LIPSCHITZ CONTINUITY PROPERTIES FOR p-ADIC SEMI-ALGEBRAIC AND SUBANALYTIC FUNCTIONS

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Abstract.   We prove that a (globally) subanalytic function $f : X \subset \mathbf{Q}_p^n \to \mathbf{Q}_p$ which is locally Lipschitz continuous with some constant C is piecewise (globally on each piece) Lipschitz continuous with possibly some other constant, where the pieces can be taken to be subanalytic. We also prove the analogous result for a subanalytic family of functions $f_y : X_y \subset \mathbf{Q}_p^n \to \mathbf{Q}_p$ depending on p-adic parameters. The statements also hold in a semi-algebraic set-up and also in a finite field extension of $\mathbf{Q}_p$. These results are p-adic analogues of results of K. Kurdyka over the real numbers. To encompass the total disconnectedness of p-adic fields, we need to introduce new methods adapted to the p-adic situation.

Introduction

In the real setting, a C^1-function on an interval in $\mathbf{R}$ which has bounded derivative is automatically Lipschitz continuous. Indeed, if $f : (a, b) \to \mathbf{R}$ satisfies $|f'(x)| \leq C$ for some C and all x, then, for any $c < d$ in (a, b), one has $|f(c) - f(d)| = |\int_c^d f'(x)dx| \leq C|c - d|$. Such a result cannot hold for general C^1-functions over the p-adics, because of total disconnectedness. Indeed, there are easy examples of locally constant (hence C^1) functions $g : X \subset \mathbf{Q}_p \to \mathbf{Q}_p$ with X open, for which there exists no partition of X into finitely many pieces such that g is Lipschitz continuous on each piece (see Example 2.2). Also, there are examples of C^1 functions $g : X \subset \mathbf{Q}_p \to \mathbf{Q}_p$, with g' identically zero, that are not locally Lipschitz continuous for any constant C, at an infinite number of points (Example 1.5). Such examples show that over p-adic fields, the relation between bounds on derivatives and local Lipschitz properties may be quite chaotic, so in order to be able to obtain significant results related to piecewise Lipschitz continuity, it seems reasonable to limit the class of p-adic functions we consider to a class of tame piecewise C^1-functions where in particular functions of this class can be described by finite amounts of data. In this paper we consider two such tame classes of p-adic sets and functions: semi-algebraic sets and functions on the one hand, and (globally) subanalytic p-adic sets and functions on the other hand. Semi-algebraic functions are a natural generalization of algebraic functions on algebraic subsets of $\mathbf{Q}_p^n$, and subanalytic functions are a further enrichment of

semi-algebraic functions with certain analytic functions. In both frameworks, the functions are more than piecewise C^1: they are even piecewise (locally) $\mathbf{Q}_p$-analytic where the pieces are $\mathbf{Q}_p$-analytic manifolds which are moreover subanalytic (resp. semi-algebraic). In these two frameworks we obtain several results about piecewise Lipschitz continuity for multi-variable functions $g : X \subset \mathbf{Q}_p^m \to \mathbf{Q}_p$, assuming only local conditions on g, like for example boundedness of the partial derivatives. Of course, a (locally) $\mathbf{Q}_p$-analytic function $f : X \to \mathbf{Q}_p$ on an open set $X \subset \mathbf{Q}_p^m$ satisfies the following local property, cf. Lemma 1.3:

> If $|\partial f(x)/\partial x_i| \leq 1$ for all $i = 1, \ldots, m$ and all x, then f is locally Lipschitz continuous with constant 1.

From this one deduces, cf. Proposition 1.4:

> Let $g : X \subset \mathbf{Q}_p^m \to \mathbf{Q}_p$ be subanalytic (resp. semi-algebraic) and C^1 on an open X such that $|\partial g(x)/\partial x_i| \leq 1$ for all $i = 1, \ldots, m$. Take any finite partition of X into subanalytic (resp. semi-algebraic) $\mathbf{Q}_p$-analytic manifolds X_i on which g is $\mathbf{Q}_p$-analytic. Then the restriction of g to X_i is locally Lipschitz continuous with constant 1 for each i.

Note that the manifolds X_i are not necessarily open in $\mathbf{Q}_p^m$. This indicates that, for functions whose domain is not necessarily open, it may be more convenient to work with the condition of local Lipschitz continuity instead of conditions on partial derivatives.

Over the reals Kurdyka [K1] obtains the following result: if a (globally) subanalytic function $f : X \subset \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitz continuous for some fixed constant C, then it is piecewise (globally on the piece) Lipschitz continuous with possibly some other constant. We prove the following p-adic analogue: if a subanalytic function $g : X \subset \mathbf{Q}_p^n \to \mathbf{Q}_p$ is locally Lipschitz continuous with constant C, then it is piecewise (globally on the piece) Lipschitz continuous for possibly some other constant. In both the real and the p-adic setting, the pieces can be taken to be subanalytic, resp. semi-algebraic if f and g are semi-algebraic. In fact, we will prove this result for a fixed finite field extension K of $\mathbf{Q}_p$, and for subanalytic families of functions instead of for individual subanalytic functions.

Let us start by explaining the real (globally) subanalytic and semi-algebraic situation, according to Kurdyka [K1], giving a rough sketch of the main arguments for functions in up to two variables. The one variable case becomes trivial by the mentioned relation between distance and the integral of the derivative once one notes that any globally subanalytic subset of $\mathbf{R}$ is a finite union of points and open intervals (compare with o-minimality). For $f : X \subset \mathbf{R}^2 \to \mathbf{R}$ a globally subanalytic function, Kurdyka proves that X can be partitioned into finitely many pieces which are o-minimal cells (sometimes called cylinders), for example of the form

$$A_1 = \big\{ (x,y) \in (a,b) \times \mathbf{R} \mid \alpha(x) < y < \beta(x) \big\},$$

or of the form

$$A_2 = \big\{ (x,y) \in (a,b) \times \mathbf{R} \mid \alpha(x) = y \big\},$$

where moreover the "boundary functions" α and β are (globally) subanalytic or semi-algebraic and have bounded derivatives. For such a decomposition in so-called

L-regular cells to exist, it is important that for each piece A separately, affine coordinates on $\mathbf{R}^2$ are chosen so that A has indeed such a nice description (see also [Pa]). Using such a decomposition into L-regular cells and affine coordinates adapted to each cell individually, the Lipschitz continuity result follows using a path integral of the derivative of f along a well-chosen path inside the cell, after noticing that any two points in an L-regular cell can be connected by a path whose length is not much bigger than the distance between the two points it connects.

Over $\mathbf{Q}_p$, it seems not to be possible to follow a strategy similar to Kurdyka's, since there is no clear notion of paths connecting two points, let alone of the length of a path, which are basic ingredients for Kurdyka's approach over the reals. More generally speaking, as far as we know there is no clear connection between integrals and distances between points in $\mathbf{Q}_p$. Hence, a new approach had to be devised. We sketch this new approach in the one and the two variable case. Let $g : X \subset \mathbf{Q}_p \to \mathbf{Q}_p$ be a subanalytic function (globally subanalytic, as always in this paper) which is locally Lipschitz continuous with constant C. We know that we can partition X into finitely many p-adic cells, all of which are very roughly of a form similar to

$$A = \left\{ t \in K \mid |\alpha| \leq |t - c| \leq |\beta|, \ t - c \in \lambda Q_{m,n} \right\},$$

with constants $n > 0$, $m > 0$, $\lambda \in K$, and where $Q_{m,n}$ is the set of all p-adic numbers of the form $p^{na}(1 + p^m x)$ for some $x \in \mathbf{Z}_p$ and some $a \in \mathbf{Z}$ (see 3.1 and 3.3 for precise statements). We call c the center of A (note that c may lie outside A, which happens precisely when $\lambda \neq 0$). We define the balls of the cell A as the collection of maximal balls (with respect to inclusion) contained in A (cf. 3.2). It then follows from a certain Jacobian property 3.11 that we can select the cells A in such a way that each ball of A is mapped to either a point or a ball under g. We refine this Jacobian property so that we can ensure that the images of the balls of A form a single cell which has moreover as collection of maximal balls precisely the collection of the images of the maximal balls in A (cf. Proposition 3.12). In particular, $g(A)$ is a cell with a certain center d. Then, roughly, distances between points in A are compared to distances to c and similarly in the range of g, using the center d. A calculation based on this comparison of distances and the Jacobian property then finishes the one variable case. Actually, these arguments also show that the analogous statement for subanalytic *families* of functions $g_y : X_y \subset \mathbf{Q}_p \to \mathbf{Q}_p$ (instead of for individual subanalytic functions) holds. Such families are used in the several variable case for the induction argument.

In the two variables case, we obtain a result vaguely reminiscent of real partitions into L-regular cells. Let $g : X \subset \mathbf{Q}_p^2 \to \mathbf{Q}_p$ be locally Lipschitz continuous with some constant C. Roughly, we partition the family $X_{x_1} := \{x_2 \mid x \in X\}$ into finitely many families of cells $A_{x_1} \subset K$ with center c and boundaries α and β now depending on x_1. We show that, after possibly switching the role of x_1 and x_2, we can ensure that the center c is Lipschitz continuous in x_1 (see Proposition 2.4). By a piecewise bi-Lipschitz transformation, we may then assume that the center is identically zero for each of the cells. This is already an important reduction, but the obstacles due to the lack of a good notion of paths and integrals to control distances remain. Instead

of working with paths as is done in the real case, we work with a finite sequence of points with given starting point and endpoint, and one could understand such a finite sequence of jumps from one point to the next as a p-adic analogue of a real "path". For such a sequence of jumps to be of use, the following is required: after each jump, one should still stay in the same cell so that one can still evaluate the function g, the total (cumulative) distance of the jumps should be comparable to the distance between the starting point and the endpoint, and the function should not vary too much at each jump so that one can control $|g(a) - g(b)|$ for any jump from a to b in the sequence. This is done in the two-variable case as follows. Let a and a' be given in A. Either $\alpha(x_1)$ has bounded derivative, and then we can use induction for the one-variable function $x_1 \mapsto g(x_1, \alpha(x_1))$ and roughly jump from a to $(a_1, \alpha(a_1)))$, then to $(a_1', \alpha(a_1')))$ and finally to a'. In the, at first sight, more difficult case where $\alpha(x_1)$ has large derivative, we invert the role of x_1 and x_2 in the parametrization of the function $g(x_1, \alpha(x_1))$, namely, we essentially work with the one-variable function $x_2 \mapsto g(\alpha^{-1}(x_2), x_2)$ and use induction for this function and then roughly make similar jumps as before: from a to $(a_1, \alpha(a_1))) = (\alpha^{-1}(b), b)$, then to $(\alpha^{-1}(b'), b') = (a_1', \alpha(a_1')))$ and finally to a', for some $b, b' \in \mathbf{Q}_p$. This "path" allow us to bound $|g(a) - g(a')|$ in terms of $|a - a'|$ as needed for Lipschitz continuity, uniformly in a and a' in the cell. Of course, some fine tuning is required in order to guarantee injectivity before one starts inverting functions like α, which is provided e.g. by Corollary 3.7. As already indicated, the bulk of the paper is concerned with a fixed finite field extension K of $\mathbf{Q}_p$. We conclude the paper by extending our main theorem to elementary extensions of K, see Proposition 4.2.

This paper arose from our work [CCL] which provides p-adic analogues of real results in [KR], [Com1,2], (see also [ComM] for a multidimensional version), and of complex results in [T]. The main result of the present paper, Theorem 2.1, is used in [CCL] to prove the existence of distinguished tangent cones of definable sets and to establish the p-adic counterpart of Thie's formula of [T].

1   Basic Terminology and Results

1.1   Let K be a fixed finite field extension of $\mathbf{Q}_p$, the field of p-adic numbers. Write $\mathcal{O}_K$ for the valuation ring of K and $\mathcal{M}_K$ for the maximal ideal of $\mathcal{O}_K$. We denote by $\mathrm{ord} : K^\times \to \mathbf{Z}$ the valuation and we set $|x| := q^{-\mathrm{ord}(x)}$ and $|0| = 0$, with q the cardinality of the residue field of K. For a tuple $x \in K^n$, we write $|x|$ for $\max_{i=1}^n |x_i|$.

A *ball* in K is a subset of the form $a + b\mathcal{O}_K$ with $a \in K$ and b in $K^\times$. Note that in this terminology, a ball is always a nonempty, open, proper subset of K.

Let π_K be a uniformizer of $\mathcal{O}_K$. For each integer $n > 0$, let $\overline{\mathrm{ac}}_n : K \to \mathcal{O}_K/(\pi_K^n)$ be the map sending 0 to 0 and nonzero x to $x\pi_K^{-\mathrm{ord}(x)} \bmod (\pi_K^n)$.

The language $\mathcal{L}$ is, consequently in the whole paper, either the subanalytic (as in e.g. [DrHM]) or the semi-algebraic language on K (Macintyre's language), with coefficients (also called parameters) from K. Hence, $\mathcal{L}$-definable means either subanalytic or semi-algebraic with parameters from K consequently throughout the paper, which corresponds to the set-up of [CCL]. For the convenience of the reader, we recall here these languages. The language of Macintyre is the language $\mathcal{L}_{\mathrm{Mac}} = \{0, 1, +, -, \cdot, \{P_n\}_{n>0}\}$, and the subanalytic language is

$$\mathcal{L}_{\mathrm{an}} = \mathcal{L}_{\mathrm{Mac}} \cup \left\{ {}^{-1}, \cup_{m>0} K\{x_1, \ldots, x_m\} \right\},$$

where P_n stands for the set of n-th powers in $K^\times$, where ${}^{-1}$ stands for the field inverse extended on 0 by $0^{-1} = 0$, where $K\{x_1, \ldots, x_m\}$ is the ring of restricted power series over K (that is, formal power series converging on R^m), and each element f of $K\{x_1, \ldots, x_m\}$ is interpreted as the restricted analytic function $K^m \to K$ given by

$$x \mapsto \begin{cases} f(x) & \text{if } x \in R^m, \\ 0 & \text{otherwise}. \end{cases} \tag{1.1.1}$$

Hence, $\mathcal{L}$ is either $\mathcal{L}_{\mathrm{Mac}}$ with parameters from K or the language $\mathcal{L}_{\mathrm{an}}$ (which automatically has all parameters from K).

DEFINITION 1.2. Given two metric spaces (X, d_X) and (Y, d_Y), where d_X denotes the metric on the set X and d_Y the metric on Y, a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $C \geq 0$ such that, for all x_1 and x_2 in X,

$$d_Y\big(f(x_1), f(x_2)\big) \leq C d_X(x_1, x_2).$$

In the above case, we also call f Lipschitz continuous with constant C, or just C-Lipschitz continuous. If there is a constant C such that locally around each $x \in X$ the function f is C-Lipschitz continuous, then f is called locally Lipschitz continuous with constant C, or just locally C-Lipschitz continuous.

In this paper, the metrics come from the p-adic norm on the spaces K^n. For general K-analytic functions (which are not necessarily $\mathcal{L}$-definable) on an open domain in K^m, there is a general link between bounded partial derivatives and local Lipschitz continuity, as follows.

LEMMA 1.3 (Lemma 1.4.6 of [CCL]). Let $X \subset K^m$ be open and let $f : X \to K$ be K-analytic, meaning that locally, f is given by convergent power series over K. Suppose that

$$\big|\partial f(x)/\partial x_i\big| \leq 1$$

for all $i = 1, \ldots, m$ and all x in X. Then f is locally Lipschitz continuous with constant 1.

If f is merely C^1 one has to be more careful, even for one variable functions (cf. Example 1.5). The next proposition is about $\mathcal{L}$-definable C^1 functions on an open X.

PROPOSITION 1.4. Let $X \subset K^m$ be open and $\mathcal{L}$-definable and let $f : X \to K$ be $\mathcal{L}$-definable. Suppose that f is C^1, and that

$$\big|\partial f(x)/\partial x_j\big| \leq 1$$

for all x in X and for all $j = 1, \ldots, m$. Let $\{X_i\}_i$ be any finite partition of X into $\mathcal{L}$-definable parts which are K-analytic manifolds on which f is K-analytic (such a partition always exists). Then the restriction of f to X_i is locally Lipschitz continuous with constant 1 for each i.

Moreover, the same results hold for $\mathcal{L}$-definable families of open $X_y \subset K^m$ and C^1 functions $f_y : X_y \to K$, where y runs over an $\mathcal{L}$-definable set Y.

Note that in Proposition 1.4, the X_i need not be open in K^m, hence they are different from the situation of Lemma 1.3.

Proof. That such a finite partition exists follows from the cell decomposition Theorem 3.3 below, but was already obtained in [DD] without cell decomposition. Take a point x_0 on some X_i. Suppose that the manifold X_i is of dimension d. By the implicit function theorem and by the non-archimedean property, there exists an open neighborhood U of x_0 in X_i and a K-bi-analytic isometry $i : U \subset K^m \to B^d$ for some ball $B \subset K$. We can finish by Lemma 1.3 applied to $f \circ i^{-1}$. Exactly the same proof works for families $f_y : X_y \to K$. Indeed, by cell decomposition and up to a finite $\mathcal{L}$-definable partition of the family X_y we may suppose that X_y is a K-analytic manifold on which f_y is K-analytic for each y. $\square$

EXAMPLE 1.5.   Proposition 1.4 has no analogue for general C^1 functions, even in just one variable, say, from K to K, using finite partitions. For example, write K as a countable disjoint union of translates of the ball $\mathcal{M}_K$,

$$K = \bigcup_{i \in \mathbf{N}} a_i + \mathcal{M}_K$$

for some choice of the $a_i \in K$. Let $f : K \to K$ send $a_i + x$ with $x \in \mathcal{M}_K$ to $g(x)$ where $g : \mathcal{M}_K \to K$ is defined as follows. Write $\mathcal{M}_K \setminus \{0\}$ as a countable union of disjoint balls of the form $b + b^3 \mathcal{O}_K$, that is,

$$\mathcal{M}_K \setminus \{0\} = \bigcup_{i \in \mathbf{N}} b_i + b_i^3 \mathcal{O}_K$$

for some choice of $b_i \in \mathcal{M}_K \setminus \{0\}$. For each integer $n > 0$, fix one of the b_i with $\mathrm{ord}(b_i) = n$ and call these fixed b_i special. For $x \in \mathcal{O}_K$, define $g(b_i + b_i^3 x)$ as 0 if b_i is non-special and as b_i^2 if b_i is special, and put $g(0) = 0$. Then g and f are C^1 and f' and g' are both identically zero. However, g is not locally C-Lipschitz continuous around 0 for any constant $C > 0$. Indeed, let B be a small ball around 0, and take a special b_i inside B close to zero. Then, there exists a non-special b_j inside B such that

$$|b_i - b_j| = q \cdot |b_i^3|,$$

with q the cardinality of the residue field of K. On the other hand,

$$\left| g(b_i) - g(b_j) \right| = |b_i^2|.$$

Hence, for f, there exists no finite partition of K which makes f locally C-Lipschitz continuous on the pieces, for any choice of C.

2 The Main Results

Theorem 2.1 (Main theorem). *Let $\varepsilon > 0$ be given. Let $f : X \subset K^m \to K$ be an $\mathcal{L}$-definable function which is locally ε-Lipschitz continuous. Then there exist $C > 0$ and a finite definable partition of X into parts A_i such that the restriction of f to A_i is (globally) C-Lipschitz continuous for each i.*

EXAMPLE 2.2.   Theorem 2.1 has no analogue for general C^1 functions, even in just one variable, say, from an open $X \subset K$ to K. For example, if X is the open $\mathcal{O}_K \setminus \{0\}$, and f sends $x \in X$ to $|x|$, where the rational number $|x|$ is seen as an element of K, then f is clearly locally constant, but, for $x_1, x_2 \in X$ with $|x_2| < |x_1|$ one has $|f(x_1) - f(x_2)| \geq |x_2|^{-1}$ which grows to infinity while $|x_1 - x_2| = |x_1|$ goes to zero whenever x_1 approaches zero in X.

In order to formulate a variant of Theorem 2.1 for families of functions, the following notation will be convenient. For $g : D \subset A \times B \to C$ a function, and for $b \in B$, write $g(\,\cdot\,,b)$ for the function which sends a with $(a,b) \in D$ to $g(a,b)$. The domain of $g(\,\cdot\,,b)$ is thus the set $\{a \in A \mid (a,b) \in D\}$ which we will denote by D_b.

Theorem 2.3 (Main theorem: parameterized version). *Let $\varepsilon > 0$ be given. Let Y be an $\mathcal{L}$-definable set. Let $f : X \subset K^m \times Y \to K$ be an $\mathcal{L}$-definable function such that for each $y \in Y$ the function $f(\,\cdot\,,y) : x \mapsto f(x,y)$ is locally ε-Lipschitz continuous on X_y. Then there exist $C > 0$ and a finite definable partition of X into parts A_i such that for each $y \in Y$ and i the restriction of $f(\,\cdot\,,y)$ to A_{iy} is (globally) C-Lipschitz continuous.*

The following proposition compares with the notion of L-regular cells on the real number field, see [K1], which goes back to A. Parusiński [P], see also the more recent [Pa]. The definition of p-adic cells and their centers will be given in section 3.

PROPOSITION 2.4 (Cells with Lipschitz continuous centers). *Let Y and $X \subset K^m \times Y$ be $\mathcal{L}$-definable. Then there exist $C > 0$, a finite partition of X into $\mathcal{L}$-definable parts A and for each part A a coordinate projection*

$$\pi : K^m \times Y \to K^{m-1} \times Y$$

such that, over $K^{m-1} \times Y$ along this projection π, the set A is a p-adic cell with center $c : \pi(A) \to K$ and such that moreover the function

$$c(\,\cdot\,,y) : (x_1, \ldots, x_{m-1}) \mapsto c(x_1, \ldots, x_{m-1}, y)$$

is C-Lipschitz continuous on $\pi(A)_y$ for each $y \in Y$.

A last, more technical result in this paper is Proposition 3.12, which, in the one-variable case, says that for an injective definable function f one can partition the domain and the range compatibly into cells in some strong sense related to the maximal balls contained in the cells.

3 Some Results Related to Cell Decomposition Over K

For integers $m > 0$ and $n > 0$, let $Q_{m,n}$ be the ($\mathcal{L}$-definable) set

$$Q_{m,n} := \left\{ x \in K^\times \mid \operatorname{ord}(x) \in n\mathbf{Z} \,,\ \overline{\mathrm{ac}}_m(x) = 1 \right\}.$$

For $\lambda \in K$ let $\lambda \cdot Q_{m,n}$ denote $\{\lambda x \mid x \in Q_{m,n}\}$. The sets $Q_{m,n}$ are a variant of Macintyre's predicates P_ℓ of ℓ-th powers; the corresponding notions of cells are slightly different but equally powerful and similar in usage. Indeed, any coset of P_ℓ is a finite disjoint union of cosets of some $Q_{m,n}$ and vice versa.

DEFINITION 3.1 (p-adic cells). Let Y be an $\mathcal{L}$-definable set. A p-adic cell $A \subset K \times Y$ over Y is a (nonempty) set of the form

$$A = \left\{(t,y) \in K \times Y \mid y \in Y',\, |\alpha(y)| \square_1 |t - c(y)| \square_2 |\beta(y)|,\, t - c(y) \in \lambda Q_{m,n}\right\}, \quad (3.1.1)$$

with Y' a K-analytic $\mathcal{L}$-definable manifold, constants $n > 0$, $m > 0$, λ in K, $\alpha, \beta : Y' \to K^\times$ and $c : Y' \to K$ all K-analytic $\mathcal{L}$-definable functions, and $\square_i$ either $<$ or no condition, such that A projects surjectively onto $Y' \subset Y$. We call c the center of the cell A, $\lambda Q_{m,n}$ the coset of A, α and β the boundaries of A, and Y' the base of A. If $\lambda = 0$ we call A a 0-cell and if $\lambda \neq 0$ we call A a 1-cell.

Note that a p-adic cell over Y is an $\mathcal{L}$-definable set which is moreover a K-analytic manifold.

PROPOSITION-DEFINITION 3.2 (Balls of cells). Let Y be $\mathcal{L}$-definable. Let $A \subset K \times Y$ be a p-adic 1-cell over Y with coset $\lambda Q_{m,n}$ and center c. Then, for each $(t,y) \in A$ with $y \in Y$, there exists a unique maximal ball $B_{t,y}$ containing t and satisfying $B_{t,y} \times \{y\} \subset A$, where maximality is under inclusion. We call the collection of balls $\{B_{t,y} \mid (t,y) \in A\}$ the balls of the cell A; for fixed $y_0 \in Y$ we call the collection of balls $\{B_{t,y_0}\}_{\{t \mid (t,y_0) \in A\}}$ the balls of the cell A above y_0. Moreover, for each $(t,y) \in A$ one has

$$B_{t,y} = \left\{w \in K \mid \operatorname{ord}(w - c(y)) = a,\, \overline{\operatorname{ac}}_m(w - c(y)) = \overline{\operatorname{ac}}_m(\lambda)\right\}$$

for a unique $a \in \mathbf{Z}$ depending on t and y. If $A \subset K \times Y$ is a p-adic 0-cell then we define the collection of balls of A to be the empty collection, that is, there are no balls of A.

Proof. The uniqueness of $B_{t,y}$ follows from the non-archimedean property. We prove existence of a maximal ball $B_{t,y}$ containing t and satisfying $B_{t,y} \times \{y\} \subset A$. Choose $(t,y) \in A$. Since the collection of balls is preserved under translation by a constant, we may suppose that $c(y) = 0$. Then $(0,y) \notin A$ since $\lambda \neq 0$ and thus $t \neq 0$. Clearly, for B_1 being the ball $B_1 = t\mathcal{O}_K$ one has that $B_1 \times \{y\}$ is not a subset of A since B_1 contains 0. One the other hand, let B_2 be the ball $t + \pi_K^m t \mathcal{O}_K$, then clearly $B_2 \times \{y\} \subset A$. Since the value group is discrete and since $t \in B_2 \subset B_1$, the existence follows. In fact, $B_{t,y} = B_2$ since for any strictly bigger ball B_3 containing B_2 there exists $t' \in B_3$ with $\overline{\operatorname{ac}}_m(t') \neq \overline{\operatorname{ac}}_m(t) = \overline{\operatorname{ac}}_m(\lambda)$. Hence, the description for $B_{t,y}$ in the proposition follows. $\square$

In the p-adic semialgebraic case, cell decomposition theorems are due to Cohen [Co] and Denef [D1,2], and they were extended in [C2] to the subanalytic setting where one can find the following version:

Theorem 3.3 (p-adic cell decomposition). Let $X \subset K^{m+1}$ and $f_j : X \to K$ be $\mathcal{L}$-definable for $j = 1, \ldots, r$. Then there exists a finite partition of X into p-adic cells A_i (over K^m) with center c_i and coset $\lambda_i Q_{m_i, n_i}$ such that

$$\left|f_j(x,t)\right| = \left|h_{ij}(x)\right| \cdot \left|(t - c_i(x))^{a_{ij}} \lambda_i^{-a_{ij}}\right|^{1/n_i}, \quad \text{for each } (x,t) \in A_i,$$

with $(x,t) = (x_1, \ldots, x_m, t)$, integers a_{ij}, and $h_{ij} : K^m \to K$ $\mathcal{L}$-definable functions
which are K-analytic on the base of A_i, $j = 1, \ldots, r$. If $\lambda_i = 0$, we use the convention
that $a_{ij} = 0$. Moreover, given $\ell, n > 0$, we can take the A_i such that moreover

$$f_j(x,t) \cdot Q_{\ell,n}$$

for $(x,t) \in A_i$ only depends on i and j (and not on (x,t)), and such that the
restriction of f_j to each A_i is K-analytic.

DEFINITION 3.4.   If f_j and the A_i are as in Theorem 3.3, then call f_j *prepared* on
the cells A_i. If the base of A_i is itself a cell on which the $h_{ij}(x)$ and the boundaries
of A_i are prepared, and so on m times, then we call A_i a *full cell* and we call f_j *fully
prepared* on the A_i. It is also clear what is meant by a full cell $A \subset K^m \times Y$ over
some $\mathcal{L}$-definable set Y, in analogy with the notion of cells over Y of Definition 3.1.

Clearly by induction on dimension (that is, on m) one can use Theorem 3.3 to
get a partition into full cells on which the f_j are fully prepared.

We formulate four basic corollaries of Theorem 3.3. The first one was originally
proven without using Theorem 3.3 in [Dr1] and [DD].

COROLLARY 3.5 (Definable Skolem functions).   Let $X \subset K^n \times K^m$ be an $\mathcal{L}$-
definable set. Then there exists an $\mathcal{L}$-definable function $f : K^n \to K^m$ such that for
each $(x,y) \in X$ with $x \in K^n$ and $y \in K^m$ the point $(x, f(x))$ lies in X.

COROLLARY 3.6 (Uniform boundedness).   Let $X \subset K^n \times K^m$ be $\mathcal{L}$-definable, with
$n, m \geq 0$. Then there exists $N > 0$ such that for all $y \in K^m$ with $X_y := \{x \in K^n \mid (x,y) \in X\}$ a finite set, one has

$$\#X_y < N \,.$$

Moreover, any discrete $\mathcal{L}$-definable set $A \subset K^n$ is finite.

COROLLARY 3.7 (Injectivity versus constancy).   Let Y and $X \subset K \times Y$ be $\mathcal{L}$-
definable sets and let $F : X \to K$ be an $\mathcal{L}$-definable function. Then there exists a
finite partition of X into $\mathcal{L}$-definable sets X_i such that for each $y \in Y$, the restriction
of $F(\,\cdot\,,y) : x \mapsto F(x,y)$ to

$$X_{iy} := \{x \in K \mid (x,y) \in X_i\}$$

is either injective or constant, where this distinction only depends on i (and not
on y).

Proof. Let $\Gamma \subset K^2 \times Y$ be the graph of F. Now let $p : \Gamma \to K \times Y$ be the coordinate
projection sending $(x, F(x,y), y)$ to $(F(x,y), y)$ and let $p' : \Gamma \to K \times Y$ be the
projection sending $(x, F(x,y), y)$ to (x,y). Apply Theorem 3.3 to Γ over $K \times Y$
according to p (that is, the cells have a cell-like description in the x-variable). For
each 1-cell Γ_i in the partition of Γ, the corresponding restriction of $F(\,\cdot\,,y)$ to $p'(\Gamma_i)_y$
is clearly locally constant. For each 0-cell Γ_i in the partition of Γ, the corresponding
restriction of $F(\,\cdot\,,y)$ to $p'(\Gamma_i)_y$ is clearly injective. One completes the proof by
Corollaries 3.6 and 3.5.   □

COROLLARY 3.8 (Cell criterion).   Let Y and $X \subset K \times Y$ be $\mathcal{L}$-definable and let
$d : Y \to K$ be an $\mathcal{L}$-definable function and $n > 0$. Suppose that for each $(t,y) \in X$

with $y \in Y$ there is a maximal ball $B_{t,y}$ containing t such that $B_{t,y} \times \{y\}$ is contained in X. Suppose further that

$$B_{t,y} = \left\{ w \in K \mid \mathrm{ord}(w - d(y)) = b_{t,y} \, , \; \overline{\mathrm{ac}}_n(w - d(y)) = \xi_{t,y} \right\}$$

for some $b_{t,y}$ in $\mathbf{Z}$ and $\xi_{t,y} \neq 0$. Then X is a finite disjoint union of p-adic 1-cells A_i with center the restriction of d to the base of A_i, and such that each ball $B_{t,y}$ appears as a ball above y of one of the cells A_i.

Proof. Since the image of $\overline{\mathrm{ac}}_n$ is finite we may suppose that $\xi_{t,y}$ is constant. Now the corollary follows from Theorem 3.3 and from Presburger cell decomposition results of [C1] in a straightforward way. Indeed, by Theorem 3.3 we may suppose that Y itself is a full cell on which d is K-analytic, with centers c_i in the respective coordinates of Y. Now one maps the set X surjectively to a Presburger subset G of some Cartesian power of $\mathbf{Z}$, by taking as component functions $\mathrm{ord}(y_i - c_i)$ if this is an integer, and constantly 0 otherwise, and the integer $b_{t,y}$ for the last component function. Now apply the Presburger cell decomposition theorem of [C1] to G. This induces back a cell decomposition of X as desired in the statement, by taking the inverse image under the constructed map. $\qquad\square$

LEMMA 3.9.   Let Y and $X \subset K \times Y$ be $\mathcal{L}$-definable. Suppose that for each $(t, y) \in X$ with $y \in Y$ there is a maximal ball $B_{t,y}$ containing t such that $B_{t,y} \times \{y\}$ is contained in X. Then X is a finite disjoint union of p-adic 1-cells A_i such that each ball $B_{t,y}$ appears as a ball above y of one of the cells A_i.

Proof. Note that each ball $B_{t,y}$ can be written as

$$B_{t,y} = \left\{ z \in K \mid \mathrm{ord}(z - w) = a_{t,y} \, , \; z - w \in Q_{1,1} \right\}, \tag{3.9.1}$$

for unique $a_{t,y} \in \mathbf{Z}$ and for (non-unique) $w \in K$. In the basic case that for each y in Y the set $X_y := \{t \in K \mid (t, y) \in X\}$ is a ball, one automatically has that $X_y = B_{t,y}$ for all $(t, y) \in X$. In this basic case define W as

$$W = \left\{ (w, y) \in K \times Y \mid \text{Equation (3.9.1) holds for } B_{t,y} \text{ and } w \right\}.$$

Now use Corollary 3.5 to find an $\mathcal{L}$-definable function $d : Y \to K$ whose graph lies in W and use Corollary 3.8 to finish this basic case.

In the general case partition X into finitely many cells X_i over Y with center c_i, coset $\lambda_i Q_{m_i,n_i}$, and base Y_i by using Theorem 3.3. Up to refining the partition $\{X_i\}$ of X, we may suppose that the following distinction only depends on i when y moves over Y_i: either $c_i(y)$ lies inside $B_{t,y}$ for some t, or, $c_i(y)$ lies outside $B_{t,y}$ for all t. Define I_1 and I_2 such that $i \in I_1$ if and only if $c_i(y)$ lies inside $B_{t,y}$ for some t, and $i \in I_2$ otherwise. Then, for $i \in I_1$ and $y \in Y_i$, let $B(i, y)$ be the ball $B_{t,y}$ containing $c_i(y)$ and define

$$A_i := \left\{ (t, y) \mid y \in Y_i \, , \; t \in B(i, y) \right\}.$$

Then the A_i for $i \in I_1$ are as in the basic case and can thus be treated. Put

$$X' = X \setminus \left(\bigcup_{i \in I_1} A_i \right).$$

It is enough to prove the statement of the lemma for X'. Fix $(t, y) \in X'$ and choose i (either in I_1 or in I_2) such that $B_{t,y}$ contains at least one of the balls of X_i

above y. (Such i must exist by the non-archimedean property and the maximality of the occurring balls.) Since by construction $c_i(y)$ lies outside $B_{t,y}$ for the fixed (t, y), there are $b_{t,y} \in \mathbf{Z}$, $m \leq m_i$, and $\lambda \in K^\times$ such that

$$B_{t,y} = \left\{ z \in K \mid \mathrm{ord}(z - c_i(y)) = b_{t,y} , \ z - c_i(y) \in \lambda Q_{m,n_i} \right\}.$$

Since there are only finitely many i, m, and cosets of Q_{m,n_i} in $K^\times$, we can finish by Corollary 3.8. $\qquad\qquad\square$

DEFINITION 3.10 (Jacobian property). Let $F : B_1 \to B_2$ be an $\mathcal{L}$-definable function with $B_1, B_2 \subset K$. Say that F *has the Jacobian property* if the following conditions (a) up to (d) hold

(a) F is a bijection $B_1 \to B_2$ and B_1 and B_2 are balls;
(b) F is C^1 on B_1; write $\mathrm{Jac}F$ for $\partial F/\partial x : B_1 \to K$;
(c) $\mathrm{ord}(\mathrm{Jac}F)$ is constant (and finite) on B_1;
(d) for all $x, y \in B_1$ with $x \neq y$, one has

$$\mathrm{ord}(\mathrm{Jac}F) + \mathrm{ord}(x - y) = \mathrm{ord}\big(F(x) - F(y)\big).$$

PROPOSITION 3.11 (Jacobian property for definable functions [CL, §6]). *Let Y and $X \subset K \times Y$ be $\mathcal{L}$-definable sets, let $F : X \to K$ be an $\mathcal{L}$-definable function. Suppose that for each $y \in Y$, the function $F(\cdot, y) : t \mapsto F(t, y)$ is injective. Then there exists a finite partition of X into p-adic cells A_i over Y such that for each i, each $y \in Y$ and each ball B of A_i above y, there is a ball $B^* \subset K$ such that the map*

$$F_B : B \to B^* : t \mapsto F(t, y)$$

is well defined and has the Jacobian property.

The following proposition is new and relies on Proposition 3.11.

PROPOSITION 3.12 (Compatible cell decompositions under a definable function F). *Let X, Y, and F be as in Proposition 3.11, where in particular $F(\cdot, y)$ is injective for each $y \in Y$. Define F_Y as the ($\mathcal{L}$-definable) function $F_Y : X \to K \times Y : (t, y) \mapsto (F(t, y), y)$. For A_i as in Proposition 3.11, write A_i' for the set $F_Y(A_i)$. Then we can choose the partition of X into cells A_i over Y as in Proposition 3.11 such that moreover each A_i' is a p-adic cell over Y, and such that for each $y \in Y$ and each ball B of A_i above y, B^* is a ball of A_i' above y, where B^* is as in Proposition 3.11. Hence, for any $y \in Y$, there is a correspondence between the balls of A_i above y and the balls of A_i' above y.*

Proof. Partition X into cells X_i as in Proposition 3.11. Up to this finite partition we may suppose that X equals X_1 which we may suppose is a 1-cell. Write X_1' for $F_Y(X_1)$. Partition X_1' into cells X_{1i}' with center d_i, coset $\lambda_i Q_{m_i,n_i}$, and base Y_i by using Theorem 3.3. For a ball $B = B_{t,y}$ of A_1 above y containing t write $B_{t,y}^*$ for the corresponding ball B^*, as given by Proposition 3.11. Up to refining the partition $\{X_{1i}'\}$ of X_1', we may suppose that the following distinction only depends on i when y moves over Y_i: either $d_i(y)$ lies inside $B_{t,y}^*$ for some t, or, $d_i(y)$ lies outside $B_{t,y}^*$ for all t. Define I_1 and I_2 such that $i \in I_1$ if and only if $d_i(y)$ lies inside $B_{t,y}^*$ for some t, and $i \in I_2$ otherwise. Then, for $i \in I_1$ and y in Y_i, let $B^*(i, y)$ be the ball $B_{t,y}^*$

containing $d_i(y)$. Let $B(i,y)$ be the ball $B_{t,y}$ where t is such that $B^*_{t,y} = B^*(i,y)$. By construction,

$$F\big(B(i,y) \times \{y\}\big) = B^*(i,y). \qquad (3.12.1)$$

For $i \in I_1$, define

$$A_i := \big\{(t,y) \mid y \in Y_i, \ t \in B(i,y)\big\}$$

and put $A'_i := F_Y(A_i)$. Apply Lemma 3.9 to A_i and A'_i for each $i \in I_1$. Then these A_i and A'_i are as required by (3.12.1) and it is thus sufficient to prove the proposition for the restriction of F to

$$X' := X \backslash \Big(\bigcup_{i \in I_1} A_i \Big).$$

Fix $(t,y) \in X'$ and choose i (either in I_1 or in I_2) such that $B^*_{t,y}$ contains at least one of the balls of X'_i above y. (Such i must exist by the maximality of the occurring balls.) Since by construction $d_i(y)$ lies outside $B^*_{t,y}$, and since $B^*_{t,y}$ contains a ball of X'_i above y, there are $b_{t,y} \in \mathbf{Z}$, $m \le m_i$, and $\lambda \in K^\times$ such that

$$B^*_{t,y} = \big\{z \in K \mid \operatorname{ord}(z - d_i(y)) = b_{t,y}, \ z - d_i(y) \in \lambda Q_{m,n_i}\big\}.$$

Since there are only finitely many cosets of Q_{m,n_i} in $K^\times$, the proposition follows from Theorem 3.3 and from the Presburger cell decomposition results of [C1] in a straightforward way (though slightly more delicately than in the proof of Corollary 3.8). Indeed, by Theorem 3.3 we may suppose that Y itself is a full cell with centers c_i in the respective coordinates of Y, and such that an extension of d_i to the whole of Y is K-analytic for each i. We may also suppose that X' projects surjectively onto Y, and that X has center c. Further one may suppose that λ and m do only depend on i, and not on (t,y). Now one maps the graph of the restriction of f to X' surjectively to a Presburger subset Γ of some Cartesian power of $\mathbf{Z}$, by taking as component functions $\operatorname{ord}(y_i - c_i)$ if this is an integer and the constant function 0 otherwise, the integer $\operatorname{ord}(t - c(y))$, and finally the pair of integers $(\operatorname{ord}(z - d_i(y)), i)$, where i is such that (z,y) lies in X'_{1i}, y lies in Y, and $z = f(t,y)$. Now apply the Presburger cell decomposition theorem of [C1] to Γ. This induces back a cell decomposition of X' as desired in the statement, by taking the inverse image under the constructed map. $\qquad \square$

Further we give a corollary of Proposition 3.11 that we will not use further on.

COROLLARY 3.13.   Let Y be an $\mathcal{L}$-definable set $F : X \subset K \times Y \to K$ be an $\mathcal{L}$-definable function such that $F(\cdot, y)$ is injective for each $y \in Y$. Then there exists a finite partition of X into $\mathcal{L}$-definable pieces X_i such that, for each i and each $y \in Y$, the restriction of $F(\cdot, y)$ to $X_{iy} := \{t \in K \mid (t,y) \in X_i\}$ or its inverse function is locally 1-Lipschitz.

Proof. Apply Proposition 3.11 to F, yielding a partition of X into cells A_i over Y. Now partition each A_i into pieces according to the condition that $|\partial F/\partial t|$ is ≤ 1, resp. > 1 on the piece. On the pieces where $|\partial F/\partial t|$ is ≤ 1 we are done by the Jacobian property which holds by construction. On a piece where $|\partial F/\partial t|$ is > 1, the inverse of $F(\cdot, y)$ is locally 1-Lipschitz by the chain rule for differentiation and the Jacobian property which holds by construction. $\qquad \square$

Note that the different possibilities for the (non-exclusive) disjunctions in Corollary 3.13 can be supposed to depend only on i (and not on y) by taking the parts X_i small enough. Indeed, the occurring conditions as local 1-Lipschitz continuity, injectivity, and so on, are $\mathcal{L}$-definable in $y \in Y$.

4   Proofs of the Main Results

Theorem 2.3 and Proposition 2.4 are proved using a joint induction on m.

Proof of Theorem 2.3 for $m = 1$.   We are given $\varepsilon > 0$, Y an $\mathcal{L}$-definable set, and $f : X \subset K \times Y \to K$ an $\mathcal{L}$-definable function such that for each $y \in Y$ the function $f(\cdot, y) : x \mapsto f(x, y)$ is locally ε-Lipschitz continuous on its natural domain $X_y := \{x \in K \mid (x, y) \in X\}$. Using Corollary 3.7, we may suppose that $f(\cdot, y)$ is injective for each y. Use Proposition 3.12 to partition X into finitely many p-adic cells X_i over Y with center c_i. By working piecewise we may suppose that $X = X_1$ and that X_1 is a 1-cell over Y. By the Jacobian property $f(\cdot, y)$ is C^1 and by local ε-Lipschitz continuity,

$$\left| \partial f(x, y) / \partial x \right| \leq \varepsilon \tag{4.0.1}$$

for all $(x, y) \in X$. By the above application of Proposition 3.12, the set

$$X' := f_Y(X),$$

with $f_Y : X \to K \times Y : (x, y) \mapsto (f(x, y), y)$, is a p-adic 1-cell with some center d_1. Since a function $g : A \subset K \to K$ is C-Lipschitz continuous if and only if $A \to K : x \mapsto g(x + a) + b$ is C-Lipschitz continuous for any constants $a, b \in K$, we may thus suppose, after translating, that c_1 and d_1 are identically zero.

Now fix $y \in Y$. Take (x_1, y) and (x_2, y) in X. If x_1 and x_2 both lie in the ball $B_{x_1, y}$, then

$$\left| (\partial f(x_1, y) / \partial x) \cdot (x_1 - x_2) \right| = \left| f(x_1, y) - f(x_2, y) \right| \tag{4.0.2}$$

by the Jacobian property. By (4.0.1) we are done and can take any $C \geq \varepsilon$.

Next suppose that $B_{x_1, y}$ and $B_{x_2, y}$ are two different balls. By our assumption that c and d are identically zero, we can write

$$B_{x_i, y} = \left\{ x \in K \mid \operatorname{ord}(x) = a_{x_i, y}, \ \overline{\mathrm{ac}}_m(x) = \overline{\mathrm{ac}}_m \lambda \right\}$$

and likewise for their images under $f(\cdot, y)$,

$$B^*_{x_i, y} = \left\{ z \in K \mid \operatorname{ord}(z) = b_{x_i, y}, \ \overline{\mathrm{ac}}_{m'}(z) = \overline{\mathrm{ac}}_{m'} \mu \right\}.$$

From these descriptions we get the inequalities

$$\operatorname{ord}\left(f(x_1, y) - f(x_2, y) \right) = \min_{i=1,2}(b_{x_i, y})$$

and

$$\min_{i=1,2}(a_{x_i, y}) = \operatorname{ord}(x_1 - x_2).$$

On the other hand by the Jacobian property (d) one has

$$m + \operatorname{ord}\left(\partial f(x_i, y) / \partial x \right) + a_{x_i, y} = m' + b_{x_i, y}$$

for $i = 1, 2$. Hence, putting this together with (4.0.1),

$$\left| f(x_1, y) - f(x_2, y) \right| = \max_{i=1,2} p^{-b_{x_i, y}} \leq \varepsilon p^{m'-m} \max_{i=1,2} p^{-a_{x_i, y}} = \varepsilon p^{m'-m} |x_1 - x_2|$$

and thus one can take any $C \geq \varepsilon \cdot \max(1, p^{m'-m})$.   □

REMARK 4.1.   The chain rule for differentiation yields the following statement for a C^1 function $f : X \subset K \times Y \to K$ on an open set X in the variables (t, y), where $y = (y_1, \ldots, y_m)$ runs over an open $Y \subset K^m$ and t over K. Suppose that the function $f(\cdot, y) : t \mapsto f(t, y)$ is injective and has C^1 inverse for each $y \in Y$. Define Z as $\{(f(t, y), y) \in K \times Y \mid (t, y) \in X\}$ and define the function $g : Z \to K$ by $(z, y) \mapsto t$ for the unique t with $f(t, y) = z$. Then one has for each $i = 1, \ldots, m$

$$\frac{\partial g(z, y)}{\partial y_i} = -\frac{\partial f(t, y)}{\partial y_i} \cdot \left(\frac{\partial f(t, y)}{\partial t}\right)^{-1}$$

where $z = f(t, y)$.

Proof of Proposition 2.4 for m using Theorem 2.3 for $m - 1$.   We will proceed by induction on m. For $m = 1$ the statement of Proposition 2.4 is trivial and hence we may suppose that $m > 1$. Up to a finite partition of X, we may assume that X itself is either a 1-cell or a 0-cell over $K^{m-1} \times Y$ along some coordinate projection $p : K^m \times Y \to K^{m-1} \times Y$, say with center c.

First suppose we are in the basic case that, for y such that X_y is nonempty, the set $p(X)_y$ is not open in K^{m-1}. By the induction hypotheses, we may suppose that $p(X)$ is a p-adic cell (over $K^{m-2} \times Y$) with center c_{m-1} such that $c_{m-1}(\cdot, y))$ is C-Lipschitz in the (relevant) variables $x_1, \ldots, x_{m-2}$ for each fixed value of $y \in Y$, and so on for in total $m - 1$ subsequent coordinate projections, up to the projection to Y. Then, after the (triangular) bi-Lipschitz continuous transformation where we replace x_{m-1} by $x_{m-1} - c_{m-1}$ and so on $m-1$ times, we may suppose that the center of the cell $p(X)$ is identically zero, and so on $m-1$ times up to the projection to Y. If we still use the name $p(X)$ for the so-obtained transformed set, there must be a coordinate x_i, for some $i = 1, \ldots, m-1$, which is identically zero on $p(X)$, and thus we can finish by Proposition 2.4 for $m - 1$.

Hence, we can focus on the more interesting case that the $p(X)_y$ are open in K^{m-1} for all $y \in Y$ and of course we may then suppose that moreover $c(\cdot, y)$ is C^1 on $p(X)_y$ for all $y \in Y$. After reordering the variables $x_1, \ldots, x_{m-1}$ and after finitely partitioning $p(X)$, we may suppose that $|\partial c/\partial x_{m-1}|$ is maximal among the $|\partial c/\partial x_i|$ on the whole of $p(X)$ for $i = 1, \ldots, m-1$. If $|\partial c/\partial x_{m-1}| \leq 1$ on the whole of $p(X)$, then we are done by Proposition 1.4 and Theorem 2.3 for $m - 1$, up to a finite partition of $p(X)$. Hence, we may further assume, up to a finite partition of $p(X)$, that $1 < |\partial c/\partial x_{m-1}|$ on the whole of $p(X)$. Using Corollary 3.7 and up to a further finite partition of $p(X)$, we may furthermore suppose that $c(x_1, \ldots, x_{m-2}, \cdot, y)$ is injective for each $(x_1, \ldots, x_{m-2}, y)$. Now partition $p(X)$ again, as follows. Use Proposition 3.12 for the map $c(x_1, \ldots, x_{m-2}, \cdot, y)$ to partition $p(X)$ into finitely many p-adic cells A_i over $K^{m-2} \times Y$, along the projection $(x_1, \ldots, x_{m-1}, y) \mapsto (x_1, \ldots, x_{m-2}, y)$. Up to such a finite partition of $p(X)$, we may suppose that $p(X) = A_1$ and that A_1 has center c_1.

First we treat the more simple case that X is a 0-cell over $p(X)$. In this case we simply invert the role of x_m and x_{m-1} in the build-up of the cell X as follows. Write $d(x_1, \ldots, x_{m-2}, \cdot, y)$ for the inverse function $(c(x_1, \ldots, x_{m-2}, \cdot, y))^{-1}$ of $c(x_1, \ldots, x_{m-2}, \cdot, y)$. Then X is also a 0-cell over $K^{m-1} \times Y$ along the projection p'

sending $(x_1, \ldots, x_m, y)$ to $(x_1, \ldots, x_{m-2}, x_m, y)$ with center d. Since d is constructed as an inverse function and if we recall the differentiation rule for inverse functions and Remark 4.1, it is clear that all partial derivatives $\partial d/\partial x_i$ for $i = 1, 2 \ldots, m-2, m$ are bounded in norm. We are done by Proposition 1.4 and Theorem 2.3 for $m - 1$, up to a finite partition of $p'(X)$.

Finally, we treat the most interesting case that X is a 1-cell. For $1 \leq i \leq m$ and $a \in K^m$ write $\hat{a}_i$ for $(a_1, \ldots, a_i)$. Fix (a, y) in X. Let $B_{a,y}$ be the unique ball (of the cell X) above $(\hat{a}_{m-1}, y)$ which contains a_m. Further, let $B_{a,y}^0$ be the unique ball of the cell $p(X)$ that contains a_{m-1} and lies above $(\hat{a}_{m-2}, y)$. By the previous application of Proposition 3.12 for the map $c(x_1, \ldots, x_{m-2}, \cdot, y)$, the image of $B_{a,y}^0$ under $c(\hat{a}_{m-2}, \cdot, y)$ is a ball $B_{a,y}^{0*}$ and one has moreover descriptions, uniformly in (a, y) in X,

$$B_{a,y} = \left\{ x_m \mid \operatorname{ord}(x_m - c(\hat{a}_{m-1}, y)) = b_{a,y}, \ \overline{\operatorname{ac}}_n(x_m - c(\hat{a}_{m-1}, y)) = \overline{\operatorname{ac}}_n \lambda \right\}$$

$$B_{a,y}^0 = \left\{ x_{m-1} \mid \operatorname{ord}(x_{m-1} - c_1(\hat{a}_{m-2}, y)) = b_{a,y}^0, \ \overline{\operatorname{ac}}_{n'}(x_{m-1} - c_1(\hat{a}_{m-2}, y)) = \overline{\operatorname{ac}}_{n'} \lambda' \right\}$$

and

$$B_{a,y}^{0*} = \left\{ z \mid \operatorname{ord}(z - e(\hat{a}_{m-2}, y)) = b_{a,y}^{0*}, \ \overline{\operatorname{ac}}_{n''}(z - e(\hat{a}_{m-2}, y)) = \overline{\operatorname{ac}}_{n''} \lambda'' \right\},$$

for some nonzero constants λ, λ', λ'', n, n', n'' coming from the descriptions of the cells, where e is the $\mathcal{L}$-definable function as given by the previous application of Proposition 3.12, and where $b_{a,y}$ only depends on the ball $B_{a,y}$, and similarly $b_{a,y}^0$ only depends on $B_{a,y}^0$ and $b_{a,y}^{0*}$ only on $B_{a,y}^{0*}$.

We will compare sizes of balls, where we call a ball B_1 strictly bigger in size than a ball B_2 if a translate of B_2 is strictly contained in B_1 and we say that B_1 and B_2 are equal in size of a translate of B_1 equals B_2. By partitioning $p(X)$ further we may assume that we are in one of the following two cases.

Case 1. The ball $B_{a,y}$ is bigger or equal in size than $B_{a,y}^{0*}$ for all (a, y) in X. Case 1 is equivalent to $b_{a,y}^{0*} \geq n - n'' + b_{a,y}$ for all (a, y) in X. Also, $B_{a,y}$ does not depend on a_{m-1} when a_{m-1} runs over $B_{a,y}^0$, but the center c itself may of course depend nontrivially on x_{m-1}. We will replace the center c by another center which depends trivially on x_{m-1}, as follows. By construction $c(\hat{a}_{m-1}, y)$ lies in $B_{a,y}^{0*}$ and the set $B_{a,y}^{0*}$ is described above. By this description, e is a kind of approximation of c and is thus a candidate to become the new center instead of c, which we show indeed to work as follows. Let ℓ be $\max(n, n'')$. Partition X into finitely many parts where $\overline{\operatorname{ac}}_\ell(x_m - e(\hat{a}_{m-2}, y))$ is constant. Next, apply Corollary 3.8 to each such part to obtain a partition of X into finitely many cells A_i with center the restriction of e to the base of A_i (Corollary 3.8 can be applied because ℓ is well chosen). Hence, up to this finite partition we can suppose that the center of the p-adic cell X does not depend on the variable x_{m-1}. With this new situation, we can go back in the proof and start reordering the variables $x_1, \ldots, x_{m-1}$ and finitely partitioning $p(X)$ such that $|\partial c/\partial x_{m-1}|$ is again maximal among the $|\partial c/\partial x_i|$ on the whole of $p(X)$ for $i = 1, \ldots, m-1$, as we did above. After finitely many recursions, we will not fall into Case 1 anymore. Indeed, if the $|\partial c/\partial x_i|$ are ≤ 1 then we are in a case treated above and so on.

Case 2. The ball $B_{a,y}^{0*}$ is strictly bigger in size than $B_{a,y}$ for all (a, y) in X. Case 2 is equivalent to $n + b_{a,y} > n'' + b_{a,y}^{0*}$ for all (a, y) in X and implies

$$B_{a,y} \subset B_{a,y}^{0*} \quad \text{for all } (a, y) \in X. \tag{4.1.1}$$

By construction, we can consider the inverse function of $c(\hat{a}_{m-2}, \cdot, y)$. Write $d(\hat{a}_{m-2}, \cdot, y)$ for the inverse function of $c(\hat{a}_{m-2}, \cdot, y)$. Then the domain of $d(\hat{a}_{m-2}, \cdot, y)$ contains in particular the ball $B_{a,y}$ by (4.1.1) and hence, we can apply $d(\hat{a}_{m-2}, \cdot, y)$ to x_m for any point $(\hat{a}_{m-2}, x_{m-1}, x_m, y)$ in X. Partition X into finitely many parts where $\overline{\mathrm{ac}}_{\ell'}(x_{m-1} - d(\hat{a}_{m-2}, x_m, y))$ is constant for some sufficiently large ℓ'. Since ℓ' is sufficiently large and by the Jacobian property which holds by the previous application of Proposition 3.12, we can apply Corollary 3.8 to each such part to obtain a partition of X into finitely many p-adic cells A_i with center the restriction of d to the base of A_i. Up to this partition, we may suppose that X is a p-adic cell over $K^{m-1} \times Y$ along the projection p' sending (x, y) to $(\hat{x}_{m-2}, x_m, y)$ with center d. Since d is constructed as an inverse function and if we recall the differentiation rule for inverse functions and Remark 4.1, it is clear that all partial derivatives $\partial d / \partial x_i$ for $i = 1, 2, \ldots, m - 2, m$ are bounded in norm. Hence we can finish by Proposition 1.4 and Theorem 2.3 for $m - 1$, up to a finite partition of $p'(X)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2.3 for $m > 1$, using Proposition 2.4 for m. We proceed by induction on m, where the case $m = 1$ is proved above in this section. We are given $\varepsilon > 0$, an $\mathcal{L}$-definable set Y and an $\mathcal{L}$-definable function $f : X \subset K^m \times Y \to K$ such that for each $y \in Y$ the function $f(\cdot, y) : x \mapsto f(x, y)$ is locally ε-Lipschitz continuous. Merely to ensure later on that partial derivatives are well defined when they appear, we may now already suppose that the X_y are K-analytic manifolds on which $f(\cdot, y)$ is K-analytic for each $y \in Y$, but we will not necessarily maintain this property throughout the proof for the parts of upcoming partitions. Use the notation $\hat{x}$ for $(x_1, \ldots, x_{m-1})$ and similarly for tuples $a = (a_1, \ldots, a_m)$ in K^m for which $\hat{a} = (a_1, \ldots, a_{m-1})$. By Theorem 2.3 for $m - 1$ and up to a finite partition of X, we may suppose that for each $(a, y) = (a_1, \ldots, a_m, y)$ in X each of the functions

$$f(\cdot, a_i, \cdot, y) : (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \mapsto f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_m, y)$$
$$\tag{4.1.2}$$

is C-Lipschitz continuous. By Theorem 3.3 we may suppose that X is a cell over $K^{m-1} \times Y$, say, with center c, along a coordinate projection $p : X \to K^{m-1} \times Y$. By Proposition 2.4 for m, we may suppose that $c(\cdot, y)$ is C-Lipschitz continuous in $\hat{x}$ for each $y \in Y$. Up to a finite partition of $p(X)$, we may suppose by Theorem 2.3 for $m - 1$ that, for each fixed value of a_m and y, the function $\hat{x} \mapsto f(\hat{x}, a_m - c(\hat{x}, y), y)$ is C-Lipschitz continuous. Hence, if we perform the bi-Lipschitz transformation which replaces x_m by $x_m - c(\hat{x}, y)$ but which preserves the other coordinates, then we see that we may suppose that

(∗) X is a cell over $K^{m-1} \times Y$ whose center is identically zero, the function $f(\cdot, a_m, y)$ is C-Lipschitz continuous in $\hat{x}$ for each $a_m \in K$ and $y \in Y$, and for all fixed $\hat{x}, y$, the function $f(\hat{x}, \cdot, y)$ is also C-Lipschitz continuous in x_m.

In the simple case that X is a 0-cell over $K^{m-1} \times Y$, we can finish by Theorem 2.3 for $m-1$, since the x_m-coordinate is identically zero on X and can be neglected. Next suppose that X is a 1-cell. Let (a, y) and (a', y) be given in X. If $|a_m| = |a'_m|$, then the point $(\hat{a}, a'_m, y)$ also lies in X by the definition of cells since the center of X is zero, and hence we can jump inside X from (a, y) to $(\hat{a}, a'_m, y)$ and finally jump further to (a', y). Calculating the images under f and controlling the distances between these points yields

$$
\begin{aligned}
\left| f(a, y) - f(a', y) \right| &= \left| f(a, y) - f(\hat{a}, a'_m, y) + f(\hat{a}, a'_m, y) - f(a', y) \right| \\
&\leq \max \left(|f(a, y) - f(\hat{a}, a'_m, y)|, \ |f(\hat{a}, a'_m, y) - f(a', y)| \right) \\
&\leq \max \left(C|a_m - a'_m|, \ C|(\hat{a}, a'_m) - a'| \right) \\
&= C|a - a'|,
\end{aligned}
$$

where the first inequality follows from the non-archimedean property, the second from property $(*)$, and the last equality follows from the definition of the norm on K^m. Let us now suppose that $|a_m| \neq |a'_m|$, say, $|a'_m| < |a_m|$. First suppose that $|x_m|$ has no lower bound in X, that is, for each $(\hat{x}, x_m, y)$ in X there exists $x'_m \in K$ arbitrarily close to 0 such that $(\hat{x}, x'_m, y)$ still lies in X. Then again the point $(\hat{a}, a'_m, y)$ lies in X by the definition of cells and since the center of X is zero. Hence we can make the same jumps as in the previous case and the same computation will hold for the same reasons. In the other case we may suppose that for each (b, y) in X there is a minimal value $e(\hat{b}, y) > 0$ among the values $|x_m|$ for all x_m with $(\hat{b}, x_m, y)$ in X. Let $\alpha(\hat{b}, y) \in K$ be such that $|\alpha(\hat{b}, y)| = e(\hat{b}, y)$. By Corollary 3.5 we may suppose that α is an $\mathcal{L}$-definable function in $(\hat{b}, y)$ whose graph lies in X.

Up to a finite partition we may suppose, by Proposition 2.4 for $m-1$, that $p(X)$ is a full cell over Y whose centers are all C-Lipschitz continuous. If for $y \in Y$, X_y is not open in K^m if nonempty, then we can can, after a natural triangular transformation, force one of the coordinates x_i for $i = 1, \ldots, m$ to be 0 on X, as in the proof of Proposition 2.4. Hence in this case, we are again done by Theorem 2.3 for $m-1$. Thus, we may suppose that X_y is open in K^m for each y.

Up to a finite partition of $p(X)$ and by Proposition 1.4, we may suppose that we are in one of the two following cases.

Case 1. The function α is locally C-Lipschitz continuous on $p(X)_y$ for each $y \in Y$. Case 1 implies that the functions $\hat{x} \mapsto f(\hat{x}, \alpha(\hat{x}), y)$ are also locally C'-Lipschitz continuous for some C'. By Theorem 2.3 for $m-1$ and up to a finite partition of $p(X)$ we may suppose that, for all y,

$$
\hat{x} \mapsto f\big(\hat{x}, \alpha(\hat{x}), y\big) \tag{4.1.3}
$$

is C-Lipschitz continuous on the whole of $p(X)$, possibly by replacing C by some bigger constant as allowed. Omitting y from the notation, we will jump from a to $(\hat{a}, \alpha(\hat{a}))$, jump further to $(\hat{a}', \alpha(\hat{a}'))$, and finally to a', where $\hat{a}'$ of course stands for $(a'_1, \ldots, a'_{m-1})$. We compute, still omitting y from the notation,

$$
\begin{aligned}
\left| f(a) - f(a') \right| &= \left| f(a) - f(\hat{a}, \alpha(\hat{a})) + f(\hat{a}, \alpha(\hat{a})) - f(\hat{a}', \alpha(\hat{a}')) + f(\hat{a}', \alpha(\hat{a}')) - f(a') \right| \\
&\leq C \max \left(|a_m - \alpha(\hat{a})|, |(\hat{a}, \alpha(\hat{a})) - (\hat{a}', \alpha(\hat{a}'))|, |\alpha(\hat{a}') - a'_m| \right)
\end{aligned}
$$

$$= C|a - a'|,$$

where the inequality holds by the C-Lipschitz continuity of the function (4.1.3) and by property $(*)$, and the last equality holds by properties of the non-archimedean norm on K^m and the facts that $|a_m| \neq |a'_m|$ and $|\alpha(\hat{a})| \leq |a_m|$ and $|\alpha(\hat{a})| \leq |a'_m|$.

Case 2. $|\partial\alpha/\partial x_i| > 1$ **on** $p(X)$ **for some** $i < m$. We may further suppose that for a certain j, $|\partial\alpha/\partial x_j|$ is maximal among the $|\partial\alpha/\partial x_i|$ on the whole of $p(X)$. For notational simplicity, suppose that $j = m - 1$ (the case $j < m - 1$ is only notationally different). Hence, by Corollary 3.7 and Proposition 3.11 we may suppose that

$$\alpha(\hat{x}_{m-2}, \cdot, y)$$

is injective and C^1 with C^1 inverse for each $(x, y) \in X$. Let $\beta(\hat{x}_{m-2}, \cdot, y)$ be the inverse of $\alpha(\hat{x}_{m-2}, \cdot, y)$. We will make exactly the same jumps as in Case 1, and establish exactly the same series of inequalities, but these inequalities will hold for different reasons than in Case 1. Write $\hat{x}_{m-2}$ for $(x_1, \ldots, x_{m-2})$. By Remark 4.1, by the differentiation rule for inverse functions, and by the above supposition that $f(\cdot, y)$ is K-analytic on X_y for each y (which still may be supposed to hold here), the function

$$F : (\hat{x}_{m-2}, x_m, y) \mapsto f\big(\hat{x}_{m-2}, \beta(\hat{x}_{m-2}, x_m, y), x_m, y\big)$$

has bounded partial derivatives along x_i for $i = 1, \ldots, m-2, m$, on its natural domain $X' \subset K^{m-1} \times Y$ (where X'_y is also open for each y, hence the partial derivatives are well defined on X'_y). By Proposition 1.4 and Theorem 2.3 for $m - 1$, after a finite partition of $p(X)$ (and hence of X'), we may suppose that $F(\cdot, y)$ is C-Lipschitz continuous on X'_y for each y. Write d for $\alpha(\hat{a}_{m-1}, y)$ and d' for $\alpha(\hat{a}'_{m-1}, y)$. Now we can compute, since $(\hat{a}_{m-1}, d, y), y) = (\hat{a}_{m-2}, \beta(\hat{a}_{m-2}, d, y), d, y)$ and again omitting y from the notation,

$$\begin{aligned}
\big|f(a) - f(a')\big| &= \big|f(a) - f(\hat{a}_{m-1}, d) + f(\hat{a}_{m-2}, \beta(\hat{a}_{m-2}, d), d) \\
&\quad - f(\hat{a}'_{m-2}, \beta(\hat{a}'_{m-2}, d'), d') + f(\hat{a}'_{m-2}, \beta(\hat{a}'_{m-2}, d'), d') - f(a')\big| \\
&\leq C \max\big(|a_m - d|, |(\hat{a}_{m-2}, \beta(\hat{a}_{m-2}, d), d) \\
&\quad - (\hat{a}'_{m-2}, \beta(\hat{a}'_{m-2}, d'), d')|, |d' - a'_m|\big) \\
&= C|a - a'|,
\end{aligned}$$

where we have used that $F(\cdot, y)$ is C-Lipschitz continuous for each y (instead of the Lipschitz continuity of (4.1.3) used in Case 1) but further similar reasons as in Case 1. Indeed, the above equations and inequalities have exactly the same meaning as in Case 1, they are only written differently to make it apparent that the Lipschitz continuity of $F(\cdot, y)$ can be used.   □

Elementary equivalent fields. We conclude with an analogue of Theorem 2.3 for p-adically closed fields. Let K_1 be a field which is elementary equivalent to K in the language $\mathcal{L}$. Then K_1 is a valued field and we write $\mathcal{O}_{K_1}$ for its valuation ring. One uses the norm notation $|\cdot|$ for the natural map from K to the ordered multiplicative semi-group $\Gamma_1 := K_1/(\mathcal{O}_{K_1}^\times)$. Using this norm, for rational $C > 0$, one can take the obvious definition for Lipschitz continuity with constant C. More

generally, for nonzero C in the divisible hull of Γ_1, there is a natural notion of Lipschitz continuity with constant C. Note that in the following result we can take C to be a rational number, which is stronger than allowing nonzero C from Γ_1.

PROPOSITION 4.2. *Let a rational number $\varepsilon > 0$ be given. Let $f : X \subset K_1^m \times Y \to K_1$ be an $\mathcal{L}(K_1)$-definable function (that is, $\mathcal{L}$-definable with parameters from the field K_1) such that for each $y \in Y$ the function $f(\,\cdot\,, y) : x \mapsto f(x, y)$ is locally ε-Lipschitz continuous on X_y, where also Y is $\mathcal{L}(K_1)$-definable. Then there exist a rational number $C > 0$ and a finite partition of X into $\mathcal{L}(K_1)$-definable parts A_i such that for each $y \in Y$ and i the restriction of $f(\,\cdot\,, y)$ to A_{iy} is (globally) C-Lipschitz continuous.*

Proof. (The proof uses a standard technique for using results like Theorem 2.3 which hold for definable families.) In both the $\mathcal{L}(K_1)$-formulas φ_X and φ_f describing X and f there appear only finitely many parameters from K_1, say $r_1, \ldots, r_s \in K_1$. Replace these parameters $r_1, \ldots, r_s$ by new variables, say $z_1, \ldots, z_s$ (that is, the z_i do not occur in the formulas φ_X and φ_f). Let the tuple of variables $z = (z_1, \ldots, z_s)$ run over K_1^s. The obvious variants of φ_X and φ_f, with the r_i replaced by the z_i, are of course interpretable in the standard p-adic field K itself. Now it is an exercise to construct $\mathcal{L}$-definable families $\tilde{X}_{\tilde{y}} \subset K^m$ of sets and functions $\tilde{f}_{\tilde{y}} : \tilde{X}_{\tilde{y}} \to K$, for some suitable parameter $\tilde{y}$ (containing in particular the z-tuple), which fall under the conditions and thus the conclusion of Theorem 2.3 in such a way that Proposition 4.2 follows when one fills in the values r_i back in for z_i.   □

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