

MOTIVIC IGUSA ZETA FUNCTIONS

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0. Introduction

Let p be a prime number and let K be a finite extension of \mathbf{Q}_p . Let R be the valuation ring of K , P the maximal ideal of R , and $\bar{K} = R/P$ the residue field of K . Let q denote the cardinality of \bar{K} , so $\bar{K} \simeq \mathbf{F}_q$. For z in K , let $\text{ord } z$ denote the valuation of z , and set $|z| = q^{-\text{ord } z}$. Let f be a non-constant element of $K[x_1, \dots, x_m]$. The p -adic Igusa local zeta function $Z(s)$ associated to f (relative to the trivial multiplicative character) is defined as the p -adic integral

$$(0.1) \quad Z(s) = \int_{R^m} |f(x)|^s |dx|,$$

for $s \in \mathbf{C}$, $\text{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^m normalized in such a way that R^m is of volume 1. For n in \mathbf{N} , set $Z_n = \{x \in R^m \mid \text{ord } f(x) = n\}$. We may express $Z(s)$ as a series

$$(0.2) \quad Z(s) = \sum_{n \geq 0} \text{vol}(Z_n) q^{-ns}.$$

Now, if we denote by X_n the image of Z_n in $(R/P^{n+1})^m$, we may rewrite the series as

$$(0.3) \quad Z(s) = \sum_{n \geq 0} \text{card}(X_n) q^{-ns - (n+1)m}$$

since $\text{vol}(Z_n) = \text{card}(X_n) q^{-(n+1)m}$.

Let k be a field of characteristic zero. M. Kontsevich recently introduced the concept of motivic integration [13] (see also [6] and 2.5), which is a $k[[t]]$ -analogue of the usual p -adic integration. This motivic integration takes values into a certain completion of a localisation of the Grothendieck ring $K_0(\text{Sch}_k)$ of algebraic varieties over k , i.e., reduced separated schemes of finite type over k (see 2.5 for more details). The ring $K_0(\text{Sch}_k)$ is generated by symbols $[S]$, for S an algebraic variety over k , with the relations $[S] = [S']$ if S is isomorphic to S' , $[S] = [S \setminus S'] + [S']$ if S' is closed in S and $[S \times S'] = [S][S']$. We set

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$\mathbf{L} := [\mathbf{A}_k^1]$. In this setting, the analogue of R^m , resp. of $(R/P^{n+1})^m$, is the k -scheme $\mathcal{L}(\mathbf{A}_k^m)$, resp. $\mathcal{L}_n(\mathbf{A}_k^m)$, which represents the functor $R \mapsto R[[t]]^m$, resp. the functor $R \mapsto (R[[t]]/t^{n+1}R[[t]])^m$, on the category of k -algebras, i.e., the k -scheme parametrizing m -tuples of series, resp. of series modulo t^{n+1} . Let f be a non-constant element of $K[x_1, \dots, x_m]$. Define Z_n as the subscheme of $\mathcal{L}(\mathbf{A}_k^m)$ of series φ such that $\text{ord}_t f(\varphi) = n$ and X_n as the image of Z_n in $\mathcal{L}_n(\mathbf{A}_k^m)$, viewed as a reduced subscheme. A natural analogue of the right-hand side of (0.3), which is a series in $\mathbf{Z}[p^{-1}][[p^{-s}]]$, is the following series in $K_0(\text{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]]$:

$$(0.4) \quad Z_{\text{geom}}(s) = \sum_{n \geq 0} [X_n] \mathbf{L}^{-ns - (n+1)m}.$$

When s takes a fixed value in \mathbf{N} , this series can be interpreted as a Kontsevich integral (see 2.5).

More generally, p -adic Igusa local zeta functions involve multiplicative characters. Let π be a fixed uniformizing parameter of R and set $\text{ac}(z) = z\pi^{-\text{ord } z}$ for z in K . For any character $\alpha : R^\times \rightarrow \mathbf{C}^\times$, one defines the p -adic Igusa local zeta function $Z(s, \alpha)$ as the integral

$$(0.5) \quad Z(s, \alpha) = \int_{R^m} \alpha(\text{ac}(f(x))) |f(x)|^s |dx|,$$

for $s \in \mathbf{C}$, $\text{Re}(s) > 0$ (see [12], [3]). To extend the definition of (0.4) to the more general situation involving characters, it is necessary to replace varieties by motives. More generally, let X be a smooth connected separated scheme of finite type over k (a field of characteristic zero), let W be a reduced subscheme of X , and let $f : X \rightarrow \mathbf{A}_k^1$ be a morphism. In the present paper, we define, for α a multiplicative character of any finite subgroup of k^\times , motivic Igusa functions $\int_W (f^s, \alpha)$. These functions live in a power series ring $K_0(\mathcal{M})[[\mathbf{L}^{-s}]]$, where $K_0(\mathcal{M})$ is a Grothendieck ring of Chow motives and \mathbf{L} is the standard Lefschetz motive (precise definitions are given in 1.1 and 2.1), and are defined as series quite similar to the right-hand side of (0.4). These motivic Igusa functions specialize, in the p -adic case with good reduction, to the usual p -adic Igusa local zeta functions (see 2.4). They also specialize to the topological zeta functions $Z_{\text{top}}(s)$ introduced by the authors in [5] (see 2.3). The functions $Z_{\text{top}}(s)$ are, heuristically, obtained as a limit as q goes to 1 of p -adic Igusa local zeta functions (in a more provocative way, one can say they are defined by integrals over $W(\mathbf{F}_1)$, the ring of Witt vectors with coefficients in the field with one element).

The content of the paper is the following. The motivic integrals $\int_W (f^s, \alpha)$ are defined in section 2. Their definition uses variants for schemes with finite group action of recent results of Gillet-Soulé [8] and Guillén-Navarro [9] on motivic Euler characteristics of schemes which are given in section 1. In Theorem 2.2.1 we give a formula for $\int_W (f^s, \alpha)$ in terms of an embedded resolution of f , a result that implies, in particular, the rationality of $\int_W (f^s, \alpha)$. We then explain the relationship with topological zeta functions, p -adic Igusa local zeta functions and motivic integration. Section 3 is devoted to functional equations. Here $X = W = \mathbf{A}_k^m$ and f is a homogenous polynomial. In this situation we are able to prove a functional equation for motivic Igusa functions when α is the trivial character. For general α the result depends upon a conjectural statement on motivic Euler characteristics of quotients, but we are able to prove it holds true if one replaces the Grothendieck group of Chow motives by the Grothendieck group of Voevodsky’s “triangulated category of geometrical motives” [28]. These functional equations are analogues in the present setting of the functional equations for p -adic Igusa local zeta functions proved in [7]. In section 4 we study the limit for $s \rightarrow -\infty$ of motivic Igusa functions and investigate its relation with nearby cycles of f at the origin. Here we are guided by analogy with [4], where the limit when $s \rightarrow -\infty$ was studied for p -adic Igusa local zeta functions, and shown to be related to the trace of some liftings of the Frobenius automorphism acting on the cohomology of Milnor fibers. More precisely, for x a closed point of the fiber $f^{-1}(0)$, we give a meaning to

$$\frac{\mathbf{L}^m}{1 - \mathbf{L}} \lim_{s \rightarrow -\infty} \int_{\{x\}} (f^s, \alpha).$$

Heuristically this limit is the “motivic incarnation” of $\chi_c(i_x^* R\psi_{f,\alpha})$, where $R\psi_{f,\alpha}$ denotes the eigenspace of nearby cycles for the eigenvalue corresponding to the character α of the semi-simple part of the monodromy. We prove in Theorem 4.2.1 that this holds, in particular, for the **C**-Hodge realization. As a corollary it follows that the whole Hodge spectrum of f at x , which is an important invariant of singularities (see [22], [23]), may be deduced from the knowledge of motivic Igusa functions.

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1. Grothendieck groups of Chow motives

1.1. Chow motives. In this section we recall material from [14], [15], [21]. We fix a base field k , and we denote by \mathcal{V}_k the category of smooth and projective k -schemes. For an object X in \mathcal{V}_k and an integer d , $\mathcal{Z}^d(X)$ denotes the free abelian group generated by irreducible subvarieties of X of codimension d . We define the rational Chow group $A^d(X)$ as the quotient of $\mathcal{Z}^d(X) \otimes \mathbf{Q}$ modulo rational equivalence. For X and Y in \mathcal{V}_k , we denote by $\text{Corr}^r(X, Y)$ the group of correspondences of degree r from X to Y . If X is purely d -dimensional, $\text{Corr}^r(X, Y) = A^{d+r}(X \times Y)$, and if $X = \coprod X_i$, $\text{Corr}^r(X, Y) = \bigoplus \text{Corr}^r(X_i, Y)$. The category \mathcal{M}_k of k -motives may be defined as follows (cf. [21]). Objects of \mathcal{M}_k are triples (X, p, n) where X is in \mathcal{V}_k , p is an idempotent (i.e., $p^2 = p$) in $\text{Corr}^0(X, X)$, and n is an integer. If (X, p, n) and (Y, q, m) are motives, then

$$\text{Hom}_{\mathcal{M}_k}((X, p, n), (Y, q, m)) = q \text{Corr}^{m-n}(X, Y) p.$$

Composition of morphisms is given by composition of correspondences. The category \mathcal{M}_k is additive, \mathbf{Q} -linear, and pseudo-abelian. There is a natural tensor product on \mathcal{M}_k , defined on objects by

$$(X, p, n) \otimes (Y, q, m) = (X \times Y, p \otimes q, n + m).$$

We denote by h the functor $h : \mathcal{V}_k^\circ \rightarrow \mathcal{M}_k$ that sends an object X to $h(X) = (X, \text{id}, 0)$ and a morphism $f : Y \rightarrow X$ to its graph in $\text{Corr}^0(X, Y)$. This functor is compatible with the tensor product and the unit motive $1 = h(\text{Spec } k)$ is the identity for the product. We denote by \mathbf{L} the Lefschetz motive $\mathbf{L} = (\text{Spec } k, \text{id}, -1)$. There is a canonical isomorphism $h(\mathbf{P}_k^1) \simeq 1 \oplus \mathbf{L}$. We denote by ${}^\vee$ the involution ${}^\vee : \mathcal{M}_k^\circ \rightarrow \mathcal{M}_k$, defined on objects by $(X, p, n)^\vee = (X, {}^t p, d - n)$ if X is purely d -dimensional, and as the transpose of correspondences on morphisms. For X in \mathcal{V}_k purely of dimension d , $h(X)^\vee = h(X) \otimes \mathbf{L}^{-d}$.

Let E be a field of characteristic zero. Replacing the Chow groups A by $A \otimes_{\mathbf{Q}} E$, one defines similarly the category $\mathcal{M}_{k,E}$ of k -motives with coefficients in E .

1.2. Grothendieck groups of Chow motives. Let $K_0(\mathcal{M}_k)$ be the Grothendieck group of the pseudo-abelian category \mathcal{M}_k . It is also the abelian group associated to the monoid of isomorphism classes of motives with respect to the addition \oplus . The tensor product on \mathcal{M}_k induces a natural ring structure on $K_0(\mathcal{M}_k)$. Let Sch_k be the category of schemes that are separated and of finite type over k . We suppose from now on that the characteristic of k is zero. The following result has been proven by Gillet and Soulé [8] and also by Guillén and Navarro Aznar [9].

Theorem 1.2.1. *Let k be a field of characteristic 0. There exists a unique map*

$$\chi_c : \text{ObSch}_k \longrightarrow K_0(\mathcal{M}_k)$$

such that

- (1) *If X is smooth and projective, $\chi_c(X)$ is equal to $[h(X)]$, the image of $h(X)$ in $K_0(\mathcal{M}_k)$.*
- (2) *If Y is a closed subscheme in a scheme X ,*

$$\chi_c(X \setminus Y) = \chi_c(X) - \chi_c(Y).$$

Let us remark that $\chi_c(\mathbf{A}_k^1) = \mathbf{L}$. Also, for X and Y in ObSch_k , we have $\chi_c(X \times Y) = \chi_c(X) \otimes \chi_c(Y)$.

The following result, due to Guillén and Navarro Aznar [9], gives, dually, the existence of motivic Euler characteristics without supports. A proper relative isomorphism $(\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$ consists of the following data : a proper morphism $f : \tilde{X} \rightarrow X$ between objects of Sch_k , reduced closed subschemes \tilde{Y} and Y of \tilde{X} and X respectively, such that \tilde{Y} is the preimage of Y in \tilde{X} , and such that the restriction of f to $\tilde{X} \setminus \tilde{Y}$ is an isomorphism onto $X \setminus Y$.

Theorem 1.2.2. *Let k be a field of characteristic 0. There exists a map*

$$\chi : \text{ObSch}_k \longrightarrow K_0(\mathcal{M}_k)$$

such that

- (1) *If X is smooth and projective, $\chi(X) = [h(X)]$.*
- (2) *If $(\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$ is a proper relative isomorphism,*

$$\chi(X) = \chi(\tilde{X}) + \chi(Y) - \chi(\tilde{Y}).$$

- (3) *If Y is a smooth divisor in a smooth scheme X ,*

$$\chi(X \setminus Y) = \chi(X) - \chi(Y) \otimes \mathbf{L}.$$

- (4) *If X is a smooth scheme purely of dimension d ,*

$$\chi(X)^\vee = \chi_c(X) \otimes \mathbf{L}^{-d}.$$

Furthermore, χ is determined by conditions (1)-(3).

The Euler characteristics χ_c and χ are compatible, with realization functors, in particular, with Euler characteristics of mixed Hodge structures on cohomology with compact support and cohomology, respectively. By additivity χ_c may be naturally extended to constructible sets.

Remark 1.2.3. We expect, but do not know how to prove, that $K_0(\mathcal{M}_{k,E})$ has no $(\mathbf{L} - 1)$ -torsion. This assertion is implied by the conjectural existence (cf. [21], p. 185) of additive functors $h^{\leq j} : \mathcal{M}_{k,E} \rightarrow \mathcal{M}_{k,E}$, $j \in \mathbf{Z}$, such that for any X in $\mathcal{M}_{k,E}$, the $h^{\leq j}(X)$ form a filtration of X with $h^{\leq -k}(X) = 0$,

$h^{\leq k}(X) = X$ for some k , and $h^{\leq j}(\mathbf{L}X) = \mathbf{L}h^{\leq j-2}(X)$ for all j . Indeed, for A in $K_0(\mathcal{M}_{k,E})$ the $h^{\leq j}(A)$'s are well defined in $K_0(\mathcal{M}_{k,E})$ and the relation $(\mathbf{L} - 1)A = 0$ implies that $h^{\leq j}(A) = \mathbf{L}h^{\leq j-2}(A)$, whence $A = 0$. A similar argument also shows, without using any conjecture, that the étale realization and the Hodge realization kill all $(\mathbf{L} - 1)^i$ -torsion in $K_0(\mathcal{M}_{k,E})$, for each i in \mathbf{N} .

1.3. Finite group action. Let G be a finite abelian group and let \hat{G} be its complex character group. We denote by $\mathcal{V}_{k,G}$ the category of smooth and projective k -schemes with G -action. Let E be a subfield of \mathbf{C} containing all the roots of unity of order dividing $|G|$. For X in $\mathcal{V}_{k,G}$ and g in G , we denote by $[g]$ the correspondence given by the graph of multiplication by g .

For α in \hat{G} we consider the idempotent

$$f_\alpha := |G|^{-1} \sum_{g \in G} \alpha^{-1}(g)[g]$$

in $\text{Corr}^0(X, X) \otimes E$, and we denote by $h(X, \alpha)$ the motive $(X, f_\alpha, 0)$ in $\mathcal{M}_{k,E}$. Clearly, for X in $\mathcal{V}_{k,G}$ purely of dimension d and α in \hat{G} , we have $h(X, \alpha)^\vee = h(X, \alpha^{-1}) \otimes \mathbf{L}^{-d}$. We will denote by $\text{Sch}_{k,G}$ the category of separated schemes of finite type over k with G -action satisfying the following condition: the G -orbit of any closed point of X is contained in an affine open subscheme. This condition is clearly satisfied for X quasiprojective and insures the existence of X/G as a scheme. Objects of $\text{Sch}_{k,G}$ will be called G -schemes. From now on all schemes with G -action that will be considered will be assumed to be G -schemes.

We will need the following variants of Theorems 1.2.1 and 1.2.2. They are proved in the appendix as a consequence of [9] and [27]. Theorem 1.3.2 will only be used in section 3.

Theorem 1.3.1. *Let k be a field of characteristic 0. There exists a unique map*

$$\chi_c : \text{ObSch}_{k,G} \times \hat{G} \longrightarrow K_0(\mathcal{M}_{k,E})$$

such that

- (1) *If X is smooth and projective with G -action, for any character α , $\chi_c(X, \alpha) = [h(X, \alpha)]$.*
- (2) *If Y is a closed G -stable subscheme in a scheme X with G -action, for any character α ,*

$$\chi_c(X \setminus Y, \alpha) = \chi_c(X, \alpha) - \chi_c(Y, \alpha).$$

- (3) If X is a scheme with G -action, U and V are G -invariant open subschemes of X , for any character α ,

$$\chi_c(U \cup V, \alpha) = \chi_c(U, \alpha) + \chi_c(V, \alpha) - \chi_c(U \cap V, \alpha).$$

Furthermore, χ_c is determined by conditions (1)-(2).

By a proper relative isomorphism of G -schemes $(\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$ we mean the following data: a proper morphism $f : \tilde{X} \rightarrow X$ of G -schemes, reduced closed G -stable subschemes \tilde{Y} and Y of \tilde{X} and X respectively, such that \tilde{Y} is the preimage of Y in \tilde{X} , and such that the restriction of f to $\tilde{X} \setminus \tilde{Y}$ is a G -isomorphism onto $X \setminus Y$.

Theorem 1.3.2. *Let k be a field of characteristic 0. There exists a map*

$$\chi : \text{Obsch}_{k,G} \times \hat{G} \longrightarrow K_0(\mathcal{M}_{k,E})$$

such that

- (1) If X is smooth and projective with G -action, $\chi(X, \alpha) = [h(X, \alpha)]$.
 (2) If $(\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$ is a proper relative isomorphism of G -schemes,

$$\chi(X, \alpha) = \chi(\tilde{X}, \alpha) + \chi(Y, \alpha) - \chi(\tilde{Y}, \alpha).$$

- (3) If Y is a smooth G -invariant divisor in a smooth scheme X with G -action,

$$\chi(X \setminus Y, \alpha) = \chi(X, \alpha) - \chi(Y, \alpha) \otimes \mathbf{L}.$$

- (4) If X is a smooth scheme with G -action purely of dimension d ,

$$\chi(X, \alpha)^\vee = \chi_c(X, \alpha^{-1}) \otimes \mathbf{L}^{-d}.$$

- (5) If X is a proper scheme with G -action,

$$\chi_c(X, \alpha) = \chi(X, \alpha).$$

- (6) If X is a scheme with G -action and U and V are G -invariant open subschemes of X , then, for any character α ,

$$\chi(U \cup V, \alpha) = \chi(U, \alpha) + \chi(V, \alpha) - \chi(U \cap V, \alpha).$$

Furthermore, χ is determined by conditions (1)-(3).

Proposition 1.3.3. *Let k be a field of characteristic 0.*

- (1) For any X in $\text{Obsch}_{k,G}$,

$$\chi_c(X) = \sum_{\alpha \in \hat{G}} \chi_c(X, \alpha).$$

- (2) Let X be in $\text{Obsch}_{k,G}$. Assume the G -action factors through a quotient $G \rightarrow H$. If α is not in the image of $\hat{H} \rightarrow \hat{G}$, then $\chi_c(X, \alpha) = 0$.

(3) Let X and Y be in $\text{ObSch}_{k,G}$ and let G act diagonally on $X \times Y$. Then

$$\chi_c(X \times Y, \alpha) = \sum_{\beta \in \hat{G}} \chi_c(X, \beta) \chi_c(Y, \alpha\beta^{-1}).$$

Proof. If X is smooth and projective with G -action, $\text{Id} = \bigoplus_{\alpha \in \hat{G}} f_\alpha$; so $h(X) = \bigoplus_{\alpha \in \hat{G}} h(X, \alpha)$. It is a direct verification that, if the G -action factors through a quotient $G \rightarrow H$ and α is not in the image of $\hat{H} \rightarrow \hat{G}$, then $f_\alpha = 0$. If Y is another smooth and projective scheme with G -action, then

$$f_\alpha(X \times Y) = \sum_{\beta \in \hat{G}} f_\beta(X) \otimes f_{\alpha\beta^{-1}}(Y).$$

Assertions (1), (2) and (3) follow by additivity of $\chi_c(\ , \alpha)$. □

1.4. Motivic Kummer sheaves. We fix an integer $d \geq 1$. We denote by $\mu_d(k)$ the group of d -roots of 1 in k and by ζ_d a fixed primitive d -th root of unity in \mathbf{C} . We assume from now on that $\mu_d(k)$ is of order d .

Let $f : X \rightarrow \mathbf{G}_{m,k}$ be a morphism in Sch_k . For any character α of order d of $\mu_d(k)$, one may define an element $[X, f^* \mathcal{L}_\alpha]$ of $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$ as follows.

The morphism $[d] : \mathbf{G}_{m,k} \rightarrow \mathbf{G}_{m,k}$ given by $x \mapsto x^d$ is a Galois covering with Galois group $\mu_d(k)$. We consider the fiber product

$$\begin{array}{ccc} \tilde{X}_{f,d} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathbf{G}_{m,k} & \xrightarrow{[d]} & \mathbf{G}_{m,k} \end{array}$$

The scheme $\tilde{X}_{f,d}$ is endowed with an action of $\mu_d(k)$, and so we can define

$$[X, f^* \mathcal{L}_\alpha] := \chi_c(\tilde{X}_{f,d}, \alpha).$$

Lemma 1.4.1. *Let $f : X \rightarrow \mathbf{G}_{m,k}$ and $g : X \rightarrow \mathbf{G}_{m,k}$ be morphisms in Sch_k . For any character α of order d of $\mu_d(k)$, the following holds:*

$$[X, (f^d g)^* \mathcal{L}_\alpha] = [X, g^* \mathcal{L}_\alpha].$$

In particular, $[X, f^{d*} \mathcal{L}_\alpha] = \chi_c(X)$.

Proof. The morphism $(x, t) \mapsto (x, tf^{-1}(x))$ induces an isomorphism of $\mu_d(k)$ -schemes $\tilde{X}_{f^d g, d} \simeq \tilde{X}_{g, d}$. For the last assertion remark that the fiber product $\tilde{X}_{1,d}$ is isomorphic as a $\mu_d(k)$ -scheme to the product $X \times \mu_d(k)$. □

Lemma 1.4.2. *Let $f : X \rightarrow \mathbf{G}_{m,k}$ and $g : Y \rightarrow X$ be morphisms in Sch_k . Assume that g is a locally trivial fibration for the Zariski topology with fiber Z . For any character α of order d of $\mu_d(k)$, the following holds:*

$$[Y, (f \circ g)^* \mathcal{L}_\alpha] = \chi_c(Z)[X, f^* \mathcal{L}_\alpha].$$

Proof. Immediate. □

Lemma 1.4.3. *Let a be an integer and let $\mu_d(k)$ act on $\mathbf{G}_{m,k}$ by multiplication by ξ^a , $\xi \in \mu_d(k)$. For any nontrivial character α of $\mu_d(k)$, $\chi_c(\mathbf{G}_{m,k}, \alpha) = 0$.*

Proof. The action of $\mu_d(k)$ on $\mathbf{G}_{m,k}$ extends to an action on \mathbf{P}_k^1 leaving fixed 0 and ∞ . So it is enough to verify that if α is a nontrivial character of $\mu_d(k)$, $[h(\mathbf{P}_k^1, \alpha)] = 0$. Now remark that, for any ξ in k^\times , the class of the graph of the multiplication by ξ in $A^1(\mathbf{P}_k^1 \times \mathbf{P}_k^1)$ is equal to the class of the diagonal; hence $f_\alpha = 1$ if α is trivial, and $f_\alpha = 0$ otherwise. □

If $f : X \rightarrow \mathbf{G}_{m,k}$ and $g : Y \rightarrow \mathbf{G}_{m,k}$ are morphisms in Sch_k , we denote by $f \otimes g$ the morphism $X \times Y \rightarrow \mathbf{G}_{m,k}$ given by multiplication of f and g . We have the following generalization of Lemma 1.4.3.

Lemma 1.4.4. *Let $g : Z \rightarrow \mathbf{G}_{m,k}$ be a morphism in Sch_k . For any character α of order d of $\mu_d(k)$ and any integer n not divisible by d ,*

$$[\mathbf{G}_{m,k} \times Z, ([n] \otimes g)^* \mathcal{L}_\alpha] = 0.$$

Proof. Set $W = (\widetilde{\mathbf{G}_{m,k} \times Z})_{[n] \otimes g, d}$. We may identify W with $\{(x, z, t) \in \mathbf{G}_{m,k} \times Z \times \mathbf{G}_{m,k} \mid x^n g(z) = t^d\}$, the action of $\mu_d(k)$ being multiplication on the last factor. Set $\delta = \text{gcd}(n, d)$, $n = \delta n'$, $d = \delta d'$, and choose integers a and b such that $an' = 1 + bd'$. If we set $w = t^{d'} x^{-n'}$, $x' = w^a x$, $t' = w^b t$, we may identify W with

$$\{(x', z, t', w) \in \mathbf{G}_{m,k} \times Z \times \mathbf{G}_{m,k} \times \mathbf{G}_{m,k} \mid g(z) = w^\delta \text{ and } x'^{n'} = t'^{d'}\}.$$

We may rewrite this as an isomorphism $W \simeq \widetilde{Z}_{g,\delta} \times \mathbf{G}_{m,k}$, the action of $\mu_d(k)$ being the product of the action on $\widetilde{Z}_{g,\delta}$ given by composition with the surjection $\xi \mapsto \xi^{d'}$, $\mu_d(k) \rightarrow \mu_\delta(k)$, with the action on $\mathbf{G}_{m,k}$ given by multiplication by ξ^a , for $\xi \in \mu_d(k)$. By Proposition 1.3.3 (3),

$$\chi_c(W, \alpha) = \sum_{\beta \in \widehat{\mu_d(k)}} \chi_c(\widetilde{Z}_{g,\delta}, \beta) \chi_c(\mathbf{G}_{m,k}, \alpha \beta^{-1}).$$

Hence, by Lemma 1.4.3 and Proposition 1.3.3 (1),

$$\chi_c(W, \alpha) = \chi_c(\widetilde{Z}_{g,\delta}, \alpha) \chi_c(\mathbf{G}_{m,k}).$$

But $\chi_c(\widetilde{Z}_{g,\delta}, \alpha) = 0$, by Proposition 1.3.3 (2), because α is of order d and $\delta < d$. □

1.5. Quotients. We discuss here motivic Euler characteristics of quotients. This part will only be used in section 3.

Lemma 1.5.1. *Let X be a smooth projective scheme with G -action, H a subgroup of G , and α a character of G/H . Assume the quotient X/H is smooth. Then $h(X/H, \alpha) \simeq h(X, \alpha \circ \varrho)$, where ϱ is the projection $G \rightarrow G/H$.*

Proof. The projection $X \rightarrow X/H$ induces by functoriality a morphism

$$h(X/H) \longrightarrow h(X)$$

in \mathcal{M}_k , which induces an isomorphism between $h(X/H, \alpha)$ and $h(X, \alpha \circ \varrho)$. \square

In view of Lemma 1.5.1 and Corollary 1.5.4, it seems quite natural to expect that the following statement holds.

Assertion 1.5.2. *If X is a scheme with G -action, H a subgroup of G , and α a character of G/H , then $\chi_c(X/H, \alpha) = \chi_c(X, \alpha \circ \varrho)$ and $\chi(X/H, \alpha) = \chi(X, \alpha \circ \varrho)$, where ϱ is the projection $G \rightarrow G/H$.*

In the paper [28], Voevodsky constructs for a perfect field k a tensor triangulated category $DM_{gm}(k)$ which he calls the triangulated category of geometrical motives. When k is of characteristic zero, he associates to any object X in Sch_k complexes $M_{gm}^c(X)$ and $M_{gm}(X)$ in $DM_{gm}(k)$. Let E be a field of characteristic zero and denote by $DM_{gm}(k)_E$ the category $DM_{gm}(k) \otimes E$. By [28] 2.2, when X is proper and smooth, $M_{gm}(X) = M_{gm}^c(X)$ and the restriction of M_{gm} to \mathcal{V}_k factorizes through an additive functor $\mathcal{M}_{k,E} \rightarrow DM_{gm}(k)_E$. Hence there is a canonical morphism of groups $\varphi : K_0(\mathcal{M}_{k,E}) \rightarrow K_0(DM_{gm}(k)_E)$. The tensor structure on $DM_{gm}(k)$ induces a ring structure on the Grothendieck group $K_0(DM_{gm}(k)_E)$ and φ is a morphism of rings. By [28] Corollary 3.5.5, the morphism φ is surjective, but it does not seem to be known whether φ is injective or not. It follows directly from the properties of χ_c and M_{gm}^c that, for any X in Sch_k , $\varphi(\chi_c(X)) = [M_{gm}^c(X)]$ (cf. [8] 3.2.4). Similarly, it follows by an easy induction on dimension and the properties of χ and M_{gm} that, for any X in Sch_k , $\varphi(\chi(X)) = [M_{gm}(X)]$. Let G be a finite abelian group and assume E contains all the roots of unity of order dividing $|G|$. Let X be an object of $Sch_{k,G}$. Then it follows from the definition of the complexes $M_{gm}^c(X)$ and $M_{gm}(X)$ that G acts on them and that they decompose in $DM_{gm}(k)_E$ into direct sums of isotypic components $M_{gm}^c(X) \simeq \bigoplus_{\alpha \in \hat{G}} M_{gm}^c(X)_\alpha$ and $M_{gm}(X) \simeq \bigoplus_{\alpha \in \hat{G}} M_{gm}(X)_\alpha$. One derives similarly as before that $\varphi(\chi_c(X, \alpha)) = [M_{gm}^c(X)_\alpha]$ and $\varphi(\chi(X, \alpha)) = [M_{gm}(X)_\alpha]$.

Lemma 1.5.3. *If X is a scheme with G -action, H is a subgroup of G , and α is a character of G/H , then there are canonical isomorphisms $M_{gm}^c(X/H)_\alpha \simeq M_{gm}^c(X)_{\alpha \circ \varrho}$ and $M_{gm}(X/H)_\alpha \simeq M_{gm}(X)_{\alpha \circ \varrho}$.*

Proof. This follows directly from the definition (such a statement is already true at the level of the Nisnevich sheaves L^c and L of [28] 4.1). \square

Corollary 1.5.4. *If X is a scheme with G -action, H is a subgroup of G , and α is a character of G/H , then $\varphi(\chi_c(X/H, \alpha)) = \varphi(\chi_c(X, \alpha \circ \varrho))$ and $\varphi(\chi(X/H, \alpha)) = \varphi(\chi(X, \alpha \circ \varrho))$. \square*

2. Motivic Igusa zeta functions

2.1. We will consider the ring of formal series $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[[\mathbf{L}^{-s}]]$. In this ring we will write $\mathbf{L}^j \cdot (\mathbf{L}^{-s})^i = \mathbf{L}^{j-si}$, when $j \in \mathbf{Z}$ and $i \in \mathbf{N}$. We will also consider the subring $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[\mathbf{L}^{-s}]_{\text{loc}}$ of the ring $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[[\mathbf{L}^{-s}]]$ generated by $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[\mathbf{L}^{-s}]$ and the series

$$(1 - \mathbf{L}^{-Ns-n})^{-1} = \sum_{i \in \mathbf{N}} \mathbf{L}^{-Nsi-ni},$$

for N and n in $\mathbf{N} \setminus \{0\}$.

Let X be a smooth and connected separated k -scheme of finite type of dimension m , $f : X \rightarrow \mathbf{A}_k^1$ be a morphism, and W be a reduced subscheme of X . We assume that $\mu_d(k)$ is of order d . For any character α of $\mu_d(k)$ of order d , we define the motivic Igusa zeta function $\int_W (f^s, \alpha)$ in $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[[\mathbf{L}^{-s}]]$ as follows.

We denote by $\mathcal{L}_n(X)$ the k -scheme that represents the functor, defined on the category of k -algebras,

$$R \mapsto \text{Mor}_{k\text{-schemes}}(\text{Spec } R[t]/t^{n+1}R[t], X),$$

for $n \geq 0$ (cf. p. 276 of S. Bosch, W. Lütkebohmert and M. Raynaud, *Neron models*, *Ergeb. Math. Grenzgeb.* (3) 21, Springer-Verlag, Berlin, 1990). We denote by $\mathcal{L}(X)$ the projective limit in the category of schemes of the schemes $\mathcal{L}_n(X)$, which exists since the transition maps are affine. Note that for any field K containing k the K -rational points of $\mathcal{L}(X)$ are the morphisms $\text{Spec } K[[t]] \rightarrow X$. For $f : X \rightarrow \mathbf{A}_k^1 = \text{Spec } k[x]$ a morphism, and n in \mathbf{N} , we define $Z_{n,f,W}$ as the reduced subscheme of $\mathcal{L}(X)$ whose K -rational points, for any field K containing k , are the morphisms $\varphi : \text{Spec } K[[t]] \rightarrow X$ sending the closed point of $\text{Spec } K[[t]]$ to a point in W , and such that $f \circ \varphi$ is exactly of order n at the origin. We denote by $X_{n,f,W}$ the image of $Z_{n,f,W}$ in $\mathcal{L}_n(X)$, viewed as a reduced subscheme of $\mathcal{L}_n(X)$, and by \bar{f} the morphism $\bar{f} : X_{n,f,W} \rightarrow \mathbf{G}_{m,k}$ that associates to φ in $X_{n,f,W}$ the constant term of the series $t^{-n}x(f \circ \varphi)$.

We define, for any character α of $\mu_d(k)$ of order d ,

$$\int_W (f^s, \alpha) := \sum_{n \in \mathbf{N}} [X_{n,f,W}, \bar{f}^* \mathcal{L}_\alpha] \mathbf{L}^{-ns-(n+1)m}$$

in $K_0(\mathcal{M}_{k, \mathbb{Q}[\zeta_d]})[[\mathbf{L}^{-s}]]$. When α is the trivial character, we write $\int_W f^s$ instead of $\int_W (f^s, \alpha)$.

Remarks.

- (1) When α is the trivial character, $\int_W f^s$ is the image of the series

$$\int_W^{\sim} f^s := \sum_{n \in \mathbb{N}} [X_{n, f, W}] \mathbf{L}^{-ns - (n+1)m}$$

in $K_0(\text{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]]$ by the natural morphism

$$K_0(\text{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]] \longrightarrow K_0(\mathcal{M}_k)[[\mathbf{L}^{-s}]]$$

induced by χ_c .

- (2) It would be interesting to investigate whether motivic Igusa functions already exist at a finer level than a Grothendieck group of Chow motives, for instance at the level of complexes of Chow motives, or objects of $DM_{gm}(k)$, or “mixed motives”.

2.2. Let D be the divisor defined by $f = 0$ in X . Let (Y, h) be a resolution of f . By this, we mean that Y is a smooth and connected k -scheme of finite type, $h : Y \rightarrow X$ is proper, that the restriction $h : Y \setminus h^{-1}(D) \rightarrow X \setminus D$ is an isomorphism, and that $(h^{-1}(D))_{\text{red}}$ has only normal crossings as a subscheme of Y . Let $E_i, i \in J$, be the irreducible (smooth) components of $(h^{-1}(D))_{\text{red}}$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ h$ on Y , and by $\nu_i - 1$ the multiplicity of E_i in the divisor of $h^* dx$, where dx is a local nonvanishing volume form, i.e., a local generator of the sheaf of differential forms of maximal degree. For $i \in J$ and $I \subset J$, we consider the schemes $E_i^\circ := E_i \setminus \bigcup_{j \neq i} E_j, E_I := \bigcap_{i \in I} E_i,$ and $E_I^\circ := E_I \setminus \bigcup_{j \in J \setminus I} E_j$. When $I = \emptyset$, we have $E_\emptyset = Y$.

Now denote by J_d the set of $I \subset J$ such that $d \mid N_i$ for all i in I and by U_d the union of the E_I° 's, with I in J_d . Let Z be locally closed in U_d . For any character α of $\mu_d(k)$ of order d , we will construct an element $[Z_{f, \alpha}]$ in $K_0(\mathcal{M}_{k, \mathbb{Q}[\zeta_d]})$ as follows. If on Z we may write $f \circ h = uv^d$ with u nonvanishing on Z , we set $[Z_{f, \alpha}] = [Z, u^* \mathcal{L}_\alpha]$. It is well defined by Lemma 1.4.1. In general we cover Z by a finite set of Z_r 's for which the previous condition holds, and we set

$$[Z_{f, \alpha}] = \sum_r [(Z_r)_{f, \alpha}] - \sum_{r_1 \neq r_2} [(Z_{r_1} \cap Z_{r_2})_{f, \alpha}] + \dots,$$

which is well defined by additivity of $\chi_c(\quad, \alpha)$.

We can now state the following result.

Theorem 2.2.1. *For any character α of $\mu_d(k)$ of order d ,*

$$\int_W (f^s, \alpha) = \mathbf{L}^{-m} \sum_{I \in J_d} [(E_I^\circ \cap h^{-1}(W))_{f, \alpha}] \prod_{i \in I} \frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - \nu_i}}{1 - \mathbf{L}^{-N_i s - \nu_i}}$$

in $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [[\mathbf{L}^{-s}]]$. In particular, $\int_W (f^s, \alpha)$ belongs to the ring $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]_{\text{loc}}$.

Proof. We set $\mathcal{L}(X, D) := \mathcal{L}(X) \setminus \mathcal{L}(D)$, and we define similarly the scheme $\mathcal{L}(Y, h^{-1}(D))$. We denote by π_n the projections $\mathcal{L}(X, D) \rightarrow \mathcal{L}_n(X)$ and $\mathcal{L}(Y, h^{-1}(D)) \rightarrow \mathcal{L}_n(Y)$, and by π the projections onto X and Y respectively. If U is a reduced subscheme of Y we set $\mathcal{L}_U(Y, h^{-1}(D)) = \pi^{-1}(U)$ and $\mathcal{L}_U(X, D) = h_*(\mathcal{L}_U(Y, h^{-1}(D)))$, and we define similarly $\mathcal{L}_{n,U}(Y)$. Moreover, for $n' \geq n$, we denote by $\pi_{n',n}$ the projections $\mathcal{L}_{n'}(X) \rightarrow \mathcal{L}_n(X)$ and $\mathcal{L}_{n'}(Y) \rightarrow \mathcal{L}_n(Y)$.

The morphism h being proper, composition with h induces a bijective morphism $h_* : \mathcal{L}(Y, h^{-1}(D)) \simeq \mathcal{L}(X, D)$, and we have a commutative diagram

$$\begin{CD} \mathcal{L}(Y, h^{-1}(D)) @>\pi_n>> \mathcal{L}_n(Y) \\ @V h_* VV @VV h_{n*} V \\ \mathcal{L}(X, D) @>\pi_n>> \mathcal{L}_n(X). \end{CD}$$

For U a reduced subscheme of Y , we set

$$X_{n,f,W,U}^{n'} := \pi_{n'}(Z_{n,f,W} \cap \mathcal{L}_U(X, D)).$$

We define

$$M_{n,f,W,U} := \mathbf{L}^{-(n-n')m} [X_{n,f,W,U}^{n'} (\bar{f} \circ \pi_{n',n})^* \mathcal{L}_\alpha],$$

where n' is big enough with respect to n . By Lemma 1.4.2, the definition of $M_{n,f,W,U}$ does not depend on n' because $\pi_{n',n}$ is a locally trivial fibration for the Zariski topology with fiber $\mathbf{A}_k^{m(n'-n)}$ and because $Z_{n,f,W} \cap \mathcal{L}_U(X, D)$ is a union of fibers of $\pi_{n'} : \mathcal{L}(X, D) \rightarrow \mathcal{L}_{n'}(X)$ when $n' \gg n$, since h_*^{-1} has “only powers of f in the denominator”. For any character α of $\mu_d(k)$ of order d , we define

$$\int_{U \cap h^{-1}(W)} h^*(f^s, \alpha) = \sum_{n \in \mathbf{N}} M_{n,f,W,U} \mathbf{L}^{-ns - (n+1)m}$$

in $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [[\mathbf{L}^{-s}]]$. The result is a direct consequence of the following proposition, by additivity of $\chi_c(\ , \alpha)$. \square

Proposition 2.2.2. *Assume $U \subset E_I^\circ$ and $f \circ h = u \prod_{i \in I} y_i^{N_i}$ on a neighborhood of U , where u is a unit on U , and $y_i = 0$ is an equation for E_i on a neighborhood of U .*

(1) If d divides N_i , for all $i \in I$, then

$$\int_{U \cap h^{-1}(W)} h^*(f^s, \alpha) = \mathbf{L}^{-m}[(U \cap h^{-1}(W))_{f, \alpha}] \prod_{i \in I} \frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - \nu_i}}{1 - \mathbf{L}^{-N_i s - \nu_i}}$$

in $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[[\mathbf{L}^{-s}]]$.

(2) If d does not divide N_i , for some $i \in I$, then

$$\int_{U \cap h^{-1}(W)} h^*(f^s, \alpha) = 0.$$

Proof. We may from the beginning assume $U \cap h^{-1}(W) = U$ and we will write $X_{n, f, U}^{n'}$ instead of $X_{n, f, W, U}^{n'}$. We will use the following lemma.

Let X, Y and F be algebraic varieties over k , and let A , resp. B , be a constructible subset of X , resp. Y . We say that a map $\pi : A \rightarrow B$ is a *piecewise trivial fibration with fiber F* , if there exists a finite partition of B in subsets S that are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and isomorphic, as a variety over k , to $S \times F$, with π corresponding under the isomorphism to the projection $S \times F \rightarrow S$. We say that the map π is a *piecewise trivial fibration over some constructible subset C of B* , if the restriction of π to $\pi^{-1}(C)$ is a piecewise trivial fibration.

Lemma 2.2.3. *Let X and Y be connected smooth schemes over a field k and let $h : Y \rightarrow X$ be a birational morphism. For e in \mathbf{N} , let Δ_e be the reduced subscheme of $\mathcal{L}(Y)$ defined by*

$$\Delta_e(K) := \{ \varphi \in Y(K[[t]]) \mid \text{ord}_t \det \mathcal{J}_\varphi = e \},$$

for any field K containing k , where \mathcal{J}_φ is the Jacobian of h at φ . For n in \mathbf{N} , let $h_{n*} : \mathcal{L}_n(Y) \rightarrow \mathcal{L}_n(X)$ be the morphism induced by h , and let $\Delta_{e, n}$ be the image of Δ_e in $\mathcal{L}_n(Y)$. If $n \geq 2e$, the following holds.

- a) The set $\Delta_{e, n}$ is a union of fibers of h_{n*} .
- b) The restriction of h_{n*} to $\Delta_{e, n}$ is a piecewise trivial fibration with fiber \mathbf{A}_k^e onto its image.

Proof. This is a special case of Lemma 3.4 of [6]. □

Let $m_i, i \in I$, be strictly positive integers with $\sum_{i \in I} m_i N_i = n$. We denote by $\tilde{Z}_{(m_i), U}$ the reduced subscheme of $\mathcal{L}_U(Y, h^{-1}(D))$ whose K -rational points $\varphi : \text{Spec } K[[t]] \rightarrow Y$, for any field K containing k , satisfy the condition that $y_i \circ \varphi$ is exactly of order m_i at the origin, for $i \in I$. We denote by $Y_{(m_i), U}$ the image of $\tilde{Z}_{(m_i), U}$ in $\mathcal{L}_{n, U}(Y)$. By Lemma 2.2.3, for n' big enough with respect to n , the set $X_{n, f, U}^{n'}$ is the disjoint finite union of the sets $h_{n'*}(\bigcup_{\mathcal{E}_e} \pi_{n', n}^{-1}(Y_{(m_i), U}))$ for $e = 0, 1, 2, \dots$, where \mathcal{E}_e is the set of all $(m_i)_{i \in I}$ with $m_i > 0$, $\sum_{i \in I} m_i N_i =$

n and $\sum_{i \in I} (\nu_i - 1)m_i = e$. Hence we deduce from Lemma 1.4.2 and Lemma 2.2.3 that

$$\int_{U \cap h^{-1}(W)} h^*(f^s, \alpha) = \sum_{m_i > 0} \mathbf{L}^{-\sum_{i \in I} (\nu_i - 1)m_i} [Y_{(m_i), U}, (\bar{f} \circ h_{n*})^* \mathcal{L}_\alpha] \mathbf{L}^{-ns - (n+1)m},$$

with $n = \sum_{i \in I} m_i N_i$. (Actually we need here the slightly stronger version of Lemma 2.2.3 obtained by replacing Δ_e by $\Delta_e \cap \mathcal{L}_U(Y, h^{-1}(D))$. But the proof of this version is the same.)

Now remark that $\gamma : Y_{(m_i), U} \rightarrow U$ is a locally trivial fibration for the Zariski topology with fibre $\mathbf{G}_m^I \times \mathbf{A}^{nm - \sum_{i \in I} m_i}$. On $Y_{(m_i), U}$ the function $\bar{f} \circ h_{n*}$ coincides with the product $(u|_U \circ \gamma) \cdot \psi$, where $\psi(\varphi)$ is the constant term of $\prod_{i \in I} y_i(\varphi(t))^{N_i} t^{-\sum_i m_i N_i}$.

So, if d divides N_i for all $i \in I$, we deduce from Lemma 1.4.1 and Lemma 1.4.2 that

$$[Y_{(m_i), U}, (\bar{f} \circ h_{n*})^* \mathcal{L}_\alpha] = [U_{f, \alpha}] (\mathbf{L} - 1)^{|I|} \mathbf{L}^{nm - \sum_{i \in I} m_i},$$

and the result follows from the previous relation.

Assume now that some N_i with $i \in I$ is not divisible by d . We may assume, shrinking U if necessary, that γ is a product. We may then identify $Y_{(m_i), U}$ with a product $\mathbf{G}_{m, k} \times Z$ in such a way that $\bar{f} \circ h_{n*} = [m] \otimes g$, with m not divisible by d (notations of 1.4) and now the result follows from Lemma 1.4.4. □

2.3. Relation with the topological zeta functions of [5]. Let us denote by $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]'_{\text{loc}}$ the subring of $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]_{\text{loc}}$ generated by the ring of polynomials $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]$ and by the quotients $(\mathbf{L} - 1)(1 - \mathbf{L}^{-Ns-n})^{-1}$, for N and n in $\mathbf{N} \setminus \{0\}$. By expanding \mathbf{L}^{-s} and $(\mathbf{L} - 1)(1 - \mathbf{L}^{-Ns-n})^{-1}$ into series in $\mathbf{L} - 1$, one gets a canonical morphism of algebras

$$\varphi : K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]'_{\text{loc}} \longrightarrow \overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [s][[(Ns + n)^{-1}]_{n, N \in \mathbf{N} \setminus \{0\}}][[\mathbf{L} - 1]],$$

where $[[\mathbf{L} - 1]]$ denotes completion with respect to the ideal generated by $\mathbf{L} - 1$ and where $\overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$ is the largest quotient of $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$ with no $(\mathbf{L} - 1)$ -torsion, cf. remark 1.2.3. Taking the quotient of $\overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [s][[(Ns + n)^{-1}]_{n, N \in \mathbf{N} \setminus \{0\}}][[\mathbf{L} - 1]]$ by the ideal generated by $\mathbf{L} - 1$, one obtains the

evaluation morphism

$$\begin{aligned} \text{ev}_{\mathbf{L}=1} : \overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [s] [(Ns + n)^{-1}]_{n, N \in \mathbf{N} \setminus \{0\}} [[\mathbf{L} - 1]] &\longrightarrow \\ (\overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) / \mathbf{L} - 1) [s] [(Ns + n)^{-1}]_{n, N \in \mathbf{N} \setminus \{0\}}. & \end{aligned}$$

For X in \mathcal{V}_k , we denote by $\chi_{\text{top}}(X)$ the usual Euler characteristic of X (say in étale $\overline{\mathbf{Q}}_\ell$ -cohomology). This induces by 1.2.3 a morphism

$$\chi_{\text{top}} : \overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) \longrightarrow \mathbf{Z},$$

which induces, since $\chi_{\text{top}}(\mathbf{L}) = 1$, a morphism

$$\begin{aligned} \chi_{\text{top}} : (\overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) / \mathbf{L} - 1) [s] [(Ns + n)^{-1}]_{n, N \in \mathbf{N} \setminus \{0\}} &\longrightarrow \\ \mathbf{Z}[s] [(Ns + n)^{-1}]_{n, N \in \mathbf{N} \setminus \{0\}}. & \end{aligned}$$

By Theorem 2.2.1, for any character α of $\mu_d(k)$ of order d , the motivic Igusa function $\int_W (f^s, \alpha)$ belongs to $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]'_{\text{loc}}$; hence we can consider the rational function $(\chi_{\text{top}} \circ \text{ev}_{\mathbf{L}=1} \circ \varphi) \left(\int_W (f^s, \alpha) \right)$ in $\mathbf{C}(s)$.

Proposition 2.3.1. *For any character α of $\mu_d(k)$ of order d ,*

$$(\chi_{\text{top}} \circ \text{ev}_{\mathbf{L}=1} \circ \varphi) \left(\int_W (f^s, \alpha) \right) = \sum_{I \in J_d} \chi_{\text{top}}(E_I^\circ \cap h^{-1}(W)) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

Proof. By Theorem 2.2.1, it is enough to check that

$$\chi_{\text{top}}([(E_I^\circ \cap h^{-1}(W))_{f, \alpha}]) = \chi_{\text{top}}(E_I^\circ \cap h^{-1}(W)).$$

This is clear, since the étale ℓ -adic realization of $[(E_I^\circ \cap h^{-1}(W))_{f, \alpha}]$ is given by cohomology with compact support of a rank one lisse sheaf on $E_I^\circ \cap h^{-1}(W)$. □

Remarks.

- (1) It follows from Proposition 2.3.1 that the topological zeta functions of [5] are obtained by specialization of motivic Igusa zeta functions. This gives another proof, not using p -adic analysis, of the main results of [5] on the invariance of topological zeta functions in the algebraic case. In fact, it is easily checked that similar arguments work also in the complex analytic case.
- (2) It might be interesting to study the functions $(\text{ev}_{\mathbf{L}=1} \circ \varphi) \left(\int_W (f^s, \alpha) \right)$ that belong to $(\overline{K}_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) / \mathbf{L} - 1)(s)$.

2.4. Relation with p -adic Igusa local zeta functions. Let p be a prime number and let K be a finite extension of \mathbf{Q}_p . Let R be the valuation ring of K , P the maximal ideal of R , and $\bar{K} = R/P$ the residue field of K . Let q denote the cardinality of \bar{K} , so that $\bar{K} \simeq \mathbf{F}_q$. For z in K , let $\text{ord } z$ denote the valuation of z , and set $|z| = q^{-\text{ord } z}$ and $\text{ac}(z) = z\pi^{-\text{ord } z}$, where π is a fixed uniformizing parameter of R . Let f be an element of $R[x_1, \dots, x_m]$ that is not zero modulo P . For any character $\alpha : R^\times \rightarrow \mathbf{C}^\times$, one defines the p -adic Igusa local zeta function $Z(s, \alpha)$ as the integral

$$Z(s, \alpha) = \int_{R^m} \alpha(\text{ac}(f(x))) |f(x)|^s |dx|,$$

for $s \in \mathbf{C}$, $\text{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^m normalized in such a way that R^m is of volume 1.

Let (Y, h) be a resolution of f as in 2.2. We say the resolution (Y, h) has good reduction mod P , if Y has a smooth model Y_R over $\text{Spec } R$ such that h extends to a morphism $Y_R \rightarrow \mathbf{A}_R^m$ and the closure of $h^{-1}(D)_{\text{red}}$ in Y_R is a relative divisor with normal crossings over Y_R . For Z closed in Y , we denote by \bar{Z} the fiber over the closed point of the closure of Z in Y_R . Hence \bar{Y} and all the \bar{E}_i 's are smooth, $\bigcup_{i \in J} \bar{E}_i$ is a divisor with normal crossings, and the schemes \bar{E}_i and \bar{E}_j have no component in common for $i \neq j$. Let (Y, h) be a resolution with good reduction mod P . For $I \subset J$ we have $\bar{E}_I = \bigcap_{i \in I} \bar{E}_i$ and we set $\bar{E}_I^\circ := \bar{E}_I \setminus \bigcup_{j \in J \setminus I} \bar{E}_j$.

Assume now that the character α is of finite order d and is trivial on $1 + P$. Choose a prime number $\ell \neq p$ and denote by \mathcal{L}_α the Kummer $\bar{\mathbf{Q}}_\ell$ -sheaf on $\mathbf{G}_{m, \bar{K}}$ associated to α viewed as a character of \bar{K}^\times (here we choose an isomorphism between the group of roots of unity in \mathbf{C} and in $\bar{\mathbf{Q}}_\ell$). Set $U = \bar{Y} \setminus \bigcup_{i \in J} \bar{E}_i$ and denote by $\nu : U \hookrightarrow \bar{Y}$ the open immersion and by $\beta : U \rightarrow \mathbf{G}_{m, \bar{K}}$ the map induced by $\bar{f} \circ \bar{h}$. Set $\mathcal{F}_\alpha = \nu_* \beta^* \mathcal{L}_\alpha$. Denote by \bar{K}^a the algebraic closure of \bar{K} and by F the geometric Frobenius automorphism.

In the good reduction case the following result gives a cohomological expression for p -adic Igusa local zeta functions.

Theorem 2.4.1 ([1][2]). *Let (Y, h) be a resolution of f with good reduction mod P . Assume the character α is of finite order d and is trivial on $1 + P$. Then*

$$Z(s, \alpha) = q^{-m} \sum_{I \in J_d} c_{I, \alpha} \prod_{i \in I} \frac{(q-1) q^{-N_i s - \nu_i}}{1 - q^{-N_i s - \nu_i}}$$

with

$$c_{I, \alpha} = \sum_i (-1)^i \text{Tr}(F, H_c^i(\bar{E}_I^\circ \otimes \bar{K}^a, \mathcal{F}_\alpha)).$$

In conclusion, in view of Theorem 2.2.1 and Theorem 2.4.1, one can state that “in the good reduction case, the p -adic Igusa local zeta functions are given by the trace of the Frobenius action on the ℓ -adic étale realization of the corresponding motivic ones”. As in the p -adic case (see e.g. [3]), there is the intriguing question whether $\int_X (f^s, \alpha)$ always belongs to $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[(1 - \mathbf{L}^{-Ns-n})^{-1}]_{(N,n) \in M}$, where M is the set of all pairs (N, n) in $(\mathbf{N} \setminus \{0\})^2$ with $\exp(2\pi i n/N)$ an eigenvalue of the monodromy action on the complex $R\psi_f$ of nearby cycles on $f^{-1}(0)$. For some recent work in the p -adic case, see [24], [25].

2.5. Relation with motivic integration. M. Kontsevich introduced in [13] the completion $\widehat{K}_0(\text{Sch}_k)$ of $K_0(\text{Sch}_k)[\mathbf{L}^{-1}]$ with respect to the filtration $F^m K_0(\text{Sch}_k)[\mathbf{L}^{-1}]$, where $F^m K_0(\text{Sch}_k)[\mathbf{L}^{-1}]$ is the subgroup of $K_0(\text{Sch}_k)[\mathbf{L}^{-1}]$ generated by $\{[S] \mathbf{L}^{-i} \mid i - \dim S \geq m\}$, and defined, for smooth X over k , a motivic integration on $\mathcal{L}(X)$ with values into $\widehat{K}_0(\text{Sch}_k)$. In the paper [6], we extended Kontsevich’s construction to semi-algebraic subsets of $\mathcal{L}(X)$ and also to the non-smooth case. The following statement is proved in [6] (Definition-Proposition 3.2).

Definition-Proposition 2.5.1. Let X be an algebraic variety over k of pure dimension m . Denote by π_n the natural morphism $\mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$. Let \mathbf{B} be the boolean algebra of all semi-algebraic subsets of $\mathcal{L}(X)$. There exists a unique map $\mu : \mathbf{B} \rightarrow \widehat{K}_0(\text{Sch}_k)$ satisfying the following three properties.

(2.5.2) If $A \in \mathbf{B}$ is stable at level n , then

$$\mu(A) = [\pi_n(A)] \mathbf{L}^{-(n+1)m}.$$

(2.5.3) If $A \in \mathbf{B}$ is contained in $\mathcal{L}(S)$ with S a closed subvariety of X with $\dim S < \dim X$, then $\mu(A) = 0$.

(2.5.4) Let A_i be in \mathbf{B} for each i in \mathbf{N} . Assume that the A_i ’s are mutually disjoint and that $A := \bigcup_{i \in \mathbf{N}} A_i$ is semi-algebraic. Then $\sum_{i \in \mathbf{N}} \mu(A_i)$ converges in $\widehat{K}_0(\text{Sch}_k)$ to $\mu(A)$.

We call this unique map the *motivic volume* on $\mathcal{L}(X)$ and denote it by $\mu_{\mathcal{L}(X)}$ or μ . Moreover we have

(2.5.5) If A and B are in \mathbf{B} , $A \subset B$, and if $\mu(B)$ belongs to the closure $F^m(\widehat{K}_0(\text{Sch}_k))$ of $F^m K_0(\text{Sch}_k)[\mathbf{L}^{-1}]$ in $\widehat{K}_0(\text{Sch}_k)$, then $\mu(A) \in F^m(\widehat{K}_0(\text{Sch}_k))$.

Hence, for A in \mathbf{B} and $\alpha : A \rightarrow \mathbf{Z} \cup \{+\infty\}$ a simple function, we can define

$$\int_A \mathbf{L}^{-\alpha} d\mu := \sum_{n \in \mathbf{Z}} \mu(A \cap \alpha^{-1}(n)) \mathbf{L}^{-n}$$

in $\widehat{K}_0(\text{Sch}_k)$, whenever the right-hand side converges in $\widehat{K}_0(\text{Sch}_k)$, in which case we say that $\mathbf{L}^{-\alpha}$ is integrable on A . If the function α is bounded from below, then $\mathbf{L}^{-\alpha}$ is integrable on A , because of (2.5.5).

Semi-algebraic subsets of $\mathcal{L}(X)$ and simple functions on semi-algebraic subsets are defined in [6], as well as the notion of stable semi-algebraic subsets of $\mathcal{L}(X)$ of level n . In particular, $\mathcal{L}(X)$ is a semi-algebraic subset of $\mathcal{L}(X)$ and, for any morphism $g : Y \rightarrow X$ of algebraic varieties over k , the image of $\mathcal{L}(Y)$ in $\mathcal{L}(X)$ under the morphism induced by g is a semi-algebraic subset of $\mathcal{L}(X)$. When X is smooth, a semi-algebraic subset of $\mathcal{L}(X)$ is stable of level n if and only if it is a union of fibers of $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$. Consider a coherent sheaf of ideals \mathcal{I} on X and denote by $\text{ord}_t \mathcal{I}$ the function $\text{ord}_t \mathcal{I} : \mathcal{L}(X) \rightarrow \mathbf{N} \cup \{+\infty\}$ given by $\varphi \mapsto \min_g \text{ord}_t g(\varphi)$, where the minimum is taken over all g in the stalk $\mathcal{I}_{\pi_0(\varphi)}$ of \mathcal{I} at $\pi_0(\varphi)$. The function $\text{ord}_t \mathcal{I}$ is a simple function. When X is smooth and \mathcal{I} is the ideal sheaf of an effective divisor D on X , the motivic integral $\int_{\mathcal{L}(X)} \mathbf{L}^{-\text{ord}_t \mathcal{I}} d\mu$ was first introduced by Kontsevich [13] and denoted by him $[\int_X e^D]$. In particular, for a morphism $f : X \rightarrow \mathbf{A}_k^1$ with divisor D and a natural number d in \mathbf{N} , we can consider the motivic integral $\int_{\pi_0^{-1}(W)} \mathbf{L}^{-\text{ord}_t \mathcal{O}(-dD)} d\mu$, for any reduced subscheme W of X , because $\pi_0^{-1}(W)$ is a semi-algebraic subset of $\mathcal{L}(X)$. It follows from Theorem 5.1 of [6] that $\int_{\pi_0^{-1}(W)} \mathbf{L}^{-\text{ord}_t \mathcal{O}(-dD)} d\mu$ belongs to the image of $K_0(\text{Sch}_k)[\mathbf{L}^{-1}, ((\mathbf{L}^i - 1)^{-1})_{i \geq 1}]$ in $\widehat{K}_0(\text{Sch}_k)$. On the other hand, the motivic Igusa function $\int_W f^s$ belongs to $K_0(\mathcal{M}_k)[\mathbf{L}^{-s}]_{\text{loc}}$ and is the natural image of a well-defined element $\int_W \widetilde{f}^s$ in $K_0(\text{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]]$; cf. remark (1) in 2.1. Moreover, the proof of Theorem 2.2.1 also shows that $\int_W \widetilde{f}^s$ belongs to $K_0(\text{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]]_{\text{loc}}$. Hence for any natural number d in \mathbf{N} , we can formally replace s by d and obtain by evaluation a well-defined element $(\int_W \widetilde{f}^s)_{|s=d}$ in $\widehat{K}_0(\text{Sch}_k)$. The following statement is a direct consequence of the definitions.

Proposition 2.5.6. *Let X be a smooth and connected separated k -scheme of finite type of pure dimension m , $f : X \rightarrow \mathbf{A}_k^1$ be a morphism, and W be a reduced subscheme of X . For any natural number d in \mathbf{N} , the equality*

$$\int_{\pi_0^{-1}(W)} \mathbf{L}^{-\text{ord}_t \mathcal{O}(-dD)} d\mu = \left(\int_W \widetilde{f}^s \right)_{|s=d}$$

holds in $\widehat{K}_0(\text{Sch}_k)$. □

3. Functional equation

3.1. We denote by $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}}$ the localization of the algebra of Laurent polynomials $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])[\mathbf{L}^s, \mathbf{L}^{-s}]$ with respect to the multiplicative set generated by the polynomials $1 - \mathbf{L}^{-Ns-n}$, for N and n in $\mathbf{N} \setminus \{0\}$. One may consider $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])[\mathbf{L}^{-s}]_{\text{loc}}$ as embedded in $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}}$. The involution $M \mapsto M^\vee$ extends to $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])$. One can extend it to a $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])$ -algebra involution on $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}}$ by setting $(\mathbf{L}^s)^\vee = \mathbf{L}^{-s}$, $(\mathbf{L}^{-s})^\vee = \mathbf{L}^s$, and $((1 - \mathbf{L}^{-Ns-n})^{-1})^\vee = -\mathbf{L}^{-Ns-n}(1 - \mathbf{L}^{-Ns-n})^{-1}$.

In this section we assume that $X = \mathbf{A}_k^m$ and f is a homogenous polynomial of degree r .

Theorem 3.1.1.

(1) *The equality*

$$\left(\int_X f^s\right)^\vee = \mathbf{L}^{-rs} \int_X f^s$$

holds in $K_0(\mathcal{M}_k)[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}}$.

(2) *Assume 1.5.2 holds. Then, for any character α of $\mu_d(k)$ of order d ,*

$$\left(\int_X (f^s, \alpha)\right)^\vee = \mathbf{L}^{-rs} \int_X (f^s, \alpha^{-1})$$

in $K_0(\mathcal{M}_k, \mathbf{Q}[\zeta_d])[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}}$.

We begin with the following lemma.

Lemma 3.1.2. *Let α be a character of $\mu_d(k)$ of order d . If d does not divide r , then $\int_X (f^s, \alpha) = 0$.*

Proof. It is enough to prove that $[X_{n,f,\mathbf{A}_k^m}, \bar{f}^* \mathcal{L}_\alpha] = 0$ if d does not divide r . From the second displayed formula in the proof of Proposition 3.2.1 below, which actually holds for any d , it follows that it suffices to prove that $[X_{n,f,\{0\}}, \bar{f}^* \mathcal{L}_\alpha] = 0$. Thus we have to show that $\int_{\{0\}} (f^s, \alpha) = 0$. Let \bar{D} , resp. D , be the divisor in \mathbf{P}_k^{m-1} , resp. \mathbf{A}_k^m , defined by $f = 0$, and let $\bar{h} : \bar{Y} \rightarrow \mathbf{P}_k^{m-1}$ be a resolution of $\bar{D} \subset \mathbf{P}_k^{m-1}$ (in the sense of 2.2). Denote by $\pi : B \rightarrow \mathbf{A}_k^m$ the blowing-up of $\{0\}$ in \mathbf{A}_k^m , and by p the natural map $p : B \rightarrow \mathbf{P}_k^{m-1}$ that is the identity on $\pi^{-1}\{0\} = \mathbf{P}_k^{m-1}$. Note that p is a locally trivial fibration for the Zariski topology with fiber \mathbf{A}_k^1 . One verifies that the natural map

$$h : \bar{Y} \times_{\mathbf{P}_k^{m-1}} B \rightarrow B \rightarrow \mathbf{A}_k^m$$

is a resolution of $D \subset \mathbf{A}_k^m$. Moreover, $h^{-1}(D) \simeq \bar{Y}$ is a component (hence equal to some E_i) of $h^{-1}(D)$ on which $f \circ h$ has multiplicity r . Thus $\int_{\{0\}} (f^s, \alpha) = 0$ when d does not divide r , by Theorem 2.2.1. □

3.2. Proof of Theorem 3.1.1. By Lemma 3.1.2 we may assume d divides r . We consider the canonical projection $\gamma : \mathbf{A}_k^m \setminus \{0\} \rightarrow \mathbf{P}_k^{m-1}$ and denote by \bar{D} the image of $D \setminus \{0\}$ in \mathbf{P}_k^{m-1} . Let $h : Y \rightarrow \mathbf{P}_k^{m-1}$ be a resolution of \bar{D} (in the sense of 2.2). As in 2.2 we denote by $E_i, i \in J$, the irreducible (smooth) components of $(h^{-1}(D))_{\text{red}}$. We define similarly integers N_i and ν_i , and E_I, J_d, U_d , etc.

We denote by U^j the open $x_j \neq 0$ in \mathbf{P}_k^{m-1} . The restriction of h to $h^{-1}(U_j)$ is a resolution of $f_j = \frac{f}{x_j}$ in U_j . For Z locally closed in $U_d, [(Z \cap h^{-1}(U_j))_{f_j, \alpha}]$ has been defined in 2.2, and by Lemma 1.4.1

$$[(Z \cap h^{-1}(U_j) \cap h^{-1}(U_{j'}))_{f_j, \alpha}] = [(Z \cap h^{-1}(U_j) \cap h^{-1}(U_{j'}))_{f_{j'}, \alpha}].$$

Thus we may define without ambiguity

$$[Z_{f, \alpha}] = \sum_j [(Z \cap h^{-1}(U_j))_{f_j, \alpha}] - \sum_{j \neq j'} [(Z \cap h^{-1}(U_j) \cap h^{-1}(U_{j'}))_{f_j, \alpha}] + \dots$$

Set $E_I^{(d)} = E_I \setminus \bigcup_{\{j\} \notin J_d} E_j$.

Proposition 3.2.1. *Assume d divides r . For any character α of $\mu_d(k)$ of order d ,*

$$\int_X (f^s, \alpha) = \frac{(\mathbf{L} - 1) \mathbf{L}^{-m}}{1 - \mathbf{L}^{-rs-m}} \sum_{I \in J_d} [(E_I^{(d)})_{f, \alpha}] \prod_{i \in I} \left(\frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - n_i}}{1 - \mathbf{L}^{-N_i s - n_i}} - 1 \right).$$

Proof. Let us first remark that

$$\int_X (f^s, \alpha) = \frac{1}{1 - \mathbf{L}^{-rs-m}} \int_{X \setminus \{0\}} (f^s, \alpha).$$

Indeed, by homogeneity of f , multiplication by t induces an isomorphism between Z_{n, f, \mathbf{A}_k^m} and $Z_{n+r, f, \{0\}}$ (notations of 2.1), from which one deduces the relation

$$\mathbf{L}^m [X_{n+r, f, \{0\}}, \bar{f}^* \mathcal{L}_\alpha] = \mathbf{L}^{rm} [X_{n, f, \mathbf{A}_k^m}, \bar{f}^* \mathcal{L}_\alpha],$$

and the equality follows.

Write $X \setminus \{0\}$ as the disjoint union of the W^j 's, for $1 \leq j \leq m$, with $W^j = \{x \in \mathbf{A}_k^m \mid x_i = 0 \text{ for } i < j \text{ and } x_j \neq 0\}$. Now $\gamma(W^j) \subset U^j$ and the restriction of γ to W^j is a trivial fibration onto its image, with fibre $\mathbf{G}_{m, k}$. Since the valuation of $f(\varphi(t))$ only depends on $\gamma(\varphi(t))$ we deduce

$$\int_{W^j} (f^s, \alpha) = (1 - \mathbf{L}^{-1}) \int_{\gamma(W^j)} (f_j^s, \alpha).$$

Since Y is the disjoint union of the $h^{-1}(\gamma(W^j))$'s, we deduce from Theorem 2.2.1, by adding up, that

$$\int_{X \setminus \{0\}} (f^s, \alpha) = (1 - \mathbf{L}^{-1}) \mathbf{L}^{-(m-1)} \sum_{I \in J_d} [(E_I^\circ)_{f, \alpha}] \prod_{i \in I} \frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - n_i}}{1 - \mathbf{L}^{-N_i s - n_i}}.$$

The result follows, because

$$\begin{aligned} \sum_{I \in J_d} [(E_I^\circ)_{f, \alpha}] \prod_{i \in I} \frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - n_i}}{1 - \mathbf{L}^{-N_i s - n_i}} &= \\ &= \sum_{I \in J_d} [(E_I^{(d)})_{f, \alpha}] \prod_{i \in I} \left(\frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - n_i}}{1 - \mathbf{L}^{-N_i s - n_i}} - 1 \right). \quad \square \end{aligned}$$

By Proposition 3.2.1 we may write

$$\int_X (f^s, \alpha) = A \sum_{I \in J_d} [(E_I^{(d)})_{f, \alpha}] \prod_{i \in I} B_i,$$

with

$$A = \frac{(\mathbf{L} - 1) \mathbf{L}^{-m}}{1 - \mathbf{L}^{-rs-m}} \quad \text{and} \quad B_i = \frac{(\mathbf{L} - 1) \mathbf{L}^{-N_i s - n_i}}{1 - \mathbf{L}^{-N_i s - n_i}} - 1.$$

Remark that $A^\vee = \mathbf{L}^{-rs} \mathbf{L}^{m-1} A$ and $B_i^\vee = \mathbf{L}^{-1} B_i$. When $\alpha = 1$ the result follows because, E_I being proper and smooth, $\chi_c(E_I)^\vee = \mathbf{L}^{|I|-m+1} \chi_c(E_I)$. For general α the result follows from the following lemma. □

Lemma 3.2.2. *Assume 1.5.2 holds. For any I in J_d ,*

$$[(E_I^{(d)})_{f, \alpha}]^\vee = \mathbf{L}^{|I|-m+1} [(E_I^{(d)})_{f, \alpha^{-1}}].$$

Proof. Let Z be locally closed in U_d . If on a neighborhood of Z we may write $f \circ h = uv^d$ with u nonvanishing, $[Z, u^* \mathcal{L}_\alpha] = \chi_c(\tilde{Z}_{u,d}, \alpha)$, where $\tilde{Z}_{u,d}$ is the cyclic cover defined in 1.4. In general the covers $\tilde{Z}_{u,d}$ can be glued together (cf. the proof of Lemma 1.4.1) to give a Galois cover $\tilde{Z}_d \rightarrow Z$ with group $\mu_d(k)$, such that $[Z_{f, \alpha}] = \chi_c(\tilde{Z}_d, \alpha)$, for any character α of order d .

Set $W^\circ = E_I^{(d)}$ and $W = E_I$.

Lemma 3.2.3. *The cover $\tilde{W}_d^\circ \rightarrow W^\circ$ extends to a ramified $\mu_d(k)$ -cover $\pi : \tilde{W}_d \rightarrow W$ that satisfies the following conditions.*

- (1) *The scheme \tilde{W}_d is proper and locally for the Zariski topology is the quotient of a smooth scheme X by a finite abelian group G , the $\mu_d(k)$ -action on \tilde{W}_d being induced from a $\mu_d(k)$ -action on X commuting with the G -action.*

- (2) *The morphism π ramifies on $\widetilde{W}_d \setminus \widetilde{W}_d^\circ$ and the $\mu_d(k)$ -action on the restriction of π to $\widetilde{W}_d \setminus \widetilde{W}_d^\circ$ factors locally for the Zariski topology through a $\mu_{d'}(k)$ -action, for some $d' < d$ dividing d .*

Proof. Let x be a closed point of $W \setminus W^\circ$. On a Zariski neighborhood Ω' of x in Y ,

$$f = u \prod_{i \in J_x} h_i^{N_i} v^d$$

with u nonvanishing on Ω' , $J_x = \{i \in J \setminus I \mid x \in E_i, d \nmid N_i\}$ and h_i local equations for E_i near x . Put $\Omega = \Omega' \cap W$. Since $\widetilde{W}_{d|\Omega \cap W^\circ}^\circ$ is given by $y^d = u \prod_{i \in J_x} h_i^{N_i}$ in $\mathbf{G}_m \times (\Omega \cap W^\circ)$, we may extend $\widetilde{W}_{d|\Omega \cap W^\circ}^\circ \rightarrow \Omega \cap W^\circ$ to $\widetilde{W}_{d|\Omega} \rightarrow \Omega$ by taking $\widetilde{W}_{d|\Omega}$ to be the normalization of the subscheme $\widetilde{W}_{d|\Omega}$ of $\mathbf{A}^1 \times \Omega$ given by $y^d = u \prod_{i \in J_x} h_i^{N_i}$, and the $\mu_d(k)$ -action extends naturally. The schemes $\widetilde{W}_{d|\Omega} \rightarrow \Omega$ glue together (cf. the proof of Lemma 1.4.1) to give a scheme $\widetilde{W}_d \rightarrow W$ with $\mu_d(k)$ -action.

Let d' be the gcd of d and the N_i 's, $i \in J_x$. We have $d' < d$. Locally for the étale topology near x , \widetilde{W}_d is the disjoint union of the normalizations of $y^{d/d'} = \varepsilon^j u' \prod_{i \in J_x} h_i^{N_i/d'}$, for $1 \leq j \leq d'$, for ε a fixed primitive d' -th root of unity and u' such that $u'^{d'} = u$. This implies that on a Zariski neighborhood of x in $W \setminus W^\circ$, the $\mu_d(k)$ -action on $\pi|_{\widetilde{W}_d \setminus \widetilde{W}_d^\circ} : \widetilde{W}_d \setminus \widetilde{W}_d^\circ \rightarrow W \setminus W^\circ$ factors through a $\mu_{d'}(k)$ -action, since the normalization of a local complete domain is again local. We still have to verify that locally for the Zariski topology \widetilde{W}_d is the quotient of a smooth scheme X by a finite abelian group G , the $\mu_d(k)$ -action being induced from a $\mu_d(k)$ -action on X commuting with the G -action. It is enough to check this for the scheme $\widetilde{W}_{d|\Omega}$ which is the normalization of the subscheme $\widetilde{W}_{d|\Omega}$ of $\mathbf{A}^1 \times \Omega$ given by $y^d = u \prod_{i \in J_x} h_i^{N_i}$. We may assume $u = 1$. Indeed, consider the étale cyclic cover of degree d , $p : \Omega' \rightarrow \Omega$ given by $u = u'^d$. Since $\widetilde{W}_{d|\Omega}$ is the quotient of the normalization of the subscheme of $\mathbf{A}^1 \times \Omega'$ given by $y^d = u'^d \prod_{i \in J_x} (h_i \circ p)^{N_i}$, we are done by using the isomorphism $\mathbf{A}^1 \times \Omega' \rightarrow \mathbf{A}^1 \times \Omega'$ given by $(y, x) \mapsto (yu'^{-1}, x)$. When $u = 1$ the scheme $\widetilde{W}_{d|\Omega}$ is the disjoint union of the normalizations of $y^{d/d'} = \varepsilon^j \prod_{i \in J_x} h_i^{N_i/d'}$, for $1 \leq j \leq d'$, for ε a fixed primitive d' -th root of unity and d' the gcd of d and the N_i 's, $i \in J_x$. Hence we may finally assume that $\widetilde{W}_{d|\Omega}$ is the normalization of the scheme $\widetilde{W}_{d|\Omega}$ given by $y^d = \prod_{i \in J_x} h_i^{N_i}$ in $\mathbf{A}^1 \times \Omega$, and that the gcd of d and the N_i 's, $i \in J_x$, is 1. Now consider the subscheme W' of $\mathbf{A}^1 \times \Omega \times \mathbf{A}^{|J_x|}$ given by $h_i = t_i^d$ and $y = \prod_{i \in J_x} t_i^{N_i}$. It is easily seen that W' is smooth. Let us denote by $\pi : W' \rightarrow \widetilde{W}_{d|\Omega}$ the morphism given by $(y, x, (t_i)) \mapsto (y, x)$ and by G the kernel of the morphism $\mu_d(k)^{|J_x|} \rightarrow \mu_d(k)$ given by $(\xi_i) \mapsto \prod_{i \in J_x} \xi_i^{N_i}$. The canonical action of G on

$\mathbf{A}^{|J_x|}$ induces an action on W' for which the morphism π is equivariant; hence π factorizes through a morphism $\pi_G : W'/G \rightarrow \widetilde{W}_d/\Omega$. Since π_G is of degree 1, the result follows. \square

We are now able to finish the proof. By Lemma 3.2.3 (2) and Proposition 1.3.3 (2), we have

$$[(E_I^{(d)})_{f,\alpha}] = \chi_c(\widetilde{W}_d, \alpha) \quad \text{and} \quad [(E_I^{(d)})_{f,\alpha^{-1}}] = \chi_c(\widetilde{W}_d, \alpha^{-1});$$

hence the result follows from the following proposition. \square

Proposition 3.2.4. *Assume 1.5.2 holds. Let W be a proper k -scheme of pure dimension m , endowed with a G -action, with G a finite abelian group. Assume that, locally for the Zariski topology, W is the quotient of a smooth k -scheme with a G -action by the action of a finite abelian group commuting with the action of G . Then, for any character α of G ,*

$$\chi_c(W, \alpha)^\vee = \mathbf{L}^{-m} \chi_c(W, \alpha^{-1}).$$

Proof. By additivity of Euler characteristics (Theorem 1.3.1 (3) and 1.3.2 (6)) and by Theorem 1.3.2 (5), we are reduced to proving that if W is a k -scheme of pure dimension m with a G -action, which is the quotient of a smooth k -scheme with a G -action by the action of a finite abelian group commuting with the action of G , then, for any character α of G , $\chi(W, \alpha)^\vee = \mathbf{L}^{-m} \chi_c(W, \alpha^{-1})$. This follows directly from Assertion 1.5.2 and Theorem 1.3.2 (4). \square

Let us denote by \mathbf{L}' the image of \mathbf{L} by the morphism

$$\varphi : K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) \longrightarrow K_0(DM_{gm}(k)_{\mathbf{Q}[\zeta_d]}).$$

One then defines a ring $K_0(DM_{gm}(k)_{\mathbf{Q}[\zeta_d]})[\mathbf{L}'^s, \mathbf{L}'^{-s}]_{\text{loc}}$ similarly as we defined the ring $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}}$, and φ extends to a morphism $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})[\mathbf{L}^s, \mathbf{L}^{-s}]_{\text{loc}} \rightarrow K_0(DM_{gm}(k)_{\mathbf{Q}[\zeta_d]})[\mathbf{L}'^s, \mathbf{L}'^{-s}]_{\text{loc}}$, which we still denote by φ .

Theorem 3.2.5. *For any character α of $\mu_d(k)$ of order d ,*

$$\varphi\left(\left(\int_X (f^s, \alpha)\right)^\vee\right) = \varphi\left(\mathbf{L}^{-rs} \int_X (f^s, \alpha^{-1})\right)$$

in $K_0(DM_{gm}(k)_{\mathbf{Q}[\zeta_d]})[\mathbf{L}'^s, \mathbf{L}'^{-s}]_{\text{loc}}$.

Proof. The proof is the same as the one of Theorem 3.2.1, using Corollary 1.5.4 instead of Assertion 1.5.2. \square

4. Limit for $s \rightarrow -\infty$ and nearby cycles

4.1. We consider in this section the subring $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]''_{\text{loc}}$ of $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [[\mathbf{L}^{-s}]]_{\text{loc}}$ generated by the subring $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$ and the series $\mathbf{L}^{-Ns-n} (1 - \mathbf{L}^{-Ns-n})^{-1}$, for N and n in $\mathbf{N} \setminus \{0\}$.

Lemma 4.1.1. *There is a well-defined ring homomorphism*

$$\text{CT} : K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]}) [\mathbf{L}^{-s}]''_{\text{loc}} \longrightarrow K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$$

that induces the identity on $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$ and that sends the series $\mathbf{L}^{-Ns-n} (1 - \mathbf{L}^{-Ns-n})^{-1}$ to -1 .

Proof. Similar to the one for the constant term of the power series expansion in T^{-1} of usual rational functions of degree ≤ 0 in T , considering \mathbf{L}^{-s} as a variable T . (Heuristically this amounts to taking the “limit” for $s \rightarrow -\infty$.) □

Definition 4.1.2. Let X be a smooth and connected k -scheme of finite type of dimension m , $f : X \rightarrow \mathbf{A}_k^1$ be a morphism and x be a closed point of $f^{-1}(0)$. Let α be a character of $\mu_d(k)$ of order d . We set

$$S_{\alpha, x} := \frac{\mathbf{L}^m}{1 - \mathbf{L}} \text{CT} \int_{\{x\}} (f^s, \alpha).$$

By Theorem 2.2.1 and Lemma 4.1.1, $S_{\alpha, x}$ is an element of $K_0(\mathcal{M}_{k, \mathbf{Q}[\zeta_d]})$ that is well defined modulo $(\mathbf{L} - 1)$ -torsion, cf. remark 1.2.3. Furthermore, for any resolution of f ,

$$S_{\alpha, x} = \sum_{I \in J_d} [(E_I^\circ \cap h^{-1}(x))_{f, \alpha}] (1 - \mathbf{L})^{|I|-1},$$

modulo $(\mathbf{L} - 1)$ -torsion.

Remark that, for almost all d , $S_{\alpha, x} = 0$. We now assume for simplicity that k contains all roots of unity and that the group of roots of unity in k is embedded in \mathbf{C} . Hence all the groups $\widehat{\mu_d(k)}$ are canonically embedded in \mathbf{Q}/\mathbf{Z} . We denote by γ the section $\mathbf{Q}/\mathbf{Z} \rightarrow [0, 1)$ and by i_x the inclusion of $\{x\}$ in $f^{-1}(0)$.

We believe that $S_{\alpha, x}$ is the “motivic incarnation” of $\chi_c(i_x^* R\psi_{f, \alpha})$. Here $R\psi_{f, \alpha}$ denotes the eigenspace of nearby cycles for the eigenvalue $\exp(2\pi i \gamma(\alpha))$ of the semi-simple part of the monodromy. We will verify in the next subsection that this is true for the \mathbf{C} -Hodge realization.

4.2. Hodge realization. We will use freely the theory of mixed Hodge modules developed by M. Saito in [16], [18]. In particular, for X a scheme of finite type over \mathbf{C} , we denote by $\text{MHM}(X)$ the abelian category of mixed Hodge modules on X . In the definition of mixed Hodge modules it is required that the underlying perverse sheaf is defined over \mathbf{Q} . To allow some more

flexibility we will also use the category $\text{MHM}'(X)$ of bifiltered \mathcal{D} -modules on X which are direct factors of objects of $\text{MHM}(X)$ as bifiltered \mathcal{D} -modules. We denote by $D^b(\text{MHM}(X))$ and $D^b(\text{MHM}'(X))$ the corresponding derived categories.

Let $f : X \rightarrow \mathbf{A}_{\mathbf{C}}^1$ be a morphism. We denote by ψ_f^H and ϕ_f^H the nearby and vanishing cycle functors for mixed Hodge modules as defined in [18] and by T_s the semi-simple part of the monodromy operator. One should note that ψ_f^H and ϕ_f^H on mixed Hodge modules correspond to $\psi_f[-1]$ and $\phi_f[-1]$ on the underlying perverse sheaves. If M is a mixed Hodge module on X we denote by $\psi_{f,\alpha}^H M$ the object of $\text{MHM}'(X)$ that corresponds to the eigenspace of T_s for the eigenvalue $\exp(2\pi i\gamma(\alpha))$. These definitions extend to the Grothendieck group of the abelian category $\text{MHM}'(X)$.

Let us recall the definition of *complex* mixed Hodge structures. A \mathbf{C} -Hodge structure of weight n is just a finite-dimensional bigraded vector space $V = \bigoplus_{p+q=n} V^{p,q}$, or, equivalently, a finite-dimensional vector space V with decreasing filtrations F^\bullet and \bar{F}^\bullet such that $V = F^p \oplus \bar{F}^q$ when $p + q = n + 1$. A mixed \mathbf{C} -Hodge structure is a finite-dimensional vector space V with an increasing filtration W and decreasing filtrations F^\bullet and \bar{F}^\bullet that induce for each $n \in \mathbf{Z}$ a \mathbf{C} -Hodge structure of weight n on $\text{Gr}_n^W V$. We denote by $K_0(\text{MHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of complex mixed Hodge structures. The Hodge realization functor induces a morphism $H : K_0(\mathcal{M}_{\mathbf{C},\mathbf{C}}) \rightarrow K_0(\text{MHS}_{\mathbf{C}})$. Remark that $H(\mathbf{L}) = \mathbf{C}(-1)$, and that H kills $(\mathbf{L} - 1)$ -torsion (cf. remark 1.2.3).

For any object K of $D^b(\text{MHM}(X))$ we denote by $\chi_c(X, K)$ the class of $Rp_!(K)$ in $K_0(\text{MHS}_{\mathbf{C}})$, where p is the projection onto $\text{Spec } \mathbf{C}$. Clearly this definition may be extended to $D^b(\text{MHM}'(X))$.

If X is smooth and connected of dimension m , we denote by $\mathbf{C}_X^H[m]$ the trivial variation of Hodge structure of weight 0.

Theorem 4.2.1. *Let X be a smooth and connected \mathbf{C} -scheme of finite type of dimension m , $f : X \rightarrow \mathbf{A}_{\mathbf{C}}^1$ be a morphism and x be a closed point of $f^{-1}(0)$. The following equality holds:*

$$H(S_{\alpha,x}) = (-1)^{m-1} \chi_c(i_x^* \psi_{f,\alpha}^H \mathbf{C}_X^H[m]).$$

Proof. We will use a resolution $h : Y \rightarrow X$ of f . We set $g = f \circ h$, $Y_0 = g^{-1}(0)_{\text{red}}$, $Y^* = Y \setminus Y_0$, $g' = g|_{Y^*}$ and we denote by $j : Y^* \hookrightarrow Y$ the inclusion morphism. We also set $D = h^{-1}(x)_{\text{red}}$, $D_I = D \cap E_I$, $D_I^\circ = D \cap E_I^\circ$, and denote by $i_D : D \hookrightarrow Y_0$ the inclusion morphism. In the derived category $D^b(\text{MHM}'(\mathbf{G}_{m,\mathbf{C}}))$ we have a decomposition

$$R[d]_* \mathbf{C}^H \simeq \bigoplus_{0 \leq j < d} \mathcal{K}_{\frac{d}{j}}$$

where the underlying sheaf of $\mathcal{K}_{\frac{j}{d}}$ has monodromy $\exp(2\pi i \frac{j}{d})$ at the origin. We set $\mathcal{F}_\alpha := j_* g'^* \mathcal{K}_{-\gamma(\alpha)}$.

Lemma 4.2.2.

- (1) *The sheaf underlying \mathcal{F}_α is locally constant of rank 1 on U_d .*
- (2) *For any Z locally closed in U_d ,*

$$\chi_c(Z, \mathcal{F}_\alpha) = H([Z_{f,\alpha}]).$$

- (3) *For $i \geq 0$, $R^i j_* g'^* \mathcal{K}_{-\gamma(\alpha)}$ is zero outside U_d .*
- (4) *For $i \geq 0$, $R^i j_{I*} \mathcal{F}_{\alpha|E_I^\circ}$ is zero outside $E_I \cap U_d$, with $j_I : E_I^\circ \hookrightarrow E_I$ the inclusion morphism.*

Proof. Assertion (1) follows from [2] Proposition 3.1. Assertion (2) follows from the fact that, if locally $g = uv^d$ with u nonvanishing on Z , then $u^* \mathcal{K}_{-\gamma(\alpha)}$ is isomorphic to $\mathcal{F}_{\alpha|Z}$. The proof of (3) and (4) are completely similar to the one of [2] Lemma 3.2. □

The functor $R^j(h_{|Y_0})_* : D^b(\text{MHM}(Y_0)) \rightarrow \text{MHM}(f^{-1}(0)_{\text{red}})$ being a cohomological functor, we have a spectral sequence (cf. [19] (2.14.3))

$$E_2^{pq} = R^p(h_{|D})_* H^q i_D^*(\psi_g^H \mathbf{C}_Y^H[m]) \implies H^{p+q} i_x^*(\psi_f^H \mathbf{C}_X^H[m]),$$

which is T_s -equivariant. Hence, because the restriction of h to D is proper, it is enough, by Proposition 4.2.2 (2), to prove the following proposition. □

Proposition 4.2.3. *With the previous notation, the following equality holds:*

$$\chi_c(D, i_D^*(\psi_{g,\alpha}^H \mathbf{C}_Y^H[m])) = (-1)^{m-1} \sum_{I \in J_d} \chi_c(D_I^\circ, \mathcal{F}_\alpha) (1 - \mathbf{C}(-1))^{|I|-1}.$$

Proof. Let N be the logarithm of the unipotent part of the monodromy and let P_N denote the primitive part with respect to N . We have the primitive decomposition

$$(4.2.3.1) \quad \bigoplus_j \text{Gr}_j^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m] \simeq \bigoplus_{k \geq 0} \bigoplus_{i=0}^k \left[N^i P_N \text{Gr}_{m-1+k}^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m] \right](i).$$

The proposition will follow from the following lemma.

Lemma 4.2.4. *For $k \geq 0$, there is a canonical isomorphism*

$$P_N \text{Gr}_{m-1+k}^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m] \simeq \bigoplus_{I \in J_d, |I|=k+1} \text{IC}_{E_I} \mathcal{F}_{\alpha|E_I^\circ}(-k),$$

with $\text{IC}_{E_I} \mathcal{F}_{\alpha|E_I^\circ}$ the intersection cohomology module on E_I with coefficients in $\mathcal{F}_{\alpha|E_I^\circ}$.

Proof. The statement for the underlying perverse sheaves is essentially Lemma 2.13 of [19]. The preprint [19] being unpublished, we reproduce the argument for the convenience of the reader. Let us denote by S the functor which to an object of MHM' associates its underlying perverse sheaf. By [17] § 3 we have a canonical isomorphism

$$(4.2.4.1) \quad S\left[\psi_{g,\alpha}^H \mathbf{C}_Y^H[m]\right] \simeq \varinjlim \text{Ker} \left(j_! g'^* \mathcal{K}_{-\gamma(\alpha),k}[m] \longrightarrow j_* g'^* \mathcal{K}_{-\gamma(\alpha),k}[m] \right),$$

where, for $k \geq 0$, $\mathcal{K}_{-\gamma(\alpha),k}$ is a local system of rank $k+1$ on $\mathbf{G}_{m,\mathbb{C}}$ whose monodromy at the origin has a unique Jordan block and eigenvalue $\exp(-2\pi i \gamma(\alpha))$, and $\mathcal{K}_{-\gamma(\alpha),0}$ is isomorphic to $S(\mathcal{K}_{-\gamma(\alpha)})$. We have a canonical isomorphism

$$\text{Gr}_{m-|I|}^W j_! g'^* \mathcal{K}_{-\gamma(\alpha),k}[m]_{|E_I^\circ} \simeq S\left[\mathcal{F}_\alpha[m - |I|]_{|E_I^\circ}\right]$$

from which one deduces the canonical isomorphism

$$\text{Gr}_{m-k}^W j_! g'^* \mathcal{K}_{-\gamma(\alpha),k}[m] \simeq S\left[\bigoplus_{I \in J_d, |I|=k} \text{IC}_{E_I} \mathcal{F}_\alpha|_{E_I^\circ}\right],$$

for $k \geq 0$, and one can check that the natural morphism

$$j_! g'^* \mathcal{F}_\alpha[m] \longrightarrow \psi_{g,\alpha}^H \mathbf{C}_Y^H[m]$$

deduced from (4.2.4.1) induces an isomorphism

$$\text{Gr}_{m-1-k}^W j_! g'^* \mathcal{K}_{-\gamma(\alpha),k}[m] \simeq S\left[N^k P_N \text{Gr}_{m-1+k}^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m]\right]$$

by loc.cit. Hence we have a canonical isomorphism

$$S\left[P_N \text{Gr}_{m-1+k}^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m]\right] \simeq S\left[\bigoplus_{I \in J_d, |I|=k+1} \text{IC}_{E_I} \mathcal{F}_\alpha|_{E_I^\circ}(-k)\right].$$

The fact that this isomorphism lifts canonically to an isomorphism between the corresponding objects of MHM' follows from [16] 3.6.10 and 5.2.16 (cf. [16] p. 990). □

By primitive decomposition (4.2.3.1),

$$\begin{aligned} \chi_c(D, i_D^*(\psi_{g,\alpha}^H \mathbf{C}_Y^H[m])) &= \sum_k \sum_{i=0}^k \chi_c(D, (i_D^* N^i P_N \text{Gr}_{m-1+k}^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m])(i)) \\ &= \sum_k \chi_c(D, i_D^* P_N \text{Gr}_{m-1+k}^W \psi_{g,\alpha}^H \mathbf{C}_Y^H[m]) \left(\sum_{i=0}^k \mathbf{C}(i) \right). \end{aligned}$$

By Lemma 4.2.4 we deduce

$$\chi_c(D, i_D^*(\psi_{g,\alpha}^H \mathbf{C}_Y^H[m])) = \sum_k \sum_{I \in J_d, |I|=k+1} \chi_c(D, i_D^* \mathbf{IC}_{E_I} \mathcal{F}_{\alpha|E_I^\circ}) \left(\sum_{i=0}^k \mathbf{C}(-i) \right).$$

By Lemma 4.2.2 we have

$$\begin{aligned} \chi_c(D, i_D^* \mathbf{IC}_{E_I} \mathcal{F}_{\alpha|E_I^\circ}) &= \chi_c(D_I, j_{I*} \mathcal{F}_{\alpha|E_I^\circ}[m - |I|]) \\ &= (-1)^{m-|I|} \sum_{\substack{I' \supset I \\ I' \in J_d}} \chi_c(D_{I'}^\circ, \mathcal{F}_{\alpha|D_{I'}^\circ}), \end{aligned}$$

where j_I denotes the inclusion $E_I^\circ \hookrightarrow E_I$.

Hence we deduce

$$\chi_c(D, i_D^*(\psi_{g,\alpha}^H \mathbf{C}_Y^H[m])) = \sum_{I \in J_d} \alpha_I \chi_c(D_I^\circ, \mathcal{F}_{\alpha|D_I^\circ}),$$

with

$$\alpha_I = \sum_{k=0}^{|I|-1} (-1)^{m-k-1} \binom{|I|}{k+1} (1 + \mathbf{C}(-1) + \dots + \mathbf{C}(-k)).$$

The result follows, since, by the binomial theorem,

$$\alpha_I = (-1)^{m-1} (1 - \mathbf{C}(-1))^{|I|-1}. \quad \square$$

Remark. Of course one deduces from Theorem 4.2.1 the analogous statement for the Betti realization, which may also be proved directly using Lemma 4.2.2 and Leray’s spectral sequence for nearby cycles. See also [4] for a related result concerning the étale realization.

4.3. Relation with the Hodge spectrum. Let us review the definition of Hodge spectrum according to [20]. Let H be a complex mixed Hodge structure with an automorphism T of order dividing d . The Hodge spectrum of (H, T) is defined as $\text{HSp}(H, T) = \sum_{\alpha \in \frac{1}{d}\mathbf{Z}} n_\alpha t^\alpha \in \mathbf{Z}[t^{-\frac{1}{d}}, t^{\frac{1}{d}}]$, with $n_\alpha = \dim \text{Gr}_F^p H_\lambda$, for $\lambda = \exp(2\pi i\alpha)$ and $p = [\alpha]$, where H_λ is the eigenspace of T with eigenvalue λ , and F is the Hodge filtration. This definition extends to the Grothendieck group of the abelian category of complex mixed Hodge structures with an automorphism T of order dividing d . Remark that $\text{HSp}(H(k), T) = t^{-k} \text{HSp}(H, T)$, where (k) is the Tate twist.

We assume now X to be a smooth and connected scheme of finite type over \mathbf{C} and dimension m . We set

$$\text{HSp}(f, x) = \sum_{j \in \mathbf{Z}} (-1)^j \text{HSp}(H^j i_x^* \psi_f^H \mathbf{C}_X^H[m], T_s)$$

and

$$\text{HSp}'(f, x) = \sum_{j \in \mathbf{Z}} (-1)^j \text{HSp}(H^j i_x^* \phi_f^H \mathbf{C}_X^H[m], T_s).$$

We have $\text{HSp}(f, x) = (-1)^{m-1} + \text{HSp}'(f, x)$. The relation with the spectrum $\text{Sp}(f, x)$ as defined in [22] and [20] (which differs from that of [23] by multiplication by t) is

$$\text{Sp}(f, x) = t^m \iota(\text{HSp}'(f, x))$$

where ι is the isomorphism such that $\iota(t^{\frac{1}{d}}) = t^{-\frac{1}{d}}$.

The following statement is a direct corollary of Theorem 4.2.1.

Corollary 4.3.1. *Let X be a smooth and connected \mathbf{C} -scheme of finite type of dimension m , $f : X \rightarrow \mathbf{A}_{\mathbf{C}}^1$ be a morphism and x be a closed point of $f^{-1}(0)$. The following equality holds:*

$$\sum_{\alpha \in \mathbf{Q}/\mathbf{Z}} \text{HSp}(H(S_{\alpha,x}), \text{Id}) t^{\gamma(\alpha)} = (-1)^{m-1} \text{HSp}(f, x). \quad \square$$

Appendix

Let G be a finite group. As in 1.3 we denote by $\text{Sch}_{k,G}$ the category of separated schemes of finite type over k with G -action such that the G -orbit of any closed point of X is contained in an affine open subscheme. We also denote by $\text{Reg}_{k,G}$ the full subcategory of smooth schemes. In this section we state some analogs with group action of resolution statements used in [9]. In [9] the statements were deduced from Hironaka's theorems [10]. Here we will use instead Villamayor's results in [27] §7 (which, as indicated in the introduction of [27], may also be deduced from [11] together with [26]).

Let us begin by some elementary observations. By the very definition, objects of $\text{Sch}_{k,G}$ have covers by G -stable affine subschemes. Furthermore, such affine subschemes may be embedded equivariantly in a smooth G -scheme. Indeed, if X is an affine G -scheme, it may be embedded in a smooth scheme Z , and the embedding $X \hookrightarrow Z^G$ given by $x \mapsto (gx)_{g \in G}$ is equivariant. Similarly any object X of $\text{Sch}_{k,G}$ admits an equivariant compactification: if Z is any compactification of X , the closure of the image of X by $x \mapsto (gx)_{g \in G}$ in Z^G gives an equivariant compactification. Remark that the equivariant Chow Lemma may be deduced directly from the existence of equivariant compactifications by the usual proof of the Chow Lemma.

By a proper relative isomorphism of G -schemes $(\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$, we mean the following data: a proper G -morphism $f : \tilde{X} \rightarrow X$ between objects of

$\text{Sch}_{k,G}$, reduced closed G -stable subschemes \tilde{Y} and Y of \tilde{X} and X respectively, such that \tilde{Y} is the preimage of Y in \tilde{X} , and such that the restriction of f to $\tilde{X} \setminus \tilde{Y}$ is a G -isomorphism onto $X \setminus Y$. If moreover X is smooth, Y is a closed subscheme that is smooth and of smaller dimension, and $f : \tilde{X} \rightarrow X$ is isomorphic to the blowing-up of X along Y , we will say that f is an elementary G -modification.

Theorem A.1. *For any reduced object X in $\text{Sch}_{k,G}$, there exists a proper relative isomorphism of G -schemes $(\tilde{X}, \tilde{Y}) \rightarrow (X, Y)$ with \tilde{X} smooth and $\dim Y, \dim \tilde{Y} < \dim X$.*

Proof. Follows directly from [27] § 7. □

We will also need the following equivariant Chow-Hironaka Lemma.

Lemma A.2. *Assume G is a finite group. Let $f : \tilde{X} \rightarrow X$ be a birational proper morphism in $\text{Reg}_{k,G}$, inducing a birational proper morphism on each irreducible component of \tilde{X} . Then there exists a G -morphism $X' \rightarrow X$ that is the composition of a finite sequence of elementary G -modifications and that factors through a proper birational G -morphism $X' \rightarrow \tilde{X}$.*

Proof. By the Chow-Hironaka Lemma one can dominate f by a projective birational map $X' \rightarrow X$ that is the blowing-up of a sheaf of ideals \mathcal{J} . Let $\mathcal{I} = g_1\mathcal{J} \cdot g_2\mathcal{J} \cdots g_n\mathcal{J}$, with g_1, \dots, g_n the elements of G . Since the blowing-up of \mathcal{I} dominates the one of \mathcal{J} , it is enough to know that it is possible to make \mathcal{I} principal by a composition of a finite sequence of elementary G -modifications, a fact that follows from [27] § 7. □

We denote by $\text{Reg}_{k,G}^2$ the category of pairs (X, U) with X in $\text{Reg}_{k,G}$ and U a G -stable open subscheme of X such that the complement $D = X \setminus U$ is a G -stable divisor with normal crossings.

Lemma A.3. *Assume G is a finite group. Let U be an object of $\text{Reg}_{k,G}$ and V be a smooth G -stable subscheme. There exists a compactification X of U such that (X, U) belongs to $\text{Reg}_{k,G}^2$ and the closure of V in X has normal crossings with $X \setminus U$.*

Proof. It is the same proof as the one in [9] Lemme 3.5, using [27] instead of [10], once it is noticed that U admits a compactification with G -action. □

A morphism $f : (\tilde{X}, \tilde{U}) \rightarrow (X, U)$ in $\text{Reg}_{k,G}^2$ will be called an elementary G -modification if $f : \tilde{X} \rightarrow X$ is the blowing-up of X along a smooth G -stable subscheme Y having normal crossings with $X \setminus U$ and furthermore $\tilde{U} = f^{-1}U$. With this terminology, the analogue of Lemma 2.7 of [9] holds in the G -equivariant setting by using [27] instead of [10].

Proof of Theorems 1.3.1 and 1.3.2. The construction and proof of unicity of χ_c satisfying conditions (1) and (2) in Theorem 1.3.1 is done just the same as for the analogue statement in [9] (Corollaire 6.4), using the previous

equivariant versions of the resolution results used in [9]. Assertion (3) is easily deduced from (2). Similarly, the construction of χ satisfying conditions (1) to (4) in Theorem 1.3.2, and determined by the first three, is done exactly as in [9] (Corollaire 6.13). Assertion (5) follows by construction, and (6) is easily deduced from (2) and (5) by induction on dimension. \square

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