mathcal{L}(X)$ and the set of germs of arcs with coefficients in K on X. We will call K-rational points of $\mathcal{L}(X)$, for K a field extension of k, arcs on X, and $\varphi(0)$ will be called the origin of the arc φ. More precisely, the scheme $\mathcal{L}(X)$ is defined as the projective limit $\mathcal{L}(X) := \varprojlim \mathcal{L}_n(X)$ in the category of k-schemes of the schemes $\mathcal{L}_n(X)$ representing the functor

$$R \longmapsto \mathrm{Mor}_{k-\text{schemes}}(R[t]/t^{n+1}R[t], X)$$

defined on the category of k-algebras. (The existence of $\mathcal{L}_n(X)$ is well known (cf. [8]) and the projective limit exists since the transition morphisms are affine.) We shall denote by π_n the canonical morphism, corresponding to truncation of arcs, $\pi_n : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$. The schemes $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ will always be considered with their reduced structure. If W is a subscheme of X, we set $\mathcal{L}(X)_W = \pi_0^{-1}(W)$.

Since, in Section 2, we shall lift arcs to Galois covers, we also have to consider 'ramified' arcs, so we define similarly, for $d \geqslant 1$ an integer, the scheme $\mathcal{L}^{1/d}(X)$ as the projective limit $\mathcal{L}^{1/d}(X) := \varprojlim \mathcal{L}_n^{1/d}(X)$ in the category of k-schemes of the schemes $\mathcal{L}_n^{1/d}(X)$ representing the functor

$$R \longmapsto \mathrm{Mor}_{k-\text{schemes}}(R[t^{1/d}]/t^{(n+1)/d}R[t^{1/d}], X)$$

defined on the category of k-algebras. Of course the schemes $\mathcal{L}^{1/d}(X)$ are all isomorphic to $\mathcal{L}(X)$. We shall still denote by π_n the canonical morphism π_n: $\mathcal{L}^{1/d}(X) \longrightarrow \mathcal{L}_n^{1/d}(X)$ and for W a subscheme of X, we set $\mathcal{L}^{1/d}(X)_W = \pi_0^{-1}(W)$.

The above definitions extend to the case where X is a reduced and separated scheme of finite type over $k[t]$. For n in $\mathbf{N}$, one defines the k-scheme $\mathcal{L}_n(X)$ as representing the functor

$$R \mapsto \mathrm{Mor}_{k[t]-\mathrm{schemes}}(\mathrm{Spec}\, R[t]/t^{n+1}R[t], X),$$

defined on the category of k-algebras, and one sets $\mathcal{L}(X) := \varprojlim \mathcal{L}_n(X)$. The existence of $\mathcal{L}_n(X)$ is well known, cf. [7] p. 276, and again the projective limit exists since the transition morphisms are affine. We shall still denote by π_n the canonical morphism $\mathcal{L}(X) \to \mathcal{L}_n(X)$.

1.3. Let X and Y be k-varieties. A function h: $\mathcal{L}(Y) \to \mathcal{L}(X)$ will be called a $k[t]$-morphism if it is induced by a morphism of $k[t]$-schemes $Y \otimes_k k[t] \to X \otimes_k k[t]$. We shall denote by the same symbol a $k[t]$-morphism and the corresponding morphism of $k[t]$-schemes.

1.4. We now introduce the concept of semi-algebraic and $k[t]$-semi-algebraic subsets of the space of arcs $\mathcal{L}(X)$. The main motivation for introducing such objects is that in general being a subset of $\mathcal{L}(X)$ defined by (Boolean combination of) algebraic conditions is not a property which is conserved by taking images, i.e. Theorem 1.5 and Proposition 1.7 (1) would not remain true when replacing 'semi-algebraic' by '(Boolean combination of) algebraic'.

From now on we will denote by $\bar{k}$ a fixed algebraic closure of k, and by $\bar{k}((t))$ the fraction field of $\bar{k}[[t]]$, where t is one variable. Let $x_1, \ldots, x_m$ be variables running over $\bar{k}((t))$ and let $\ell_1, \ldots, \ell_r$ be variables running over $\mathbf{Z}$. A *semi-algebraic* (resp. $k[t]$-*semi-algebraic*) condition $\theta(x_1, \ldots, x_m; \ell_1, \ldots, \ell_r)$ is a finite boolean combination of conditions of the form

(1) $\mathrm{ord}_t f_1(x_1, \ldots, x_m) \geqslant \mathrm{ord}_t f_2(x_1, \ldots, x_m) + L(\ell_1, \ldots, \ell_r)$,

(2) $\mathrm{ord}_t f_1(x_1, \ldots, x_m) \equiv L(\ell_1, \ldots, \ell_r) \pmod{d}$,

and

(3) $h(\overline{ac}(f_1(x_1, \ldots, x_m)), \ldots, \overline{ac}(f_{m'}(x_1, \ldots, x_m))) = 0$,

where f_i are polynomials with coefficients in k (resp. f_i are polynomials with coefficients in $k[t]$), h is a polynomial with coefficients in k, L is a polynomial of degree $\leqslant 1$ over $\mathbf{Z}$, $d \in \mathbf{N}$, and $\overline{ac}(x)$ is the coefficient of lowest degree of x in $\bar{k}((t))$ if $x \neq 0$, and is equal to 0 otherwise. Here we use the convention that $\infty + \ell = \infty$ and $\infty \equiv \ell \bmod d$, for all $\ell \in \mathbf{Z}$. In particular, the algebraic (resp. $k[t]$-algebraic) condition $f(x_1, \ldots, x_m) = 0$ is a semi-algebraic (resp. $k[t]$-semi-algebraic) condition, for f a polynomial over k (resp. $k[t]$).

A subset of $\bar{k}((t))^m \times \mathbf{Z}^r$ defined by a semi-algebraic (resp. $k[t]$-semi-algebraic) condition is called *semi-algebraic* (resp. $k[t]$-*semi-algebraic*). One defines similarly semi-algebraic and $k[t]$-semi-algebraic subsets of $K((t))^m \times \mathbf{Z}^r$ for K an algebraically closed field containing $\bar{k}$.

A function $\alpha : \bar{k}((t))^m \times \mathbf{Z}^n \to \mathbf{Z}$ is called *simple* (resp. $k[t]$-*simple*) if its graph is semi-algebraic (resp. $k[t]$-semi-algebraic).

We will use in an essential way the following result on quantifier elimination due to J. Pas [15].

THEOREM 1.5. *If θ is a semi-algebraic (resp. $k[t]$-semi-algebraic) condition, then*

$$(\exists x_1 \in \bar{k}((t)))\ \theta(x_1, \ldots, x_m; \ell_1, \ldots, \ell_r)$$

is semi-algebraic (resp. $k[t]$-semi-algebraic). Furthermore, for any algebraically closed field K containing $\bar{k}$,

$$(\exists x_1 \in K((t)))\ \theta(x_1, \ldots, x_m; \ell_1, \ldots, \ell_r)$$

is also semi-algebraic (resp. $k[t]$-semi-algebraic) and may be defined by the same conditions (i.e. independently of K).

1.6. Let X be an algebraic variety over k. For $x \in \mathcal{L}(X)$, we denote by k_x the residue field of x on $\mathcal{L}(X)$, and by $\tilde{x}$ the corresponding rational point $\tilde{x} \in \mathcal{L}(X)(k_x) = X(k_x[[t]])$. When there is no danger of confusion we will often write x instead of $\tilde{x}$. A *semi-algebraic family of semi-algebraic subsets* (resp. $k[t]$-*semi-algebraic family of* $k[t]$-*semi-algebraic subsets*) (for $n = 0$ a semi-algebraic subset (resp. $k[t]$-semi-algebraic subset)) A_ℓ, $\ell \in \mathbf{N}^n$, of $\mathcal{L}(X)$ is a family of subsets A_ℓ of $\mathcal{L}(X)$ such that there exists a covering of X by affine Zariski open sets U with

$$A_\ell \cap \mathcal{L}(U) = \Big\{x \in \mathcal{L}(U) \;\Big|\; \theta(h_1(\tilde{x}), \ldots, h_m(\tilde{x}); \ell)\Big\},$$

where $h_1, \ldots, h_m$ are regular functions on U and θ is a semi-algebraic condition (resp. $k[t]$-semi-algebraic condition). Here the h_i's and θ may depend on U and $h_i(\tilde{x})$ belongs to $k_x[[t]]$.

Let A be a semi-algebraic subset (resp. $k[t]$-semi-algebraic subset) of $\mathcal{L}(X)$. A function $\alpha : A \times \mathbf{Z}^n \to \mathbf{Z} \cup \{\infty\}$ is called *simple* (resp. $k[t]$-*simple*) if the family of subsets $\{x \in \mathcal{L}(X) \mid \alpha(x, \ell_1, \ldots, \ell_n) = \ell_{n+1}\}$, $(\ell_1, \ldots, \ell_{n+1}) \in \mathbf{N}^{n+1}$, is a semi-algebraic family of semi-algebraic subsets (resp. a $k[t]$-semi-algebraic family of $k[t]$-semi-algebraic subsets) of $\mathcal{L}(X)$.

We will use the following consequences of Theorem 1.5.

PROPOSITION 1.7. (1) *If X and Y are algebraic varieties over k, $f\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ is a $k[t]$-morphism and A is a $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$, then $f(A)$ is a $k[t]$-semi-algebraic subset of $\mathcal{L}(Y)$.*

(2) *If X is an algebraic variety over k and A is a $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$, then $\pi_n(A)$ is a constructible subset of $\mathcal{L}_n(X)$.*

Proof. (1) is a direct consequence of Theorem 1.5. The proof of (2) is similar to the proof of Proposition 2.3 in [8]. □

1.8. By replacing t by $t^{1/d}$ in the definition, one defines similarly semi-algebraic (resp. $k[t]$-semi-algebraic) subsets of $\mathcal{L}^{1/d}(X)$.

1.9. We denote by $\mathcal{M}$ the Abelian group generated by symbols $[S]$, for S a variety over k, with the relations $[S]=[S']$ if S and S' are isomorphic and $[S]=[S']+[S\setminus S']$ if S' is closed in S. There is a natural ring structure on $\mathcal{M}$, the product being induced by the Cartesian product of varieties, and to any constructible set S in some variety one naturally associates a class $[S]$ in $\mathcal{M}$. We denote by $\mathcal{M}_{\mathrm{loc}}$ the localization $\mathcal{M}_{\mathrm{loc}} := \mathcal{M}[\mathbf{L}^{-1}]$ with $\mathbf{L} := [\mathbf{A}^1_k]$. We denote by $F^m\mathcal{M}_{\mathrm{loc}}$ the subgroup generated by $[S]\mathbf{L}^{-i}$ with $\dim S - i \leqslant -m$, and by $\widehat{\mathcal{M}}$ the completion of $\mathcal{M}_{\mathrm{loc}}$ with respect to the filtration $F^{\cdot}$. We will also denote by $F^{\cdot}$ the filtration induced on $\widehat{\mathcal{M}}$. We denote by $\overline{\mathcal{M}}_{\mathrm{loc}}$ the image of $\mathcal{M}_{\mathrm{loc}}$ in $\widehat{\mathcal{M}}$.

1.10. In fact, for technical reasons appearing in the proof of Lemma 3.4, we shall need to consider the following quotient $\mathcal{M}_{/}$ of $\mathcal{M}$, which is defined by adding the relation

$$[V/G]=[V],$$

for every vector space V over k endowed with a linear action of a finite group G. We shall still denote by $\mathbf{L}$ the class of the affine line and, replacing $\mathcal{M}$ by $\mathcal{M}_{/}$, one defines similarly as above rings $\mathcal{M}_{\mathrm{loc}/}$ and $\widehat{\mathcal{M}}_{/}$.

1.11. Let A be a $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$. We call A *weakly stable at level* $n\in\mathbf{N}$ if A is a union of fibers of $\pi_n\colon \mathcal{L}(X)\to\mathcal{L}_n(X)$. We call A *weakly stable* if it is stable at some level n. Note that weakly stable $k[t]$-semi-algebraic subsets form a Boolean algebra. Let X, Y and F be algebraic varieties over k, and let A, resp. B, be a constructible subset of X, resp. Y. We say that a map $\pi\colon A\to B$ is a *piecewise morphism* if there exists a finite partition of the domain of π into locally closed subvarieties of X such that the restriction of π to any of these subvarieties is a morphism of schemes. We say that a map $\pi\colon A\to B$ is a *piecewise trivial fibration with fiber* F, if there exists a finite partition of B in subsets S which are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and isomorphic, as a variety over k, to $S\times F$, with π corresponding under the isomorphism to the projection $S\times F\to S$. We say that the map π is a *piecewise trivial fibration over* some constructible subset C of B, if the restriction of π to $\pi^{-1}(C)$ is a piecewise trivial fibration onto C. One defines similarly the notion of a *pievewise vector bundle of rank e*.

Let X be an algebraic variety over k of pure dimension d (in particular we assume that X is nonempty) and let A be a $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$. We call A *stable at level* $n\in\mathbf{N}$, if A is weakly stable at level n and $\pi_{m+1}(\mathcal{L}(X))\to\pi_m(\mathcal{L}(X))$

is a piecewise trivial fibration over $\pi_m(A)$ with fiber $\mathbf{A}_k^d$ for all $m \geqslant n$. We call A *stable* if it stable at some level n.

LEMMA 1.12. *Let X be an algebraic variety over k of pure dimension d, and let A be a $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$. There exists a reduced closed subscheme S of $X \otimes k[t]$, with $\dim_{k[t]} S < \dim X$, and a $k[t]$-semi-algebraic family A_i, $i \in \mathbf{N}$, of $k[t]$-semi-algebraic subsets of A such that $\mathcal{L}(S) \cap A$ and the A_i's form a partition of A, each A_i is stable at some level n_i, and*

$$\lim_{i \to \infty} (\dim \pi_{n_i}(A_i) - (n_i + 1)\, d) = -\infty.$$

Moreover, if $\alpha : \mathcal{L}(X) \to \mathbf{Z}$ is a $k[t]$-simple function, we can choose the partition such that α is constant on each A_i.

Proof. The proof is literally the same as the one of Lemma 3.1 of [8], noticing that Lemma 4.4 of [8] also holds for a closed subscheme S of $X \otimes k[t]$ with $\dim_{k[t]} S < d$. □

Let X be an algebraic variety over k of pure dimension d. Denote by $\mathbf{B}^t$ the set of all $k[t]$-semi-algebraic subsets of $\mathcal{L}(X)$, and by $\mathbf{B}_0^t$ the set of all A in $\mathbf{B}^t$ which are stable. Clearly there is a unique additive measure $\tilde{\mu}$: $\mathbf{B}_0^t \longrightarrow \mathcal{M}_{\mathrm{loc}}$ satisfying $\tilde{\mu}(A) = [\pi_n(A)]\, \mathbf{L}^{-(n+1)d}$, when A is stable at level n.

DEFINITION–PROPOSITION 1.13. Let X be an algebraic variety over k of pure dimension d. Let $\mathbf{B}^t$ be the set of all $k[t]$-semi-algebraic subsets of $\mathcal{L}(X)$. There exists a unique map μ: $\mathbf{B}^t \to \widehat{\mathcal{M}}$ satisfying the following three properties.

(1) If $A \in \mathbf{B}^t$ is stable at level n, then $\mu(A) = [\pi_n(A)]\mathbf{L}^{-(n+1)d}$.
(2) If $A \in \mathbf{B}^t$ is contained in $\mathcal{L}(S)$ with S a reduced closed subscheme of $X \otimes k[t]$ with $\dim_{k[t]} S < \dim X$, then $\mu(A) = 0$.
(3) Let A_i be in $\mathbf{B}^t$ for each i in $\mathbf{N}$. Assume that the A_i's are mutually disjoint and that $A := \bigcup_{i \in \mathbf{N}} A_i$ is $k[t]$-semi-algebraic. Then $\sum_{i \in \mathbf{N}} \mu(A_i)$ converges in $\widehat{\mathcal{M}}$ to $\mu(A)$.

Moreover we have:

(4) If A and B are in $\mathbf{B}^t$, $A \subset B$ and if $\mu(B) \in F^m\widehat{\mathcal{M}}$, then $\mu(A) \in F^m\widehat{\mathcal{M}}$.

This unique map μ is called the *motivic measure* on $\mathcal{L}(X)$ and is denoted by $\mu_{\mathcal{L}(X)}$ or μ. For A in $\mathbf{B}^t$ and α: $A \to \mathbf{Z} \cup \{\infty\}$ a $k[t]$-simple function, one defines the motivic integral

$$\int_A \mathbf{L}^{-\alpha} \mathrm{d}\mu := \sum_{n \in \mathbf{Z}} \mu(A \cap \alpha^{-1}(n))\, \mathbf{L}^{-n}$$

in $\widehat{\mathcal{M}}$, whenever the right-hand side converges in $\widehat{\mathcal{M}}$, in which case we say that $\mathbf{L}^{-\alpha}$ is integrable on A. If the function α is bounded from below, then $\mathbf{L}^{-\alpha}$ is integrable on A, because of (4).

Proof. The proof of Definition–Proposition 3.2 of [8] generalizes to the present case because Lemma 4.3 of [8] holds also for X a scheme of finite type over $k[t]$ (replacing 'dimension' by 'relative dimension'), and because Lemma 2.4 of [8] holds for 'semi-algebraic' replaced by '$k[t]$-semi-algebraic' (cf. Lemma A.3 below), both with identically the same proofs. Note that we have to replace Lemma 3.1 of [8] by Lemma 1.12. □

1.14. Let h: $\mathcal{L}(Y) \to \mathcal{L}(X)$ be a $k[t]$-morphism with Y and X of pure dimension d. Let y be a closed point of $\mathcal{L}(Y) \setminus \mathcal{L}(Y_{\mathrm{sing}})$ and denote by φ the corresponding morphism φ: $\operatorname{Spec} K[[t]] \to Y$, with K a field extension of k. We define an element $\operatorname{ord}_t \mathcal{J}_h(y)$ in $\mathbf{N} \cup \{\infty\}$, the order of the Jacobian of h at y, as follows. Consider the $K[[t]]$-module $M = \varphi^*(\Omega^d_Y)$ and set $L := M \otimes_{K[[t]]} K((t))$. Here by Ω^d we mean dth exterior power of the sheaf of differentials. The image $\tilde{M}$ of M in the $K((t))$-vector space L is a lattice of rank 1. One may also consider the image $\tilde{N}$ of the module $\varphi^* h^*(\Omega^d_{X \otimes k[t] | k[t]})$ in L. If $\tilde{N}$ is nonzero, $\tilde{N} = t^n \tilde{M}$ for some n in $\mathbf{N}$ and one sets $\operatorname{ord}_t \mathcal{J}_h(y) = n$. When $\tilde{N} = 0$, one sets $\operatorname{ord}_t \mathcal{J}_h(y) = \infty$.

Similarly assume Y is irreducible and let ω be an element in $\Omega^d_Y \otimes_k k(Y)$. Denote by Λ the $K[[t]]$-submodule of L generated by $\varphi^*(\omega)$. If Λ is nonzero, $\Lambda = t^n \tilde{M}$ for some n in $\mathbf{Z}$ and one sets $\operatorname{ord}_t \omega(y) = n$. When $\Lambda = 0$, one sets $\operatorname{ord}_t \omega(y) = \infty$.

LEMMA 1.15. *Let X and Y be k-varieties of pure dimension d and let $h : \mathcal{L}(Y) \to \mathcal{L}(X)$ be a $k[t]$-morphism. Then the function $y \mapsto \operatorname{ord}_t \mathcal{J}_h(y)$ is $k[t]$-simple on $\mathcal{L}(Y) \setminus \mathcal{L}(Y_{\mathrm{sing}})$. Similarly, if Y is irreducible and ω belongs to $\Omega^d_Y \otimes_k k(Y)$, the function $y \mapsto \operatorname{ord}_t \omega(y)$ is $k[t]$-simple on $\mathcal{L}(Y) \setminus \mathcal{L}(Y_{\mathrm{sing}})$.*

Proof. Direct verification. □

Under the preceding assumptions, we extend the functions $\operatorname{ord}_t \mathcal{J}_h(y)$ and $\operatorname{ord}_t \omega(y)$ by ∞ to a $k[t]$-simple function on $\mathcal{L}(Y)$.

THEOREM 1.16 (Change of variables formula). *Let X and Y be algebraic varieties over k, of pure dimension d. Let h: $\mathcal{L}(Y) \to \mathcal{L}(X)$ be a $k[t]$-morphism. Let A and B be $k[t]$-semi-algebraic subsets of $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, respectively. Assume that h induces a bijection between B and A. Then, for any $k[t]$-simple function α: $A \to \mathbf{Z} \cup \{\infty\}$ such that $\mathbf{L}^{-\alpha}$ is integrable on A, we have*

$$\int_A \mathbf{L}^{-\alpha} \mathrm{d}\mu = \int_B \mathbf{L}^{-\alpha \circ h - \operatorname{ord}_t \mathcal{J}_h(y)} \mathrm{d}\mu.$$

Proof. By resolution of singularities we may assume that Y is smooth. If h is induced by a proper birational morphism from Y to X, then Theorem 1.16 is a direct consequence of Lemma 3.4 of [8]. In the general case it is a direct consequence of Lemma 1.12 and the following Lemma 1.17. □

For X a variety and e in $\mathbf{N}$, we set

$$\mathcal{L}^{(e)}(X) := \mathcal{L}(X) \setminus \pi_e^{-1}(\mathcal{L}_e(X_{\text{sing}})).$$

We call a subset A of $\mathcal{L}(X)$ cylindrical at level n if $A = \pi_n^{-1}(C)$, with C a *constructible* subset of $\mathcal{L}_n(X)$. We say that A is cylindrical if it is cylindrical at some level n.

LEMMA 1.17. *Let X and Y be algebraic varieties over k, of pure dimension d, with Y smooth. Let h: $\mathcal{L}(Y) \to \mathcal{L}(X)$ be a $k[t]$-morphism. Let $B \subset \mathcal{L}(Y)$ be cylindrical and put $A = h(B)$. Assume that* $\text{ord}_t \mathcal{J}_h(\varphi)$ *has constant value $e < \infty$ for all $\varphi \in B$, and that $A \subset \mathcal{L}^{(e')}(X)$ for some e' in $\mathbf{N}$. Then A is cylindrical. Morever, if the restriction of h to B is injective, then, for $n \in \mathbf{N}$ large enough, we have the following:*

(a) *If φ and φ' belong to B and $\pi_n(h(\varphi)) = \pi_n(h(\varphi'))$, then $\pi_{n-e}(\varphi) = \pi_{n-e}(\varphi')$.*
(b) *The morphism h_{n*}: $\pi_n(B) \to \pi_n(A)$ induced by h is a piecewise trivial fibration with fiber $\mathbf{A}_k^e$.*

Proof. Let n in $\mathbf{N}$ be large enough. We may assume that B is cylindrical at level $n-e$. That A is cylindrical at level n is an easy consequence of the following assertion:

(a″) For all φ in B and x in $\mathcal{L}(X)$, with $\pi_n(h(\varphi)) = \pi_n(x)$, there exists y in $\mathcal{L}(Y)$ with $h(y) = x$ and $\pi_{n-e}(\varphi) = \pi_{n-e}(y)$ (whence $y \in B$, since B is cylindrical at level $n-e$).

The proof of (a″) is the same as the proof of assertion (a″) in Lemma 3.4 of [8]. (Note that with the notation of loc. cit. B is contained in $\Delta_{e,e'}$.) Assertion (a) is a direct consequence of (a″), taking $x = h(\varphi')$ and using the injectivity of $h_{|B}$. It remains to prove (b). Because of (a), we may assume that X and Y are affine. Let s: $\mathcal{L}_n(X) \to \mathcal{L}(X)$ be a section of the projection π_n: $\mathcal{L}(X) \to \mathcal{L}_n(X)$ such that the restriction of $\pi_{n+e} \circ s$ to $\pi_n(A)$ is a piecewise morphism. The existence of such a section has been shown in the proof of Lemma 3.4 of [8]. Since A is cylindrical at level n, $s(\pi_n(A))$ is contained in A. Let θ be the mapping

$$\theta\colon \pi_n(A) \longrightarrow B\colon \qquad x \longmapsto h^{-1}(s(x)).$$

We will prove the following assertion:

(c) The map $\pi_n \circ \theta : \pi_n(A) \to \pi_n(B)$ is a piecewise morphism.

Using (c), the proof of (b) is the same as in the proof of Lemma 3.4 in [8], except that we have to replace the assertion that θ in loc. cit. is a piecewise morphism by the slightly weaker assertion (c) above.

It only remains to prove (c). Let x be in $\pi_n(A)$ and y in $\pi_n(B)$. Using assertion (a) we see that $y = (\pi_n \circ \theta)(x)$ if and only if there exists $\tilde{y}$ in $\pi_{n+e}(B)$ such that $y = \pi_n(\tilde{y})$ and $h_{n+e*}(\tilde{y}) = \pi_{n+e}(s(x))$. Thus, the graph of the map $\pi_n \circ \theta$ is constructible and assertion (c) follows from the next lemma. □

LEMMA 1.18. *Let X and Y be algebraic varieties over k and let U and V be constructible subsets of X and Y respectively. If f: $U \to V$ is a map whose graph is a constructible subset of $X \times Y$, then f is a piecewise morphism.*

Proof. Well known. □

Remark 1.19. All the material in this section (before 1.18) generalizes to 'X and Y algebraic varieties' replaced by 'X and Y separated reduced schemes of finite type over $k[t]$'. In that case X_{sing} denotes the locus of points at which X is not smooth over $k[t]$, 'dimension' has to be replaced by 'relative dimension over $k[t]$', and in 1.17 one replaces the hypothesis 'Y smooth' by '$Y \otimes k(t)$ smooth'. Moreover one can also work with schemes over $k[[t]]$ instead of over $k[t]$, replacing everywhere $k[t]$ by $k[[t]]$. The proofs remain essentially the same, but since this is not needed in the present paper, we do not give details here.

2. Study at the Origin

2.1. Let $d \geqslant 1$ be an integer and let k be field of characteristic 0 containing all dth roots of unity. Let G be a finite subgroup of $\mathrm{GL}_n(k)$ of order d. We fix a primitive dth root of unity ξ in k. We denote by $\mathrm{Conj}(G)$ the set of conjugacy classes in G. We let G act on $\mathbf{A}_k^n$ and we consider the morphism of schemes h: $\tilde{X} = \mathbf{A}_k^n \longrightarrow X = \mathbf{A}_k^n/G$. We denote by 0 the origin in $\tilde{X}$ and X. Let $\tilde{\Delta}$ be the closed subvariety of $\tilde{X}$ consisting of the closed points having a nontrivial stabilizer and let Δ be its image in X (the discriminant). We denote by $\mathcal{L}(X)^g$ (resp. $\mathcal{L}^{1/d}(\tilde{X})^g$) the complement of $\mathcal{L}(\Delta)$ (resp. $\mathcal{L}^{1/d}(\tilde{\Delta})$) in $\mathcal{L}(X)$ (resp. $\mathcal{L}^{1/d}(\tilde{X})$), and define similarly $\mathcal{L}(X)_W^g$ (resp. $\mathcal{L}^{1/d}(\tilde{X})_W^g$) when W is a subscheme of X (resp. $\tilde{X}$).

Let φ be a geometric point of $\mathcal{L}(X)_0^g$. So φ is given by a morphism φ: $\mathrm{Spec}\, K[[t]] \to X$ with K an algebraically closed overfield of k. The generic point of the image of φ is in $X \setminus \Delta$ and the special point is 0. We can lift φ to a morphism $\tilde{\varphi}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Spec}\, K[[t^{1/d}]] & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\ \downarrow & & \downarrow h \\ \mathrm{Spec}\, K[[t]] & \xrightarrow{\varphi} & X. \end{array} \tag{2.1.1}$$

There is a unique element γ in G such that

$$\tilde{\varphi}(\xi t^{1/d}) = \gamma\, \tilde{\varphi}(t^{1/d}). \tag{2.1.2}$$

If we change $\tilde{\varphi}$ in the diagram (2.1.1), γ will be replaced by a conjugate. If we denote by $\mathcal{L}(X)_{0,\gamma}^g$ the set of φ's in $\mathcal{L}(X)_0^g$ such that there exists $\tilde{\varphi}$ satisfying (2.1.2), we have $\mathcal{L}(X)_{0,\gamma}^g = \mathcal{L}(X)_{0,\gamma'}^g$ for γ and γ' in the same conjugacy class, and we have a decomposition $\mathcal{L}(X)_0^g = \coprod \mathcal{L}(X)_{0,\gamma}^g$, for γ running over a set of representatives of the conjugacy classes.

For each γ in G, choose a basis b_γ in which the matrix of γ is diagonal, and denote by $\xi^{e_{\gamma,i}}$, the diagonal coefficients, with $1 \leqslant e_{\gamma,i} \leqslant d$, $1 \leqslant i \leqslant n$.

LEMMA 2.2. *Let γ be in G. A point $\tilde{\varphi}$ in $\mathcal{L}^{1/d}(\tilde{X})^g$ projects to a point in $\mathcal{L}(X)_{0,\gamma}^g$ if and only if it is in the G-orbit of a point in $\mathcal{L}^{1/d}(\tilde{X})^g$ of the form*

$$\tilde{\varphi}(t^{1/d}) = (t^{e_{\gamma,1}/d}\varphi_1(t), \ldots, t^{e_{\gamma,n}/d}\varphi_n(t)) \tag{2.2.3}$$

in the basis b_γ.

Proof. It follows from (2.1.2) that a point of $\mathcal{L}^{1/d}(\tilde{X})^g$ which projects to a point in $\mathcal{L}(X)_{0,\gamma}^g$ is in the G-orbit of a point of the form (2.2.3). To conclude observe that, in the basis b_γ, G-invariant polynomials are sums of monomials of the form $x_1^{m_1} \ldots x_n^{m_n}$, with d dividing $\sum_{1 \leqslant i \leqslant n} e_{\gamma,i} m_i$. □

2.3. We consider the morphism of $k[t]$-schemes

$$\tilde{\lambda}\colon \mathbf{A}^n_{k[t]} \longrightarrow X \otimes k[t] \qquad (x_1, \ldots, x_n) \longmapsto h(t^{e_{\gamma,1}/d}x_1, \ldots, t^{e_{\gamma,n}/d}x_n),$$

where $x_1, \ldots, x_n$ are the affine coordinates corresponding to the basis b_γ. This is indeed a $k[t]$-morphism, since, in the basis b_γ, G-invariant polynomials are sums of monomials of the form $x_1^{m_1} \cdots x_n^{m_n}$, with d dividing $\sum_{1 \leqslant i \leqslant n} e_{\gamma,i} m_i$. The morphism $\tilde{\lambda}$ induces a $k[t]$-morphism $\tilde{\lambda}_*\colon \mathcal{L}(\mathbf{A}^n_k) \to \mathcal{L}(X)_0$. Note that Lemma 2.2 implies that

$$\mathcal{L}(X)_{0,\gamma}^g = \tilde{\lambda}_*(\mathcal{L}(\mathbf{A}^n_k)) \cap \mathcal{L}(X)^g. \tag{2.3.4}$$

PROPOSITION 2.4. *For every γ in G, $\mathcal{L}(X)_{0,\gamma}^g$ is a $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$.*

Proof. This follows directly from (2.3.4) and Proposition 1.7 (1). □

2.5. For γ in G we denote by G_γ the centralizer of γ in G. It follows from Theorem 1.5 that $\mathcal{L}(\mathbf{A}^n_k)/G_\gamma$ is a semi-algebraic subset of $\mathcal{L}(\mathbf{A}^n_k/G_\gamma)$.

LEMMA 2.6. *The morphism $\tilde{\lambda}$ is invariant under the action of G_γ on $\mathbf{A}^n_{k[t]}$. Moreover the fibers of $\tilde{\lambda}_*$ above $\mathcal{L}(X)_{0,\gamma}^g$ are G_γ-orbits.*

Proof. The first assertion is clear because the eigenspaces of γ are invariant subspaces under the action of G_γ. Next we prove the second assertion. Let $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n)$ be in a same fiber of $\tilde{\lambda}_*$ above $\mathcal{L}(X)_{0,\gamma}^g$, and set

$$\tilde{\varphi} = (t^{e_{\gamma,1}/d}x_1, \ldots, t^{e_{\gamma,n}/d}x_n) \quad \text{and} \quad \tilde{\varphi}' = (t^{e_{\gamma,1}/d}x'_1, \ldots, t^{e_{\gamma,n}/d}x'_n).$$

Then (2.1.2) holds for $\tilde{\varphi}$, and also for $\tilde{\varphi}$ replaced by $\tilde{\varphi}'$. There exists σ in G such that $\tilde{\varphi}' = \sigma(\tilde{\varphi})$. Hence, (2.1.2) also holds for $\tilde{\varphi}$ and γ replaced by $\sigma(\tilde{\varphi}) = \tilde{\varphi}'$ and $\sigma\gamma\sigma^{-1}$ respectively. Thus $\sigma = \sigma\gamma\sigma^{-1}$ and $\sigma \in G_\gamma$. But the equality $\sigma(\tilde{\varphi}) = \tilde{\varphi}'$ implies that $\sigma(x) = x'$. □

By Lemma 2.6, $\tilde{\lambda}$ induces a morphism of $k[t]$-schemes

$$\lambda\colon (\mathbf{A}_k^n/G_\gamma)\otimes k[t]\longrightarrow X\otimes k[t].$$

The morphism λ induces a $k[t]$-morphism

$$\lambda_*\colon \mathcal{L}(\mathbf{A}_k^n/G_\gamma)\longrightarrow\mathcal{L}(X).$$

2.7. Considering $\mathcal{L}(\mathbf{A}_k^n)/G_\gamma$ as a (semi-algebraic) subset of $\mathcal{L}(\mathbf{A}_k^n/G_\gamma)$ we have by (2.3.4) and Lemma 2.6 that λ_* induces a bijection between $\mathcal{L}(\mathbf{A}_k^n)/G_\gamma\cap\lambda_*^{-1}(\mathcal{L}(X)^g)$ and $\mathcal{L}(X)^g_{0,\gamma}$.

3. Motivic Gorenstein Measure of Quotients

3.1. Let X be an irreducible normal algebraic variety over k of dimension d and assume X is Gorenstein with at most canonical singularities at each point. Hence, there exists ω_X in $\Omega_X^d\otimes_k k(X)$ generating Ω_X^d at each smooth point of X, and, since X is canonical, $\operatorname{div} h^*(\omega_X)$ is effective for any resolution $h\colon Y\to X$. So by pulling back to Y and using the change of variables formula Lemma 3.4 of [8], we see that $\mathbf{L}^{-\mathrm{ord}_t\omega_X}$ is integrable on $\mathcal{L}(X)$ (see 1.14 for the definition of $\mathrm{ord}_t\omega_X$). Furthermore, the function $\mathrm{ord}_t\omega_X$ does not depend on the choice of ω_X. Hence, one may define the *motivic Gorenstein measure* $\mu^{\mathrm{Gor}}(A)$ of a $k[t]$-semi-algebraic subset A of $\mathcal{L}(X)$ as

$$\mu^{\mathrm{Gor}}(A):=\int_A \mathbf{L}^{-\mathrm{ord}_t\omega_X}\,\mathrm{d}\mu_{\mathcal{L}(X)}$$

in $\widehat{\mathcal{M}}$.

3.2. Let $d\geqslant 1$ be an integer and let k be field of characteristic 0 containing all dth roots of unity. Let G be a finite subgroup of $\mathrm{SL}_n(k)$ of order d. Set $X=\mathbf{A}_k^n/G$ and let $h\colon \mathbf{A}_k^n\to X$ be the projection. The variety X has only canonical Gorenstein singularities and we can take ω_X in $\Omega^n_{X/k}\otimes k(X)$ such that $h^*(\omega_X)=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$.

For γ in G, recall the weight $w(\gamma)$ of γ is defined as $w(\gamma):=\sum_{1\leqslant i\leqslant n}e_{\gamma,i}/d$, where the $e_{\gamma,i}$'s are as in 2.1, i.e. $1\leqslant e_{\gamma,i}\leqslant d$ and $\xi^{e_{\gamma,i}}$ are the eigenvalues of γ for $i=1,\ldots,n$. Note that $w(\gamma)\in\mathbf{N}\setminus\{0\}$, since $G\subset\mathrm{SL}_n(k)$.

LEMMA 3.3. *For any γ in G, we have*

$$\mu^{\mathrm{Gor}}(\mathcal{L}(X)^g_{0,\gamma})=\mathbf{L}^{-w(\gamma)}\,\mu^{\mathrm{Gor}}_{\mathcal{L}(\mathbf{A}_k^n/G_\gamma)}(\mathcal{L}(\mathbf{A}_k^n)/G_\gamma).$$

Proof. Let λ be as in Section 2. Direct verification yields $\lambda^*(\omega_X)=t^{w(\gamma)}\omega_{\mathbf{A}_k^n/G_\gamma}$. The lemma follows now from 2.7 and Theorem 1.16 (with h replaced by λ_*). □

A reason for considering the measure $\mu^{\mathrm{Gor}}_{\mathcal{L}(X)}$ instead of $\mu_{\mathcal{L}(X)}$ is given by the next lemma. It is also at that place that it seems necessary to work in the ring $\widehat{\mathcal{M}}_{/}$ instead of just $\widehat{\mathcal{M}}$.

LEMMA 3.4. *The image of* $\mu^{\mathrm{Gor}}_{\mathcal{L}(X)}(\mathcal{L}(\mathbf{A}^n_k)/G)$ *in* $\widehat{\mathcal{M}}_{/}$ *is equal to* 1.

Proof. Let M be a large integer. For e in $\mathbf{N}$, we consider the subset $\Delta_{e,M}$ of $\mathcal{L}(\mathbf{A}^n_k)$ consisting of all points φ in $\mathcal{L}(\mathbf{A}^n_k)$ such that $\mathrm{ord}_t\mathcal{J}_h(\varphi) = e$ and $h(\varphi) \in \mathcal{L}^{(M)}(X)$. Note that $(\mathrm{ord}_t\omega_X) \circ h = -\mathrm{ord}_t\mathcal{J}_h$, because $\mathrm{ord}_t h^*(\omega_X) = \mathrm{ord}_t(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = 0$. Thus

$$\mu^{\mathrm{Gor}}_{\mathcal{L}(X)}(\mathcal{L}(\mathbf{A}^n_k)/G) = \sum_{e=0}^{M} \mathbf{L}^e \mu_{\mathcal{L}(X)}(h(\Delta_{e,M})) + R_M,$$

with $\lim_{M\to\infty} R_M = 0$, since $\mathbf{L}^{-\mathrm{ord}_t\omega_X}$ is integrable on $\mathcal{L}(X)$. By the first assertion of Lemma 1.17 and by Lemma 3.5 below, for m in $\mathbf{N}$ large enough with respect to M, we have for all $e \leqslant M$ that $h(\Delta_{e,M})$ is stable at level m and that $[\pi_m(h(\Delta_{e,M}))] = \mathbf{L}^{-e}[\pi_m(\Delta_{e,M})/G]$. Hence

$$\begin{aligned}\mu^{\mathrm{Gor}}_{\mathcal{L}(X)}(\mathcal{L}(\mathbf{A}^n_k)/G) &= \sum_{e=0}^{M} [\pi_m(\Delta_{e,M})/G]\,\mathbf{L}^{-(m+1)n} + R_M\\ &= [\pi_m(\cup_{e=0,\dots,M}\Delta_{e,M})/G]\,\mathbf{L}^{-(m+1)n} + R_M\\ &= [\pi_m(\mathcal{L}(\mathbf{A}^n_k))/G]\,\mathbf{L}^{-(m+1)n} + R'_M,\end{aligned}$$

with $\lim_{M\to\infty} R'_M = 0$ (because of Lemma 4.4 of [8]). The lemma follows now, since $\pi_m(\mathcal{L}(\mathbf{A}^n_k))/G$ is isomorphic to $\mathbf{A}^{(m+1)n}_k/G$, the G-action on $\mathbf{A}^{(m+1)n}_k$ being the diagonal one, and the image of $\mathbf{A}^{(m+1)n}_k/G$ in $\widehat{\mathcal{M}}_{/}$ is equal to $\mathbf{L}^{(m+1)n}$ (it is here that we use the fact that we work in $\widehat{\mathcal{M}}_{/}$ instead of $\widehat{\mathcal{M}}$). □

LEMMA 3.5. *Let* $Y = \mathbf{A}^d_k$ *and* $X = \mathbf{A}^d_k/G$, *with* G *a finite subgroup of* $\mathrm{GL}_d(k)$. *Denote by* h: $\mathcal{L}(Y) \to \mathcal{L}(X)$ *the natural projection. Let* $B \subset \mathcal{L}(Y)$ *be cylindrical and stable under the* G*-action. Set* $A = h(B)$. *Assume that* $\mathrm{ord}_t\mathcal{J}_h(\varphi)$ *has constant value* $e < \infty$ *for all* $\varphi \in B$, *and that* $A \subset \mathcal{L}^{(e')}(X)$ *for some* e' *in* $\mathbf{N}$. *Then, for* $n \in \mathbf{N}$ *large enough, we have the following:*

(a) *If* $\varphi \in B$, $\varphi' \in \mathcal{L}(Y)$ *and* $\pi_n(h(\varphi)) = \pi_n(h(\varphi'))$, *then* $\pi_{n-e}(\varphi)$ *and* $\pi_{n-e}(\varphi')$ *have the same image in* $\mathcal{L}_{n-e}(Y)/G$.

(b) *The morphism* h_{n*}: $\pi_n(B)/G \to \pi_n(A)$ *induced by* h *may be endowed with the structure of a piecewise vector bundle of rank* e.

(c) $[\pi_n(B)/G] = \mathbf{L}^e\,[\pi_n(A)]$.

Proof. Since assertion (a) is a direct consequence of assertion (a″) in the proof of Lemma 1.17, taking $x = h(\varphi')$, and assertion (c) follows from (b), it remains to prove (b).

By the first assertion in the statement of Lemma 1.17, A is cylindrical at level n, taking n large enough. In order to prove (b), we may assume that $\pi_n(A)$ is a locally closed subvariey of $\mathcal{L}_n(X)$. The inverse image of $\pi_n(A)$ under the natural map $\mathcal{L}_n(Y)/G \to \mathcal{L}_n(X)$ is locally closed in $\mathcal{L}_n(Y)/G$ and is equal to $\pi_n(B)/G$ by assertion (a) and the fact that B is cylindrical at level $n-e$, for n large enough. Hence, $\pi_n(B)/G$ is a locally closed subvariety of $\mathcal{L}_n(Y)/G$, and $\pi_n(B)$ is a locally closed subvariety of $\mathcal{L}_n(Y)$.

Next we prove the following assertion:

(d) The stabilizer of G acting on $\pi_{n-e}(B)$ is trivial at every point of $\pi_{n-e}(B)$.

Let $\sigma \in G \setminus \{1\}$ and set $\Delta_\sigma = \{y \in Y \mid \sigma(y) = y\}$. Since $\mathrm{ord}_t \mathcal{J}_h \neq \infty$ on B, we have $B \cap \mathcal{L}(\Delta_\sigma) = \emptyset$. Hence, B is contained in $\cup_{m\in \mathbf{N}}(\mathcal{L}(Y) \setminus \pi_m^{-1}(\mathcal{L}_m(\Delta_\sigma)))$. Thus, since B is cylindrical, Lemma A.3 implies that B is contained in $\mathcal{L}(Y) \setminus \pi_m^{-1}(\mathcal{L}_m(\Delta_\sigma))$ when m is large enough. This concludes the proof of assertion (d).

Our next step is to construct a section of the morphism h_{n*}: $\pi_n(B)/G \to \pi_n(A)$. Let s: $\mathcal{L}_n(X) \to \mathcal{L}(X)$ be a section of the projection π_n: $\mathcal{L}(X) \to \mathcal{L}_n(X)$ such that the restriction of $\pi_{n+e} \circ s$ to $\pi_n(A)$ is a piecewise morphism. The existence of such a section s has been shown in the proof of Lemma 1.17. Note that $s(\pi_n(A))$ is contained in A, since $\pi_n(A)$ is cylindrical at level n. Denote by θ the map

$$\theta\colon\ \pi_n(A) \longrightarrow B/G\colon \quad x \longmapsto h^{-1}(s(x)) \mod G,$$

and set

$$\overline{\theta} = \tilde{\pi}_n \circ \theta\colon\ \pi_n(A) \longrightarrow \pi_n(B)/G\colon \quad x \longmapsto \theta(x) \mod t^{n+1},$$

where $\tilde{\pi}_n$: $B/G \to \pi_n(B)/G$ is the projection. Clearly $\overline{\theta}$ is a section of h_{n*}. One proves that $\overline{\theta}$ is a piecewise morphism by exactly the same argument as for assertion (c) in the proof of Lemma 1.17, replacing B, $\pi_n(B)$, and $\pi_{n+e}(B)$ by their quotient under the action of G.

By (d), the natural morphism p: $\pi_n(B) \to \pi_n(B)/G$ is étale. We consider the fiber product

$$\widetilde{\pi_n(A)} := \pi_n(A) \times_{\pi_n(B)/G} \pi_n(B).$$

The strategy of proof is to construct a G-equivariant morphism γ: $\pi_n(B) \to \widetilde{\pi_n(A)}$, such that the following diagram is commutative,

$$\begin{array}{ccc} \pi_n(B)/G & \underset{\overline{\theta}}{\overset{h_{n*}}{\rightleftarrows}} & \pi_n(A) \\ {\scriptstyle p}\uparrow & & \uparrow \\ \pi_n(B) & \underset{}{\overset{\gamma}{\rightleftarrows}} & \widetilde{\pi_n(A)}, \end{array} \tag{3.5.5}$$

then to show it may be endowed with the structure of a piecewise vector bundle of rank e, and finally to conclude by étale descent.

We first construct the mapping γ. Let φ be a point in $\pi_n(B)$. It follows from (a) that there exists a lifting $\tilde{\varphi}$ in $\pi_n(B)$ of $\overline{\theta}(h_{n*}(p(\varphi)))$ such that $\varphi \equiv \tilde{\varphi} \mod t^{n+1-e}$.

Furthermore, by (d), the lifting $\tilde{\varphi}$ is uniquely determined by φ. We set

$$\gamma(\varphi) := (h_{n*}(p(\varphi)), \tilde{\varphi}).$$

Clearly, the graph of γ is constructible, hence, by Lemma 1.18, γ is a piecewise morphism. We shall show later that, as soon as $\overline{\theta}$ is a morphism and $\pi_n(B)$ is smooth, γ is an actual morphism. Now take a point $(a, \tilde{\varphi})$ in $\widetilde{\pi_n(A)}$. We have $a = h_{n*}(p(\tilde{\varphi}))$ and $\tilde{\varphi}$ is a lifting of $\overline{\theta}(a)$. Hence, the conditions for a point φ to be in the fiber $\gamma^{-1}(a, \tilde{\varphi})$ are that $\varphi \equiv \tilde{\varphi} \mod t^{n+1-e}$ and $h(\varphi) \equiv h(\tilde{\varphi}) \mod t^{n+1}$. Rewriting the first condition as $\varphi = \tilde{\varphi} + t^{n+1-e}u$, with a unique u in $\mathcal{L}_{e-1}(\mathbf{A}_k^d)$, the fiber $\gamma^{-1}(a, \tilde{\varphi})$ can be determined by rewriting the condition

$$h(\tilde{\varphi} + t^{n+1-e}u) \equiv h(\tilde{\varphi}) \mod t^{n+1}$$

using the Taylor expansion of h at $\tilde{\varphi}$. In this way, using again that n is large enough and that B is cylindrical at level $n - e$, we find that

$$\gamma^{-1}(a, \tilde{\varphi}) = \left\{ \tilde{\varphi} + t^{n+1-e}(u_0 + u_1 t + \cdots + u_{e-1}t^{e-1}) \;\middle|\; L_{\tilde{\varphi}}(u_0, \ldots, u_{e-1}) = 0 \right\},$$

where $L_{\tilde{\varphi}}(u_0, \ldots, u_{e-1}) = 0$ is a system of linear homogeneous equations whose coefficients are regular functions of $\tilde{\varphi} \in \mathcal{L}_n(Y)$.

We refer to [8] 3.4 (3) for more details. Moreover, the solution space of this linear system has dimension e, since the Jacobian matrix of h at any point in $\pi_n^{-1}(\tilde{\varphi})$ is equivalent over $\bar{k}[[t]]$ to a diagonal matrix with diagonal elements t^{e_1}, t^{e_2}, ..., satisfying $e = e_1 + e_2 + \cdots$, cf. [8] 3.4 (4).

In order to prove (b), we may assume that $\pi_n(A)$ is a locally closed smooth subvariety of $\mathcal{L}_n(X)$ and that $\overline{\theta}$ is a morphism, provided that from now on we only assume B is cylindrical at level n and that we do not anymore increase n, which could destroy the property of $\overline{\theta}$ to be a morphism. When $k = \mathbf{C}$, we see from our previous discussion about $\gamma^{-1}(a, \tilde{\varphi})$, that $\pi_n(B)$ is locally bianalytically isomorphic to $\pi_n(A) \times \mathbf{C}^e$. Hence, $\pi_n(B)$ is smooth for any k. Now let us prove that γ is a morphism. When $k = \mathbf{C}$, it is easy to see that γ is continuous, hence is a morphism, since its domain is smooth and it is a piecewise morphism. Thus, by the Lefschetz principle, it follows that γ is a morphism, for any k. The fact that it may be endowed with the structure of a vector bundle of rank e follows from the above description of the fibers. Now by étale descent (Hilbert's Theorem 90, see, e.g., [14] p. 124), we deduce that h_{n*} may be endowed with the structure of a vector bundle of rank e. □

We can now prove the main result.

THEOREM 3.6. *Let $d \geqslant 1$ be an integer and let k be field of characteristic 0 containing all dth roots of unity. Let G be a finite subgroup of* $\mathrm{SL}_n(k)$ *of order d, so G acts on $\mathbf{A}_k^n$. Consider the quotient $X := \mathbf{A}_k^n/G$.*

(1) *For any* γ *in* G, *we have*

$$\mu^{\mathrm{Gor}}(\mathcal{L}(X)^{g}_{0,\gamma}) = \mathbf{L}^{-w(\gamma)}$$

in $\widehat{\mathcal{M}}_{/}$.

(2) *The relation*

$$\mu^{\mathrm{Gor}}(\mathcal{L}(X)_0) = \sum_{[\gamma]\in \mathrm{Conj}(G)} \mathbf{L}^{-w(\gamma)}$$

holds in the ring $\widehat{\mathcal{M}}_{/}$, *where* Conj$(G)$ *denotes the set of conjugacy classes in* G.

Proof. The first statement is a direct consequence of Lemmas 3.3 and 3.4 with G, X replaced by G_γ, $\mathbf{A}^n_k/G_\gamma$. The second follows then, using the decomposition

$$\mathcal{L}(X)^g_0 = \coprod \mathcal{L}(X)^g_{0,\gamma}$$

and the fact that $\mu^{\mathrm{Gor}}(\mathcal{L}(X)_0 \setminus \mathcal{L}(X)^g_0) = 0$. □

3.7. Keeping the above notations, we now assume that G is a finite subgroup of $\mathrm{GL}_n(k)$, instead of $\mathrm{SL}_n(k)$. Notice that now the weight $w(\gamma) \in \mathbf{Q}$ of an element γ in G might not be integral and that ω_X might not exist. To remedy this we consider the function $\alpha_X\colon \mathcal{L}(X) \to \mathbf{Q}\cup\{\infty\}$ which is defined by $\alpha_X(\varphi) = -\mathrm{ord}_t \mathcal{J}_h(\tilde{\varphi})$ for any $\tilde{\varphi}$ in $\mathcal{L}^{1/d}(\mathbf{A}^n_k)$ with $h(\tilde{\varphi}) = \varphi$. Clearly $\alpha_X = \mathrm{ord}_t\omega_X$ when $G \subset \mathrm{SL}_n(k)$. We define the *motivic orbifold measure* $\mu^{\mathrm{orb}}(A)$ of a $k[t]$-semi-algebraic subset A of $\mathcal{L}(X)$ as

$$\mu^{\mathrm{orb}}(A) := \int_A \mathbf{L}^{-\alpha_X}\, \mathrm{d}\mu_{\mathcal{L}(X)} \in \widehat{\mathcal{M}}[\mathbf{L}^{1/d}].$$

Theorem 3.6 remains true for $G \subset \mathrm{GL}_n(k)$ if we replace μ^{Gor} by μ^{orb} and $\widehat{\mathcal{M}}_{/}$ by $\widehat{\mathcal{M}}_{/}[\mathbf{L}^{1/d}]$. Indeed the proofs remain basically the same, replacing $\mathrm{ord}_t\omega_X$ by α_X. At the same time one verifies that the integrals $\mu^{\mathrm{orb}}(A)$ converge. At the level of Hodge realization a similar result is contained in Section 7 of [6]. Indeed, with the notation of loc. cit., $H(\mu^{\mathrm{orb}}(\mathcal{L}(X))) = E_{st}(X, \Delta_X; u, v)$.

3.8. More generally we may consider a smooth irreducible algebraic variety $\tilde{X}$ endowed with an effective action of a finite group G of order d. We assume the field k contains all dth roots of unity. We shall also assume that every G-orbit is contained in an affine open subset of $\tilde{X}$ and we denote by X the quotient variety $\tilde{X}/G$. Using the previous methods, it is possible to express $\mu^{\mathrm{orb}}(\mathcal{L}(X))$ in terms of weights associated to the group action along the orbifold strata, similarly as in Section 7 of [6], cf. [13].

4. Chow Motives and Realizations

4.1. We denote by $\mathcal{V}_k$ the category of smooth and projective k-schemes. For an object X in $\mathcal{V}_k$ and an integer d, we denote by $A^d(X)$ the Chow group of codimension

d cycles with rational coefficients modulo rational equivalence. Objects of the category CHM_k of (rational) k-motives are triples (X, p, n) where X is in $\mathcal{V}_k$, p is an idempotent (i.e. $p^2 = p$) in the ring of correspondences $\mathrm{Corr}^0(X, X)$ $(= A^d(X \times X)$ when X is of pure dimension d), and n is an integer. If (X, p, n) and (Y, q, m) are motives, then

$$\mathrm{Hom}_{\mathrm{CHM}_k}((X, p, n), (Y, q, m)) = q\,\mathrm{Corr}^{m-n}(X, Y)\,p.$$

Here $\mathrm{Corr}^r(X, Y)$ is the group of correspondences of degree r from X to Y (which is $A^{d+r}(X \times Y)$ when X is of pure dimension d). Composition of morphisms is given by composition of correspondences. The category CHM_k is additive, $\mathbf{Q}$-linear, and pseudo-Abelian, and there is a natural tensor product on CHM_k. We denote by h the functor $h: \mathcal{V}_k^\circ \to \mathrm{CHM}_k$ which sends an object X to $h(X) = (X, \mathrm{id}, 0)$ and a morphism $f: Y \to X$ to its graph in $\mathrm{Corr}^0(X, Y)$. We denote by $\mathbf{L}$ the Lefschetz motive $\mathbf{L} = (\mathrm{Spec}\, k, \mathrm{id}, -1)$. There is a canonical isomorphism $h(\mathbf{P}^1_k) \simeq 1 \oplus \mathbf{L}$.

Let $K_0(\mathrm{CHM}_k)$ be the Grothendieck group of the pseudo-Abelian category CHM_k. It is also the Abelian group associated to the monoid of isomorphism classes of k-motives with respect to the addition $\oplus$. The tensor product on CHM_k induces a natural ring structure on $K_0(\mathrm{CHM}_k)$. For m in $\mathbf{Z}$, let $F^m K_0(\mathrm{CHM}_k)$ denote the subgroup of $K_0(\mathrm{CHM}_k)$ generated by $h(S, f, i)$, with $i - \dim S \geqslant m$. This gives a filtration of the ring $K_0(\mathrm{CHM}_k)$ and we denote by $\widehat{K}_0(\mathrm{CHM}_k)$ the completion of $K_0(\mathrm{CHM}_k)$ with respect to this filtration.

Gillet and Soulé [10] and Guillén and Navarro Aznar [11] proved the following result.

THEOREM 4.2. *Let k be a field of characteristic 0. There exists a unique map χ_c which to any variety X over k associates $\chi_c(X)$ in $K_0(\mathrm{CHM}_k)$ such that*

(1) *If X is smooth and projective, $\chi_c(X) = [h(X)]$.*
(2) *If Y is a closed reduced subscheme in a variety X*

$$\chi_c(X \setminus Y) = \chi_c(X) - \chi_c(Y).$$

(3) *If X is a variety, U and V are open reduced subschemes of X,*

$$\chi_c(U \cup V) = \chi_c(U) + \chi_c(V) - \chi_c(U \cap V).$$

(4) *If X and Y are varieties*

$$\chi_c(X \times Y) = \chi_c(X)\,\chi_c(Y).$$

Furthermore, χ_c is determined by conditions (1)–(2).

Hence, χ_c induces a morphism of rings $\chi_c: \mathcal{M} \to K_0(\mathrm{CHM}_k)$ with $\chi_c(\mathbf{L}) = \mathbf{L}$ and extends to a morphism $\widehat{\chi}_c: \widehat{\mathcal{M}} \to \widehat{K}_0(\mathrm{CHM}_k)$.

4.3. Recall that the Hodge polynomial of an algebraic variety S defined over a subfield of $\mathbf{C}$ is the polynomial

$$H(S; u, v) := \sum_{p,q} e^{p,q}(S)\, u^p v^q$$

with

$$e^{p,q}(S) := \sum_{i \geqslant 0} (-1)^i h^{p,q}(H^i_c(S, \mathbf{C})),$$

where $h^{p,q}(H^i_c(S, \mathbf{C}))$ denotes the rank of the (p, q)-Hodge component of the ith cohomology group with compact supports. One defines similarly the Hodge polynomial of Chow motives. It follows from a weight argument, cf. [8] and [9], that the Hodge polynomial H factorizes (hence also the Euler Characteristic Eu) through the image of $K_0(\mathrm{CHM}_k)$ in $\widehat{K}_0(\mathrm{CHM}_k)$.

4.4. The following proposition shows that the morphisms χ_c and $\widehat{\chi}_c$ factorize through $\mathcal{M}_/$ and $\widehat{\mathcal{M}}_/$ respectively.

PROPOSITION 4.5. *Let V be a finite dimensional vector space over k and let G be a finite subgroup of* $\mathrm{GL}(V)$. *Then the following equality holds:* $\chi_c(V/G) = \chi_c(V)$.

Proof. We will use the functor h_c of [11] which to a variety X over k associates an object $h_c(X)$ of the homotopy category $\mathrm{Ho}(C^b(\mathrm{CHM}_k))$ of bounded complexes of objects in CHM_k, such that $\chi_c(X)$ is the Euler characteristic of $h_c(X)$. Consider the functor τ: $\mathrm{CHM}_k \to \mathrm{Ho}(C^b(\mathrm{CHM}_k))$ which to an object M associates the complex in $\mathrm{Ho}(C^b(\mathrm{CHM}_k))$ which is zero in nonzero degree and is equal to M in degree 0. It follows from the identity $h(\mathbf{P}^1_k) \simeq 1 \oplus \mathbf{L}$ in CHM_k and the definitions that $h_c(V)$ is isomorphic to $\tau(\mathbf{L}^{\dim V})$ in $\mathrm{Ho}(C^b(\mathrm{CHM}_k))$. By Corollary 5.3 of [1], $h_c(V/G)$ is a direct factor of $h_c(V)$ in $\mathrm{Ho}(C^b(\mathrm{CHM}_k))$. The functor τ being fully faithful and $\mathbf{L}^r$ being indecomposable, it follows that $h_c(V/G)$ is zero or equal to $\tau(\mathbf{L}^{\dim V})$. Using a realization, for instance the Betti realization, one obtains that $h_c(V/G) = \tau(\mathbf{L}^{\dim V})$, and the result follows. □

5. Relation with Resolution of Singularities and the McKay Correspondence

Let X be an algebraic variety over k of pure dimension d, and let h: $Y \to X$ be a resolution of singularities of X. By this we mean Y is a smooth algebraic variety over k, h is birational, proper and defined over k, and the exceptional locus E of h has normal crossings, meaning that the k-irreducible components of E are smooth and intersect transversally. Let us denote the k-irreducible components of E by E_i, $i \in J$. For $I \subset J$, set $E_I = \bigcap_{i \in I} E_i$ and $E^\circ_I = E_I \setminus \bigcup_{j \notin I} E_j$. Assume now X is Gorenstein with at most canonical singularities at each point and consider

ω_X in $\Omega^d_X \otimes_k k(X)$ generating Ω^d_X at each smooth point of X. For i in I, we denote by $v_i - 1$ the length of $\Omega^d_Y/h^*\omega_X\mathcal{O}_Y$ at the generic point of E_i.

Let W be a closed subvariety of X. By Lemma 3.3 of [8] (cf. Proposition 6.3.2 of [8]), the following formula holds in $\widehat{\mathcal{M}}$:

$$\mu^{\mathrm{Gor}}(\pi_0^{-1}(W)) = \mathbf{L}^{-d} \sum_{I \subset J} [E_I^\circ \cap h^{-1}(W)] \prod_{i \in I} \frac{\mathbf{L} - 1}{\mathbf{L}^{v_i} - 1}\,. \qquad (*)$$

Now we can specialize to the case where $X = \mathbf{A}^n_k/G$ with G a finite subgroup of $\mathrm{SL}_n(k)$ and $W = \{0\}$. Theorem 3.6 may now be rephrased as follows:

THEOREM 5.1. *Let $d \geqslant 1$ be an integer and let k be field of characteristic 0 containing all dth roots of unity. Let G be a finite subgroup of $\mathrm{SL}_n(k)$ of order d. Let h: $Y \to X$ be a resolution of $X = \mathbf{A}^n_k/G$. Then the following relation holds in $\widehat{\mathcal{M}}_{/}$:*

$$\mathbf{L}^{-n} \sum_{I \subset J} [E_I^\circ \cap h^{-1}(0)] \prod_{i \in I} \frac{\mathbf{L} - 1}{\mathbf{L}^{v_i} - 1} = \sum_{[\gamma] \in \mathrm{Conj}(G)} \mathbf{L}^{-w(\gamma)}. \qquad \square$$

In particular, if the resolution h is crepant, i.e. all the v_i's are equal to 1, we get as a corollary the following form of the McKay correspondence (cf. [16]).

COROLLARY 5.2. *Let h: $Y \to X$ be a crepant resolution of $X = \mathbf{A}^n_k/G$. Then the following relation holds in $\widehat{\mathcal{M}}_{/}$:*

$$[h^{-1}(0)] = \sum_{[\gamma] \in \mathrm{Conj}(G)} \mathbf{L}^{n-w(\gamma)}. \qquad \square$$

By passing to the Hodge realization, cf. 4.3, one obtains in particular the following form of the McKay correspondence, which was conjectured by Reid in [16] and proved by Batyrev in [6], see also [4,17].

COROLLARY 5.3. *Let h: $Y \to X$ be a crepant resolution of $X = \mathbf{A}^n_k/G$. Then*

$$H(h^{-1}(0)) = \sum_{[\gamma] \in \mathrm{Conj}(G)} (uv)^{n-w(\gamma)} \quad \textit{and} \quad \mathrm{Eu}(h^{-1}(0))) = \mathrm{card}\,\mathrm{Conj}(G). \qquad \square$$

Remark 5.4. Within the framework of 3.7, when A is of the form $\pi_0^{-1}(W)$, one may express $\mu^{\mathrm{orb}}(A)$ in terms of a resolution of X in a way completely similar to $(*)$, replacing the integers v_i by rational numbers v_i^* similarly defined with the help of α_X, cf. [13].

Appendix: Measurable Subsets of $\mathcal{L}(X)$

Let X be an algebraic variety of pure dimension d over a field k of characteristic zero. We develop here the theory of measurable subsets of $\mathcal{L}(X)$. When X is smooth, a measure theory for $\mathcal{L}(X)$ in the case of the Hodge realization has been considered by Batyrev in [5].

A.1. We call a cylindrical subset A of $\mathcal{L}(X)$ stable at level $n \in \mathbf{N}$ if A is cylindrical at level n and $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a piecewise trivial fibration over $\pi_m(A)$ with fiber $\mathbf{A}_k^d$ for all $m \geqslant n$. We call A stable if it is stable at some level n.

Denote by $\mathbf{C}_0$ the family of stable cylindrical subsets of $\mathcal{L}(X)$ and by $\mathbf{C}$ the Boolean algebra of cylindrical subsets of $\mathcal{L}(X)$. Since there might exist cylindrical subsets of $\mathcal{L}(X)$ which are not semi-algebraic, we cannot apply the motivic measure μ of Section 1 to elements of $\mathbf{C}$ or $\mathbf{C}_0$. Some precautions are necessary.

A.2. Clearly there exists a unique additive measure $\tilde{\mu} : \mathbf{C}_0 \longrightarrow \mathcal{M}_{\mathrm{loc}}$ satisfying

$$\tilde{\mu}(A) = [\pi(A)]\,\mathbf{L}^{-(n+1)d}$$

when $A \in \mathbf{C}_0$ is stable at level n. For A in $\mathbf{C}$, we define

$$\mu(A) = \lim_{e \to \infty} \tilde{\mu}(A \cap \mathcal{L}^{(e)}(X)) \in \widehat{\mathcal{M}}.$$

Indeed, $A \cap \mathcal{L}^{(e)}(X)$ is stable by Lemma 4.1 of [8], and the limit exists in $\widehat{\mathcal{M}}$ by Lemma 4.4 of loc. cit. Moreover if $A \in \mathbf{C}_0$ then $\mu(A)$ is the image in $\widehat{\mathcal{M}}$ of $\tilde{\mu}(A)$. Clearly μ is additive on $\mathbf{C}$, and even σ-additive because of the following lemma, which first appeared in [8] Lemma 2.4 for weakly stable semi-algebraic subsets, with a proof which actually holds also for cylindrical subsets. A different proof is given in Theorem 6.6 of [5].

LEMMA A.3. *Let A_i, $i \in \mathbf{N}$, be a family of cylindrical subsets of $\mathcal{L}(X)$. Suppose that $A := \cup_{i \in \mathbf{N}} A_i$ is cylindrical. Then A equals the union of a finite number of the A_i's.*

A.4. We consider on $\widehat{\mathcal{M}}$ the norm $\|\cdot\|$ defined by

$$\|\cdot\| : \quad \widehat{\mathcal{M}} \longrightarrow \mathbf{R}_{\geqslant 0}: \qquad a \longmapsto \|a\| := 2^{-n},$$

where n is the largest n such that $a \in F^n\widehat{\mathcal{M}}$.

For all a, b in $\widehat{\mathcal{M}}$, we have $\|ab\| \leqslant \|a\|\,\|b\|$ and $\|a+b\| \leqslant \max(\|a\|, \|b\|)$.

Note also that, for all A, B in $\mathbf{C}$, we have

$$\|\mu(A \cup B)\| \leqslant \max(\|\mu(A)\|, \|\mu(B)\|)$$

and $\|\mu(A)\| \leqslant \|\mu(B)\|$ when $A \subset B$.

For A and B subsets of the same set, we use the notation $A \triangle B$ for $A \cup B \setminus A \cap B$.

DEFINITION A.5. We say that a subset A of $\mathcal{L}(X)$ is *measurable* if, for every positive real number ε, there exists a sequence of cylindrical subsets $A_i(\varepsilon)$, $i \in \mathbf{N}$, such that

$$\left(A\triangle A_0(\varepsilon)\right)\subset\bigcup_{i\geqslant 1}A_i(\varepsilon),$$

and $||\mu(A_i(\varepsilon))||\leqslant\varepsilon$ for all $i\geqslant 1$. We say that A is *strongly measurable* if moreover we can take $A_0(\varepsilon)\subset A$.

THEOREM A.6. *If A is a measurable subset of $\mathcal{L}(X)$, then*

$$\mu(A):=\lim_{\varepsilon\to 0}\mu(A_0(\varepsilon))$$

exists in $\widehat{\mathcal{M}}$ and is independent of the choice of the sequences $A_i(\varepsilon)$, $i\in\mathbf{N}$.

Proof. This is proved in exactly the same way as Theorem 6.18 of [5] using Lemma A.3. □

For A a measurable subset of $\mathcal{L}(X)$, we shall call $\mu(A)$ the motivic measure of A.

One should remark that obviously any cylindrical subset of $\mathcal{L}(X)$ is strongly measurable and that the measurable subsets of $\mathcal{L}(X)$ form a Boolean algebra. Note also that if A_i, $i\in\mathbf{N}$, is a sequence of measurable subsets of $\mathcal{L}(X)$ with $\lim_{i\to\infty}||\mu(A_i)||=0$, then $\cup_{i\in\mathbf{N}}A_i$ is measurable.

Since Lemma 4.4 of [8] also holds for a closed subscheme S of $X\otimes k[t]$ with $\dim_{k[t]}S<d$, we see that, for such an S, the subset $\mathcal{L}(S)$ of $\mathcal{L}(X)$ is a measurable subset of $\mathcal{L}(X)$ of measure 0. Using Lemma 1.12, it follows that any $k[t]$-semi-algebraic subset of $\mathcal{L}(X)$ is strongly measurable, with the same measure as in Section 1.

For a measurable subset A of $\mathcal{L}(X)$ and a function α: $A\to\mathbf{Z}\cup\{\infty\}$, we say that $\mathbf{L}^{-\alpha}$ is *integrable* ot that α is *exponentially integrable* if the fibers of α are measurable and if the motivic integral

$$\int_A\mathbf{L}^{-\alpha}\mathrm{d}\mu:=\sum_{n\in\mathbf{Z}}\mu(A\cap\alpha^{-1}(n))\mathbf{L}^{-n}$$

converges in $\widehat{\mathcal{M}}$.

PROPOSITION A.7. (i) *Let A_i, $i\in\mathbf{N}$, be a family of measurable subsets of $\mathcal{L}(X)$. Assume the sets A_i are mutually disjoint and that $A:=\cup_{i\in\mathbf{N}}A_i$ is measurable. Then $\sum_{i\in\mathbf{N}}\mu(A_i)$ converges in $\widehat{\mathcal{M}}$ to $\mu(A)$.*

(ii) *If A and B are measurable subsets of $\mathcal{L}(X)$ and if $A\subset B$, then $||\mu(A)||\leqslant||\mu(B)||$.*

Proof. Straightforward exercise, using Lemma A.3. □

THEOREM A.8. *Let X and Y be algebraic varieties over k of pure dimension d, and let h: $\mathcal{L}(Y)\to\mathcal{L}(X)$ be a $k[t]$-morphism. If $B\subset\mathcal{L}(Y)$ is measurable, resp. strongly measurable, then $h(B)\subset\mathcal{L}(X)$ is also measurable, resp. strongly measurable.*

Proof. We may assume that Y is irreducible. Set

$$\Delta := \mathcal{L}(Y_{\rm sing}) \cup h^{-1}(\mathcal{L}(X_{\rm sing})) \cup \Big\{ y \in \mathcal{L}(Y) \;\Big|\; {\rm ord}_t \mathcal{J}_h(y) = \infty \Big\}.$$

We may assume there exists a closed subscheme S of $Y \otimes k[t]$ with $\dim_{k[t]} S < d$ such that Δ is contained in $\mathcal{L}(S)$, because otherwise $h(\mathcal{L}(Y))$ and $h(B)$ have measure zero. Since B is measurable and Δ is contained in cylindrical subsets C of $\mathcal{L}(Y)$ with $||\mu(C)||$ arbitrary small, we see that, for every $\varepsilon > 0$, there exists cylindrical subsets $B_i(\varepsilon)$, $i \in \mathbf{N}$, of $\mathcal{L}(Y)$, such that $B_0(\varepsilon) \cap \Delta = \emptyset$, $B \triangle B_0(\varepsilon) \subset \cup_{i \geqslant 1} B_i(\varepsilon)$, and $||\mu(B_i(\varepsilon))|| < \varepsilon$ for all $i \geqslant 1$. Moreover, when B is strongly measurable we can take $B_0(\varepsilon) \subset B$. Hence, $h(B) \triangle h(B_0(\varepsilon)) \subset \cup_{i \geqslant 1} h(B_i(\varepsilon))$. This implies the theorem, since by Lemma A.9 below, $h(B_0(\varepsilon))$ is cylindrical and, for $i \geqslant 1$, $h(B_i(\varepsilon))$ is contained in a cylindrical subset $A_i(\varepsilon)$ of $\mathcal{L}(X)$ with $||\mu(A_i(\varepsilon))|| \leqslant \max(||\mu(B_i(\varepsilon))||, \varepsilon) \leqslant \varepsilon$.

LEMMA A.9. *Let X and Y be algebraic varieties over k, of pure dimension d, and let h: $\mathcal{L}(Y) \to \mathcal{L}(X)$ be a $k[t]$-morphism. Let B be a cylindrical subset of $\mathcal{L}(Y)$. Then the following holds:*

(a) *For every $\varepsilon > 0$, $h(B)$ is contained in a cylindrical subset A of $\mathcal{L}(X)$ with $||\mu(A)|| \leqslant \max(||\mu(B)||, \varepsilon)$.*
(b) *Assume $B \cap \mathcal{L}(Y_{\rm sing}) = \emptyset$, $h(B) \cap \mathcal{L}(X_{\rm sing}) = \emptyset$, and ${\rm ord}_t \mathcal{J}_h(y)$ is nowhere equal to ∞ on B. Then $h(B)$ is cylindrical.*

Proof. (a) First assume that $||\mu(B)|| = 0$. Then, since B is cylindrical, we have $B \subset \mathcal{L}(Y_{\rm sing})$ and $h(B)$ is contained in some $\mathcal{L}(S)$, with S a closed subscheme of $X \otimes k[t]$, with $\dim_{k[t]} S < d$. This yields assertion (a) when $||\mu(B)|| = 0$. Now suppose that $||\mu(B)|| \neq 0$. Take e in $\mathbf{N}$ large enough to insure that $||\mu_{\mathcal{L}(X)}(\mathcal{L}(X) \setminus \mathcal{L}^{(e)}(X))|| \leqslant ||\mu_{\mathcal{L}(Y)}(B)||$. We may assume that $h(B)$ is contained in $\mathcal{L}^{(e)}(X)$. Now we choose $n \geqslant e$ large enough with respect to e to insure that B is cylindrical at level n and that $\mathcal{L}^{(e)}(Y)$ and $\mathcal{L}^{(e)}(X)$ are cylindrically stable at level n. Set $A := \pi_n^{-1}(\pi_n(h(B)))$ and note that A is cylindrical at level n, since $\pi_n(h(B))$ is constructible. Moreover, A is contained in $\mathcal{L}^{(e)}(X)$, since $h(B)$ is contained in $\mathcal{L}^{(e)}(X)$ and $n \geqslant e$. Hence, A is cylindrically stable at level n. Thus

$$\mu(A) = [\pi_n(h(B))]\, \mathbf{L}^{-(n+1)d} \quad \text{and} \quad ||\mu(A)|| \leqslant 2^{-(n+1)d + \dim \pi_n(B)}. \tag{A.9.1}$$

Since $||\mu(B)|| \neq 0$, we have, for e large enough and for n large enough with respect to e, that

$$\dim\Big(\pi_n(B \cap \mathcal{L}^{(e)}(Y))\Big) > \dim\Big(\pi_n(\mathcal{L}(Y) \setminus \mathcal{L}^{(e)}(Y))\Big),$$

and hence $||\mu(B)|| = 2^{-(n+1)d + \dim \pi_n(B)}$. Together with (A.9.1), this yields assertion (a).

(b) Using resolution of singularities, we may assume that Y is smooth. By Lemma A.3, there exists e' in $\mathbf{N}$ such that B is contained in $h^{-1}(\mathcal{L}^{(e')}(X))$ and ${\rm ord}_t \mathcal{J}_h$ is bounded on B. Assertion (b) follows now from the first part of Lemma 1.17. □

THEOREM A.10 (Change of variables formula). *Let X and Y be algebraic varieties over k, of pure dimension d. Let h:* $\mathcal{L}(Y) \to \mathcal{L}(X)$ *be a* $k[t]$*-morphism and let A and B be strongly measurable subsets of* $\mathcal{L}(X)$ *and* $\mathcal{L}(Y)$*, respectively. Assume that h induces a bijection between B and A. Then, for any exponentially integrable function* $\alpha: A \to \mathbf{Z} \cup \{\infty\}$*, the function* $B \to \mathbf{Z} \cup \{\infty\}: y \mapsto \alpha(h(y)) + \mathrm{ord}_t \mathcal{J}_h(y)$ *is exponentially integrable and*

$$\int_A \mathbf{L}^{-\alpha} \mathrm{d}\mu = \int_B \mathbf{L}^{-\alpha\circ h - \mathrm{ord}_t \mathcal{J}_h(y)} \mathrm{d}\mu.$$

Proof. Reasoning as in the proof of Theorem A.8, we reduce to the case where B is cylindrical and satisfies $B \cap \Delta = \emptyset$, with

$$\Delta := \mathcal{L}(Y_{\mathrm{sing}}) \cup h^{-1}(\mathcal{L}(X_{\mathrm{sing}})) \cup \left\{ y \in \mathcal{L}(Y) \;\middle|\; \mathrm{ord}_t \mathcal{J}_h(y) = \infty \right\}.$$

For this reduction we use the assumption that B is strongly measurable to insure that the cylinder $B_0(\varepsilon)$ in A.8 is contained in B, so that the restriction of h to $B_0(\varepsilon)$ is injective. Next we can reduce to the case where Y is smooth, using resolution of singularities. Since $B \cap \Delta = \emptyset$, it follows from Lemma A.3 that there exists e' in $\mathbf{N}$ such that B is contained in $h^{-1}(\mathcal{L}^{(e')}(X))$ and that $\mathrm{ord}_t \mathcal{J}_h$ is bounded on B. Thus we may as well assume that $\mathrm{ord}_t \mathcal{J}_h$ has constant value e on B and the theorem follows now from Lemma 1.17 (b). □

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