AX-KOCHEN-ERŠOV THEOREMS FOR P-ADIC INTEGRALS AND MOTIVIC INTEGRATION

by

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1. Introduction

This paper is concerned with extending classical results à la Ax-Kochen-Eršov to p-adic integrals in a motivic framework. The first section is expository, starting from Artin's conjecture and the classical work of Ax, Kochen, and Eršov and ending with recent work of Denef and Loeser giving a motivic version of the results of Ax, Kochen, and Eršov. In that section we have chosen to adopt a quite informal style, since the reader will find precise technical statements of more general results in later sections. We also explain the cell decomposition Theorem of Denef-Pas and how it leads to a quick proof of the results of Ax, Kochen, and Eršov. In sections 3, 4 and 5, we present our new, general construction of motivic integration, in the framework of constructible motivic functions. This has been announced in [**5**] and [**6**] and is developed in the paper [**7**]. In the last two sections we explain the relation to p-adic integration and we announce general Ax-Kochen-Eršov Theorems for integrals depending on parameters. We conclude the paper by discussing briefly the relevance of our results to the study of orbital integrals and the Fundamental Lemma.

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2. From Ax-Kochen-Eršov to motives

2.1. Artin's conjecture. — Let i and d be integers. A field K is said to be $C_i(d)$ if every homogeneous polynomial of degree d with coefficients in K in d^i+1 (effectively appearing) variables has a non trivial zero in K. Note we could replace "in d^i+1 variables" by "in at least d^i+1 variables" in that definition. When the field K is $C_i(d)$ for every d we say it is C_i. For instance for a field K to be C_0 means to be algebraically closed, and all finite fields are C_1, thanks to the Chevalley-Warning

Theorem. Also, one can prove without much trouble that if the field K is C_i then the fields $K(X)$ and $K((X))$ are C_{i+1}. It follows in particular that the fields $\mathbf{F}_q((X))$ are C_2.

***2.2. Conjecture* (Artin).** — *The p-adic fields $\mathbf{Q}_p$ are C_2.*

In 1965 Terjanian [**29**] gave an example of homogeneous form of degree 4 in $\mathbf{Q}_2$ in $18 > 4^2$ variables having only trivial zeroes in $\mathbf{Q}_2$, thus giving a counterexample to Artin's Conjecture. Let us briefly recall Terjanian's construction, refering to [**29**] and [**10**] for more details. The basic idea is the following: if f is a homogeneous polynomial of degree 4 in 9 variables with coeffiecients in $\mathbf{Z}$, such that, for every x in $\mathbf{Z}^9$, if $f(x) \equiv 0 \mod 4$, then 2 divides x, then the polynomial in 18 variables $h(x, y) = f(x) + 4f(y)$ will have no non trivial zero in $\mathbf{Q}_2$. An example of such a polynomial f is given by

(2.2.1) $$f = n(x_1, x_2, x_3) + n(x_4, x_5, x_6) + n(x_7, x_8, x_9)$$

with

(2.2.2) $$n(X, Y, Z) = X^2YZ + XY^2Z + XYZ^2 + X^2 + Y^2 + Z^2 - X^4 - Y^4 - Z^4.$$

At about the same time, Ax and Kochen proved that, if not true, Artin's conjecture is asymptotically true in the following sense:

***2.3. Theorem* (Ax-Kochen).** — *An integer d being fixed, all but finitely many fields $\mathbf{Q}_p$ are $C_2(d)$.*

2.4. Some Model Theory. — In fact, Theorem 2.3 is a special instance of the following, much more general, statement:

***2.5. Theorem* (Ax-Kochen-Eršov).** — *Let φ be a sentence in the language of rings. For all but finitely prime numbers p, φ is true in $\mathbf{F}_p((X))$ if and only if it is true in $\mathbf{Q}_p$. Moreover, there exists an integer N such that for any two local fields K, K' with isomorphic residue fields of characteristic $> N$ one has that φ is true in K if and only if it is true in K'.*

By a sentence in the language of rings, we mean a formula, without free variables, built from symbols 0, $+$, $-$, 1, $\times$, symbols for variables, logical connectives $\wedge$, $\vee$, $\neg$, quantifiers $\exists$, $\forall$ and the equality symbol $=$. It is very important that in this language, any given natural number can be expressed - for instance 3 as $1 + 1 + 1$- but that quantifiers running for instance over natural numbers are not allowed. Given a field k, we may interpret any such formula φ in k, by letting the quantifiers run over k, and, when φ is a sentence, we may say whether φ is true in k or not. Since for a field to be $C_2(d)$ for a fixed d may be expressed by a sentence in the language of rings, we see that Theorem 2.3 is a special case of Theorem 2.5. On the other hand, it is for instance impossible to express by a single sentence in the language of rings for a field to be algebraically closed.

In fact, it is natural to introduce here the language of valued fields. It is a language with two sorts of variables. The first sort of variables will run over the valued field and the second sort of variables will run over the value group. We shall use the language of rings over the valued field variables and the language of ordered abelian groups $0, +, -, \geq$ over the value group variables. Furthemore, there will be an additional functional symbol ord, going from the valued field sort to the value group sort, which will be interpreted as assigning to a non zero element in the valued field its valuation.

***2.6. Theorem* (Ax-Kochen-Eršov).** — *Let K and K' be two henselian valued fields of residual characteristic zero. Assume their residue fields k and k' and their value groups Γ and Γ' are elementary equivalent, that is, they have the same set of true sentences in the rings, resp. ordered abelian groups, language. Then K and K' are elementary equivalent, that is, they satisfy the same set of formulas in the valued fields language.*

We shall explain a proof of Theorem 2.5 after Theorem 2.10. Let us sketch how Theorem 2.5 also follows from Theorem 2.6. Indeed this follows directly from the classical ultraproduct construction. Let φ be a given sentence in the language of valued fields. Suppose by contradiction that for each r in $\mathbf{N}$ there exist two local fields K_r, K'_r with isomorphic residue field of characteristic $> r$ and such that φ is true in K_r and false in K'_r. Let U be a non principal ultrafilter on $\mathbf{N}$. Denote by $\mathbf{F}_U$ the corresponding ultraproduct of the residue fields of K_r, r in $\mathbf{N}$. It is a field of characteristic zero. Now let K_U and K'_U be respectively the ultraproduct relative to U of the fields K_r and K'_r. They are both henselian with residue field $\mathbf{F}_U$ and value group $\mathbf{Z}_U$, the ultraproduct over U of the ordered group $\mathbf{Z}$. Hence certainly Theorem 2.6 applies to K_U and K'_U. By the very ultraproduct construction, φ is true in K_U and false in K'_U, which is a contradiction.

2.7. In this paper, we shall in fact consider, instead of the language of valued fields, what we call a language of Denef-Pas, $\mathcal{L}_{\mathrm{DP}}$. It it is a language with 3 sorts, running respectively over valued field, residue field, and value group variables. For the first 2 sorts, the language is the ring language and for the last sort, we take any extension of the language of ordered abelian groups. For instance, one may choose for the last sort the Presburger language $\{+, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \mathbf{N},\ n > 1\}$, where $\equiv_n$ denote equivalence modulo n. We denote the corresponding Denef-Pas language by $\mathcal{L}_{\mathrm{DP,P}}$. We also have two additional symbols, ord as before, and a functional symbol ac, going from the valued field sort to the residue field sort.

A typical example of a structure for that language is the field of Laurent series $k((t))$ with the standard valuation $\mathrm{ord} : k((t))^\times \to \mathbf{Z}$ and ac defined by $\mathrm{ac}(x) = xt^{-\mathrm{ord}\,(x)} \mod t$ if $x \neq 0$ in $k((t))$ and by $\mathrm{ac}(0) = 0$.[(1)] Also, we shall usually add to the language constant symbols in the first, resp. second, sort for every element of $k((t))$ resp. k, thus considering formulas with coefficients in $k((t))$, resp. k, in

[(1)]Technically speaking, any function symbol of a first order language must have as domain a product of sorts; a concerned reader may choose an arbitrary extension of ord to the whole field K; sometimes we will use $\mathrm{ord}_0 : K \to \mathbf{Z}$ which sends 0 to 0 and nonzero x to $\mathrm{ord}\,(x)$.

the valued field, resp. residue field, sort. Similarly, any finite extension of $\mathbf{Q}_p$ is naturally a structure for that language, once a uniformizing parameter ϖ has been chosen; one just sets $\mathrm{ac}(x) = x\varpi^{-\mathrm{ord}\,(x)} \mod \varpi$ and $\mathrm{ac}(0) = 0$. In the rest of the paper, for $\mathbf{Q}_p$ itself, we shall always take $\varpi = p$.

We now consider a valued field K with residue field k and value group $\mathbf{Z}$. We assume k is of characteristic zero, K is henselian and admits an angular component map, that is, a map $\mathrm{ac} : K \to k$ such that $\mathrm{ac}(0) = 0$, ac restricts to a multiplicative morphism $K^\times \to k^\times$, and on the set $\{x \in K, \mathrm{ord}\,(x) = 0\}$, ac restricts to the canonical projection to k. We also assume that $(K, k, \Gamma, \mathrm{ord}\,, \mathrm{ac})$ is a structure for the language $\mathcal{L}_{\mathrm{DP}}$.

We call a subset C of $K^m \times k^n \times \mathbf{Z}^r$ definable if it may be defined by a $\mathcal{L}_{\mathrm{DP}}$-formula. We call a function $h : C \to K$ definable if its graph is definable.

2.8. Definition. — Let $D \subset K^m \times k^{n+1} \times \mathbf{Z}$ and $c : K^m \times k^n \to K$ be definable. For ξ in k^n, we set

$$A(\xi) = \Big\{(x, t) \in K^m \times K \,\Big|\, (x, \xi, \mathrm{ac}(t - c(x, \xi)), \mathrm{ord}_0(t - c(x, \xi))) \in D\},$$

where $\mathrm{ord}_0(x) = \mathrm{ord}\,(x)$ for $x \neq 0$ and $\mathrm{ord}_0(0) = 0$. If for every ξ and ξ' in k^n with $\xi \neq \xi'$, we have $A(\xi) \cap A(\xi') = \emptyset$, then we call

$$A = \bigcup_{\xi \in k^n} A(\xi) \tag{2.8.1}$$

a cell in $K^m \times K$ with parameters ξ and center $c(x, \xi)$.

Now can state the following version of the cell decomposition Theorem of Denef and Pas:

***2.9. Theorem* (Denef-Pas [26])**. — *Consider functions $f_1(x, t)$, …, $f_r(x, t)$ on $K^m \times K$ which are polynomials in t with coefficients definable functions from K^m to K. Then, $K^m \times K$ admits a finite partition into cells A with parameters ξ and center $c(x, \xi)$, such that, for every ξ in k^n, (x, t) in $A(\xi)$, and $1 \leq i \leq r$, we have,*

$$\mathrm{ord}_0 f_i(x, t) = \mathrm{ord}_0 h_i(x, \xi)(t - c(x, \xi))^{\nu_i} \tag{2.9.1}$$

and

$$\mathrm{ac} f_i(x, t) = \xi_i, \tag{2.9.2}$$

where the functions $h_i(x, \xi)$ are definable and ν_i, n are in $\mathbf{N}$ and where $\mathrm{ord}_0(x) = \mathrm{ord}\,(x)$ *for $x \neq 0$ and* $\mathrm{ord}_0(0) = 0$.

Using Theorem 2.9 it is not difficult to prove by induction on the number of valued field variables the following quantifier elimination result (in fact, Theorems 2.9 and 2.10 have a joint proof in [**26**]):

***2.10. Theorem* (Denef-Pas [26])**. — *Let K be a valued field satisfying the above conditions. Then, every formula in $\mathcal{L}_{\mathrm{DP}}$ is equivalent to a formula without quantifiers running over the valued field variables.*

Let us now explain why Theorem 2.5 follows easily from Theorem 2.10. Let U be a non principal ultrafilter on $\mathbf{N}$. Let K_r and K'_r be local fields for every r in $\mathbf{N}$, such that the residue field of K_r is isomorphic to the residue field of K'_r and has characteristic $> r$. We consider again the fields K_U and K'_U that are respectively the ultraproduct relative to U of the fields K_r and K'_r. By the argument we already explained it is enough to prove that these two fields are elementary equivalent. Clearly they have isomorphic residue fields and isomorphic value groups (isomorphic as ordered groups). Furthermore they both satisfy the hypotheses of Theorem 2.10. Consider a sentence true for K_U. Since it is equivalent to a sentence with quantifiers running only over the residue field variables and the value group variables, it will also be true for K'_U, and vice versa.

Note that the use of cell decomposition to prove Ax-Kochen-Eršov type results goes back to P.J. Cohen [**8**].

2.11. From sentences to formulas. — Let φ be a formula in the language of valued fields or, more generally, in the language $\mathcal{L}_{\mathrm{DP,P}}$ of Denef-Pas. We assume that φ has m free valued field variables and no free residue field nor value group variables. For every valued field K which is a structure for the language $\mathcal{L}_{\mathrm{DP}}$, we denote by $h_\varphi(K)$ the set of points $(x_1, \ldots, x_m)$ in K^m such that $\varphi(x_1, \ldots, x_m)$ is true.

When $m = 0$, φ is a sentence and $h_\varphi(K)$ is either the one point set or the empy set, depending on whether φ is true in K or not. Having Theorem 2.5 in mind, a natural question is to compare $h_\varphi(\mathbf{Q}_p)$ with $h_\varphi(\mathbf{F}_p((t)))$.

An answer is provided by the following statement:

***2.12. Theorem* (Denef-Loeser [15]).** — *Let φ be a formula in the language $\mathcal{L}_{\mathrm{DP,P}}$ with m free valued field variables and no free residue field nor value group variables. There exists a virtual motive M_φ, canonically attached to φ, such that, for almost all prime numbers p, the volume of $h_\varphi(\mathbf{Q}_p)$ is finite if and only if the volume of $h_\varphi(\mathbf{F}_p((t)))$ is finite, and in this case they are both equal to the number of points of M_φ in $\mathbf{F}_p$.*

Here we have chosen to state Theorem 2.12 in an informal, non technical way. A detailed presentation of more general results we recently obtained is given in § 7. A few remarks are necessary in order to explain the statement of Theorem 2.12. Firstly, what is meant by volume? Let d be an integer such that for almost all p, $h_\varphi(\mathbf{Q}_p)$ is contained in $X(\mathbf{Q}_p)$, for some subvariety of dimension d of $\mathbf{A}^m_{\mathbf{Q}}$. Then the volume is taken with respect to the canonical d-dimensional measure (cf. § 6 and 7). Implicit in the statement of the Theorem is the fact that $h_\varphi(\mathbf{Q}_p)$ and $h_\varphi(\mathbf{F}_p((t)))$ are measurable (at least for almost all p for the later one). Originally, cf. [**15**] [**16**] [**17**], the virtual motive M_φ lies in a certain completion of the ring $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ explained in 5.7 (in particular, $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ is a subring of the Grothendieck ring of Chow motives with rational coefficients), but it now follows from the new construction of motivic integration developed in [**7**] that we can take

M_φ in the ring obtained from $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ by inverting the Lefschetz motive $\mathbf{L}$ and $1 - \mathbf{L}^{-n}$ for $n > 0$.

One should note that even for $m = 0$, Theorem 2.12 gives more information than Theorem 2.5, since it says that for almost all p the validity of φ in $\mathbf{Q}_p$ and $\mathbf{F}_p((t))$ is governed by the virtual motive M_φ. Finally, let us note that Theorem 2.5 naturally extends to integrals of definable functions as will be explained in § 7.

The proof of Theorem 2.12 is based on motivic integration. In the next sections we shall give a quick overview of the new general construction of motivic integration given in [**7**], that allows one to integrate a very general class of functions, constructible motivic functions. These results have already been announced in a condensed way in the notes [**5**] and [**6**]; here, we are given the opportunity to present them more leisurely and with some more details.

3. Constructible motivic functions

3.1. Definable subassignments. — Let φ be a formula in the language $\mathcal{L}_{\mathrm{DP,P}}$ with coefficients in $k((t))$, resp. k, in the valued field, resp. residue field, sort, having say respectively m, n, and r free variables in the various sorts. To such a formula φ we assign, for every field K containing k, the subset $h_\varphi(K)$ of $K((t))^m \times K^n \times \mathbf{Z}^r$ consisting of all points satisfying φ. We shall call the datum of such subsets for all K definable (sub)assignments. In analogy with algebraic geometry, where the emphasis is not put anymore on equations but on the functors they define, we consider instead of formulas the corresponding subassignments (note $K \mapsto h_\varphi(K)$ is in general not a functor). Let us make these definitions more precise.

First, we recall the definition of subassignments, introduced in [**15**]. Let $F : \mathcal{C} \to \mathrm{Ens}$ be a functor from a category $\mathcal{C}$ to the category of sets. By a subassignment h of F we mean the datum, for every object C of $\mathcal{C}$, of a subset $h(C)$ of $F(C)$. Most of the standard operations of elementary set theory extend trivially to subassignments. For instance, given subassignments h and h' of the same functor, one defines subassignments $h \cup h'$, $h \cap h'$ and the relation $h \subset h'$, etc. When $h \subset h'$ we say h is a subassignment of h'. A morphism $f : h \to h'$ between subassignments of functors F_1 and F_2 consists of the datum for every object C of a map $f(C) : h(C) \to h'(C)$. The graph of f is the subassignment $C \mapsto \mathrm{graph}(f(C))$ of $F_1 \times F_2$.

Next, we explain the notion of definable subassignments. Let k be a field and consider the category F_k of fields containing k. We denote by $h[m, n, r]$ the functor $F_k \to \mathrm{Ens}$ given by $h[m, n, r](K) = K((t))^m \times K^n \times \mathbf{Z}^r$. In particular, $h[0, 0, 0]$ assigns the one point set to every K. To any formula φ in $\mathcal{L}_{\mathrm{DP,P}}$ with coefficients in $k((t))$, resp. k, in the valued field, resp. residue field, sort, having respectively m, n, and r free variables in the various sorts, we assign a subassignment h_φ of $h[m, n, r]$, which associates to K in F_k the subset $h_\varphi(K)$ of $h[m, n, r](K)$ consisting of all points satisfying φ. We call such subassignments definable subassignements. We denote by Def_k the category whose objects are definable subassignments of some $h[m, n, r]$, morphisms in Def_k being morphisms of subassignments $f : h \to h'$ with h

and h' definable subassignments of $h[m, n, r]$ and $h[m', n', r']$ respectively such that the graph of f is a definable subassignment. Note that $h[0, 0, 0]$ is the final object in this category.

If S is an object of Def_k, we denote by Def_S the category of morphisms $X \to S$ in Def_k. If $f : X \to S$ and $g : Y \to S$ are in Def_S, we write $X \times_S Y$ for the product in Def_S defined as $K \mapsto \{(x, y) \in X(K) \times Y(K) | f(x) = g(y)\}$, with the natural morphism to S. When $S = h[0, 0, 0)]$, we write $X \times Y$ for $X \times_S Y$. We write $S[m, n, r]$ for $S \times h[m, n, r]$, hence, $S[m, n, r](K) = S(K) \times K((t))^m \times K^n \times \mathbf{Z}^r$. By a point x of S we mean a pair (x_0, K) with K in F_k and x_0 a point of $S(K)$. We denote by $|S|$ the set of points of S. For such x we then set $k(x) = K$. Consider a morphism $f : X \to S$, with X and S respectively definable subassignments of $h[m, n, r]$ and $h[m', n', r']$. Let $\varphi(x, s)$ be a formula defining the graph of f in $h[m+m', n+n', r+r']$. Fix a point (s_0, K) of S. The formula $\varphi(x, s_0)$ defines a subassignment in Def_K. In this way we get for s a point of S a functor "fiber at s" $i_s^* : \mathrm{Def}_S \to \mathrm{Def}_{k(s)}$.

3.2. Constructible motivic functions. — In this subsection we define, for S in Def_k, the ring $\mathcal{C}(S)$ of constructible motivic functions on S. The main goal of this construction is that, as we will see in section 4, motivic integrals with parameters in S are constructible motivic functions on S. In fact, in the construction of a measure, as we all know since studying Lebesgue integration, positive functions often play a basic fundamental role. This the reason why we also introduce the semiring $\mathcal{C}_+(S)$ of positive[(2)] constructible motivic functions. A technical novelty occurs here: $\mathcal{C}(S)$ is the ring associated to the semiring $\mathcal{C}_+(S)$, but the canonical morphism $\mathcal{C}_+(S) \to \mathcal{C}(S)$ has in general no reason to be injective.

Basically, $\mathcal{C}_+(S)$ and $\mathcal{C}(S)$ are built up from two kinds of functions. The first type consists of elements of a certain Grothendieck (semi)ring. Recall that in "classical" motivic integration as developed in [**14**], the Grothendieck ring $K_0(\mathrm{Var}_k)$ of algebraic varieties over k plays a key role. In the present setting the analogue of the category of algebraic varieties over k is the category of definable subassignments of $h[0, n, 0]$, for some n, when $S = h[0, 0, 0]$. Hence, for a general S in Def_k, it is natural to consider the subcategory RDef_S of Def_S whose objects are definable subassignments Z of $S \times h[0, n, 0]$, for variable n, the morphism $Z \to S$ being induced by the projection on S. The Grothendieck semigroup $SK_0(\mathrm{RDef}_S)$ is the quotient of the free semigroup on isomorphism classes of objects $[Z \to S]$ in RDef_S by relations $[\emptyset \to S] = 0$ and $[(Y \cup Y') \to S] + [(Y \cap Y') \to S] = [Y \to S] + [Y' \to S]$. We also denote by $K_0(\mathrm{RDef}_S)$ the corresponding abelian group. Cartesian product induces a unique semiring structure on $SK_0(\mathrm{RDef}_S)$, resp. ring structure on $K_0(\mathrm{RDef}_S)$.

There are some easy functorialities. For every morphism $f : S \to S'$, there is a natural pullback by $f^* : SK_0(\mathrm{RDef}_{S'}) \to SK_0(\mathrm{RDef}_S)$ induced by the fiber product. If $f : S \to S'$ is a morphism in $\mathrm{RDef}_{S'}$, composition with f induces a morphism $f_! : SK_0(\mathrm{RDef}_S) \to SK_0(\mathrm{RDef}_{S'})$. Similar constructions apply to K_0. That one can view elements of $SK_0(\mathrm{RDef}_S)$ as functions on S (which we even would like to

(2) Or maybe better, non negative.

integrate), is illustrated in section 6 on p-adic integration and in the introduction of [**7**], in the part on integration against Euler characteristic over the reals.

The second type of functions are certain functions with values in the ring

$$A = \mathbf{Z}\Big[\mathbf{L}, \mathbf{L}^{-1}, \Big(\frac{1}{1-\mathbf{L}^{-i}}\Big)_{i>0}\Big], \tag{3.2.1}$$

where, for the moment, $\mathbf{L}$ is just considered as a symbol. Note that a definable morphism $\alpha : S \to h[0,0,1]$ determines a function $|S| \to \mathbf{Z}$, also written α, and a function $|S| \to A$ sending x to $\mathbf{L}^{\alpha(x)}$, written $\mathbf{L}^{\alpha}$. We consider the subring $\mathcal{P}(S)$ of the ring of functions $|S| \to A$ generated by constants in A and by all functions α and $\mathbf{L}^{\alpha}$ with $\alpha : S \to \mathbf{Z}$ definable morphisms. Now we should define positive functions with values in A. For every real number $q > 1$, let us denote by $\vartheta_q : A \to \mathbf{R}$ the morphism sending $\mathbf{L}$ to q. We consider the subsemigroup A_+ of A consisting of elements a such that $\vartheta_q(a) \geq 0$ for all $q > 1$ and we define $\mathcal{P}_+(S)$ as the semiring of functions in $\mathcal{P}(S)$ taking their values in A_+.

Now we explain how to put together these two type of functions. For Y a definable subassignment of S, we denote by $\mathbf{1}_Y$ the function in $\mathcal{P}(S)$ taking the value 1 on Y and 0 outside Y. We consider the subring $\mathcal{P}^0(S)$ of $\mathcal{P}(S)$, resp. the subsemiring $\mathcal{P}^0_+(S)$ of $\mathcal{P}_+(S)$, generated by functions of the form $\mathbf{1}_Y$ with Y a definable subassignment of S, and by the constant function $\mathbf{L} - 1$. We have canonical morphisms $\mathcal{P}^0(S) \to K_0(\mathrm{RDef}_S)$ and $\mathcal{P}^0_+(S) \to SK_0(\mathrm{RDef}_S)$ sending $\mathbf{1}_Y$ to $[Y \to S]$ and $\mathbf{L}-1$ to the class of $S \times (h[0,1,0] \setminus \{0\})$ in $K_0(\mathrm{RDef}_S)$ and in $SK_0(\mathrm{RDef}_S)$, respectively. To simplify notation we shall denote by $\mathbf{L}$ and $\mathbf{L}-1$ the class of $S[0,1,0]$ and $S \times (h[0,1,0] \setminus \{0\})$ in $K_0(\mathrm{RDef}_S)$ and in $SK_0(\mathrm{RDef}_S)$.

We may now define the semiring of positive constructible functions as

$$\mathcal{C}_+(S) = SK_0(\mathrm{RDef}_S) \otimes_{\mathcal{P}^0_+(S)} \mathcal{P}_+(S) \tag{3.2.2}$$

and the ring of constructible functions as

$$\mathcal{C}(S) = K_0(\mathrm{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S). \tag{3.2.3}$$

If $f : S \to S'$ is a morphism in Def_k, one shows in [**7**] that the morphism f^* may naturally be extended to a morphism

$$f^* : \mathcal{C}_+(S') \longrightarrow \mathcal{C}_+(S). \tag{3.2.4}$$

If, furthermore, f is a morphism in $\mathrm{RDef}_{S'}$, one shows that the morphism $f_!$ may naturally be extended to

$$f_! : \mathcal{C}_+(S) \longrightarrow \mathcal{C}_+(S'). \tag{3.2.5}$$

Similar functorialities exist for $\mathcal{C}$.

3.3. Constructible motivic "Functions". — In fact, we shall need to consider not only functions as we just defined, but functions defined almost everywhere in a given dimension, that we call Functions. (Note the capital in Functions.)

We start by defining a good notion of dimension for objects of Def_k. Heuristically, that dimension corresponds to counting the dimension only in the valued field variables, without taking in account the remaining variables. More precisely, to any algebraic subvariety Z of $\mathbf{A}^m_{k((t))}$ we assign the definable subassignment h_Z of $h[m, 0, 0]$ given by $h_Z(K) = Z(K((t)))$. The Zariski closure of a subassignment S of $h[m, 0, 0]$ is the intersection W of all algebraic subvarieties Z of $\mathbf{A}^m_{k((t))}$ such that $S \subset h_Z$. We define the dimension of S as $\dim S := \dim W$. In the general case, when S is a subassignment of $h[m, n, r]$, we define $\dim S$ as the dimension of the image of S under the projection $h[m, n, r] \to h[m, 0, 0]$.

One can prove, using Theorem 2.9 and results of van den Dries [**18**], the following result, which is by no means obvious:

3.4. Proposition. — *Two isomorphic objects of* Def_k *have the same dimension.*

For every non negative integer d, we denote by $\mathcal{C}_+^{\leq d}(S)$ the ideal of $\mathcal{C}_+(S)$ generated by functions $\mathbf{1}_Z$ with Z definable subassignments of S with $\dim Z \leq d$. We set $C_+(S) = \oplus_d C_+^d(S)$ with $C_+^d(S) := \mathcal{C}_+^{\leq d}(S)/\mathcal{C}_+^{\leq d-1}(S)$. It is a graded abelian semigroup, and also a $\mathcal{C}_+(S)$-semimodule. Elements of $C_+(S)$ are called positive constructible Functions on S. If φ is a function lying in $\mathcal{C}_+^{\leq d}(S)$ but not in $\mathcal{C}_+^{\leq d-1}(S)$, we denote by $[\varphi]$ its image in $C_+^d(S)$. One defines similarly $C(S)$ from $\mathcal{C}(S)$.

One of the reasons why we consider functions which are defined almost everywhere originates in the differentiation of functions with respect to the valued field variables: one may show that a definable function $c : S \subset h[m, n, r] \to h[1, 0, 0]$ is differentiable (in fact even analytic) outside a definable subassignment of S of dimension $< \dim S$. In particular, if $f : S \to S'$ is an isomorphism in Def_k, one may define a function $\mathrm{ordjac} f$, the order of the jacobian of f, which is defined almost everywhere and is equal almost everywhere to a definable function, so we may define $\mathbf{L}^{-\mathrm{ordjac} f}$ in $C_+^d(S)$ when S is of dimension d. In 5.2, we shall define $\mathbf{L}^{-\mathrm{ordjac} f}$ using differential forms.

4. Construction of the general motivic measure

Let k be a field of characteristic zero. Given S in Def_k, we define S-integrable Functions and construct pushforward morphisms for these:

4.1. Theorem. — *Let k be a field of characteristic zero and let S be in* Def_k. *There exists a unique functor $Z \mapsto \mathrm{I}_S C_+(Z)$ from* Def_S *to the category of abelian semigroups, the functor of S-integrable Functions, assigning to every morphism $f : Z \to Y$ in* Def_S *a morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ such that for every Z in* Def_S, *$\mathrm{I}_S C_+(Z)$ is a graded subsemigroup of $C_+(Z)$ and $\mathrm{I}_S C_+(S) = C_+(S)$, satisfying the following list of axioms (A1)-(A8).*

(A1a) **(Naturality)**
If $S \to S'$ is a morphism in Def_k and Z is an object in Def_S, then any S'-integrable Function φ in $C_+(Z)$ is S-integrable and $f_!(\varphi)$ is the same, considered in $\mathrm{I}_{S'}$ or in I_S.

(A1b) **(Fubini)**

A positive Function φ on Z is S-integrable if and only if it is Y-integrable and $f_!(\varphi)$ is S-integrable.

(A2) **(Disjoint union)**
If Z is the disjoint union of two definable subassignments Z_1 and Z_2, then the isomorphism $C_+(Z) \simeq C_+(Z_1) \oplus C_+(Z_2)$ induces an isomorphism $\mathrm{I}_S C_+(Z) \simeq \mathrm{I}_S C_+(Z_1) \oplus \mathrm{I}_S C_+(Z_2)$, under which $f_! = f_{|Z_1 !} \oplus f_{|Z_2 !}$.

(A3) **(Projection formula)**
For every α in $\mathcal{C}_+(Y)$ and every β in $\mathrm{I}_S C_+(Z)$, $\alpha f_!(\beta)$ is S-integrable if and only if $f^*(\alpha)\beta$ is, and then $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.

(A4) **(Inclusions)**
If $i : Z \hookrightarrow Z'$ is the inclusion of definable subassignments of the same object of Def_S, $i_!$ is induced by extension by zero outside Z and sends injectively $\mathrm{I}_S C_+(Z)$ to $\mathrm{I}_S C_+(Z')$.

(A5) **(Integration along residue field variables)**
Let Y be an object of Def_S and denote by π the projection $Y[0, n, 0] \to Y$. A Function $[\varphi]$ in $C_+(Y[0, n, 0])$ is S-integrable if and only if, with notations of 3.2.5, $[\pi_!(\varphi)]$ is S-integrable and then $\pi_!([\varphi]) = [\pi_!(\varphi)]$.

Basically this axiom means that integrating with respect to variables in the residue field just amounts to taking the pushforward induced by composition at the level of Grothendieck semirings.

(A6) **(Integration along Z-variables)** Basically, integration along $\mathbf{Z}$-variables corresponds to summing over the integers, but to state precisely (A6), we need to perform some preliminary constructions.

Let us consider a function in φ in $\mathcal{P}(S[0, 0, r])$, hence φ is a function $|S| \times \mathbf{Z}^r \to A$. We shall say φ is S-integrable if for every $q > 1$ and every x in $|S|$, the series $\sum_{i \in \mathbf{Z}^r} \vartheta_q(\varphi(x, i))$ is summable. One proves that if φ is S-integrable there exists a unique function $\mu_S(\varphi)$ in $\mathcal{P}(S)$ such that $\vartheta_q(\mu_S(\varphi)(x))$ is equal to the sum of the previous series for all $q > 1$ and all x in $|S|$. We denote by $\mathrm{I}_S \mathcal{P}_+(S[0, 0, r])$ the set of S-integrable functions in $\mathcal{P}_+(S[0, 0, r])$ and we set

$$\mathrm{I}_S \mathcal{C}_+(S[0, 0, r]) = \mathcal{C}_+(S) \otimes_{\mathcal{P}_+(S)} \mathrm{I}_S \mathcal{P}_+(S[0, 0, r]). \tag{4.1.1}$$

Hence $\mathrm{I}_S \mathcal{P}_+(S[0, 0, r])$ is a sub-$\mathcal{C}_+(S)$-semimodule of $\mathcal{C}_+(S[0, 0, r])$ and μ_S may be extended by tensoring to

$$\mu_S : \mathrm{I}_S \mathcal{C}_+(S[0, 0, r]) \to \mathcal{C}_+(S). \tag{4.1.2}$$

Now we can state (A6):
Let Y be an object of Def_S and denote by π the projection $Y[0, 0, r] \to Y$. A Function $[\varphi]$ in $C_+(Y[0, 0, r])$ is S-integrable if and only if there exists φ' in $\mathcal{C}_+(Y[0, 0, r])$ with $[\varphi'] = [\varphi]$ which is Y-integrable in the previous sense and such that $[\mu_Y(\varphi')]$ is S-integrable. We then have $\pi_!([\varphi]) = [\mu_Y(\varphi')]$.

(A7) **(Volume of balls)** It is natural to require (by analogy with the p-adic case) that the volume of a ball $\{z \in h[1,0,0] | \mathrm{ac}(z-c) = \alpha, \mathrm{ac}(z-c) = \xi\}$, with α in $\mathbf{Z}$, c in $k((t))$ and ξ non zero in k, should be $\mathbf{L}^{-\alpha-1}$. (A7) is a relative version of that statement:
Let Y be an object in Def_S and let Z be the definable subassignment of $Y[1,0,0]$ defined by $\mathrm{ord}\,(z-c(y)) = \alpha(y)$ and $\mathrm{ac}(z-c(y)) = \xi(y)$, with z the coordinate on the $\mathbf{A}^1_{k((t))}$-factor and α, ξ, c definable functions on Y with values respectively in $\mathbf{Z}$, $h[0,1,0] \setminus \{0\}$, and $h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbf{L}^{-\alpha-1}[\mathbf{1}_Y]$ is, and then $f_!([\mathbf{1}_Z]) = \mathbf{L}^{-\alpha-1}[\mathbf{1}_Y]$.

(A8) **(Graphs)** This last axiom expresses the pushforward for graph projections. It relates volume and differentials and is a special case of the change of variables Theorem 4.2.
Let Y be in Def_S and let Z be the definable subassignment of $Y[1,0,0]$ defined by $z - c(y) = 0$ with z the coordinate on the $\mathbf{A}^1_{k((t))}$-factor and c a morphism $Y \to h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbf{L}^{(\mathrm{ordjac} f)\circ f^{-1}}$ is, and then $f_!([\mathbf{1}_Z]) = \mathbf{L}^{(\mathrm{ordjac} f)\circ f^{-1}}$.

Once Theorem 4.1 is proved, one may proceed as follows to extend the constructions from C_+ to C . One defines $\mathrm{I}_S C(Z)$ as the subgroup of $C(Z)$ generated by the image of $\mathrm{I}_S C_+(Z)$. One shows that if $f : Z \to Y$ is a morphism in Def_S, the morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ has a natural extension $f_! : \mathrm{I}_S C(Z) \to \mathrm{I}_S C(Y)$.
The relation of Theorem 4.1 with motivic integration is the following. When S is equal to $h[0,0,0]$, the final object of Def_k, one writes $\mathrm{I}C_+(Z)$ for $\mathrm{I}_S C_+(Z)$ and we shall say integrable for S-integrable, and similarly for C. Note that $\mathrm{I}C_+(h[0,0,0]) = C_+(h[0,0,0]) = SK_0(\mathrm{RDef}_k) \otimes_{\mathbf{N}[\mathbf{L}-1]} A_+$ and that $\mathrm{I}C(h[0,0,0]) = K_0(\mathrm{RDef}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A$. For φ in $\mathrm{I}C_+(Z)$, or in $\mathrm{I}C(Z)$, one defines the motivic integral $\mu(\varphi)$ by $\mu(\varphi) = f_!(\varphi)$ with f the morphism $Z \to h[0,0,0]$. Working in the more general framework of Theorem 4.1 to construct μ appears to be very convenient for inductions occuring in the proofs. Also, it is not clear how to characterize μ alone by existence and unicity properties. Note also, that one reason for the statement of Theorem 4.1 to look somewhat cumbersone, is that we have to define at once the notion of integrability and the value of the integral.

The proof of Theorem 4.1 is quite long and involved. In a nutshell, the basic idea is the following. Integration along residue field variables is controlled by (A5) and integration along $\mathbf{Z}$-variables by (A6). Integration along valued field variables is constructed one variable after the other. To integrate with respect to one valued field variable, one may, using (a variant of) the cell decomposition Theorem 2.9 (at the cost of introducing additional new residue field and $\mathbf{Z}$-variables), reduce to the case of cells which is covered by (A7) and (A8). An important step is to show that this is independent of the choice of a cell decomposition. When one integrates with respect to more than one valued field variable (one after the other) it is crucial to show that it is independent of the order of the variables, for which we use a notion of bicells.

In this new framework, we have the following general form of the change of variables Theorem, generalizing the corresponding statements in [**14**] and [**15**].

4.2. Theorem. — *Let $f : X \to Y$ be an isomorphism between definable subassignments of dimension d. For every function φ in $\mathcal{C}_+^{\leq d}(Y)$ having a non zero class in $C_+^d(Y)$, $[f^*(\varphi)]$ is Y-integrable and $f_![f^*(\varphi)] = \mathbf{L}^{(\mathrm{ordjac} f)\circ f^{-1}}[\varphi]$. A similar statement holds in C.*

4.3. Integrals depending on parameters. — One pleasant feature of Theorem 4.1 is that it generalizes readily to the relative setting of integrals depending on parameters.

Indeed, let us fix Λ in Def_k playing the role of a parameter space. For S in Def_Λ, we consider the ideal $\mathcal{C}^{\leq d}(S \to \Lambda)$ of $\mathcal{C}_+(S)$ generated by functions $\mathbf{1}_Z$ with Z definable subassignment of S such that all fibers of $Z \to \Lambda$ are of dimension $\leq d$. We set

$$C_+(S \to \Lambda) = \bigoplus_d C_+^d(S \to \Lambda) \tag{4.3.1}$$

with

$$C_+^d(S \to \Lambda) := \mathcal{C}_+^{\leq d}(S \to \Lambda)/\mathcal{C}_+^{\leq d-1}(S \to \Lambda). \tag{4.3.2}$$

It is a graded abelian semigroup (and also a $\mathcal{C}_+(S)$-semimodule). If φ belongs to $\mathcal{C}_+^{\leq d}(S \to \Lambda)$ but not to $\mathcal{C}_+^{\leq d-1}(S \to \Lambda)$, we write $[\varphi]$ for its image in $C_+^d(S \to \Lambda)$. The following relative analogue of Theorem 4.1 holds.

4.4. Theorem. — *Let k be a field of characteristic zero, let Λ be in Def_k, and let S be in Def_Λ. There exists a unique functor $Z \mapsto \mathrm{I}_S C_+(Z \to \Lambda)$ from Def_S to the category of abelian semigroups, assigning to every morphism $f : Z \to Y$ in Def_S a morphism $f_{!\Lambda} : \mathrm{I}_S C_+(Z \to \Lambda)) \to \mathrm{I}_S C_+(Y \to \Lambda))$ satisfying properties analogue to (A0)-(A8) obtained by replacing $C_+(\_)$ by $C_+(\_ \to \Lambda)$ and* ordjac *by its relative analogue* ordjac_Λ[3].

Note that $C_+(\Lambda \to \Lambda) = \mathcal{C}_+(\Lambda)$ (and also $\mathrm{I}_\Lambda C_+(\Lambda \to \Lambda) = C_+(\Lambda \to \Lambda)$. Hence, given $f : Z \to \Lambda$ in Def_Λ, we may define the relative motivic measure with respect to Λ as the morphism

$$\mu_\Lambda := f_{!\Lambda} : \mathrm{I}_\Lambda C_+(Z \to \Lambda) \longrightarrow \mathcal{C}_+(\Lambda). \tag{4.4.1}$$

By the following statement, μ_Λ indeed corresponds to integration along the fibers over Λ:

(3) Defined similarly as ordjac, but using relative differential forms.

4.5. Proposition. — *Let φ be a Function in $C_+(Z \to \Lambda)$. It belongs to $\mathrm{I}_\Lambda C_+(Z \to \Lambda)$ if and only if for every point λ in Λ, the restriction φ_λ of φ to the fiber of Z at λ is integrable. The motivic integral of φ_λ is then equal to $i_\lambda^*(\mu_\Lambda(\varphi))$, for every λ in Λ.*

Similarly as in the absolute case, one can also define the relative analogue $C(S \to \Lambda)$ of $C(S)$, and extend the notion of integrability and the construction of $f_{!\Lambda}$ to this setting.

5. Motivic integration in a global setting and comparison with previous constructions

5.1. Definable subassignments on varieties. — Objects of Def_k are by construction affine, being subassignments of functors $h[m, n, r] : F_k \to \mathrm{Ens}$ given by $K \mapsto K((t))^m \times K^n \times \mathbf{Z}^r$. We shall now consider their global analogues and extend the previous constructions to the global setting.

Let $\mathcal{X}$ be a variety over $k((t))$, that is, a reduced and separated scheme of finite type over $k((t))$, and let X be a variety over k. For r an integer ≥ 0, we denote by $h[\mathcal{X}, X, r]$ the functor $F_k \to \mathrm{Ens}$ given by $K \mapsto \mathcal{X}(K((t))) \times X(K) \times \mathbf{Z}^r$. When $X = \operatorname{Spec} k$ and $r = 0$, we write $h[\mathcal{X}]$ for $h[\mathcal{X}, X, r]$. If $\mathcal{X}$ and X are affine and if $i : \mathcal{X} \hookrightarrow \mathbf{A}^m_{k((t))}$ and $j : X \hookrightarrow \mathbf{A}^n_k$ are closed immersions, we say a subassignment h of $h[\mathcal{X}, X, r]$ is definable if its image by the morphism $h[\mathcal{X}, X, r] \to h[m, n, r]$ induced by i and j is a definable subassignment of $h[m, n, r]$. This definition does not depend on i and j. More generally, we shall say a subassignment h of $h[\mathcal{X}, X, r]$ is definable if there exist coverings $(\mathcal{U}_i)$ and (U_j) of $\mathcal{X}$ and X by affine open subsets such that $h \cap h[\mathcal{U}_i, U_j, r]$ is a definable subassignment of $h[\mathcal{U}_i, U_j, r]$ for every i and j. We get in this way a category GDef_k whose objects are definable subassignments of some $h[\mathcal{X}, X, r]$, morphisms being definable morphisms, that is, morphisms whose graphs are definable subassignments.

The category Def_k is a full subcategory of GDef_k. Dimension as defined in 3.3 may be directly generalized to objects of GDef_k and Proposition 3.4 still holds in GDef_k. Also, if S is an object in GDef_k, our definitions of RDef_S, $\mathcal{C}_+(S)$, $\mathcal{C}(S)$, $C_+(S)$ and $C(S)$ extend.

5.2. Definable differential forms and volume forms. — In the global setting, one does not integrate functions anymore, but volume forms. Let us start by introducing differential forms in the definable framework. Let h be a definable subassignment of some $h[\mathcal{X}, X, r]$. We denote by $\mathcal{A}(h)$ the ring of definable morphisms $h \to h[\mathbf{A}^1_{k((t))}]$. Let us define, for i in $\mathbf{N}$, the $\mathcal{A}(h)$-module $\Omega^i(h)$ of definable i-forms on h. Let $\mathcal{Y}$ be the closed subset of $\mathcal{X}$, which is the Zariski closure of the image of h under the projection $\pi : h[\mathcal{X}, X, r] \to h[\mathcal{X}]$. We denote by $\Omega^i_{\mathcal{Y}}$ the sheaf of algebraic i-forms on $\mathcal{Y}$, by $\mathcal{A}_{\mathcal{Y}}$ the Zariski sheaf associated to the presheaf $U \mapsto \mathcal{A}(h[U])$ on

$\mathcal{Y}$, and by $\Omega^i_{h[\mathcal{Y}]}$ the sheaf $\mathcal{A}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Omega^i_{\mathcal{Y}}$. We set

$$\Omega^i(h) := \mathcal{A}(h) \otimes_{\mathcal{A}(h[\mathcal{Y}])} \Omega^i_{h[\mathcal{Y}]}(\mathcal{Y}), \tag{5.2.1}$$

the $\mathcal{A}(h[\mathcal{Y}])$-algebra structure on $\mathcal{A}(h)$ given by composition with π.

We now assume h is of dimension d. We denote by $\mathcal{A}^<(h)$ the ideal of functions in $\mathcal{A}(h)$ that are zero outside a definable subassignment of dimension $< d$. There is a canonical morphism of abelian semi-groups $\lambda : \mathcal{A}(h)/\mathcal{A}^<(h) \to C^d_+(h)$ sending the class of a function f to the class of $\mathbf{L}^{-\mathrm{ord}\, f}$, with the convention $\mathbf{L}^{-\mathrm{ord}\, 0} = 0$. We set $\tilde{\Omega}^d(h) = \mathcal{A}(h)/\mathcal{A}^<(h) \otimes_{\mathcal{A}(h)} \Omega^d(h)$, and we define the set $|\tilde{\Omega}|_+(h)$ of definable positive volume forms as the quotient of the free abelian semigroup on symbols (ω, g) with ω in $\tilde{\Omega}^d(h)$ and g in $C^d_+(h)$ by relations $(f\omega, g) = (\omega, \lambda(f)g)$, $(\omega, g+g') = (\omega, g)+(\omega, g')$ and $(\omega, 0) = 0$, for f in $\mathcal{A}(h)/\mathcal{A}^<(h)$. We write $g|\omega|$ for the class (ω, g), in order to have $g|f\omega| = g\mathbf{L}^{-\mathrm{ord}\, f}|\omega|$. The $\mathcal{C}_+(h)$-semimodule structure on $C^d_+(h)$ induces after passing to the quotient a structure of semiring on $C^d_+(h)$ and $|\tilde{\Omega}|_+(h)$ is naturally endowed with a structure of $C^d_+(h)$-semimodule. We shall call an element $|\omega|$ in $|\tilde{\Omega}|_+(h)$ a gauge form if it is a generator of that semimodule. One should note that in the present setting gauge forms always exist, which is certainly not the case in the usual framework of algebraic geometry. Indeed, gauge forms always exist locally (that is, in suitable affine charts), and in our definable world there is no difficulty in gluing local gauge forms to global ones. One may define similarly $|\tilde{\Omega}|(h)$, replacing C^d_+ by C^d, but we shall only consider $|\tilde{\Omega}|_+(h)$ here.

If h is definable subassignment of dimension d of $h[m, n, r]$, one may construct, similarly as Serre [**28**] in the p-adic case, a canonical gauge form $|\omega_0|_h$ on h. Let us denote by $x_1, \ldots, x_m$ the coordinates on $\mathbf{A}^m_{k((t))}$ and consider the d-forms $\omega_I := dx_{i_1} \wedge \cdots \wedge dx_{i_d}$ for $I = \{i_1, \ldots, i_d\} \subset \{1, \ldots m\}$, $i_1 < \cdots < i_d$, and their image $|\omega_I|_h$ in $|\tilde{\Omega}|_+(h)$. One may check there exists a unique element $|\omega_0|_h$ of $|\tilde{\Omega}|_+(h)$, such that, for every I, there exists definable functions with integral values α_I, β_I on h, with β_I only taking as values 1 and 0, such that $\alpha_I + \beta_I > 0$ on h, $|\omega_I|_h = \beta_I \mathbf{L}^{-\alpha_I}|\omega_0|_h$ in $|\tilde{\Omega}|_+(h)$, and such that $\inf_I \alpha_I = 0$.

If $f : h \to h'$ is a morphism in GDef_k with h and h' of dimension d and all fibers of dimension 0, there is a mapping $f^* : |\tilde{\Omega}|_+(h') \to |\tilde{\Omega}|_+(h)$ induced by pull-back of differential forms. This follows from the fact that f is "analytic" outside a definable subassignment of dimension $d-1$ of h. If, furthermore, h and h' are objects in Def_k, one defines $\mathbf{L}^{-\mathrm{ordjac} f}$ by

$$f^*|\omega_0|_{h'} = \mathbf{L}^{-\mathrm{ordjac} f}|\omega_0|_h. \tag{5.2.2}$$

If $\mathcal{X}$ is a $k((t))$-variety of dimension d, and $\mathcal{X}^0$ is a $k[[t]]$-model of $\mathcal{X}$, it is possible to define an element $|\omega_0|$ in $|\tilde{\Omega}|_+(h[\mathcal{X}])$, which depends only on $\mathcal{X}^0$, and which is characterized by the following property: for every open U^0 of $\mathcal{X}^0$ on which the $k[[t]]$-module $\Omega^d_{U^0|k[[t]]}(U^0)$ is generated by a nonzero form ω, $|\omega_0||_{h[U^0 \otimes \mathrm{Spec}\, k((t))]} = |\omega|$ in $|\tilde{\Omega}|_+(h[U^0 \otimes \mathrm{Spec}\, k((t))])$.

5.3. Integration of volume forms and Fubini Theorem. — Now we are ready to construct motivic integration for volume forms. In the affine case, using canonical gauge forms, one may pass from volume forms to Functions in top dimension, and vice versa. More precisely, let $f : S \to S'$ be a morphism in Def_k, with S of dimension s and S' of dimension s'. Every positive form α in $|\tilde{\Omega}|_+(S)$ may be written $\alpha = \psi_\alpha |\omega_0|_S$ with ψ_α in $C^s_+(S)$. We shall say α is f-integrable if ψ_α is f-integrable and we then set

$$f_!^{\mathrm{top}}(\alpha) := \{f_!(\psi_\alpha)\}_{s'} |\omega_0|_{S'}, \tag{5.3.1}$$

$\{f_!(\psi_\alpha)\}_{s'}$ denoting the component of $f_!(\psi_\alpha)$ lying in $C^{s'}_+(S')$.

Consider now a morphism $f : S \to S'$ in GDef_k. The previous construction may be globalized as follows. Assume there exist isomorphisms $\varphi : T \to S$ and $\varphi' : T' \to S'$ with T and T' in Def_k. We denote by $\tilde{f}$ the morphism $T \to T'$ such that $\varphi' \circ \tilde{f} = f \circ \varphi$. We shall say α in $|\tilde{\Omega}|_+(S)$ is f-integrable if $\varphi^*(\alpha)$ is $\tilde{f}$-integrable and we define then $f_!^{\mathrm{top}}(\alpha)$ by the relation

$$\tilde{f}_!^{\mathrm{top}}(\varphi^*(\alpha)) = \varphi'^*(f_!^{\mathrm{top}}(\alpha)). \tag{5.3.2}$$

It follows from Theorem 4.2 that this definition is independent of the choice of the isomorphisms φ and φ'. By additivity, using affine charts, the previous construction may be extended to any morphism $f : S \to S'$ in GDef_k, in order to define the notion of f-integrability for a volume form α in $|\tilde{\Omega}|_+(S)$, and also, when α is f-integrable, the fiber integral $f_!^{\mathrm{top}}(\alpha)$, which belongs to $|\tilde{\Omega}|_+(S')$. When $S = h[0,0,0]$, we shall say integrable instead of f-integrable, and we shall write $\int_S \alpha$ for $f_!^{\mathrm{top}}(\alpha)$.

In this framework, one may deduce from (A1b) in Theorem 4.1 the following general form of Fubini Theorem for motivic integration:

***5.4. Theorem* (Fubini Theorem).** — *Let $f : S \to S'$ be a morphism in* GDef_k. *Assume S is of dimension s, S' is of dimension s', and that the fibers of f are all of dimension $s - s'$. A positive volume form α in $|\tilde{\Omega}|_+(S)$ is integrable if and only if it is f-integrable and $f_!^{\mathrm{top}}(\alpha)$ is integrable. When this holds, then*

$$\int_S \alpha = \int_{S'} f_!^{\mathrm{top}}(\alpha). \tag{5.4.1}$$

5.5. Comparison with classical motivic integration. — In the definition of Def_k, RDef_k and GDef_k, instead of considering the category F_k of all fields containing k, one could as well restrict to the subcategory ACF_k of algebraically closed fields containing k and define categories $\mathrm{Def}_{k,\mathrm{ACF}_k}$, etc. In fact, it is a direct consequence of Chevalley's constructibility theorem that $K_0(\mathrm{RDef}_{k,\mathrm{ACF}_k})$ is nothing else than the Grothendieck ring $K_0(\mathrm{Var}_k)$ considered in [**14**]. It follows that there is a canonical morphism $SK_0(\mathrm{RDef}_k) \to K_0(\mathrm{Var}_k)$ sending $\mathbf{L}$ to the class of $\mathbf{A}^1_k$, which we shall still denote by $\mathbf{L}$. One can extend this morphism to a morphism $\gamma : SK_0(\mathrm{RDef}_k) \otimes_{\mathbf{N}[\mathbf{L}-1]} A_+ \to K_0(\mathrm{Var}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A$. By considering the series expansion of $(1 - \mathbf{L}^{-i})^{-1}$, one defines a canonical morphism $\delta : K_0(\mathrm{Var}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A \to \widehat{\mathcal{M}}$, with $\widehat{\mathcal{M}}$ the completion of $K_0(\mathrm{Var}_k)[\mathbf{L}^{-1}]$ considered in [**14**].

Let X be an algebraic variety over k of dimension d. Set $\mathcal{X}^0 := X \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$ and $\mathcal{X} := \mathcal{X}^0 \otimes_{\mathrm{Spec}\, k[[t]]} \mathrm{Spec}\, k((t))$. Consider a definable subassignment W of $h[\mathcal{X}]$ in the language $\mathcal{L}_{\mathrm{DP,P}}$, with the restriction that constants in the valued field sort that appear in formulas defining W in affine charts defined over k belong to k (and not to $k((t))$). We assume $W(K) \subset \mathcal{X}(K[[t]])$ for every K in F_k. With the notation of [**14**], formulas defining W in affine charts define a semialgebraic subset of the arc space $\mathcal{L}(X)$ in the corresponding chart, by Theorem 2.10 and Chevalley's constructibility theorem. In this way we assign canonically to W a semialgebraic subset $\tilde{W}$ of $\mathcal{L}(X)$. Similarly, let α be a definable function on W taking integral values and satisfying the additional condition that constants in the valued field sort, appearing in formulas defining α can only belong to k. To any such function α we may assign a semialgebraic function $\tilde{\alpha}$ on $\tilde{W}$.

5.6. Theorem. — *Under the former hypotheses, $|\omega_0|$ denoting the canonical volume form on $h[\mathcal{X}]$, for every definable function α on W with integral values satisfying the previous conditions and bounded below, $\mathbf{1}_W \mathbf{L}^{-\alpha}|\omega_0|$ is integrable on $h[\mathcal{X}]$ and*

$$(\delta \circ \gamma)\Big(\int_{h[\mathcal{X}]} \mathbf{1}_W \mathbf{L}^{-\alpha}|\omega_0|\Big) = \int_{\tilde{W}} \mathbf{L}^{-\tilde{\alpha}} d\mu', \tag{5.6.1}$$

μ' denoting the motivic measure considered in [**14**].

It follows from Theorem 5.6 that, for semialgebraic sets and functions, the motivic integral constructed in [**14**] in fact already exists in $K_0(\mathrm{Var}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A$, or even in $SK_0(\mathrm{Var}_k) \otimes_{\mathbf{N}[\mathbf{L}-1]} A_+$, with $SK_0(\mathrm{Var}_k) = SK_0(\mathrm{RDef}_{k,\mathrm{ACF}_k})$, the Grothendieck semiring of varieties over k.

5.7. Comparison with arithmetic motivic integration. — Similarly, instead of ACF_k, we may also consider the category PFF_k of pseudo-finite fields containing k. Let us recall that a pseudo-finite field is a perfect field F having a unique extension of degree n for every n in a given algebraic closure and such that every geometrically irreducible variety over F has a F-rational point. By restriction from F_k to PFF_k we can define categories $\mathrm{Def}_{k,\mathrm{PFF}_k}$, etc. In particular, the Grothendieck ring $K_0(\mathrm{RDef}_{k,\mathrm{PFF}_k})$ is nothing else but what is denoted by $K_0(\mathrm{PFF}_k)$ in [**16**] and [**17**].

In the paper [**15**], arithmetic motivic integration was taking its values in a certain completion $\hat{K}_0^v(\mathrm{Mot}_{k,\bar{\mathbf{Q}}})_{\mathbf{Q}}$ of a ring $K_0^v(\mathrm{Mot}_{k,\bar{\mathbf{Q}}})_{\mathbf{Q}}$. Somewhat later it was remarked in [**16**] and [**17**] one can restrict to the smaller ring $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$, the definition of which we shall now recall.

The field k being of characteristic 0, there exists, by [**19**] and [**20**], a unique morphism of rings $K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k)$ sending the class of a smooth projective variety X over k to the class of its Chow motive. Here $K_0(\mathrm{CHMot}_k)$ denotes the Grothendieck ring of the category of Chow motives over k with rational coefficients. By definition, $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ is the image of $K_0(\mathrm{Var}_k)$ in $K_0(\mathrm{CHMot}_k)$ under this morphism. [Note that the definition of $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ given in [**16**] is not clearly equivalent and should be replaced by the one given above.] In [**16**] and

[**17**], the authors have constructed, using results from [**15**], a canonical morphism $\chi_c : K_0(\mathrm{PFF}_k) \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ as follows:

***5.8. Theorem* (Denef-Loeser [16] [17]).** — *Let k be a field of characteristic zero. There exists a unique ring morphism*

$$\chi_c : \mathrm{K}_0(\mathrm{PFF}_k) \longrightarrow \mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q} \tag{5.8.1}$$

satisfying the following two properties:

(i) *For any formula φ which is a conjunction of polynomial equations over k, the element $\chi_c([\varphi])$ equals the class in $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ of the variety defined by φ.*

(ii) *Let X be a normal affine irreducible variety over k, Y an unramified Galois cover of X, that is, Y is an integral étale scheme over X with $Y/G \cong X$, where G is the group of all endomorphisms of Y over X, and C a cyclic subgroup of the Galois group G of Y over X. For such data we denote by $\varphi_{Y,X,C}$ a ring formula whose interpretation, in any field K containing k, is the set of K-rational points on X that lift to a geometric point on Y with decomposition group C (i.e., the set of points on X that lift to a K-rational point of Y/C, but not to any K-rational point of Y/C' with C' a proper subgroup of C). Then*

$$\chi_c([\varphi_{Y,X,C}]) = \frac{|C|}{|\mathrm{N}_G(C)|}\chi_c([\varphi_{Y,Y/C,C}]),$$

where $\mathrm{N}_G(C)$ is the normalizer of C in G.

Moreover, when k is a number field, for almost all finite places $\mathfrak{P}$, the number of rational points of $(\chi_c([\varphi]))$ in the residue field $k(\mathfrak{P})$ of k at $\mathfrak{P}$ is equal to the cardinality of $h_\varphi(k(\mathfrak{P}))$.

The construction of χ_c has been recently extended to the relative setting by J. Nicaise [**24**].

5.9. The arithmetical measure takes its values in a certain completion $\hat{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ of the localisation of $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ with respect to the class of the affine line. There is a canonical morphism $\hat{\gamma} : SK_0(\mathrm{RDef}_k) \otimes_{\mathbf{N}[\mathbf{L}-1]} A_+ \to K_0(\mathrm{PFF}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A$. Considering the series expansion of $(1 - \mathbf{L}^{-i})^{-1}$, the map χ_c induces a canonical morphism $\tilde{\delta} : K_0(\mathrm{PFF}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A \to \hat{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$.

Let X be an algebraic variety over k of dimension d. Set $\mathcal{X}^0 := X \otimes_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$, $\mathcal{X} := \mathcal{X}^0 \otimes_{\mathrm{Spec}\, k[[t]]} \mathrm{Spec}\, k((t))$, and consider a definable subassignment W of $h[\mathcal{X}]$ satisfying the conditions in 5.5. Formulas defining W in affine charts allow to define, in the terminology and with the notation in [**15**], a definable subassignment of $h_{\mathcal{L}(X)}$ in the corresponding chart, and we may assign canonically to W a definable subassignment $\tilde{W}$ of $h_{\mathcal{L}(X)}$ in the sense of [**15**].

***5.10. Theorem*.** — *Under the previous hypotheses and with the previous notations, $\mathbf{1}_W|\omega_0|$ is integrable on $h[\mathcal{X}]$ and*

$$(\tilde{\delta} \circ \hat{\gamma})\Big(\int_{h[\mathcal{X}]} \mathbf{1}_W|\omega_0|\Big) = \nu(\tilde{W}), \tag{5.10.1}$$

ν denoting the arithmetical motivic measure as defined in [**15**].

In particular, it follows from Theorem 5.10 that in the present setting the arithmetical motivic integral constructed in [**15**] already exists in $K_0(\mathrm{PFF}_k) \otimes_{\mathbf{Z}[\mathbf{L}]} A$ (or even in $SK_0(\mathrm{PFF}_k) \otimes_{\mathbf{N}[\mathbf{L}-1]} A_+$), without completing further the Grothendieck ring and without considering Chow motives (and even without inverting additively all elements of the Grothendieck semiring).

6. Comparison with p-adic integration

In the next two sections we present new results on specialization to p-adic integration and Ax-Kochen-Eršov Theorems for integrals with parameters. We plan to give complete details in a future paper.

6.1. P-adic definable sets. — We fix a finite extension K of $\mathbf{Q}_p$ together with an uniformizing parameter ϖ_K. We denote by R_K the valuation ring and by k_K the residue field, $k_K \simeq \mathbf{F}_{q(K)}$ for some power $q(K)$ of p. Let φ be a formula in the language $\mathcal{L}_{\mathrm{DP,P}}$ with coefficients in K in the valued field sort and coefficients in k_K in the residue field sort, with m free variables in the valued field sort, n free variables in the residue field sort and r free variables in the value group sort. The formula φ defines a subset Z_φ of $K^m \times k_K^n \times \mathbf{Z}^r$ (recall that since we have chosen ϖ_K, K is endowed with an angular component mapping). We call such a subset a p-adic definable subset of $K^m \times k_K^n \times \mathbf{Z}^r$. We define morphisms between p-adic definable subsets similarly as before: if S and S' are p-adic definable subsets of $K^m \times k_K^n \times \mathbf{Z}^r$ and $K^{m'} \times k_K^{n'} \times \mathbf{Z}^{r'}$ respectively, a morphism $f : S \to S'$ will be a function $f : S \to S'$ whose graph is p-adic definable.

6.2. P-adic dimension. — By the work of Scowcroft and van den Dries [**27**], there is a good dimension theory for p-adic definable subsets of K^m. By Theorem 3.4 of [**27**], a p-adic definable subset A of K^m has dimension d if and only its Zariski closure has dimension d in the sense of algebraic geometry. For S a p-adic definable subset of $K^m \times k_K^n \times \mathbf{Z}^r$, we define the dimension of S as the dimension of its image S' under the projection $\pi : S \to K^m$.

More generally if $f : S \to S'$ is a morphism of p-adic definable subsets, one defines the relative dimension of f to be the maximum of the dimensions of the fibers of f.

6.3. Functions. — Let S be a p-adic definable subset of $K^m \times k_K^n \times \mathbf{Z}^r$. We shall consider the $\mathbf{Q}$-algebra $\mathcal{C}_K(S)$ generated by functions of the form α and q^α with α a $\mathbf{Z}$-valued p-adic definable function on S. For $S' \subset S$ a p-adic definable subset, we write $\mathbf{1}_{S'}$ for the characteristic function of S' in $\mathcal{C}_K(S)$.

For $d \geq 0$ an integer, we denote by $\mathcal{C}_K^{\leq d}(S)$ the ideal of $\mathcal{C}(S)$ generated by all functions $\mathbf{1}_{S'}$ with S' a p-adic definable subset of S of dimension $\leq d$. Similarly to what we did before, we set

$$C^d_K(S) := \mathcal{C}^{\leq d}_K(S)/\mathcal{C}^{\leq d-1}_K(S) \quad \text{and} \quad C_K(S) := \bigoplus_d C^d_K(S). \tag{6.3.1}$$

Also, similarly as before, we have relative variants of the above definitions. If $f : Z \to S$ is a morphism between p-adic definable subsets, we define $\mathcal{C}^{\leq d}_K(Z \to S)$, $C^d_K(Z \to S)$ and $C_K(Z \to S)$ by replacing dimension by relative dimension.

6.4. ***P*-adic measure.** — Let S be a p-adic definable subset of K^m of dimension d. By the construction of [**30**] based on [**28**], bounded p-adic definable subsets A of S have a canonical d-dimensional volume $\mu^d_K(A)$ in $\mathbf{R}$.

Now let S be a p-adic definable subset of $K^m \times k^n_K \times \mathbf{Z}^r$ of dimension d and S' its image under the projection $\pi : S \to K^m$. We define the measure μ_d on S as the measure induced by the product measur on $S' \times k^n_K \times \mathbf{Z}^r$ of the d-dimensional volume μ^d_K on the factor S' and the counting measure on the factor $k^n_K \times \mathbf{Z}^r$. When S is of dimension $< d$ we declare μ^d_K to be identically zero.

We call φ in $\mathcal{C}_K(S)$ integrable on S if φ is integrable against μ_d and we denote the integral by $\mu^d_K(\varphi)$.

One defines $\mathrm{I}C^d_K(S)$ as the abelian subgroup of $C^d_K(S)$ consisting of the classes of integrable functions in $\mathcal{C}_K(S)$. The measure μ^d_K induces a morphism of abelian groups $\mu^d_K : \mathrm{I}C^d_K(S) \to \mathbf{R}$.

More generally if $\varphi = \varphi \mathbf{1}_{S'}$, where S' has dimension $i \leq d$, we say φ is i-integrable if its restriction φ' to S' is integrable and we set $\mu^i_K(\varphi) := \mu^i_K(\varphi')$. One defines $\mathrm{I}C^i_K(S)$ as the abelian subgroup of $C^i_K(S)$ of the classes of i-integrable functions in $\mathcal{C}_K(S)$. The measure μ^i_K induces a morphism of abelian groups $\mu^i_K : \mathrm{I}C^i_K(S) \to \mathbf{R}$. Finally we set $\mathrm{I}C_K(S) := \bigoplus_i \mathrm{I}C^i_K(S)$ and we define $\mu_K : \mathrm{I}C_K(S) \to \mathbf{R}$ to be the sum of the morphisms μ^i_K. We call elements of $C_K(S)$, resp. $\mathrm{I}C_K(S)$, constructible Functions, resp. integrable constructible Functions on S.

Also, if $f : S \to \Lambda$ is a morphism of p-adic definable subsets, we shall say an element φ in $C_K(S \to \Lambda)$ is integrable if the restriction of φ to every fiber of f is an integrable constructible Function and we denote by $\mathrm{I}C_K(S \to \Lambda)$ the set of such Functions.

We may now reformulate Denef's basic Theorem on p-adic integration (Theorem 1.5 in [**13**], see also [**11**]):

***6.5. Theorem* (Denef).** — *Let $f : S \to \Lambda$ be a morphism of p-adic definable subsets. For every integrable constructible Function φ in $C_K(S \to \Lambda)$, there exists a unique function $\mu_{K,\Lambda}(\varphi)$ in $\mathcal{C}(\Lambda)$ such that, for every point λ in Λ,*

$$\mu_{K,\Lambda}(\varphi)(\lambda) = \mu_K(\varphi_{|f^{-1}(\lambda)}). \tag{6.5.1}$$

Strictly speaking, this is not the statement that one finds in [**13**], but the proof sketched there extends to our setting.

6.6. Pushforward. — It is possible to define, for every morphism $f : S \to S'$ of p-adic definable subsets, a natural pushforward morphism

$$f_! : \mathrm{I}C_K(S) \longrightarrow \mathrm{I}C_K(S') \tag{6.6.1}$$

satisfying similar properties as in Theorem 4.1. This may be done along similar lines as what we did in the motivic case using Denef's p-adic cell decomposition [**12**] instead of Denef-Pas cell decomposition. Note however that much less work is required in this case, since one already knows what the p-adic measure is! In particular, when f is the projection on the one point definable subset one recovers the p-adic measure μ_K. Also in the relative setting we have natural pushforward morphisms

$$f_{!\Lambda} : \mathrm{I}C_K(S \to \Lambda) \longrightarrow \mathrm{I}C_K(S' \to \Lambda), \tag{6.6.2}$$

for $f : S \to S'$ over Λ, and one recovers the relative p-adic measure $\mu_{K,\Lambda}$ when f is the projection to Λ.

6.7. Comparison with p-adic integration. — Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathcal{A}_{\mathcal{O}}$ be the collection of all the p-adic completions of k and of all finite field extensions of k. In this section and in section 7.2 we let $\mathcal{L}_{\mathcal{O}}$ be the language $\mathcal{L}_{\mathrm{DP,P}}(\mathcal{O}[[t]])$, that is, the language $\mathcal{L}_{\mathrm{DP,P}}$ with coefficients in k for the residue field sort and coefficients in $\mathcal{O}[[t]]$ for the valued field sort, and, all definable subassignments, definable morphisms, and motivic constructible functions will be with respect to this language. To stress the fact that our language is $\mathcal{L}_{\mathcal{O}}$ we use the notation $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$ for Def, and similarly for $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$, $\mathrm{Def}_S(\mathcal{L}_{\mathcal{O}})$ and so on.

For K in $\mathcal{A}_{\mathcal{O}}$ we write k_K for its residue field with $q(K)$ elements, R_K for its valuation ring and ϖ_K for a uniformizer of R_K.

Let us choose for a while, for every definable subassignment S in $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$, a $\mathcal{L}_{\mathcal{O}}$-formula ψ_S defining S. We shall write $\tau(S)$ to denote the datum (S, ψ_S). Similarly, for any element φ of $\mathcal{C}(S)$, $C(s)$, $IC(S)$, and so on, we choose a finite set $\psi_{\varphi,i}$ of formulas needed to determine φ and we write $\tau(\varphi)$ for $(\varphi, \{\psi_{\varphi,i}\}_i)$.

Let S be a definable subassignment of $h[m,n,r]$ in $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$ with $\tau(S) = (S, \psi_S)$. Let K be in $\mathcal{A}_{\mathcal{O}}$. One may consider K as an $\mathcal{O}[[t]]$-algebra via the morphism

$$\lambda_{\mathcal{O},K} : \mathcal{O}[[t]] \to K : \sum_{i \in \mathbf{N}} a_i t^i \mapsto \sum_{i \in \mathbf{N}} a_i \varpi_K^i, \tag{6.7.1}$$

hence, if one interprets elements a of $\mathcal{O}[[t]]$ as $\lambda_{\mathcal{O},K}(a)$, the formula ψ_S defines a p-adic definable subset $S_{K,\tau}$ of $K^m \times k_K^n \times \mathbf{Z}^r$.

If now $\tau(S) = (S, \psi_S)$ is replaced by $\tau'(S) = (S, \psi'_S)$ with ψ'_S another $\mathcal{L}_{\mathcal{O}}$-formula defining S, it follows, from a small variant of Proposition 5.2.1 of [**15**] (a result of Ax-Kochen-Eršov type that uses ultraproducts and follows from the Theorem of Denef-Pas), that there exists an integer N such that $S_{K,\tau} = S_{K,\tau'}$ for every K in $\mathcal{A}_{\mathcal{O}}$ with residue field characteristic $\mathrm{char} k_K \geq N$. (Note however that this number N can be arbitrarily large for different τ'.)

Let us consider the quotient

$$\prod_{K\in\mathcal{A}_{\mathcal{O}}} \mathcal{C}_K(S_{K,\tau}) / \sum_N \prod_{\substack{K\in\mathcal{A}_{\mathcal{O}}\\ \mathrm{char} k_K < N}} \mathcal{C}_K(S_{K,\tau}), \tag{6.7.2}$$

consisting of families indexed by K of elements of $\mathcal{C}_K(S_{K,\tau})$, two such families being identified if for some $N > 0$ they coincide for $\mathrm{char} k_K \geq N$. It follows from the above remark that it is independent of τ (more precisely all these quotients are canonically isomorphic), so we may denote it by

$$\prod{}' \mathcal{C}_K(S_K). \tag{6.7.3}$$

One defines similarly $\prod' C_K(S_K)$, $\prod' \mathrm{I}C_K(S_K)$, etc.

Now take W in $\mathrm{RDef}_S(\mathcal{L}_{\mathcal{O}})$. It defines a p-adic definable subset $W_{K,\tau}$ of $S_{K,\tau} \times (k_K)^\ell$, for some ℓ, for every K in $\mathcal{A}_{\mathcal{O}}$. We may now consider the function $\psi_{W,K,\tau}$ on $S_{K,\tau}$ assigning to a point x the number of points mapping to it in $W_{K,\tau}$, that is, $\psi_{W,K,\tau}(x) = \mathrm{card}(W_{K,\tau} \cap (\{x\} \times k_K^\ell))$. Similarly as before, if we take another function τ', we have $\psi_{W,K,\tau} = \psi_{W,K,\tau'}$ for every K in $\mathcal{A}_{\mathcal{O}}$ with residue field characteristic $\mathrm{char} k_K \geq N$, hence we get in this way an arrow $\mathrm{RDef}_S(\mathcal{L}_{\mathcal{O}}) \to \prod' \mathcal{C}_K(S_K)$ which factorizes through a ring morphism $K_0(\mathrm{RDef}_S(\mathcal{L}_{\mathcal{O}})) \to \prod' \mathcal{C}_K(S_K)$. If we send $\mathbf{L}$ to $q(K)$, one can extend uniquely this morphism to a ring morphism

$$\Gamma : \mathcal{C}(S, \mathcal{L}_{\mathcal{O}}) \longrightarrow \prod{}' \mathcal{C}_K(S_K). \tag{6.7.4}$$

Since Γ preserves the (relative) dimension of support on those factors K with $\mathrm{char} k_K$ big enough, Γ induces the morphisms

$$\Gamma : C(S, \mathcal{L}_{\mathcal{O}}) \longrightarrow \prod{}' C_K(S_K) \tag{6.7.5}$$

and

$$\Gamma : C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) \longrightarrow \prod{}' C_K(S_K \to \Lambda_K), \tag{6.7.6}$$

for $S \to \Lambda$ a morphism in $\mathrm{Def}_K(\mathcal{L}_{\mathcal{O}})$.

The following comparison Theorem says that the morphism Γ commutes with pushforward. In more concrete terms, given an integrable function φ in $C(S \to \Lambda, \mathcal{L}_{\mathcal{O}})$, for almost all p, its specialization φ_K to any finite extension K of $\mathbf{Q}_p$ in $\mathcal{A}_{\mathcal{O}}$ is integrable, and the specialization of the pushforward of φ is equal to the pushforward of φ_K.

6.8. Theorem. — *Let* Λ *be in* $\mathrm{Def}_K(\mathcal{L}_{\mathcal{O}})$ *and let* $f : S \to S'$ *be a morphism in* $\mathrm{Def}_\Lambda(\mathcal{L}_{\mathcal{O}})$. *The morphism*

$$\Gamma : C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) \to \prod{}' C_K(S_K \to \Lambda_K) \tag{6.8.1}$$

induces a morphism

$$\Gamma : \mathrm{I}C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) \to \prod{}' \mathrm{I}C_K(S_K \to \Lambda_K) \tag{6.8.2}$$

(and similarly for S'), and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{I}C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) & \xrightarrow{\Gamma} & \prod{}' \mathrm{I}C_K(S_K \to \Lambda_K) \\ \downarrow{\scriptstyle f_{!\Lambda}} & & \downarrow{\scriptstyle \prod' f_{K,\Lambda_K!}} \\ \mathrm{I}C(S' \to \Lambda, \mathcal{L}_{\mathcal{O}}) & \xrightarrow{\Gamma} & \prod{}' \mathrm{I}C_K(S'_K \to \Lambda_K), \end{array}$$

with $f_K : S_K \to S'_K$ the morphism induced by f and where the map $\prod' f_{K,\Lambda_K!}$ is induced by the maps $f_{K,\Lambda_K!} : \mathrm{I}C_K(S_K \to \Lambda_K) \to \mathrm{I}C_K(S'_K \to \Lambda_K)$.

Sketch of proof. — The image of φ in $\mathrm{I}C(S \to \Lambda, \mathcal{L}_{\mathcal{O}})$ under $f_{!\Lambda}$ can be calculated by taking an appropriate cell decomposition of the occurring sets, adapted to the occurring functions (as in [**7**] and inductively applied to all valued field variables). Such calculation is independent of the choice of cell decomposition by the unicity statement of Theorem 4.1. By the Ax-Kochen-Eršov principle for the language $\mathcal{L}_{\mathcal{O}}$ implied by Theorem 2.10, this cell decomposition determines, for K in $\mathcal{A}_{\mathcal{O}}$ with $\mathrm{char} k_K$ sufficiently large, a cell decomposition à la Denef (in the formulation of Lemma 4 of [**4**]) of the K-component of these sets, adapted to the K-component of the functions occuring here, where thus the same calculation can be pursued. That this calculation is actually the same follows from the fact that p-adic integration satisfies properties analogue to the axioms of Theorem 4.1. □

In particular, we have the following statement, which says that, given an integrable function φ in $C(S \to \Lambda, \mathcal{L}_{\mathcal{O}})$, for almost all p, its specialization φ_F to any finite extension F of $\mathbf{Q}_p$ in $\mathcal{A}_{\mathcal{O}}$ is integrable, and the specialization of the motivic integral $\mu(\varphi)$ is equal to the p-adic integral of φ_F:

6.9. Theorem. — *Let $f : S \to \Lambda$ be a morphism in $\mathrm{Def}_K(\mathcal{L}_{\mathcal{O}})$. The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{I}C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) & \xrightarrow{\Gamma} & \prod{}' \mathrm{I}C_K(S_K \to \Lambda_K) \\ \downarrow{\scriptstyle \mu_{\Lambda}} & & \downarrow{\scriptstyle \prod' \mu_{K,\Lambda_K}} \\ \mathcal{C}(\Lambda, \mathcal{L}_{\mathcal{O}}) & \xrightarrow{\Gamma} & \prod{}' \mathcal{C}_K(\Lambda_K). \end{array}$$

7. Reduction mod p and a motivic Ax-Kochen-Eršov Theorem for integrals with parameters

7.1. Integration over $\mathbf{F}_q((t))$. — Consider now the field $K = \mathbf{F}_q((t))$ with valuation ring R_K and residue field $k_K = \mathbf{F}_q$ with $q = q(K)$ a prime power. One may define $\mathbf{F}_q((t))$-definable sets similarly as in 6.1. Little is known about the structure of these $\mathbf{F}_q((t))$-definable sets, but, for any subset A of K^m, not necessarly definable, we may still define the dimension of A as the dimension of its Zariski closure. Similarly as in 6.2, one extends that definition to any subset A of $K^m \times k_K^n \times \mathbf{Z}^r$ and define the relative dimension of a mapping $f : A \to \Lambda$, with Λ any subset of

$K^{m'} \times k_K^{n'} \times \mathbf{Z}^{r'}$. When A is $\mathbf{F}_q((t))$-definable, one can define a $\mathbf{Q}$-algebra $\mathcal{C}_K(A)$ as in 6.3, but since no analogue of Theorem 6.5 is known in this setting, we shall consider, for A any subset of $K^m \times k_K^n \times \mathbf{Z}^r$, the $\mathbf{Q}$-algebra $\mathcal{F}_K(A)$ of all functions $A \to \mathbf{Q}$. For $d \geq 0$ an integer, we denote by $\mathcal{F}_K^{\leq d}(A)$ the ideal of functions with support of dimension $\leq d$. We set $F_K^d(A) := \mathcal{F}_K^{\leq d}(A)/\mathcal{F}_K^{\leq d-1}(A)$ and $F_K(A) := \oplus_d F_K^d(A)$. One defines similarly relative variants $\mathcal{F}_K^{\leq d}(A \to \Lambda)$, $F_K^d(A \to \Lambda)$ and $F_K(A \to \Lambda)$, for $f : A \to A'$ as above.

Let A be a subset of K^m with Zariski closure $\bar{A}$ of dimension d. We consider the canonical d-dimensional measure μ_K^d on $\bar{A}(K)$ as in [**25**]. We say a function φ in $\mathcal{F}_K(A)$ is integrable if it is measurable and integrable with respect to the measure μ_K^d. Now we may proceed as in 6.4 to define, for A a subset of $K^m \times k_K^n \times \mathbf{Z}^r$, $\mathrm{I}F_K(A)$ and $\mu_K : \mathrm{I}F_K(A) \to \mathbf{R}$. Also, if $f : A \to \Lambda$ is a mapping as before, one defines $\mathrm{I}F_K(A \to \Lambda)$ as Functions whose restrictions to all fibers lie in $\mathrm{I}F_K$. We denote by $\mu_{K,\Lambda}$ the unique mapping $\mathrm{I}F_K(A \to \Lambda) \to \mathcal{F}(\Lambda)$ such that, for every φ in $\mathrm{I}F_K(A \to \Lambda)$ and every point λ in Λ, $\mu_{K,\Lambda}(\varphi)(\lambda) = \mu_K(\varphi_{|f^{-1}(\lambda)})$.

7.2. Reduction mod p**.** — We go back to the notation of 6.7. In particular, k denotes a number field with ring of integers $\mathcal{O}$, $\mathcal{A}_{\mathcal{O}}$ denotes the set of all p-adic completions of k and of all the finite field extensions of k, and $\mathcal{L}_{\mathcal{O}}$ stands for the language $\mathcal{L}_{\mathrm{DP,P}}(\mathcal{O}[[t]])$. We also use the map τ as defined in section 6.7.

Let $\mathcal{B}_{\mathcal{O}}$ be the set of all local fields over $\mathcal{O}$ of positive characteristic. As for $\mathcal{A}_{\mathcal{O}}$, we use for every K in $\mathcal{B}_{\mathcal{O}}$ the notation k_K for its residue field with $q(K)$ elements, R_K for its valuation ring and ϖ_K for a uniformizer of R_K.

Let S be a definable subassignment of $h[m,n,r]$ in $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$ and let $\tau(S)$ be (S, ψ_S) with ψ_S a $\mathcal{L}_{\mathcal{O}}$-formula. Similarly as for $\mathcal{A}_{\mathcal{O}}$, since every K in $\mathcal{B}_{\mathcal{O}}$ is an $\mathcal{O}[[t]]$-algebra under the morphism

$$\lambda_{\mathcal{O},K} : \mathcal{O}[[t]] \to K : \sum_{i \in \mathbf{N}} a_i t^i \mapsto \sum_{i \in \mathbf{N}} a_i \varpi_K^i, \tag{7.2.1}$$

interpreting any element a of $\mathcal{O}[[t]]$ as $\lambda_{\mathcal{O},K}(a)$, ψ_S defines a K-definable subset $S_{K,\tau}$ of $K^m \times k_K^n \times \mathbf{Z}^r$. Again by a small variant of Proposition 5.2.1 of [**15**], for any other τ' we have for every K in $\mathcal{B}_{\mathcal{O}}$ with $\mathrm{char} k_K$ big enough that $S_{K,\tau} = S_{K,\tau'}$, hence, may define, similarly as in 6.7,

$$\prod{}' \mathcal{F}_K(S_K). \tag{7.2.2}$$

to be the quotient

$$\prod_{K \in \mathcal{B}_{\mathcal{O}}} \mathcal{F}_K(S_{K,\tau}) / \sum_N \prod_{\substack{K \in \mathcal{B}_{\mathcal{O}} \\ \mathrm{char} k_K < N}} \mathcal{F}_K(S_{K,\tau}), \tag{7.2.3}$$

and similarly for $\prod{}' F_K(S_K)$, $\prod{}' \mathrm{I}F_K(S_K)$, etc.

Similarly as in 6.7, one may define ring morphisms

$$\hat{\Gamma} : \mathcal{C}(S, \mathcal{L}_{\mathcal{O}}) \longrightarrow \prod{}' \mathcal{F}_K(S_K), \tag{7.2.4}$$

$$\hat{\Gamma} : C(S, \mathcal{L}_{\mathcal{O}}) \longrightarrow \prod{}' F_K(S_K) \tag{7.2.5}$$

and

$$\hat{\Gamma} : C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) \longrightarrow \prod{}' F_K(S_K \to \Lambda_K), \tag{7.2.6}$$

for $S \to \Lambda$ a morphism in $\mathrm{Def}_K(\mathcal{L}_{\mathcal{O}})$.

The following statement is a companion to Theorem 6.9 and has an essentially similar proof.

7.3. Theorem. — *Let $f : S \to \Lambda$ be a morphism in $\mathrm{Def}_K(\mathcal{L}_{\mathcal{O}})$. The morphism*

$$\hat{\Gamma} : C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) \to \prod{}' F_K(S_K \to \Lambda_K) \tag{7.3.1}$$

induces a morphism

$$\hat{\Gamma} : \mathrm{I}C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) \to \prod{}' \mathrm{I}F_K(S_K \to \Lambda_K) \tag{7.3.2}$$

and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{I}C(S \to \Lambda, \mathcal{L}_{\mathcal{O}}) & \xrightarrow{\hat{\Gamma}} & \prod{}' \mathrm{I}F_K(S_K \to \Lambda_K) \\ \downarrow{\scriptstyle \mu_\Lambda} & & \downarrow{\scriptstyle \prod{}' \mu_{K,\Lambda_K}} \\ \mathcal{C}(\Lambda, \mathcal{L}_{\mathcal{O}}) & \xrightarrow{\hat{\Gamma}} & \prod{}' \mathcal{F}_K(\Lambda_K). \end{array}$$

7.4. Ax-Kochen-Eršov Theorems for motivic integrals. — We keep the notation of 6.7 and 7.2. Let S, resp. Λ, be definable subassignments of $h[m, n, r]$, resp. $h[m', n', r']$, in the language $\mathcal{L}_{\mathcal{O}}$ and consider a definable (in the language $\mathcal{L}_{\mathcal{O}}$) morphism $f : S \to \Lambda$. Since we are interested in integrals along the fibers of f, there is no restriction in assuming, and we shall do so, that $\Lambda = h[m', n', r']$. We set $\Lambda(\mathcal{O}) := \mathcal{O}[[t]]^{m'} \times k^{n'} \times \mathbf{Z}^{r'}$.

A first attempt to get Ax-Kochen-Eršov Theorems for motivic integrals is by comparing values. This is achieved as follows. To every point λ in $\Lambda(\mathcal{O})$ we may assign, for all K in $\mathcal{A}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}}$, a point λ_K in $(R_K)^{m'} \times k_K^{n'} \times \mathbf{Z}^{r'}$, by using the maps $\lambda_{\mathcal{O},K}$ on the $\mathcal{O}[[t]]^{m'}$-factor and reduction modulo $\mathrm{char} k_K$ for the $k^{n'}$-factor.

Let φ be in $C(S \to \Lambda, \mathcal{L}_{\mathcal{O}})$. With a slight abuse of notation, we shall write φ_K for the component at K in $C_K(S_K \to \Lambda_K)$ of $\Gamma(\varphi)$, resp. of $\hat{\Gamma}(\varphi)$, for K with $\mathrm{char} k_K$ big, in $\mathcal{A}_{\mathcal{O}}$, resp. in $\mathcal{B}_{\mathcal{O}}$. We shall use similar notations for φ in $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$.

Over the final subassignment $h[0, 0, 0]$ the morphisms Γ and $\hat{\Gamma}$ have quite simple descriptions. Indeed, the morphism $\hat{\gamma}$ of 5.9 induces a ring morphism

$$\gamma' : \mathcal{C}(h_{\mathrm{Spec}\, k}, \mathcal{L}_{\mathcal{O}}) \longrightarrow K_0(\mathrm{PFF}_K) \otimes_{\mathbf{Z}[\mathbf{L}]} A. \tag{7.4.1}$$

One the other hand, note that $\mathcal{C}_K(\text{point}) \simeq \mathcal{F}_{K'}(\text{point}) \simeq \mathbf{Q}$ for K in $\mathcal{A}_{\mathcal{O}}$ and K' in $\mathcal{B}_{\mathcal{O}}$. The morphism $\chi_c : K_0(\mathrm{PFF}_k) \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ from 5.7 induces a ring morphism

$$\delta' : K_0(\mathrm{PFF}_K) \otimes_{\mathbf{Z}[\mathbf{L}]} A \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q} \otimes_{\mathbf{Z}[\mathbf{L}]} A. \tag{7.4.2}$$

Note that, for K in $\mathcal{A}_{\mathcal{O}}$, resp. in $\mathcal{B}_{\mathcal{O}}$, with char$K$ big enough, the K-component of $\Gamma(\alpha)$, resp. of $\hat{\Gamma}(\alpha)$, is equal to the trace of the Frobenius at k_K acting on an étale realisation of $(\delta' \circ \gamma')(\alpha)$, for α in $\mathcal{C}(h_{\operatorname{Spec} k}, \mathcal{L}_{\mathcal{O}})$. In particular, one deduces the following statement:

7.5. Lemma. — *Let ψ be a function in $\mathcal{C}(\Lambda, \mathcal{L}_{\mathcal{O}})$. Then, for every λ in $\Lambda(\mathcal{O})$, there exists an integer N such that, for every K_1 in $\mathcal{A}_{\mathcal{O}}$, K_2 in $\mathcal{B}_{\mathcal{O}}$ with $k_{K_1} \simeq k_{K_2}$ and* char$k_{K_1} > N$*,*

$$\psi_{K_1}(\lambda_{K_1}) = \psi_{K_2}(\lambda_{K_2}), \tag{7.5.1}$$

which also is equal to $(i_\lambda^*(\psi))_{K_1}$ and to $(i_\lambda^*(\psi))_{K_2}$.

From Lemma 7.5, Theorem 6.9 and Theorem 7.3 one deduces immediatly:

7.6. Theorem. — *Let $f : S \to \Lambda$ be as above. Let φ be a Function in* I$C(S \to \Lambda, \mathcal{L}_{\mathcal{O}})$*. Then, for every λ in $\Lambda(\mathcal{O})$, there exists an integer N such that for all K_1 in $\mathcal{A}_{\mathcal{O}}$, K_2 in $\mathcal{B}_{\mathcal{O}}$ with $k_{K_1} \simeq k_{K_2}$ and* char$k_{K_1} > N$

$$\mu_{K_1}(\varphi_{K_1 | f_{K_1}^{-1}(\lambda_{K_1})}) = \mu_{K_2}(\varphi_{K_2 | f_{K_2}^{-1}(\lambda_{K_2})}), \tag{7.6.1}$$

which also equals $(\mu_\Lambda(\varphi))_{K_1}(\lambda_{K_1})$ and $(\mu_\Lambda(\varphi))_{K_2}(\lambda_{K_2})$.

Note that Theorem 2.12 is a corollary of Theorem 7.6 when $(m', n', r') = (0, 0, 0)$. In fact, Theorem 7.6 is not really satisfactory when $(m', n', r') \neq (0, 0, 0)$, since it is not uniform with respect to λ. The following example shows that this unavoidable: take $k = \mathbf{Q}$, $S = \Lambda = h[1, 0, 0]$, f the identity and $\varphi = \mathbf{1}_{S \setminus \{0\}}$ in I$C(S \to \Lambda) = \mathcal{C}(S)$. Take K_1 in $\mathcal{A}_{\mathcal{O}}$ and K_2 in $\mathcal{B}_{\mathcal{O}}$. We have $\varphi_{K_1}(\lambda_{K_1}) = \varphi_{K_2}(\lambda_{K_2})$ for $\lambda \neq 0$ in $\mathbf{Z}$ only if the characteristic of K_2 does not divide λ.

Hence, instead of comparing values of integrals depending on parameters, we better compare the integrals as functions, which is done as follows:

7.7. Theorem. — *Let $f : S \to \Lambda$ be as above. Let φ be a Function in* I$C(S \to \Lambda, \mathcal{L}_{\mathcal{O}})$*. Then, there exists an integer N such that for all K_1 in $\mathcal{A}_{\mathcal{O}}$, K_2 in $\mathcal{B}_{\mathcal{O}}$ with $k_{K_1} \simeq k_{K_2}$ and* char$k_{K_1} > N$

$$\mu_{K_1, \Lambda_{K_1}}(\varphi_{K_1}) = 0 \quad \textit{if and only if} \quad \mu_{K_2, \Lambda_{K_2}}(\varphi_{K_2}) = 0. \tag{7.7.1}$$

Proof. — Follows directly from Theorem 6.9, Theorem 7.3, and Theorem 7.8. □

7.8. Theorem. — *Let ψ be in $\mathcal{C}(\Lambda, \mathcal{L}_{\mathcal{O}})$. Then, there exists an integer N such that for all K_1 in $\mathcal{A}_{\mathcal{O}}$, K_2 in $\mathcal{B}_{\mathcal{O}}$ with $k_{K_1} \simeq k_{K_2}$ and* char$k_{K_1} > N$

$$\psi_{K_1} = 0 \quad \textit{if and only if} \quad \psi_{K_2} = 0. \tag{7.8.1}$$

7.9. Remark. — Thanks to results of Cunningham and Hales [**9**], Theorem 7.7 applies to the orbital integrals occuring in the Fundamental Lemma. Hence, it follows from Theorem 7.7 that the Fundamental Lemma holds over function fields of large characteristic if and only if it holds for p-adic fields of large characteristic. (Note that the Fundamental Lemma is about the equality of two integrals, or, which amounts to the same, their difference to be zero.) In the special situation of the Fundamental Lemma, a more precise comparison result has been proved by Waldspurger [**31**] by representation theoretic techniques. Let us recall that the Fundamental Lemma for unitary groups has been proved recently by Laumon and Ngô [**23**] for functions fields.

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