TRANSFER PRINCIPLE FOR THE FUNDAMENTAL LEMMA

RAF CLUCKERS, THOMAS HALES, AND FRANÇOIS LOESER

Introduction

The purpose of this paper is to explain how the general transfer principle of Cluckers and Loeser [9][11] may be used in the study of the fundamental lemma. We use here the word "transfer" in a sense originating with the Ax-Kochen-Eršov transfer principle in logic. Transfer principles in model theory are results that transfer theorems from one field to another. The transfer principle of [9][11] is a general result that transfers theorems about identities of p-adic integrals from one collection of fields to others. These general transfer principles are reviewed in Theorems 2.8.3 and 10.2.3. The main purpose of this article is to explain how the identities of various fundamental lemmas fall within the scope of these general transfer principles. Consequently, once the fundamental lemma has been established for one collection of fields (for example, fields of positive characteristic), it is also valid for others (fields of characteristic zero). Precise statements appear in Theorems 9.3.1 (for the fundamental lemma), 9.3.2 (for the weighted fundamental lemma), and Section 10.3 (for the Jacquet-Ye relative fundamental lemma).

In general terms, the aim of this paper is to show that certain theorems about fields in positive characteristic imply that the same theorems hold in characteristic zero. To make this aim precise, we need to state the theorems in such a way that the field is not fixed from the outset, but appears as a parameter that can be supplied at a later stage. Stated in these terms, it is natural to turn to model theory for an answer, because model theory gives a separation of language from structure; a first-order language expresses the theorems, and structures later supply the fields. The particular form of each theorem is an identity of integrals. Thus, a model theoretic account of p-adic integration is needed. This is precisely what motivic integration provides.

In an unfortunate clash of terminology, the word "transfer" in the context of the fundamental lemma has come to mean the matching of smooth functions on a reductive group with those on an endoscopic group. We have nothing to say about transfer in that sense. For example, Waldspurger's article from 1997 "Le lemme fondamental implique le transfert" is completely unrelated (insofar as it is possible for two articles on the fundamental lemma to be unrelated).

During the preparation of this paper R. Cluckers was a postdoctoral fellow of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.).

The research of T. Hales was supported in part by NSF grant 0503447. He would also like to thank E.N.S. for its hospitality.

The project was partially supported by ANR grant 06-BLAN-0183.

The intended audience for this paper being that of mathematicians working in the areas of automorphic forms, representation theory, arithmetic geometry, and Galois representations, we tried our best to make all definitions and statements from other fields that are used in this paper understandable without any prerequisite. In particular, we start the paper by giving a quick presentation of first-order languages and the Denef-Pas language and an overview on motivic constructible functions and their integration according to [8], before stating the general transfer principle. The bulk of the paper consists in proving the definability of the various data occurring in the fundamental lemma. Once this is achieved, it is not difficult to deduce our main result in 9.3, stating that the transfer principle holds for the integrals occurring in the fundamental lemma, which is of special interest in view of the recent advances by Laumon and Ngô [28] and Ngô [30].

Other results concerning the transfer principle for the fundamental lemma appear in [12], [35], [36].

We thank Michael Harris for inviting an expository paper on this topic for the book "Stabilisation de la formule des traces, variétés de Shimura, et applications arithmétiques."

1. First order languages and the Denef-Pas language

1.1. Languages. A (first order) language L consists of an enumerable infinite set of symbols of variables $\mathcal{V} = \{v_0, \cdots, v_n, \cdots\}$, logical symbols $\neg$ (negation), $\wedge$ (or), $\vee$ (and), $\implies$, $\iff$, $\forall$ and $\exists$, together with two suites of sets $\mathcal{F}_n$ and $\mathcal{R}_n$, $n \in \mathbb{N}$. Elements of $\mathcal{F}_n$ will be symbols of n-ary functions, elements of $\mathcal{R}_n$ symbols of n-ary relations. A 0-ary function symbol will be called a constant symbol. The language L consists of the union of these sets of symbols.

1.2. Terms. The set $T(L)$ of terms of the language L is defined in the following way: variable symbols and constant symbols are terms and if f belongs to $\mathcal{F}_n$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \cdots, t_n)$ is also a term. A more formal definition of $T(L)$ is to view it as a subset of the set of finite words on L. For instance, to the word $ft_1 \ldots t_n$ corresponds the term $f(t_1, \cdots, t_n)$. One defines the weight of a function as its arity -1 and we give to symbols of variables the weight -1. The weight of a finite word on L is the sum of the weights of its symbols. Then terms correspond exactly to finite words on variable and function symbols of total weight -1 whose strict initial segments are of weight ≥ 0, when nonempty. If t is a term one writes $t = t[w_0, \ldots, w_n]$ to mean that all variables occurring in t belong to the w_i's.

1.3. Formulas. An atomic formula is an expression of the form $R(t_1, \ldots, t_n)$ with R an n-ary relation symbol and t_i terms. The set of formulas in L is the smallest set containing atomic formulas and such that if M and N are formulas then $\neg M$, $(M \wedge N)$, $(M \vee N)$, $(M \implies N)$, $(M \iff N)$, $\forall v_n M$ et $\exists v_n M$ are formulas. Formulas may also be defined as certain finite words on L. (Parentheses are just a way to rewrite terms and formulas in a more handy way, as opposed to writing them as finite words on L).

Let v be a variable symbol occurring in a formula F. If F is atomic we say all occurrences of v in F are free. If $F = \neg G$ the free occurrences of v in F are those in G. Free occurrences of v in $(F\alpha G)$ are those in F and those in G where α is either $\neg, \wedge, \vee, \implies$, or $\iff$. If $F = \forall w G$ or $\exists w G$ with $w \neq v$, free occurrences of v in F are those in G. When $v = w$, no occurrence of v in F free. Non free occurrences of a variable are said to be bound. Free variables in a formula F are those having at least one free occurrence. A sentence is a formula with no free variable.

We write $F[w_0, \dots, w_n]$ if all free variables in F belong to the w_i's (supposed to be distinct).

1.4. **Interpretation in a structure.** Let L be a language. An L-structure $\mathfrak{M}$ is a set M endowed for every n-ary function symbol f in L with a function $f^{\mathfrak{M}} : M^n \to M$ and for every n-ary relation R with a subset $R^{\mathfrak{M}}$ of M^n.

If $t[w_0, \dots, w_n]$ is a term and $a_0, \dots, a_n$ belong to M, we denote by $t[a_0, \dots, a_n]$ the interpretation of t in M defined by interpreting w_i by a_i. Namely, the interpretation of the term w_i is a_i, that of the constant symbol c is $c^{\mathfrak{M}}$, and that of the term $f(t_1, \dots t_r)$ is $f^{\mathfrak{M}}(t_1[a_0, \dots, a_n], \dots, t_r[a_0, \dots, a_n])$.

Similarly, if $F[w_1, \dots, w_n]$ is a formula and $a_1, \dots, a_n$ belong to M, there is a natural way to interpret w_i as a_i in M in the formula F, yielding a statement $F[a_1, \dots, a_n]$ about the tuple $(a_1, \dots, a_n)$ in M which is either true or false. One says that $(a_1, \dots, a_n)$ satisfies F in $\mathfrak{M}$, and writes $\mathfrak{M} \models F[a_1, \dots, a_n]$, if the statement $F[a_1, \dots, a_n]$ obtained by interpreting w_i as a_i is satisfied (true) in M. For instance, when $F = R(t_1, \dots, t_r)$, then $\mathfrak{M} \models F[a_1, \dots, a_n]$ if and only if $R^{\mathfrak{M}}(t_1[a_1, \dots, a_n], \dots, t_r[a_1, \dots, a_n])$ holds, and for a formula $F[w_0, \dots, w_n]$, one has $\mathfrak{M} \models (\forall w_0 F)[a_1, \dots, a_n]$, if and only if for every a in M, $\mathfrak{M} \models F[a, a_1, \dots, a_n]$.

When the language contains the binary relation symbol of equality $=$, one usually assumes L-structures to be equalitarian, that is, that the relation $=^{\mathfrak{M}}$ coincides with the equality relation on the set M. From now on we shall denote in the same way symbols f and R and their interpretation $f^{\mathfrak{M}}$ and $R^{\mathfrak{M}}$. In particular we may identify constant symbols and their interpretation in M which are elements of M.

We shall also be lax with the names of variables and allow other names, like x_i, x, y.

1.5. **Some examples.** Let us give some examples of languages we shall use in this paper. It is enough to give the symbols which are not variables nor logical.

For the language of abelian groups these symbols consist of the constant symbol 0, the two binary function symbols $+$, $-$ and equality. The language of ordered abelian groups is obtained by adding a binary relation symbol $<$, the language of rings by adding symbols 1 and $\cdot$ (with the obvious arity). Hence a structure for the ring language is just a set with interpretations for the symbols 0, 1, $+$, $-$ and $\cdot$. This set does not have to be a ring (but it will be in all cases we shall consider).

If S is a set, then by the ring language with coefficients in S, we mean that we add S to the set of constants in the language. For instance any ring containing S will be a structure for that language.

Note that there is a sentence φ in the ring language such that a structure $\mathfrak{M}$ satisfies φ if and only if $\mathfrak{M}$ is a field, namely the conjunction of the field axioms (there is a finite number of such axioms, each expressible by a sentence in the ring language). On the other hand, one can show there is no sentence in the ring language expressing for a field to be algebraically closed. Of course, given a natural number $n > 0$, there is a sentence expressing that every degree n polynomial has a root namely

$$\forall a_0 \forall a_1 \cdots \forall a_n \exists x (a_0 = 0 \vee a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0). \tag{1.5.1}$$

Note that here x^i is an abbreviation for $x \cdot x \cdot x \cdots x$ (i times). It is important to notice that we are not allowed to quantify over n here, since it does not correspond to a variable in the structure we are considering.

1.6. **The Denef-Pas language.** We shall need a slight generalization of the notion of language, that of many sorted languages (in fact 3-sorted language). In a 3-sorted language we have 3 sorts of variables symbols, and for relation and function symbols one should specify the type of the variables involved and for functions also the type of the value of f. A structure for a 3-sorted language will consist of 3 sets M_1, M_2 and M_3 together with interpretations of the non logical, non variable symbols. For instance if f is a binary function symbol, with first variable of type 2, second variable of type 3, and value of type 3, its interpretation will be a function $M_2 \times M_3 \to M_3$.

Let us fix a field k of characteristic 0 and consider the following 3-sorted language, the Denef-Pas language $\mathcal{L}_{\mathrm{DP}}$. The 3 sorts are respectively called the valued field sort, the residue field sort, and the value group sort. The language will consist of the disjoint union of the language of rings with coefficients in $k((t))$ restricted to the valued field sort, of the language of rings with coefficients in k restricted to the residue field sort and of the language of ordered groups restricted to the value group sort, together with two additional symbols of unary functions $\overline{\mathrm{ac}}$ and ord from the valued field sort to the residue field and valued groups sort, respectively. [In fact, the definition we give here is different from that in [8], where for the value group sort in $\mathcal{L}_{\mathrm{DP}}$ symbols $\equiv_n$ for equivalence relation modulo n, $n > 1$ in $\mathbb{N}$, are added, but since this does not change the category of definable objects, this change has no consequence on our statements.]

An example of an $\mathcal{L}_{\mathrm{DP}}$-structure is $(k((t)), k, \mathbb{Z})$ with $\overline{\mathrm{ac}}$ interpreted as the function $\overline{\mathrm{ac}} : k((t)) \to k$ assigning to a series its first nonzero coefficient if not zero, zero otherwise, and ord interpreted as the valuation function $\mathrm{ord} : k((t)) \setminus \{0\} \to \mathbb{Z}$. (There is a minor divergence here, easily fixed, since $\mathrm{ord}\, 0$ is not defined.) More generally, for any field K containing k, $(K((t)), K, \mathbb{Z})$ is naturally an $\mathcal{L}_{\mathrm{DP}}$-structure. For instance $\overline{\mathrm{ac}}(x^2 + (1 + t^3)y) - 5z^3$ and $\mathrm{ord}(x^2 + (1 + t^3)y) - 2w + 1$ are terms in $\mathcal{L}_{\mathrm{DP}}$, $\forall x \exists z \neg(\overline{\mathrm{ac}}(x^2 + (1 + t^3)y) - 5z^3 = 0)$ and $\forall x \exists w (\mathrm{ord}(x^2 + (1 + t^3)y) = 2w + 1)$ are formulas.

2. Integration of constructible motivic functions

2.1. **The category of definable objects.** Let φ be a formula in the language $\mathcal{L}_{\mathrm{DP}}$ having respectively m, n, and r free variables in the various sorts. To such a formula

φ we assign, for every field K containing k, the subset $h_\varphi(K)$ of $K((t))^m \times K^n \times \mathbb{Z}^r$ consisting of all points satisfying φ, that is,

$$h_\varphi(K) := \Big\{(x, \xi, \eta) \in K((t))^m \times K^n \times \mathbb{Z}^r; (K((t), K, \mathbb{Z}) \models \varphi(x, \xi, \eta)\Big\}. \tag{2.1.1}$$

We shall call the datum of such subsets for all K definable (sub)assignments. In analogy with algebraic geometry, where the emphasis is not put anymore on equations but on the functors they define, we consider instead of formulas the corresponding subassignments (note $K \mapsto h_\varphi(K)$ is in general not a functor).

More precisely, let $F : C \to \mathrm{Set}$ be a functor from a category C to the category of sets. By a subassignment h of F we mean the datum, for every object C of C, of a subset $h(C)$ of $F(C)$. Most of the standard operations of elementary set theory extend trivially to subassignments. For instance, given subassignments h and h' of the same functor, one defines subassignments $h \cup h'$, $h \cap h'$ and the relation $h \subset h'$, etc. When $h \subset h'$ we say h is a subassignment of h'. A morphism $f : h \to h'$ between subassignments of functors F_1 and F_2 consists of the datum for every object C of a map

$$f(C) : h(C) \to h'(C). \tag{2.1.2}$$

The graph of f is the subassignment

$$C \mapsto \mathrm{graph}(f(C)) \tag{2.1.3}$$

of $F_1 \times F_2$. Let k be a field and consider the category F_k of fields containing k. (To avoid any set-theoretical issues, we fix a Grothendieck universe $\mathcal{U}$ containing k and we define F_k as the small category of all fields in $\mathcal{U}$ containing k.)

We denote by $h[m, n, r]$ the functor $F_k \to \mathrm{Set}$ given by

$$h[m, n, r](K) = K((t))^m \times K^n \times \mathbb{Z}^r. \tag{2.1.4}$$

In particular, $h[0, 0, 0]$ assigns the one point set to every K. We sometimes write $\mathbb{Z}^r$ for $h[0, 0, r]$. Thus, to any formula φ in $\mathcal{L}_{\mathrm{DP}}$ having respectively m, n, and r free variables in the various sorts, corresponds a subassignment h_φ of $h[m, n, r]$ by (2.1.1). Such subassignments are called definable subassignments.

We denote by Def_k the category whose objects are definable subassignments of some $h[m, n, r]$, morphisms in Def_k being morphisms of subassignments $f : h \to h'$ with h and h' definable subassignments of $h[m, n, r]$ and $h[m', n', r']$ respectively such that the graph of f is a definable subassignment. Note that $h[0, 0, 0]$ is the final object in this category.

For example, for each $m \geq 2$, there is a definable subassignment of $h[m^2, 0, 0]$ that assigns to each field K containing k the set of elliptic elements of $GL(m, K((t)))$. This subassignment is not a functor, because there is no inclusion of elliptic elements corresponding to the inclusion of groups $GL(m, K((t))) \subset GL(m, K'((t)))$ under a general field extension K'/K.

For an example of a subassignment of $h[1, 0, 0]$ that fails to be definable, consider the functor that assigns to each field K the set of all roots of unity in $K((t))$. Another simple example of a subassignment of $h[0, 1, 0]$ that is not definable is the one that assigns to a field K the subset of K consisting of elements that are transcendental over k.

2.2. **First sketch of construction of the motivic measure.** The construction in [8] relies in an essential way on a cell decomposition theorem due to Denef and Pas [31]. Let us introduce the notion of cells. Fix coordinates $x = (x', z)$ on $h[m+1, n, r]$ with x' running over $h[m, n, r]$ and z over $h[1, 0, 0]$.

A 0-cell in $h[m+1, n, r]$ is a definable subassignment Z^0_A defined by

$$x' \in A \quad \text{and} \quad z = c(x') \tag{2.2.1}$$

with A a definable subassignment of $h[m, n, r]$ and c a morphism $A \to h[1, 0, 0]$.

A 1-cell in $h[m+1, n, r]$ is a definable subassignment Z^1_A defined by

$$x' \in A, \quad \overline{\mathrm{ac}}(z - c(x')) = \xi(x') \quad \text{and} \quad \mathrm{ord}(z - c(x')) = \alpha(x') \tag{2.2.2}$$

with A a definable subassignment of $h[m, n, r]$, c, ξ and α morphisms from A to $h[1, 0, 0]$, $h[0, 1, 0] \setminus \{0\}$ and $h[0, 0, 1]$, respectively.

The Denef-Pas Cell Decomposition Theorem states that, after adding a finite number of auxiliary parameters in the residue field and value group sorts, every definable subassignment becomes a finite disjoint union of cells:

Theorem 2.2.1 (Denef-Pas Cell Decomposition [31]). *Let A be a definable subassignment $h[m+1, n, r]$. After adding a finite number of auxiliary parameters in the residue field and value group sorts, A is a finite disjoint union of cells, that is, there exists an embedding*

$$\lambda : h[m+1, n, r] \longrightarrow h[m+1, n+n', r+r'] \tag{2.2.3}$$

such that the composition of λ with the projection to $h[m+1, n, r]$ is the identity on A and such that $\lambda(A)$ is a finite disjoint union of cells.

An example of a subassignment that can be transformed into a 1-cell by adding auxiliary parameters appears in Section 2.7.

The construction of the motivic measure $\mu(A)$ for a definable subassignment A of $h[m, n, r]$ goes roughly as follows (more details, including a description of the semiring in which the measure takes values, will be given in 2.6). The cell decomposition theorem expresses a definable subassignment (in $h[m, n, r]$, $m > 0$) as a disjoint union of cells. The measure of a definable subassignment is defined to be the sum of the measures of its cells. In turn, a cell in $h[m, n, r]$ is expressed in terms of a definable subassignment B in $h[m-1, n', r']$ and auxiliary data. The measure of a cell can then be defined recursively in terms of the "smaller" definable subassignment B. The base case of the recursive definition is $m = 0$ (with larger values of n and r). When $m = 0$, one may consider the counting measure on the $\mathbb{Z}^r$-factor and the tautological measure on the $h[0, n, 0]$-factor, assigning to a definable subassignment of $h[0, n, 0]$ its class in $\mathscr{C}_+(\text{point})$ (a semiring defined in the next section). The whole point is to check that the construction is invariant under permutations of valued field coordinates. This is the more difficult part of the proof and is essentially equivalent to a form of the motivic Fubini Theorem.

2.3. **Constructible functions.** For X in Def_k we now define the semiring $\mathscr{C}_+(X)$, resp ring $\mathscr{C}(X)$, of non negative constructible motivic functions, resp. constructible motivic functions.

One considers the category Def_X whose objects are morphisms $Y \to X$ in Def_k, morphisms being morphisms $Y \to Y'$ compatible with the projections to X. We write $X[m,n,r]$ for $X \times h[m,n,r]$. Of interest to us will be the subcategory RDef_X of Def_X whose objects are definable subassignments of $X \times h[0,n,0]$, for variable n. We shall denote by $SK_0(\mathrm{RDef}_X)$ the free abelian semigroup on isomorphism classes of objects of RDef_X modulo the additivity relation

$$[Y] + [Y'] = [Y \cup Y'] + [Y \cap Y']. \tag{2.3.1}$$

It is endowed with a natural semiring structure. One defines similarly the Grothendieck ring $K_0(\mathrm{RDef}_X)$, which is the ring associated to the semiring $SK_0(\mathrm{RDef}_X)$. Proceeding this way, we only defined "half" of $\mathscr{C}_+(X)$ and $\mathscr{C}(X)$.

To get the remaining "half" one considers the ring

$$\mathbb{A} := \mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1-\mathbb{L}^{-i}}\Big)_{i>0}\Big]. \tag{2.3.2}$$

For q a real number > 1, we denote by ϑ_q the ring morphism

$$\vartheta_q : \mathbb{A} \longrightarrow \mathbb{R} \tag{2.3.3}$$

sending $\mathbb{L}$ to q and we consider the semiring

$$\mathbb{A}_+ := \Big\{x \in \mathbb{A} \mid \vartheta_q(x) \geq 0, \forall q > 1\Big\}. \tag{2.3.4}$$

We denote by $|X|$ the set of points of X, that is, the set of pairs (x_0, K) with K in F_k and $x_0 \in X(K)$, and we consider the subring $\mathcal{P}(X)$ of the ring of functions $|X| \to \mathbb{A}$ generated by constants in $\mathbb{A}$ and by all functions α and $\mathbb{L}^\alpha$ with $\alpha : X \to \mathbb{Z}$ definable morphisms. We define $\mathcal{P}_+(X)$ as the semiring of functions in $\mathcal{P}(X)$ taking their values in $\mathbb{A}_+$. These are the second "halves".

To glue the two "halves", one proceed as follows. One denotes by $\mathbb{L} - 1$ the class of the subassignment $x \neq 0$ of $X \times h[0,1,0]$ in $SK_0(\mathrm{RDef}_X)$, resp $K_0(\mathrm{RDef}_X)$. One considers the subring $\mathcal{P}^0(X)$ of $\mathcal{P}(X)$, resp. the subsemiring $\mathcal{P}^0_+(X)$ of $\mathcal{P}_+(X)$, generated by functions of the form $\mathbf{1}_Y$ with Y a definable subassignment of X (that is, $\mathbf{1}_Y$ is the characteristic function of Y), and by the constant function $\mathbb{L} - 1$. We have canonical morphisms $\mathcal{P}^0(X) \to K_0(\mathrm{RDef}_X)$ and $\mathcal{P}^0_+(X) \to SK_0(\mathrm{RDef}_X)$. We may now set

$$\mathscr{C}_+(X) = SK_0(\mathrm{RDef}_X) \otimes_{\mathcal{P}^0_+(X)} \mathcal{P}_+(X)$$

and

$$\mathscr{C}(X) = K_0(\mathrm{RDef}_X) \otimes_{\mathcal{P}^0(X)} \mathcal{P}(X).$$

There are some easy functorialities. For every morphism $f : S \to S'$, there is a natural pullback by $f^* : SK_0(\mathrm{RDef}_{S'}) \to SK_0(\mathrm{RDef}_S)$ induced by the fiber product. If $f : S \to S'$ is a morphism in $\mathrm{RDef}_{S'}$, composition with f induces a morphism $f_! : SK_0(\mathrm{RDef}_S) \to SK_0(\mathrm{RDef}_{S'})$. Similar constructions apply to K_0. If $f : S \to S'$ is a morphism in Def_k, one shows in [8] that the morphism f^* may naturally be extended to a morphism

$$f^* : \mathscr{C}_+(S') \longrightarrow \mathscr{C}_+(S). \tag{2.3.5}$$

If, furthermore, f is a morphism in $\mathrm{RDef}_{S'}$, one shows that the morphism $f_!$ may naturally be extended to

$$f_! : \mathscr{C}_+(S) \longrightarrow \mathscr{C}_+(S'). \tag{2.3.6}$$

Similar functorialities exist for $\mathscr{C}$.

The semiring $\mathscr{C}_+(X)$ can be understood heuristically as follows. The right-hand side of the semiring consists of "raw integrands." Experience with p-adic integration has guided the choice of integrands on the right-hand side. We decided what functions we wanted to integrate and then built a semiring generated by precisely those functions. The variable $\mathbb{L}$ replaces the cardinality q of the residue field that appears in p-adic integrals. For example, the functions $\mathbb{L}^\alpha$ give a constructible counterpart to the common p-adic integrand consisting of the absolute value of a multivariate polynomial $|p(x_1, \ldots, x_m)|^k = q^{-k\,\mathrm{ord}(p(x_1,\ldots,x_m))}$. The left-hand side of the semiring (together with the ring $\mathbb{A}$) can be viewed as the "storage registers" for evaluated integrals. Imprecisely stated, integration consists of moving raw integrands from the right-hand and storing the answers on the left-hand side or in $\mathbb{A}_+$. The subsemiring $\mathcal{P}^0_+(X)$ and its morphisms into the two halves encode a table of integrals, matching each raw integrand with its evaluated integral. The two halves are combined into a single semiring to allow for multiple integrals; the value of the first integral becomes the integrand of the next integral. The denominators $(1-\mathbb{L}^{-i})$ in the semiring $\mathbb{A}_+$ correspond to the denominators $(1-q^{-i})$ that frequently appear in p-adic integrals when geometric series are summed.

2.4. **Taking care of dimensions.** In fact, we shall need to consider not only functions as we just defined, but functions defined almost everywhere in a given dimension, that we call $\mathscr{F}$unctions. (Note the calligraphic capital in $\mathscr{F}$unctions.)

To motivate why dimension should matter, it helps to recall that in the case of a local field F, there is a separate Haar measure (that is, translation-invariant regular Borel measure) on F^d, for each d. A set $X \subset F^d$ of positive measure with respect to the Haar measure on F^d will have measure zero with respect to the Haar measure on F^{d+1} under an embedding $F^d \subset F^{d+1}$. Hence, the appropriate measure for X depends on its dimension. This suggests that we should filter the semiring of constructible functions by dimension.

We start by defining a good notion of dimension for objects of Def_k. Heuristically, that dimension corresponds to counting the dimension only in the valued field variables, without taking in account the remaining variables. More precisely, to any algebraic subvariety Z of $\mathbb{A}^m_{k((t))}$ we assign the definable subassignment h_Z of $h[m,0,0]$ given by $h_Z(K) = Z(K((t)))$. The Zariski closure of a subassignment S of $h[m,0,0]$ is the intersection W of all algebraic subvarieties Z of $\mathbb{A}^m_{k((t))}$ such that $S \subset h_Z$. We define the dimension of S as $\dim S := \dim W$. In the general case, when S is a subassignment of $h[m,n,r]$, we define $\dim S$ as the dimension of the image of S under the projection $h[m,n,r] \to h[m,0,0]$. One can prove that isomorphic objects of Def_k have the same dimension.

For every non negative integer d, we denote by $\mathscr{C}^{\leq d}_+(S)$ the ideal of $\mathscr{C}_+(S)$ generated by functions $\mathbf{1}_Z$ with Z definable subassignments of S with $\dim Z \leq d$. We set $C_+(S) = \oplus_d C_+^d(S)$ with $C_+^d(S) := \mathscr{C}_+^{\leq d}(S)/\mathscr{C}_+^{\leq d-1}(S)$. It is a graded abelian semigroup, and also a $\mathscr{C}_+(S)$-semimodule. Elements of $C_+(S)$ are called positive constructible $\mathscr{F}$unctions on S. If φ is a function lying in $\mathscr{C}_+^{\leq d}(S)$ but not in $\mathscr{C}_+^{\leq d-1}(S)$, we denote by $[\varphi]$ its image in $C_+^d(S)$. One defines similarly $C(S)$ from $\mathscr{C}(S)$.

One of the reasons why we consider functions which are defined almost everywhere originates in the differentiation of functions with respect to the valued field variables: one may show that a definable function $c : S \subset h[m, n, r] \to h[1, 0, 0]$ is differentiable (in fact even analytic) outside a definable subassignment of S of dimension $< \dim S$. In particular, if $f : S \to S'$ is an isomorphism in Def_k, one may define a function $\mathrm{ordjac} f$, the order of the jacobian of f, which is defined almost everywhere and is equal almost everywhere to a definable function, so we may define $\mathbb{L}^{-\mathrm{ordjac} f}$ in $C_+^d(S)$ when S is of dimension d.

2.5. **Push-forward.** Let k be a field of characteristic zero. Given S in Def_k, we define S-integrable $\mathscr{F}$unctions and construct pushforward morphisms for these $\mathscr{F}$unctions by means of Theorem 2.5.1. Roughly, the notion of being S-integrable for a constructible $\mathscr{F}$unction means that one can integrate all variables out until one is left with a function on S, where this resulting function on S need not be further integrable over S itself. The pushforward morphisms for the S-integrable $\mathscr{F}$unctions in idea correspond to taking integrals while taking care of dimensions of the support: if one integrates over $\mathbb{Z}$ as in (A6), the idea is summation over $\mathbb{Z}$ and the dimension of the support is left unchanged (recall that the dimension lives in the valued field); if one integrates over an open in the valued field as in (A7), one perfoms an actual motivic integral analogous to a p-adic integral and the dimension of the support goes down by 1; if one integrates so to say on a curve as in (A8), one performs a basic change of variables which of course preserves the dimension of the support; and, if one integrates over the residue field as in (A5) the idea is not to lose any relevant information and one just takes the whole class in a Grothendieck semiring of the part one wants to measure, the dimension of the support again being preserved. Having the pushforward morphisms after this theorem allows one to use them to integrate variables out, to send an S-integrable $\mathscr{F}$unction to one living on another domain by change of variables, or, as a combination of these, (possibly mimicking the usage of Leray-differential forms) to transform an S-integrable $\mathscr{F}$unction φ on Z into an S-integrable $\mathscr{F}$unction $f_!(\varphi)$ on Y, where Z and Y are the domains seen as objects in Def_S, and $f : Z \to Y$ is merely required to make a commutative diagram over S. The crucial property that relates φ and $f_!(\varphi)$ is the ability to integrate all variables out up to arriving to a constructible $\mathscr{F}$unction on S (by S-integrability) which is the same when calculated from φ or from $f_!(\varphi)$ by the functoriality properties.

Thus, Theorem 2.5.1 defines at the same time the notion of S-integrability for constructible $\mathscr{F}$unctions on domains Z over S, their integrals, and more generally all their transforms through the pushforward morphisms. Although all the axioms interact in the proof of the theorem, one can specify that property (A1a) and the Fubini-Tonelli property (A1b) are crucial for the definition of S-integrability, and that property (A3) can be used to pull a factor out of an integral, as is done in the example in 2.7.

Theorem 2.5.1 (Cluckers-Loeser [8])**.** *Let k be a field of characteristic zero and let S be in* Def_k. *There exists a unique functor $Z \mapsto \mathrm{I}_S C_+(Z)$ from* Def_S *to the category of abelian semigroups, the functor of S-integrable $\mathscr{F}$unctions, assigning to every morphism $f : Z \to Y$ in* Def_S *a morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ such that for every Z in* Def_S*, $\mathrm{I}_S C_+(Z)$ is a graded subsemigroup of $C_+(Z)$ and $\mathrm{I}_S C_+(S) = C_+(S)$, satisfying the following list of axioms (A1)-(A8).*

(A1a) **(Naturality)**
If $S \to S'$ is a morphism in Def_k and Z is an object in Def_S, then any S'-integrable $\mathscr{F}$unction φ in $C_+(Z)$ is S-integrable and $f_!(\varphi)$ is the same, considered in $\mathrm{I}_{S'}$ or in I_S.

(A1b) **(Fubini)**
A positive $\mathscr{F}$unction φ on Z is S-integrable if and only if it is Y-integrable and $f_!(\varphi)$ is S-integrable.

(A2) **(Disjoint union)**
If Z is the disjoint union of two definable subassignments Z_1 and Z_2, then the isomorphism $C_+(Z) \simeq C_+(Z_1) \oplus C_+(Z_2)$ induces an isomorphism $\mathrm{I}_S C_+(Z) \simeq \mathrm{I}_S C_+(Z_1) \oplus \mathrm{I}_S C_+(Z_2)$, under which $f_! = f_{|Z_1 !} \oplus f_{|Z_2 !}$.

(A3) **(Projection formula)**
For every α in $\mathscr{C}_+(Y)$ and every β in $\mathrm{I}_S C_+(Z)$, $\alpha f_!(\beta)$ is S-integrable if and only if $f^*(\alpha)\beta$ is, and then $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.

(A4) **(Inclusions)**
If $i : Z \hookrightarrow Z'$ is the inclusion of definable subassignments of the same object of Def_S, then $i_!$ is induced by extension by zero outside Z and sends $\mathrm{I}_S C_+(Z)$ injectively to $\mathrm{I}_S C_+(Z')$.

(A5) **(Integration along residue field variables)**
Let Y be an object of Def_S and denote by π the projection $Y[0, n, 0] \to Y$. A $\mathscr{F}$unction $[\varphi]$ in $C_+(Y[0, n, 0])$ is S-integrable if and only if, with notations of (2.3.6), $[\pi_!(\varphi)]$ is S-integrable and then $\pi_!([\varphi]) = [\pi_!(\varphi)]$.

Basically this axiom means that integrating with respect to variables in the residue field just amounts to taking the pushforward induced by composition at the level of Grothendieck semirings.

(A6) **(Integration along $\mathbb{Z}$-variables)** Basically, integration along $\mathbb{Z}$-variables corresponds to summing over the integers, but to state precisely (A6), we need to perform some preliminary constructions.

Let us consider a function in φ in $\mathcal{P}(S[0, 0, r])$, hence φ is a function $|S| \times \mathbb{Z}^r \to A$. We shall say φ is S-integrable if for every $q > 1$ and every x in $|S|$, the series $\sum_{i \in \mathbb{Z}^r} \vartheta_q(\varphi(x, i))$ is summable. One proves that if φ is S-integrable there exists a unique function $\mu_S(\varphi)$ in $\mathcal{P}(S)$ such that $\vartheta_q(\mu_S(\varphi)(x))$ is equal to the sum of the previous series for all $q > 1$ and all x in $|S|$. We denote by $\mathrm{I}_S\mathcal{P}_+(S[0,0,r])$ the set of S-integrable functions in $\mathcal{P}_+(S[0,0,r])$ and we set

$$\mathrm{I}_S\mathscr{C}_+(S[0,0,r]) = \mathscr{C}_+(S) \otimes_{\mathcal{P}_+(S)} \mathrm{I}_S\mathcal{P}_+(S[0,0,r]). \tag{2.5.1}$$

Hence $\mathrm{I}_S\mathcal{P}_+(S[0,0,r])$ is a sub-$\mathscr{C}_+(S)$-semimodule of $\mathscr{C}_+(S[0,0,r])$ and μ_S may be extended by tensoring to

$$\mu_S : \mathrm{I}_S\mathscr{C}_+(S[0,0,r]) \to \mathscr{C}_+(S). \tag{2.5.2}$$

Now we can state (A6):
Let Y be an object of Def_S and denote by π the projection $Y[0,0,r] \to Y$. A $\mathscr{F}$unction $[\varphi]$ in $C_+(Y[0,0,r])$ is S-integrable if and only if there exists φ' in $\mathscr{C}_+(Y[0,0,r])$ with $[\varphi'] = [\varphi]$ which is Y-integrable in the previous sense and such that $[\mu_Y(\varphi')]$ is S-integrable. We then have $\pi_!([\varphi]) = [\mu_Y(\varphi')]$.

(A7) **(Volume of balls)** It is natural to require (by analogy with the p-adic case) that the volume of a ball $\{z \in h[1,0,0] \mid \mathrm{ord}(z-c) = \alpha, \overline{\mathrm{ac}}(z-c) = \xi\}$, with α in $\mathbb{Z}$, c in $k((t))$ and ξ non zero in k, should be $\mathbb{L}^{-\alpha-1}$. (A7) is a relative version of that statement:
Let Y be an object in Def_S and let Z be the definable subassignment of $Y[1,0,0]$ defined by $\mathrm{ord}(z - c(y)) = \alpha(y)$ and $\overline{\mathrm{ac}}(z - c(y)) = \xi(y)$, with z the coordinate on the $\mathbb{A}^1_{k((t))}$-factor and α, ξ, c definable functions on Y with values respectively in $\mathbb{Z}$, $h[0,1,0] \setminus \{0\}$, and $h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbb{L}^{-\alpha-1}[\mathbf{1}_Y]$ is, and then $f_!([\mathbf{1}_Z]) = \mathbb{L}^{-\alpha-1}[\mathbf{1}_Y]$.

(A8) **(Graphs)** This last axiom expresses the pushforward for graph projections. It relates volume and differentials and is a special case of the change of variables Theorem 2.6.1.
Let Y be in Def_S and let Z be the definable subassignment of $Y[1,0,0]$ defined by $z - c(y) = 0$ with z the coordinate on the $\mathbb{A}^1_{k((t))}$-factor and c a morphism $Y \to h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbb{L}^{(\mathrm{ordjac} f)\circ f^{-1}}$ is, and then $f_!([\mathbf{1}_Z]) = \mathbb{L}^{(\mathrm{ordjac} f)\circ f^{-1}}$.

Once Theorem 2.5.1 is proved, one may proceed as follows to extend the constructions from C_+ to C. One defines $\mathrm{I}_S C(Z)$ as the subgroup of $C(Z)$ generated by the image of $\mathrm{I}_S C_+(Z)$. One shows that if $f : Z \to Y$ is a morphism in Def_S, the morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ has a natural extension

$$f_! : \mathrm{I}_S C(Z) \to \mathrm{I}_S C(Y). \tag{2.5.3}$$

The proof of Theorem 2.5.1 is quite long and involved. In a nutshell, the basic idea is the following. Integration along residue field variables is controlled by (A5) and integration along $\mathbb{Z}$-variables by (A6). Integration along valued field variables is constructed one variable after the other. To integrate with respect to one valued field variable, one may, using (a variant of) the cell decomposition Theorem 2.2.1 (at the cost of introducing additional new residue field and $\mathbb{Z}$-variables), reduce to the case of cells which is covered by (A7) and (A8). An important step is to show that this is independent of the choice of a cell decomposition. When one integrates with respect to more than one valued field variable (one after the other) it is crucial to show that it is independent of the order of the variables, for which we use a notion of bicells.

2.6. **Motivic measure.** The relation of Theorem 2.5.1 with motivic integration is the following. When S is equal to $h[0,0,0]$, the final object of Def_k, one writes $\mathrm{I}C_+(Z)$ for $\mathrm{I}_S C_+(Z)$ and we shall say integrable for S-integrable, and similarly for C. Note that $\mathrm{I}C_+(h[0,0,0]) = C_+(h[0,0,0]) = SK_0(\mathrm{RDef}_k)\otimes_{\mathbb{N}[\mathbb{L}-1]}\mathbb{A}_+$ and that $\mathrm{I}C(h[0,0,0]) = K_0(\mathrm{RDef}_k)\otimes_{\mathbb{Z}[\mathbb{L}]}\mathbb{A}$. For φ in $\mathrm{I}C_+(Z)$, or in $\mathrm{I}C(Z)$, one defines the motivic integral $\mu(\varphi)$ by $\mu(\varphi) = f_!(\varphi)$ with f the morphism $Z \to h[0,0,0]$.

Let X be in Def_k of dimension d. Let φ be a function in $\mathscr{C}_+(X)$, or in $\mathscr{C}(X)$. We shall say φ is integrable if its class $[\varphi]_d$ in $C_+^d(X)$, resp. in $C^d(X)$, is integrable, and we shall set

$$\mu(\varphi) = \int_X \varphi\, d\mu = \mu([\varphi]_d).$$

Similarly as in the p-adic case, cf. [19] p.112, one may develop the integration on global (non affine) objects endowed with a differential form of top degree once the following Change of Variables Theorem 2.6.1 is established:

Theorem 2.6.1 (Cluckers-Loeser [8])**.** *Let $f : Y \to X$ be an isomorphism in* Def_k. *For any integrable function φ in $\mathscr{C}_+(X)$ or $\mathscr{C}(X)$,*

$$\int_X \varphi d\mu = \int_Y \mathbb{L}^{-\mathrm{ord}\,\mathrm{jac}(f)} f^*(\varphi) d\mu.$$

Also, the construction we outlined of the motivic measure carries over almost literally to a relative setting: one can develop a relative theory of motivic integration: integrals depending on parameters of functions in $\mathscr{C}_+$ or $\mathscr{C}$ still belong to $\mathscr{C}_+$ or $\mathscr{C}$ as functions of these parameters.

More specifically, if $f : X \to \Lambda$ is a morphism and φ is a function in $\mathscr{C}_+(X)$ or $\mathscr{C}(X)$ that is relatively integrable (a notion defined in [8]), one constructs in [8] a function

$$\mu_\Lambda(\varphi) \tag{2.6.1}$$

in $\mathscr{C}_+(\Lambda)$, resp. $\mathscr{C}(\Lambda)$, whose restriction to every fiber of f coincides with the integral of φ restricted to that fiber.

2.7. **An extended example.** Let us give an example of a definable subassignment that can be transformed into a 1-cell by adding auxiliary parameters. Let φ be the formula with a free variable z of the valued field sort:

$$z \neq 0 \ \wedge\ \mathrm{ord}(z) \geq 0 \ \wedge\ \exists\, t\, (z = t^2).$$

$Z = h_\varphi \subset h[1,0,0]$ is the subassignment of nonzero squares in the valued field with nonnegative valuation. Under the embedding

$$\lambda : h[1,0,0] \to h[1,1,1], \quad z \mapsto (z, \overline{\mathrm{ac}}(z), \mathrm{ord}(z)),$$

$W = \lambda(h_\varphi)$ is a 1-cell with defining functions

$$c(\zeta,\eta) = 0, \quad \xi(\zeta,\eta) = \zeta, \quad \alpha(\zeta,\eta) = \eta.$$

on $S \times N$, where $S \subset h[0,1,0]$ and $N \subset h[0,0,1]$ are defined by the conditions

$$\zeta \neq 0 \ \wedge\ \exists\zeta_1(\zeta_1^2 = \zeta) \quad \text{and} \quad \eta \geq 0 \ \wedge\ \exists\eta_1(2\eta_1 = \eta), \tag{2.7.1}$$

respectively. That is, W is defined by the conditions 2.7.1 and

$$\mathrm{ord}(z) = \eta \ \wedge\ \overline{\mathrm{ac}}(z) = \zeta.$$

To continue with this example, let us compute the motivic integral over Z of a particular constructible function $\mathbb{L}^{-3\mathrm{ord}(z)}$:

$$\begin{aligned}
\int_Z \mathbb{L}^{-3\mathrm{ord}(z)} d\mu(z) &= \int_W \mathbb{L}^{-3\eta} d\mu(z,\zeta,\eta) && \text{(A5), (A6)}\\
&= \int_{S\times N}\int_{W\to S\times N} \mathbb{L}^{-3\eta} d\mu(z)\, d\mu(\zeta,\eta) && \text{(A1)}\\
&= \int_{S\times N} \mathbb{L}^{-3\eta} \int_{W\to S\times N} \mathbf{1}_W\, d\mu(z)\, d\mu(\zeta,\eta) && \text{(A3)}\\
&= \int_{S\times N} \mathbb{L}^{-3\eta}\mathbb{L}^{-\eta-1}\, d\mu(\zeta,\eta) && \text{(A7)}\\
&= [\mathbf{1}_S] \int_N \mathbb{L}^{-4\eta-1} d\mu(\eta) && \text{(A3), (A5)}\\
&= \frac{[\mathbf{1}_S]}{\mathbb{L}(1-\mathbb{L}^{-8})} && \text{(A6)}
\end{aligned}$$

This integral should be compared with the corresponding p-adic integral over the set of integer squares with respect to the Haar measure, normalized so that the ring of integers has volume 1:

$$\int_{|z|\leq 1\ z=t^2} |z|^4 \frac{dz}{|z|} = \frac{\sigma}{q(1-q^{-8})},$$

where q is the cardinality of the residue field of the local field F, and σ is the cardinality of the set of nonzero squares in the residue field. (We have assumed that the residual characteristic is not 2.) The result of (A6) in the motivic calculation should be compared with the same infinite sum of real numbers that appears in the p-adic integral:

$$\sum_{i=0,\ i=2j}^{\infty} q^{-4i-1} = \frac{1}{q(1-q^{-8})}, \quad q > 1.$$

2.8. **The transfer principle.** We are now in the position of explaining how motivic integrals specialize to p-adic integrals and may be used to obtain a general transfer principle allowing to transfer relations between integrals from $\mathbb{Q}_p$ to $\mathbb{F}_p((t))$ and vice-versa.

We shall assume from now on that k is a number field with ring of integers $\mathcal{O}$. We denote by $\mathcal{A}_\mathcal{O}$ the set of p-adic completions of all finite extensions of k and by $\mathcal{B}_\mathcal{O}$ the set of all local fields of characteristic > 0 which are $\mathcal{O}$-algebras.

For F in $\mathcal{C}_\mathcal{O} := \mathcal{A}_\mathcal{O} \cup \mathcal{B}_\mathcal{O}$, we denote by

- R_F the valuation ring
- M_F the maximal ideal

- k_F the residue field
- $q(F)$ the cardinal of k_F
- ϖ_F a uniformizing parameter of R_F.

There exists a unique morphism $\overline{\mathrm{ac}} : F^\times \to k_F^\times$ extending $R_F^\times \to k_F^\times$ and sending ϖ_F to 1. We set $\overline{\mathrm{ac}}(0) = 0$. For $N > 0$, we denote by $\mathcal{A}_{\mathcal{O},N}$ the set of fields F in $\mathcal{A}_\mathcal{O}$ such that k_F has characteristic $> N$, and similarly for $\mathcal{B}_{\mathcal{O},N}$ and $\mathscr{C}_{\mathcal{O},N}$. To be able to interpret our formulas to fields in $\mathscr{C}_\mathcal{O}$, we restrict the language $\mathcal{L}_{\mathrm{DP}}$ to the sub-language $\mathcal{L}_\mathcal{O}$ for which coefficients in the valued field sort are assumed to belong to the subring $\mathcal{O}[[t]]$ of $k((t))$. We denote by $\mathrm{Def}(\mathcal{L}_\mathcal{O})$ the sub-category of Def_k of objects definable in $\mathcal{L}_\mathcal{O}$, and similarly for functions, etc. For instance, for S in $\mathrm{Def}(\mathcal{L}_\mathcal{O})$, we denote by $\mathscr{C}(S, \mathcal{L}_\mathcal{O})$ the ring of constructible functions on S definable in $\mathcal{L}_\mathcal{O}$.

We consider F as a $\mathcal{O}[[t]]$-algebra via

$$\lambda_{\mathcal{O},F} : \sum_{i\in\mathbb{N}} a_i t^i \longmapsto \sum_{i\in\mathbb{N}} a_i \varpi_F^i. \tag{2.8.1}$$

Hence, if we interpret a in $\mathcal{O}[[t]]$ by $\lambda_{\mathcal{O},F}(a)$, every $\mathcal{L}_\mathcal{O}$-formula φ defines for F in $\mathscr{C}_\mathcal{O}$ a subset φ_F of some $F^m \times k_F^n \times \mathbb{Z}^r$. One proves that if two $\mathcal{L}_\mathcal{O}$-formulas φ and φ' define the same subassignment X of $h[m, n, r]$, then $\varphi_F = \varphi'_F$ for F in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$. This allows us to denote by X_F the subset defined by φ_F, for F in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$. Similarly, every $\mathcal{L}_\mathcal{O}$-definable morphism $f : X \to Y$ specializes to a function $f_F : X_F \to Y_F$ for F in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$.

We now explain how φ in $\mathscr{C}(X, \mathcal{L}_\mathcal{O})$ can be specialized to $\varphi_F : X_F \to \mathbb{Q}$ for F in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$. Let us consider φ in $K_0(\mathrm{RDef}_X(\mathcal{L}_\mathcal{O}))$ of the form $[\pi : W \to X]$ with W in $\mathrm{RDef}_X(\mathcal{L}_\mathcal{O})$. For F in $\mathscr{C}_{\mathcal{O},N}$ with $N \gg 0$, we have $\pi_F : W_F \to X_F$, so we may define $\varphi_F : X_F \to \mathbb{Q}$ by

$$x \longmapsto \mathrm{card}\left(\pi_F^{-1}(x)\right). \tag{2.8.2}$$

For φ in $\mathcal{P}(X)$, we specialize $\mathbb{L}$ into q_F and $\alpha : X \to \mathbb{Z}$ into $\alpha_F : X_F \to \mathbb{Z}$. By tensor product we get $\varphi \mapsto \varphi_F$ for φ in $\mathscr{C}(X, \mathcal{L}_\mathcal{O})$. Note that, under that construction, functions in $\mathscr{C}_+(X, \mathcal{L}_\mathcal{O})$ specialize into non negative functions.

Let F be in $\mathscr{C}_\mathcal{O}$ and A be a subset of $F^m \times k_F^n \times \mathbb{Z}^r$. We consider the Zariski closure $\bar{A}$ of the projection of A into $\mathbb{A}_F^m$. One defines a measure μ on A by restriction of the product of the canonical (Serre-Oesterlé) measure on $\bar{A}(F)$ with the counting measure on $k_F^n \times \mathbb{Z}^r$.

Fix a morphism $f : X \to \Lambda$ in $\mathrm{Def}(\mathcal{L}_\mathcal{O})$ and consider φ in $\mathscr{C}(X, \mathcal{L}_\mathcal{O})$. One can show that if φ is relatively integrable, then, for $N \gg 0$, for every F in $\mathscr{C}_{\mathcal{O},N}$, and for every λ in Λ_F, the restriction $\varphi_{F,\lambda}$ of φ_F to $f_F^{-1}(\lambda)$ is integrable.

We denote by $\mu_{\Lambda_F}(\varphi_F)$ the function on Λ_F defined by

$$\lambda \longmapsto \mu(\varphi_{F,\lambda}). \tag{2.8.3}$$

The following theorem says that motivic integrals specialize to the corresponding integrals over local fields of high enough residue field characteristic.

Theorem 2.8.1 (Specialization, Cluckers-Loeser [10] [11])**.** *Let $f : S \to \Lambda$ be a morphism in* $\mathrm{Def}(\mathcal{L}_O)$*. Let φ be in $\mathscr{C}(S, \mathcal{L}_O)$ and relatively integrable with respect to f. For $N \gg 0$, for every F in $\mathscr{C}_{O,N}$, we have*

$$(\mu_\Lambda(\varphi))_F = \mu_{\Lambda_F}(\varphi_F). \tag{2.8.4}$$

We are now ready to state the following abstract transfer principle:

Theorem 2.8.2 (Abstract transfer principle, Cluckers-Loeser [10] [11])**.** *Let φ be in $\mathscr{C}(\Lambda, \mathcal{L}_O)$. There exists N such that for every F_1, F_2 in $\mathscr{C}_{O,N}$ with $k_{F_1} \simeq k_{F_2}$,*

$$\varphi_{F_1} = 0 \quad \textit{if and only if} \quad \varphi_{F_2} = 0. \tag{2.8.5}$$

Putting together the two previous theorems, one immediately gets:

Theorem 2.8.3 (Transfer principle for integrals with parameters, Cluckers-Loeser [10] [11])**.** *Let $S \to \Lambda$ and $S' \to \Lambda$ be morphisms in* $\mathrm{Def}(\mathcal{L}_O)$*. Let φ and φ' be relatively integrable functions in $\mathscr{C}(S, \mathcal{L}_O)$ and $\mathscr{C}(S', \mathcal{L}_O)$, respectively. There exists N such that for every F_1, F_2 in $\mathscr{C}_{O,N}$ with $k_{F_1} \simeq k_{F_2}$,*

$$\mu_{\Lambda_{F_1}}(\varphi_{F_1}) = \mu_{\Lambda_{F_1}}(\varphi'_{F_1}) \quad \textit{if and only if} \quad \mu_{\Lambda_{F_2}}(\varphi_{F_2}) = \mu_{\Lambda_{F_2}}(\varphi'_{F_2}).$$

In the special case where $\Lambda = h[0, 0, 0]$ and φ and φ' are in $\mathscr{C}(S, \mathcal{L}_O)$ and $\mathscr{C}(S', \mathcal{L}_O)$, respectively, this follows from previous results of Denef-Loeser [13].

Remark 2.8.4. *The previous constructions and statements may be extended directly - with similar proofs - to the global (non affine) setting.*

Note that when $S = S' = \Lambda = h[0, 0, 0]$, one recovers the classical

Theorem 2.8.5 (Ax-Kochen-Eršov [6] [14])**.** *Let φ be a first order sentence* (*that is, a formula with no free variables*) *in the language of rings. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.*

A major triumph of model theory is the application of this theorem to Artin's conjecture about forms over $\mathbb{Q}_p$ [5]. Artin conjectured that for every positive natural number d and every prime p, if $n > d^2$, then every homogeneous polynomial of degree d over $\mathbb{Q}_p$ in n variables has a nontrivial zero in $\mathbb{Q}_p$. Ax and Kochen used their result to transfer known results from positive characteristic to characteristic zero and thus to prove that Artin's conjecture holds asymptotically. That is, for each d, Artin's conjecture holds over $\mathbb{Q}_p$, except for possible failure as the prime p runs over a finite set $A(d)$.

2.9. In view of Theorem 2.8.3, since we are interested in the behavior of integrals for sufficiently large primes p, we shall assume throughout this paper, whenever it is useful to do so, that p is sufficiently large. *This remains a standing assumption throughout this paper.*

Remark 2.9.1. *Let $N > 0$ be an integer and let $\mathcal{L}_O(1/N)$ be the language $\mathcal{L}_O$ with one extra constant symbol to denote the rational number $1/N$. Then the above statements in section 2 remain valid if one works with $\mathcal{L}_O(1/N)$ instead of with $\mathcal{L}_O$, where now the conditions of big enough residue field characteristic mean in* *particular that the residue field characteristic is bigger than N. Indeed,* $\mathcal{L}_O(1/N)$ *is a definitional expansion of* $\mathcal{L}_O$ *in the sense that both languages give exactly the same definable subassignments and definable morphisms, hence they also yield the same rings and semirings resp. the same groups and semigroups of constructible functions and constructible* $\mathscr{F}$*unctions.*

2.10. Tensoring with $\mathbb{Q}$, with $\mathbb{R}$, or with $\mathbb{C}$. Since Arthur's weight function involves volumes (see below), it is useful to work with $\mathscr{F}$unctions in $C(X)\otimes\mathbb{R}$ instead of in $C(X)$ and so on, for definable subassignments X. One can work similarly to tensor with $\mathbb{Q}$ or $\mathbb{C}$. Once we have the direct image operators and integration operators of Theorem 2.5.1, of (2.5.3), and of (2.6.1), this is easily done as follows. Let $f : Z \to Y$ be a morphism in Def_S. Clearly $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are flat $\mathbb{Z}$-modules (as localizations, resp. direct limits of $\mathbb{Z}$, resp. of flat modules). This allows us to view $\mathrm{I_S C(Z)}\otimes_{\mathbb{Z}}\mathbb{R}$ as a submodule of $C(Z)\otimes_{\mathbb{Z}}\mathbb{R}$, and similarly for relative integrable functions. Naturally, $f_!$ extends to a homomorphism $f_!\otimes\mathbb{R} : \mathrm{I_S C(Z)}\otimes\mathbb{R} \to \mathrm{I_S C(Y)}\otimes\mathbb{R}$, and similarly for relative integrable functions. Of course, the study of semirings of constructible functions tensored with $\mathbb{R}$ does not have any additional value since $\mathbb{R}$ contains -1. On the other hand, $f^*\otimes\mathbb{R}$ has a natural meaning as a homomorphism from $C(Y)\otimes\mathbb{R}$ to $C(Z)\otimes\mathbb{R}$. In this setting, the analogue of the Change of Variables Theorem 2.6.1 clearly remains true. A function φ in $\mathscr{C}(X,\mathcal{L}_O)\otimes\mathbb{R}$ can be specialized to $\varphi_F : X_F \to \mathbb{R}$ for F in $\mathscr{C}_{O,N}$ when $N \gg 0$ as in section 2.8, by tensoring with $\mathbb{R}$, where the values of φ_F now lie in $\mathbb{R}$ instead of in $\mathbb{Q}$. With this notation, the analogues in the $\otimes\mathbb{R}$-setting of Theorems 2.8.1, 2.8.2, and 2.8.3 and the analogues of Remarks 2.8.4 and 2.9.1 naturally hold.

In the applications to the Fundamental Lemma, the space S will be the Chevalley space of the endoscopic group and the S-$\mathscr{F}$unction will be the difference of a stable orbital integral and a κ-orbital integral.

3. Definability of Field Extensions

3.1. Translations into the Denef-Pas language. The Denef-Pas language has noteworthy limits to its expressive power. For example, as previously noted, there is no sentence in the language of rings that expresses that a field is algebraically closed. There is no formula φ with one free variable of the residue field sort such that $h_\varphi(K) \subset K$ is the set of roots of unity in K. There is no formula φ with two free variables of the value group sort that consists of pairs (i, j) with $i^2 = j$, because although the language of ordered groups includes a binary addition symbol, it does not include a multiplication symbol or power symbol i^2.

Some mathematical notions cannot be expressed directly in the language, but an adequate translation or proxy can be found within the language. For example, even though there is no cubic power symbol ξ^3 in the language of rings, the expansion as a product $\xi\cdot\xi\cdot\xi$ is a perfectly adequate representation of the cube within the language of rings. Similarly, if we encounter an abstract vector space of dimension n over the residue field sort, it will be represented concretely by the subassignment $h[0,n,0]$. The endomorphism ring of the vector space will be represented concretely as the subassignment $h[0,n^2,0]$ with free variables ξ_{ij}, $i, j = 1,\ldots,n$.

Much of the discussion that follows consists of translating mathematical concepts related to the fundamental lemma into basic terms that can be expressed in the Denef-Pas language.

3.2. Fields extensions. As an example of translating mathematical concepts into more basic terms, we consider the problem of expressing the arithmetic of a field extension E/F in the Denef-Pas language. Because of limitations of the language, we cannot work directly with a field extension E/F. However, a finite field extension E/F has a proxy in the Denef-Pas language, which is obtained by working through the minimal polynomial m of $e \in E$ that generates E/F, and an explicit basis $\{1, x, \ldots, x^{r-1}\}$ of $F[x]/(m) = E$. The minimal polynomial

$$x^r + a_{r-1}x^{r-1} + \cdots a_0 \tag{3.2.1}$$

can be identified with its list of coefficients

$$(a_{r-1}, \ldots, a_0). \tag{3.2.2}$$

In this paper, general finite field extensions will only appear as bound variables in formulas. Moreover, the degree of the extensions will always be fixed. Thus, a statements "there exists a field extension of degree r such that . . ." can be translated into "there exist $a_{r-1}, \ldots, a_0$ such that $x^r + a_{r-1}x^{r-1} + \cdots + a_0$ is irreducible and such that . . .." When $r > 1$, the statement "$x^r + a_{r-1}x^{r-1} + \cdots + a_0$ is irreducible" can be further expanded into the language of rings as "there do not exist $b_0, \ldots, b_{2r}$ and $c_0, \ldots, c_{2r}$ such that

$$a_j = \sum_{i=0}^{j} b_i c_{j-i}, \text{ for } j = 0, \ldots, 2r$$

where $a_r = 1$, $b_r = c_r = 0$, and $a_j = b_j = c_j = 0$ for $j > r$."

After identifying the field extension with F^r through (3.2.2), we can define field automorphisms by linear maps on F^r that respect the field operations. Thus, field automorphisms are definable, and the condition that E/F is a Galois extension is definable.

3.3. A free parameter. We will see that all of the constructions in this paper can be arranged so that the only field extensions that are "free" in a formula are unramified of fixed degree r. These can be described in a field-independent way by a minimal polynomial $x^r - a$, where a satisfies a Denef-Pas condition that it is a unit such that $x^r - a$ is irreducible. This construction introduces a parameter a to whatever formulas involve an unramified field extension. The free parameter a will be used for instance when we describe unramified unitary groups.

If $m(x) = x^r - a$ defines a field extension $F_{a,r}$ of degree r of F, then we may represent a field extension E of F of degree k containing $F_{a,r}$ by data $(b_{k-1}, \ldots, b_0, \phi)$, where $b_i \in F$ are the coefficients of a minimal polynomial $b(y)$ of an element generating E/F, and $\phi \in M_{rk}(F)$ is a matrix giving the embedding of $F^r \to F^k$, corresponding to

$$F_{a,r} = F[x]/(m(x)) \to F[y]/(b(y)). \tag{3.3.1}$$

4. Definability of Unramified Reductive Groups

4.1. Split groups. The classification of split connected reductive groups is independent of the field F. Isomorphism classes of split reductive groups are in bijective correspondence with root data:

$$D = (X^*, \Phi, X_*, \Phi^\vee), \tag{4.1.1}$$

consisting of the character group of a Cartan subgroup, the set of roots, the cocharacter group, and the set of coroots. The root data are considered up to an obvious equivalence.

In particular, we may realize each split reductive group over $\mathbb{Q}$. Fix once and for all, a rational faithful representation $\rho = \rho_D$

$$\rho : G \to GL(V) \tag{4.1.2}$$

of each reductive group over $\mathbb{Q}$, attached to root data D. Fix a basis of V over $\mathbb{Q}$. There exists a formula $\varphi_{\rho,D}$ in the Denef-Pas language (in fact a formula in the language of rings) that describes the zero set of $\rho(G)$.

4.2. Quasi-split groups. An unramified reductive group over a local field F is a quasi-split connected reductive group that splits over an unramified extension of F. The classification of unramified reductive groups is independent of the field F. Isomorphism classes of unramified reductive groups are in bijective correspondence with pairs (D, θ), where D is the root data for the corresponding split group and θ is an automorphism of finite order of D that preserves a set of simple roots in Φ.

The quasi-split group G is obtained by an outer twist of the corresponding split group G^* as follows. Suppose that θ has order r. Let $F_{a,r}$ be the unramified extension of F of degree r defined by the polynomial $x^r - a$. Let $(B, T, \{X_\alpha\})$ be a splitting of G^*, consisting of a Borel subgroup B, Cartan T, and root vectors $\{X_\alpha\}$ all defined over $\mathbb{Q}$. The automorphism θ of the root data determines a unique automorphism of G^* preserving the splitting. We let θ denote this automorphism of G^* as well. The automorphism θ acts on root vectors by $\theta(X_\alpha) = X_{\theta\alpha}$.

For any A-algebra F, we have that $G(A)$ is the set of fixed points of $G(A \otimes F_{a,r})$ under the map $\theta \circ \tau$, where

$$\tau : G^*(A \otimes F_{a,r}) \to G^*(A \otimes F_{a,r}) \tag{4.2.1}$$

extends the Frobenius automorphism of $F_{a,r}/F$. Through the representation ρ of G^*, the fixed-point condition can be expressed on the corresponding matrix groups. In particular, $G(F)$ can be explicitly realized as the fixed points of a map $\theta \circ \tau$ on $\rho(G^*(F_{a,r})) \subset M_n(F_{a,r})$, the set of n by n matrices with coefficients in $F_{a,r}$.

To express the group G by a formula in the Denef-Pas language, we introduce a free parameter a as described in Section 3, which is the proxy in the Denef-Pas language for an unramified extension of degree r. Under the identification $F_{a,r} \to F^r$, the fixed point condition becomes a ring condition on $M_n(F) \otimes F^r$.

We find that for any pair (D, θ) (and fixed ρ), there exists a formula $\varphi = \varphi_{D,\theta}$ in the Denef-Pas language with $1 + r^2 + n^2 r$ free variables such that for any p-adic field F $\varphi(a, \tau, g)$ is true exactly when a is a unit such that $x^r - a$ is irreducible over F, $F_{a,r} = F[x]/(x^r - a)$, $\tau \in M_r(F)$ is a generator of the Galois group of the field extension $F_{a,r}/F$ (under the identification $F_{a,r} = F^r$), and $g \in M_n(F) \otimes F^r$ is identified with an element $g \in G(F) \subset M_n(F_{a,r})$ (under the identification of F^r with $F_{a,r}$ determined by a).

We stress that this construction differs from the usual global to local specialization. If we take a unitary group defined over $\mathbb{Q}$ with conjugation from a quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, then the corresponding p-adic group at completions of $\mathbb{Q}$ can be split or inert depending on the prime. By making a (or d) a parameter to the formula, we can insure that the formula $\varphi_{D,\theta}$ defines the unramified unitary group *at every finite place*, and not merely at the inert primes.

By a similar construction, we obtain formulas in the Denef-Pas language for unramified reductive Lie algebras. If the group splits over a non-trivial unramified extension, there will be corresponding parameters, a and τ.

5. Galois Cohomology

5.1. Cocycles. Once we have expressed field extensions and Galois groups within the Denef-Pas language, we may do some rudimentary Galois cohomology.

We follow various conventions when working with Galois cohomology groups. We never work directly with the cohomology groups. Rather, we represent each class in a cohomology group $H^r(\mathrm{Gal}(E/F), A)$ explicitly as a cocycle

$$b \in Z^r(\mathrm{Gal}(E/F), A),$$

viewed as a tuple $b = (b_1, b_2, \ldots)$ of elements of A, indexed by a natural number $i = 1, \ldots, \mathrm{card}(\mathrm{Gal}(E/F))$, for a given enumeration $\sigma_1, \sigma_2, \ldots$ of the elements of the Galois group. The group law of the Galois group is not given; it is to be encoded into the formula for the cocycle condition:

$$(\sigma_i \sigma_j = \sigma_k) \implies (b_i \sigma_i(b_j) = b_k) \text{ for } i, j, k = 1, \ldots, \mathrm{card}(\mathrm{Gal}(E/F)).$$

We express that two cocycles b, b' are cohomologous by means of an existential quantifier: there exists a coboundary c such that $b = b'c$ (as tuples with componentwise multiplication).

The module A will always be one that can be expressed directly in the Denef-Pas language. For example, A may be a free $\mathbb{Z}$-module of finite rank, a definable subgroup of $\mathrm{GL}_n(E)$, or a set of elements in the coordinate ring $\mathbb{Q}[M_n]$ of the space of $n \times n$ matrices. Similarly, we restrict ourselves to actions of $\mathrm{Gal}(E/F)$ on modules A that can be expressed in our first-order language.

For example, if T is a torus that splits over an extension E/F of p-adic fields, instead of the group $H^1(F, T)$, we work with the group of cocycles $Z^1(\mathrm{Gal}(E/F), T(E))$, where $T(E)$ is represented explicitly as an affine algebraic group of invertible n by n matrices. Instead of $H^1(F, X^*(T))$, we work with $Z^1(\mathrm{Gal}(E/F), X^*(T))$, where $X^*(T)$ is viewed as a subset of the coordinate ring of T. Even more concretely, we may view T as a closed subset of M_n for some n, and represent elements of the coordinate ring of T as polynomials in n^2 variables.[1] The cup-product of $b \in Z^1(\mathrm{Gal}(E/F), T(E))$ with $p \in Z^1(\mathrm{Gal}(E/F), X^*(T))$ into $Z^2(\mathrm{Gal}(E/F), E^\times)$ is

[1] Since $X^*(T)$ is a free $\mathbb{Z}$-module of finite rank, it might be tempting to represent $X^*(T)$ within the value group sort of the Denef-Pas language. This is the wrong way to proceed!

the tuple whose coordinates are $p_{\sigma'}(b_\sigma)$, where the pairing is obtained by evaluating polynomials $p_{\sigma'}$ at values b_σ.

5.2. Tate-Nakayama pairing. The canonical isomorphism of the Brauer group

$$H^2(F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z} \tag{5.2.1}$$

relies on the Frobenius automorphism of unramified field extensions of F. We avoid the Frobenius automorphism and work with an explicitly chosen generator τ of an unramified extension.

The Tate-Nakayama pairing

$$H^1(F, T) \times H^1(F, X^*(T)) \to \mathbb{Q}/\mathbb{Z} \tag{5.2.2}$$

can be translated into the Denef-Pas language as a collection of predicates

$$\mathrm{TN}_{\ell,k}(a, \tau, b, p)$$

with free variables

$$b \in Z^1(\mathrm{Gal}(E/F), T(E)), \quad p \in Z^1(\mathrm{Gal}(E/F), X^*(T)),$$

and a generator τ of an unramified field extensions $F_{a,r}/F$, defined by parameter a. The extension E/F is assumed to be Galois, and r must be sufficiently large with respect to the degree of the extension E/F. The collection of predicates are indexed by $\ell, k \in \mathbb{N}$, with $k \neq 0$.

The predicate $\mathrm{TN}_{\ell,k}(a, \tau, b, p)$ asserts that the Tate-Nakayama cup product pairing of b and p is the class $\ell/k \in \mathbb{Q}/\mathbb{Z}$ in the Brauer group. In more detail, we construct the predicate by a rather literal translation of the Brauer group isomorphism into the language of Denef and Pas. We review Tate-Nakayama, to make it evident that the Denef-Pas language is all that is needed. There is no harm in assuming that $F_{a,r} \subset E$. Let $c \in Z^2(\mathrm{Gal}(E/F), E^\times)$ be the cup-product of b and p. There exists a coboundary d and a cocycle $c' \in Z^2(\mathrm{Gal}(F_{a,r}/F), F_{a,r}^\times)$ such that $c = c''d$, where c'' is the inflation of c' to $\mathrm{Gal}(E/F)$. Set $A = Z^2(\mathrm{Gal}(F_{a,r}/F), \mathbb{Z})$. Let $\mathrm{ord}(c') \in A$ be the tuple obtained by applying the valuation to each coordinate. Associated with the short exact sequence

$$1 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 1 \tag{5.2.3}$$

is a connecting homomorphism

$$\mathrm{Hom}(\mathrm{Gal}(F_{a,r}/F), \mathbb{Q}/\mathbb{Z}) = Z^1(\mathrm{Gal}(F_{a,r}/F), \mathbb{Q}/\mathbb{Z}) \to A. \tag{5.2.4}$$

Let $e(\ell, k)$ be the image in A of the homomorphism from $\mathrm{Gal}(F_{a,r}/F)$ to $\mathbb{Q}/\mathbb{Z}$ that sends the chosen generator τ to ℓ/k. (The predicate $\mathrm{TN}_{\ell,k}$ is defined to be false if k does not divide $r\ell$.) Finally, the predicate $\mathrm{TN}_{\ell,k}$ asserts that $e(\ell, k) = \mathrm{ord}(c') \in A$.

6. Weights

6.1. The weight function. Weighted orbital integrals can also be brought into the framework of constructible functions.

Let G be a connected reductive group over a p-adic field F and let M be a Levi subgroup of G. Let $\mathcal{P}(M)$ be the set of parabolic subgroups P of G that have a Levi decomposition $P = M_P N_P$ with $M = M_P$.

Arthur defines a real-valued weight function $w_M(x) = w_M^G(x)$ on the group $G(F)$. We recall the general form of this function. The function is defined as the value at $\lambda = 0$ of a smooth function

$$v_M(x,\lambda) = \sum_{P\in\mathcal{P}(M)} v_P(x,\lambda)\theta_P(\lambda)^{-1}, \tag{6.1.1}$$

whose terms are indexed by $P \in \mathcal{P}(M)$.

The parameter λ lies in a finite-dimensional real vector space $i\mathbf{a}_M^*$, where $\mathbf{a}_M^*$ is the dual of

$$\mathbf{a}_M = \mathrm{Hom}(X(M)_{rat}, \mathbb{R}), \tag{6.1.2}$$

and where $X(M)_{rat}$ is the group of F-rational characters of M. The function

$$\theta_P(\lambda) \in S^p[\mathbf{a}_{M,\mathbb{C}}] \tag{6.1.3}$$

is a nonzero homogeneous polynomial of some degree p. The degree p of this form is independent of $P \in \mathcal{P}(M)$.

The factor $v_P(x,\lambda)$ is defined as follows. We may assume a choice of a hyperspecial maximal compact subgroup K of $G(F)$ such that is admissible in the sense of Arthur [3, p.9]. Then the Iwasawa decomposition takes the form

$$G(F) = N_P(F)M_P(F)K. \tag{6.1.4}$$

Then for $x = nmk \in G(F)$, set $H_P(x) = H_M(m)$, where $H_M(m)$ is defined by the condition

$$\langle H_M(m), \chi\rangle = -\mathrm{ord}\chi(m) \tag{6.1.5}$$

for all $\chi \in X(M)_{rat}$. The function $v_P(\lambda, x)$ is then defined to be

$$v_P(\lambda, x) = e^{-\lambda(H_P(x))}. \tag{6.1.6}$$

The function $v_M(\lambda, x)$ defined by Equation 6.1.1 is not obviously a smooth function of $\lambda \in i\mathbf{a}_M^*$, because the individual summands $v_P(\lambda, x)/\theta_P(\lambda)$ do not extend continuously to $\lambda = 0$. But according to a theorem of Arthur, it is smooth.

Arthur gives the function in the following alternative form. Fix any generic λ. Let t be a real parameter. The denominator is homogeneous of degree p, so $\theta_P(t\lambda) = t^p\theta_P(\lambda)$. We compute the limit of $v_M(t\lambda, x)$ as t tends to zero by applying l'Hôpital's rule p times. The result is

$$v_M(x) = \sum_{P\in\mathcal{P}} \frac{(-1)^p(\lambda(H_P(x)))^p}{p!\theta_P(\lambda)}. \tag{6.1.7}$$

The right-hand side appears to depend on λ, but in fact it is constant as function of λ.

6.2. Weights and constructible functions. Let us recall our context for unramified groups. Associated to root data D and an automorphism θ of D that preserves positive roots, there is a formula $\varphi_{D,\theta}(a,\tau,g)$ in $1 + r^2 + n^2r$ variables where a determines an unramified field extension $F_{a,r}/F$ of degree r, τ is a generator of the Galois group of $F_{a,r}/F$, and g is an element of the quasi-split group determined by the root data D, automorphism θ, field extension $F_{a,r}/F$, and τ. This is a definable subassignment $\tilde{G} = \tilde{G}_{D,\theta}$.

For each p-adic field F, a and τ, the definable subassignment $\tilde{G} = \tilde{G}_{D,\theta}$, gives a reductive group G and a hyperspecial maximal compact subgroup K (by taking the integer points of G).

We pick Levi factors in standard position in the usual way. We fix a splitting $(B, T, \{X_\alpha\})$ of the split group G as in Section 4. We may take parabolic subgroups P containing B and Levi factors M generated by T and by a subset of the roots vectors $\{X_\alpha \mid \alpha \in S\}$, with S a subset of the set of simple roots. For each subset S of simple roots, we have a definable subassignment $\tilde{M}_S \subset \tilde{G}$, defined by a formula $\varphi_{S,D,\theta}(a, \tau, m)$, with a and τ as before, and m constrained to be an element of the Levi factor $M_{S,a,\tau}(F)$ of the reductive group $G_{a,\tau}$ attached to D, θ, a, τ. If S is the set of all simple roots, then $\tilde{M}_S = \tilde{G}$.

Lemma 6.2.1. *Let (D, θ) be root data and an automorphism as above. For each subset S of the simple roots, There is a constructible function u_S on $\tilde{G}_{D,\theta}$ such that for for any p-adic field F,*

$$u_S(a, \tau, g) = v_M(g), \tag{6.2.1}$$

where $M = M_{S,a,\tau}$.

Proof. Constructible functions form a ring. Therefore if we express u_S as a polynomial in functions that are known to be constructible, then it follows that u_S itself is constructible.

The character group $X(M)_{rat}$ is independent of the field F. The coefficients of the form θ_P are independent of the field F. We may compute v_M from Equation 6.1.7 with respect to any sufficiently generic λ. In particular, we may choose $\lambda \in X(M)_{rat}$, once for all fields F. Then $\theta_P(\lambda)$ is a non-zero number that does not depend on F.

By definition, the function H_M on $M(F)$ is a vector of valuations of polynomial expressions in the matrix coefficients of m. The linear form $\lambda(H_M(m))$ is then clearly a constructible function.

Consider the function $\lambda(H_P(g)) = \lambda(H_M(m))$, where $g = nmk$. If we add the parameters, a, τ, its graph is

$$\{((a, \tau, g), \ell) \in \tilde{G} \times \mathbb{Z} \mid \exists\, k \in K,\ m \in M, n \in N_P,\ g = nmk,\ \ell = \lambda(H_M(m))\}. \tag{6.2.2}$$

This is a definable subassignment. Hence,

$$(a, \tau, g) \mapsto \lambda(H_P(g)), \quad P = P_{a,\tau} \tag{6.2.3}$$

is a constructible function.

As the function $v_M(g)$ is constructed as a polynomial in $\lambda(H_P(g))$ with rational coefficients, as P ranges over $\mathcal{P}(M)$, it is now clear that it can be lifted to a constructible function $u_S(a, \tau, g)$, according to the description of constructible functions recalled in Section 2.3. □

7. Measures

7.1. Differential forms. Let G be a split reductive group over $\mathbb{Q}$. Let B be a Borel subgroup of G and T a Cartan subgroup contained in B. Let N be the unipotent radical of B and Let N' be the unipotent radical of the Borel subgroup opposite to B through T. The big cell TNN' is a Zariski open subset of G. The Haar measure of G is the measure attached to the differential form, expressed on the open cell as

$$\omega_G = d^*t \wedge dn \wedge dn'. \tag{7.1.1}$$

where d^*t, dn, dn' are differential forms of top degree on T, N, and N', which are bi-invariant by the actions of T, N, and N', respectively. The radical N (and N') can be identified with affine spaces and differential forms dn come from a choice of root vectors for the algebra N.

$$dn = dx_1 \wedge \cdots \wedge dx_k. \tag{7.1.2}$$

We pick the root vectors to respect the rational structure of G. If we identify T with a split torus $\mathbb{G}_m^r$, then d^*t takes the form of a multiplicatively invariant form

$$\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_r}{t_r}. \tag{7.1.3}$$

For any p-adic field, we obtain a Haar measure $|\omega_G|$ on $G(F)$ as the measure attached to the form ω_G.

Similarly, for an endoscopic group H, there is a Haar measure on $H(F)$ obtained from a similarly constructed differential form ω_H. According to calculations of Langlands and Shelstad of the Shalika germ associated with regular unipotent conjugacy classes ([27], [33]), the invariant measures on cosets $T\backslash G$ and $T\backslash H$ that are be used for the fundamental lemma have the form

$$|d_T t|\backslash|\omega_G|, \quad |d_T t|\backslash|\omega_H|, \tag{7.1.4}$$

where $|d_T t|$ is a Haar measure on T, considered as a Cartan subgroup of both G and H.

Lemma 7.1.1. *There is a definable subassignment $\tilde{G}_{ell}$ of regular semisimple elliptic elements of $\tilde{G}_{D,\theta}$. More generally, for each S subset of simple roots, defining a standard Levi subgroup, there is a definable subassignment $\tilde{G}_{S,ell}$ of regular semisimple elements of $\tilde{G}_{G,\theta}$ that are conjugate to an element of $\tilde{M}_S$ and that are elliptic in $\tilde{M}_S$. Similarly, there are definable subassignments $\tilde{\mathbf{g}}_{S,ell}$ of regular semisimple elements of $\tilde{g}$.*

Proof. For each a, τ, the constraints on g are that it is not conjugate to a standard Levi subgroup of $G_{a,\tau}$, that there is a field extension (of some fixed degree k) over which g can be diagonalized, and that the Weyl discriminant is nonzero. These conditions are all readily expressed as a formula in the Denef-Pas language. The proof for the other statements are similar. □

A conjugacy class is said to be bounded if each element γ in that conjugacy class belongs to a compact subgroup of $G(F)$. Arthur states the weighted fundamental lemma (at the group level) in terms of bounded conjugacy classes. Similarly, we have the following lemma:

Lemma 7.1.2. *There is a definable subassignment $\tilde{G}_{bd}$ of bounded semisimple conjugacy classes of $\tilde{G}_{D,\theta}$.*

7.2. Weyl integration formula. The Weyl integration formula can be used to fix normalizations of measures. By [34, page 36], the Weyl integration formula takes the form

$$\begin{aligned}\int_{\mathbf{g}} f(X)dX &= \textstyle\sum_T |W(G,T)|^{-1} \int_{\mathbf{t}} |D^{\mathbf{g}}(X)| \times \\ &\quad \int_{G/T} f(\mathrm{Ad}\, xX) dx\, dX.\end{aligned} \tag{7.2.1}$$

The measure dX is the Serre-Oesterlé measure, which is an additive Haar measure on $\mathbf{g}$. This is compatible with the invariant form ω_G on G in the sense of [12]. The sum runs over conjugacy classes of Cartan subgroups. The factor $|W(G,T)|$ is the order of the Weyl group for T; and $|D^{\mathbf{g}}(X)|$ is the usual discriminant factor. The measure dx is the quotient measure normalized by ω_G on G and dX on $\mathbf{t}$.

Let $\mathbf{c}_G = \mathrm{spec}\,(R^G)$, where R is the coordinate ring of $\mathbf{g}$, and R^G is the subring of G-invariants. We have a morphism $\mathbf{g} \to \mathbf{c}_G$, coming from the inclusion $R^G \subset R$. For $\mathbf{t} \subset \mathbf{g}$, the morphism $\mathbf{t} \to \mathbf{g} \to \mathbf{c}_G$ is $W(G,T)$-to-1 on the set of regular semisimple elements. The fiber over a regular semisimple element $\gamma \in \mathbf{c}_G$ is the stable orbit of γ in $\mathbf{g}$. Comparing this to the Weyl integration formula, we see that the choice of measure on fibers of the morphism $\mathbf{g} \to \mathbf{c}_G$ determined by the Serre-Oesterlé measures on $\mathbf{g}$ and on $\mathbf{c}_G$ is an invariant measure dx. Similar comments apply to endoscopic groups H of G.

These observations allow us to conclude that invariant measures on stable conjugacy classes are compatible with the general framework of [11].

8. The Langlands-Shelstad Transfer Factor

8.1. Langlands dual group. In this section we present a few facts related to the Langlands-Shelstad transfer factor. The Langlands-Shelstad transfer factor was originally defined for pairs of elements in a group G and a fixed endoscopic group. By considering the limiting behavior of this transfer factor near the identity element of the group, we obtain a transfer factor on the Lie algebra.

The fundamental lemma for the Lie algebra and the Lie group are closely related. See [16] for the relation in the unweighted case and the Appendix to [36] for a sketch of the relation in the full weighted case of the fundamental lemma.

We work with the Lie algebra version of the transfer factor. The Lie algebra transfer factor avoids the additional complications of multiplicative characters that occur in the Lie group transfer factor.

Let F be a p-adic field. We fix an unramified connected reductive group G over F with Lie algebra $\mathbf{g}$. The L-group of G may be written in the form

$$ {}^L G = \hat{G} \rtimes \Gamma, \tag{8.1.1}$$

where Γ is the Galois group of a splitting field of G. The group Γ acts by automorphisms of $\hat{G}$ that fix a splitting $(\hat{T}, \hat{B}, \{X_\alpha\})$.

8.2. The parameter s and endoscopy. Let (s,ρ) be an unramified endoscopic datum for G. The element s belongs to $\hat{G}/Z(\hat{G})$. Taking its preimage in $\hat{G}$ and replacing the endoscopic datum by another isomorphic to it, we may take $s \in \hat{T}$ to be a semisimple element, whose connected centralizer is defined to be $\hat{H}$. We may assume that s has finite order.

By definition $\rho : \mathrm{Gal}(F_{a,r}/F) \to \mathrm{Out}(\hat{H})$ is a homomorphism for some unramified extension $F_{a,r}/F$ into the group of outer automorphisms of H. See [24, sec.7] for a review of endoscopic data. The expository paper [17] gives some additional details in the context of the fundamental lemma.

Let T be a maximally split Cartan subgroup of H and $\hat{T}$ its dual, which we identify with the Cartan subgroup containing s. We may assume further that $s \in \hat{T}^{\mathrm{Gal}(F_{a,r}/F)}$. Let $\mathbf{t}$ be the Lie algebra of $\hat{T}$. The exponential short exact sequence

(8.2.1) $$1 \to X_*(\hat{T}) \to \mathbf{t} \to \hat{T} \to 1,$$

where $\mathbf{t} \to \hat{T}$ is the exponential map $\lambda \mapsto \exp(2\pi i\lambda)$, gives a connecting homomorphism

(8.2.2) $$\hat{T}^{\mathrm{Gal}(F_{a,r}/F)} \to H^1(F, X^*(T)).$$

The image of s is obtained explicitly as follows. Write $s = \exp(2\pi i\lambda/k)$ for some $\lambda \in X_*(T)$ and $k > 0$. Then the cocycle $\mu_\sigma \in Z^1(F, X_*(\hat{T}))$ is given by the equation

(8.2.3) $$k\mu_\sigma = (\sigma(\lambda) - \lambda), \quad \sigma \in \mathrm{Gal}(F_{a,r}/F).$$

The action of $\mathrm{Gal}(F_{a,r}/F)$ on $X_*(T)$ comes through the action of $W_H \rtimes \langle\theta\rangle$ on $X_*(T)$ by means of a homomorphism

(8.2.4) $$\mathrm{Gal}(F_{a,r}/F) \to W_H \rtimes \langle\theta\rangle$$

(where W_H is the Weyl group of H). Instead of the cocycle μ_σ, we prefer to work with the corresponding cocycle

(8.2.5) $$\mu_w \in Z^1(W_H \rtimes \langle\theta\rangle, X_*(\hat{T}))$$

(8.2.6) $$k\mu_w = (w(\lambda) - \lambda), \quad w \in W_H \rtimes \langle\theta\rangle.$$

From the point of view of definability in the Denef-Pas language, a complex parameter s is problematical. However, the parameter $\lambda \in X^*(T)$ presents no difficulties, so we discard s and work directly with λ and the fixed natural number k. We consider $X^*(T)$ as within the coordinate ring of T. Through an explicit matrix representation of H, for any $\mu \in X^*(T)$, we may choose a polynomial P_μ representing μ in the coordinate ring of $n \times n$ matrices. The polynomials satisfy $P_\mu P_{\mu'} = P_{\mu+\mu'} \mod I$, where I is the ideal of T.

Equation 8.2.6 can be rewritten as a collection of polynomial identities

(8.2.7) $$P_w^k = P_{w\lambda} P_{-\lambda} \mod I, \quad w \in W_H \rtimes \langle\theta\rangle,$$

where $P_w = P_{\mu_w}$. The collection of polynomials $\{P_w\}$ then serve as the proxy for the complex parameter s.

8.3. The normalization of transfer factors. Langlands and Shelstad defined the transfer factor, up to a scalar factor. There is a unique choice of scalar factor for the transfer factor that is compatible with the fundamental lemma. Following Kottwitz, this normalization of the transfer factor is based on a Kostant section associated to a regular nilpotent element. Let $\mathbf{g}$ be the Lie algebra of a split reductive group G. Let $\mathbf{c}_G$ be the space defined in Section 7. There is a natural morphism $\mathbf{g} \to \mathbf{c}_G$. Kostant defines a section $\mathbf{c}_G \to \mathbf{g}$ of this morphism [22]. The construction of this section depends on a choice X of regular nilpotent element. Kostant's construction, which is based on $\mathbf{sl}_2$ triples, can easily be carried out in the context of the Denef-Pas language. In fact, once the element X is given, the construction requires only the elementary theory of rings.

We fix a regular nilpotent element as follows. Fix a splitting $(\mathbf{b}, \mathbf{t}, \{X_\alpha\})$ of $\mathbf{g}$ defined over $\mathbb{Q}$. Pick $X \in \mathbf{b}$ such that

$$X = \sum x_\alpha X_\alpha, \tag{8.3.1}$$

where x_α is a unit for every simple root α.

8.4. The pairing. The Lie algebra version of the Langlands-Shelstad transfer factor $\Delta(\gamma_H, \gamma)$ is defined for pairs $\gamma \in \mathbf{g}$ and $\gamma_H \in \mathbf{c}_H$. It has the form $q^{m(\gamma_H,\gamma)}\Delta_0(\gamma_H, \gamma)$, where Δ_0 is a root of unity, or zero. (It is defined to be zero on the set where γ_H, γ are not matching elements.)

Normalize the transfer factor so that $\Delta_0(\gamma_H, \gamma) = 1$ if γ lies in the Kostant section associated to X. By [16], this normalization is independent of the choice of such X (for sufficiently large primes, as usual). Thus, we may take the quantifier over X to be either an existential or a universal quantifier, ranging over all such nilpotent elements.

Let $\tilde{\mathbf{c}}_H \times \tilde{\mathbf{g}}_{D,\theta}$ be the definable subassignment with free variables $(a, \tau, \gamma_H, \gamma)$ with defining condition that γ belongs to the Lie algebra of $\tilde{G}_{D,\theta,a,\tau}$ and γ_H belongs to the quotient $\mathbf{c}_H$ for the corresponding Lie algebra of its endoscopic group $H_{a,\tau}$.

Lemma 8.4.1. *There is a constructible function on $\tilde{\mathbf{c}}_H \times \tilde{\mathbf{g}}_{D,\theta}$ that specializes to the function*

$$q^{m(\gamma_H,\gamma)}. \tag{8.4.1}$$

Proof. This is trivial. The constructible function $\mathbb{L}$ specializes to q. The ring of constructible functions contains functions of the form $\mathbb{L}^m$, with

$$m : \tilde{\mathbf{h}} \times \tilde{\mathbf{g}}_{D,\theta} \to \mathbb{Z} \tag{8.4.2}$$

definable. The function m is definable, because it is the valuation of a polynomial in the matrix coefficients of γ and γ_H. □

Let $\mathbf{k}$ be a lattice in $\mathbf{g}$ corresponding to a hyperspecial maximal compact subgroup of G. Let $\mathbf{k}_H$ be such a lattice in H. Let γ be a regular semi-simple element in $\mathbf{g}$. Assume that it is the image of some $\gamma_H \in \mathbf{k}_H$. Let $\gamma_0 \in \mathbf{g}$ be an element in the Kostant section with the same image as γ in $\mathbf{c}$. Let $\mathrm{inv}(\gamma_0, \gamma)$ be the invariant attached to γ and κ the character defined by the endoscopic data so that

$\Delta_0(\gamma_H, \gamma) = \langle \mathrm{inv}(\gamma_H, \gamma), \kappa \rangle$. The construction depends on an element $g \in G(\bar{F})$ such that Ad $g(\gamma) = \gamma_0$.

8.5. Description of the transfer factor. We arrive at the following statements that summarize the value of the transfer factor for elliptic unramified transfer factors. The description we give is rather verbose. In particular, we write out the coboundaries explicitly rather than taking classes in cohomology. This is intentional to prepare us for the proof that the transfer factor can be expressed in the Denef-Pas language.

Recall that a natural number k is used to define the complex parameter s in terms of λ. The value of the transfer factor is a k^{th} root of unity. For $\ell \in \mathbb{N}$, let $D_\ell(a, \tau, \gamma_H, \gamma_G)$ be the set of elements in $\mathbf{h} \times \mathbf{g}$ such that γ_H is a G-regular semisimple element of $\mathbf{c}_H$, γ_G is a regular semisimple element of $\mathbf{g}$, and $\Delta_0(\gamma_H, \gamma_G) = \exp(2\pi\ell/k)$.

We fix a natural number r that is large enough that the unramified extension $F_{a,r}$ of F of degree r splits G and H, etc.

We fix a natural number $N = N_r$ that is large enough that for every Cartan subgroup in G there exists a field extension of degree dividing N containing $F_{a,r}$ that splits the Cartan subgroup. As we are only interested in the tame situation, we may assume that the residue characteristic p does not divides N.

Let $T_H \subset G$ be the fixed Cartan subgroup of G, obtained by transfer of the maximally split Cartan subgroup of H. $\Delta_0(\gamma_H, \gamma) = 0$ if and only if γ and γ_H do not have the same image in $\mathbf{c}_G$.

The set $D_\ell(a, \tau, \gamma_H, \gamma)$ is described by the following list of conditions. The parameters a and τ are as above.

(1) Let γ_0 be the element in $\mathbf{g}$, constructed as the transfer of γ_H to $\mathbf{g}$, lying in the Kostant section we have fixed.
(2) There exists a Galois field extension of degree E/F of degree N that contains a subfield isomorphic to $F_{a,r}$.
(3) There exists $g \in G(E)$ such that Ad $g(\gamma) = \gamma_0$. Let $t_\sigma = g^{-1}\sigma(g) \in T_0(E)$, for $\sigma \in \mathrm{Gal}(E/F)$, where T_0 is the centralizer of γ_0.
(4) There exists $h \in G(E)$ such that $\mathrm{ad}(h)T_0 = T_H$. Let $t'_\sigma \in Z^1(E/F, T_H^*)$ be the cocycle $\mathrm{ad}(h)t_\sigma$, with a twisted action of $\mathrm{Gal}(E/F)$ on T_H obtained by transporting the action of $\mathrm{Gal}(E/F)$ to T_H via $\mathrm{ad}(h)$. The general properties of endoscopy imply that the cocycle $Z^1(F_{a,r}/F, X^*(T_H))$, defined by the collection of polynomials P_w, also yields a cocycle $\mu_\sigma \in Z^1(E/F, X^*(T_H^*))$ for this twisted action.
(5) The predicate $\mathrm{TN}_{\ell,k}(a, \tau, t'_*, \mu_*)$ of Section 5 expressing the Tate-Nakayama duality holds.

This description of the transfer factor is equivalent to descriptions found elsewhere. We have changed the presentation slightly by working with the polynomials P_w rather than the complex parameter. Beyond that, our description is essentially the standard one. We have written the transfer factor in this form to make it apparent that nothing beyond first-order logic, basic ring arithmetic, and a valuation map are required.

We have provided justification for the following statement. It relies on the fixed natural number k that is part of our setup.

Lemma 8.5.1. *There is a definable subassignment $D_\ell(a, \tau, \gamma_H, \gamma_G)$ with the following interpretation. The parameter a defines an unramified field extension $F_{a,r}$ of degree r, and τ is a generator of $\mathrm{Gal}(F_{a,r}/F)$. The parameter γ_H ranges over G-regular elements in $\mathbf{c}_H = \mathbf{h}^H$ of the Lie algebra $\mathbf{h}$ of the endoscopic group $H_{a,\tau}$. The parameter γ_G is a regular semisimple element in the Lie algebra $\mathbf{g}$ of the reductive group $G_{a,\tau}$. The elements γ_H and γ_G are matching elements, and the transfer factor takes the value $\Delta_0(\gamma_H, \gamma_G) = \exp(2\pi i\ell/k)$.*

We remark that the definition of D_ℓ can be described in a smaller language with two sorts, one for the valued field and another for the value group. The residue field sort and the angular component map do not enter into the description of D_ℓ.

Moreover, we have only made light use of the valuation. It is used once to fix the choice of nilpotent element that is used to construct the Kostant section. The valuation is used once again in the predicate $\mathrm{TN}_{\ell,k}$ when working with the Brauer group.

Proof. This follows directly from the description of the transfer factor in terms of ring operations and quantifiers, as provided. □

We have not made explicit estimates of the complexity of the formulas in the Denef-Pas language that define the subassignments D_ℓ. For any n, we may define $q(n)$ to be the smallest integer q such that every subassignment D_ℓ of every unramified group of rank at most n can be defined by a formula in the Denef-Pas language that contains at most q quantifiers. How rapidly does $q(n)$ grow with n?

8.6. Weighted orbital integrals. Let G be an unramified reductive group with hyperspecial maximal compact subgroup K. Let M be a Levi subgroup that is in good position with respect to K. Let M' be an unramified elliptic endoscopic group of M. Write $\mathbf{g}$, $\mathbf{k}$, $\mathbf{m}$, and so forth, for the corresponding Lie algebras. Let 1_K be the normalized unit of the Hecke algebra. The normalization appropriate for our choice of measures is discussed in [17].

Let $\ell' \in \mathbf{c}_{M'}$ be a G-regular element. The image ℓ_M of ℓ' in $\mathbf{c}_M$ determines a stable conjugacy class $C_M(\ell')$ in $\mathbf{m}$, given as the fiber in $\mathbf{m}$ over $\ell_M \in \mathbf{c}_M$. The image ℓ_G of ℓ' in $\mathbf{c}_G$ gives determines a stable conjugacy class $C_G(\ell')$ in $\mathbf{g}$. Any element $x \in C_G(\ell')$ permits a representation

$$x = \mathrm{Ad}\; g\,\gamma \tag{8.6.1}$$

with $g \in G(F)$ and $\gamma \in C_M(\ell')$. The map

$$x \mapsto (\gamma_x, g_x) = (\gamma, g) \tag{8.6.2}$$

is well-defined up to conjugacy in $M(F)$:

$$(\gamma_x, g_x) \mapsto (\mathrm{Ad}\; m\,\gamma_x, g_x m^{-1}). \tag{8.6.3}$$

In particular, the function

$$x \mapsto \Delta_{M',M}(\ell', \gamma_x) v_M(g_x) 1_K(x) \tag{8.6.4}$$

is well-defined, where $\Delta_{M',M}$ is the normalized transfer factor for reductive group M and endoscopic group M'. We have already established the constructibility of Δ and v_M. The $\mathscr{F}$unction (8.6.4) involves no more than Δ, v_M and additional existential quantifiers of the valued field sort, corresponding to the existence of the representation of Equation (8.6.1). In particular, $\mathscr{F}$unction (8.6.4) is constructible. (Here and elsewhere, we are slightly loose in our use of the term constructible. The precise sense of constructibility is stated in Lemmas 6.2.1 and 8.5.1.)

Define the weighted orbital integral to be

$$J^G_{M,M'}(\ell') = \int_{C_G(\ell')} \Delta_{M',M}(\ell', \gamma_x) v_M(g_x) 1_K(x), \tag{8.6.5}$$

where the integral is with respect to the quotient of the Serre-Oesterlé measures on $\mathbf{m}$ and $\mathbf{c}_M$.

8.7. The example $SL(n)$. As an example, we describe the constructible function for the transfer factor of $SL(n)$, when n factors as $n = mr$. Let E/F be an unramified extension of p-adic fields of degree r. The group $SL(n)$ over F has an endoscopic group H, where $H(F)$ is the subgroup of $GL(m, E)$ of elements whose determinant has norm 1. A basis of E/F gives an embedding of H into $SL(n)$. For the fundamental lemma, we are concerned with the transfer factor evaluated at an element in $SL(n)$ that is the image of some $\gamma \in GL(m, O_E)$, under this embedding.

Suppose that f and g are monic polynomials with roots $x_1, \dots, x_m$ and $y_1, \dots, y_{m'}$, respectively in an algebraic closure. The resultant $R(f, g)$ of f and g is the product $\prod_{i,j}^{m,m'}(x_i - y_j)$. If γ_1, γ_2 are elements of Mat_ℓ, then let $R(\gamma_1, \gamma_2) = R(f_{\gamma_1}, f_{\gamma_2})$, where f_γ is the characteristic polynomial of γ. For $\gamma \in \mathrm{Mat}_m(O_E)$, define

$$\Delta^{m,1}_{E/F}(\gamma) = \Big| \prod_{\sigma\neq\tau\in \mathrm{Gal}(E/F)} R(\sigma(\gamma), \tau(\gamma))\Big|^{\frac{1}{2}}.$$

When r is odd, define $\Delta^{m,2}_{E/F}(\gamma) = 1$. When r is even, let σ_+ denote the element of order 2 in $\mathrm{Gal}(E/F)$. Let η_E denote the unramified character of $E^\times$ of order 2. Define

$$\Delta^{m,2}_{E/F}(\gamma) = \eta_E\Big(R(\gamma, \sigma_+(\gamma))\Big).$$

Let $\Delta^m_{E/F}(\gamma) = \Delta^{m,1}_{E/F}(\gamma)\,\Delta^{m,2}_{E/F}(\gamma)$. By [15], this is the transfer factor (on the group).

To express the set

$$D_\pm = \{\gamma \mid \Delta^{m,2}_{E/F}(\gamma) = \pm 1\}$$

(when r is even) as a formula in the Denef-Pas language, pick a unit a such that $x^r - a$ is irreducible, and write elements of an unramified extension of degree r as linear combinations of $\{1, x, \dots, x^{r-1}\}$. The automorphism σ_+ is given by $x \mapsto -x$. The element γ is a matrix

$$\gamma = (\gamma_{ij}), \quad \gamma_{ij} = \sum_{k=0}^{r-1} \gamma_{ijk} x^k.$$

The resolvant is a determinant formed from the coefficients γ_{ij} of the characteristic polynomial of γ. Since E/F is unramified, $\eta_E(x)$ is 1, exactly when the valuation of the norm $N_{E/F}x \in F^\times$ is a multiple of $2r$. Furthermore, the norm of the resolvant is a polynomial $n(\gamma_{ijk})$ in the coefficients γ_{ijk}. We obtain a formula in the Denef-Pas language of the form

$$D_+ = \{\gamma = (\gamma_{ijk}) \mid \exists l \in \mathbb{Z}. \quad \mathrm{ord}\,(n(\gamma_{ijk})) = 2rl\}$$

The formula for D_- is obtained similarly. The function

$$\Delta^{m,1}_{E/F}(\gamma) = q^{e(\gamma)}$$

may appear to have a fractional exponent, but in fact, it is an integer. Our description of $\Delta^{m,1}_{E/F}$ leads to a similar expression for $e(\gamma)$ as the valuation of a polynomial in the coefficients γ_{ijk}.

9. The statement of the Fundamental Lemma

9.1. Weighted fundamental lemma. In [4, Conj. 5.1], Arthur conjectures a weighted form of the fundamental lemma. In this section, we review his formulation of the conjecture. The weighted case includes the standard fundamental lemma of Langlands and Shelstad as a special case. (For the formulation of the fundamental lemma in the twisted weighted context, see Arthur's appendix to [37].)

We continue to work with the Lie algebras of endoscopic groups rather than the endoscopic groups themselves, although this makes very little difference for our purposes.

Our work is now almost complete. We have already described all of the constituents of the fundamental lemma.

Let G be an unramified reductive group with hyperspecial maximal compact subgroup K. Let M be a Levi subgroup that is in good position with respect to K. Let M' be an unramified elliptic endoscopic group of M. Write **g**, **k**, **m**, and so forth, for the corresponding Lie algebras.

To state the weighted fundamental lemma, we need the following additional data. For each G, M, M', there is a set of endoscopic data $\mathcal{E}_{M'}(G)$, defined in [2, Sec. 4]. The set $\mathcal{E}_{M'}$ is defined by data in the dual group that is independent of the p-adic field. (The action of the Galois group Γ on the dual group data can be replaced with the action of the automorphism θ.) For each, $G' \in \mathcal{E}_{M'}(G)$, there is a rational number number $\iota_{M'}(G, G') \in \mathbb{Q}$, that is independent of the p-adic field.

We define a function $s^G_M(\ell)$ recursively. Assume that $s^{G'}_M$ has been defined for all G' (with Levi subgroup M) such that $\dim G' < \dim G$. Then, set

$$s^G_M(\ell) = J^G_{M,M}(\ell) - \sum_{G' \neq G} \iota_M(G, G') s^{G'}_M(\ell). \tag{9.1.1}$$

The sum runs over $\mathcal{E}_M(G) \setminus \{G\}$. This definition is coherent, because each group $G' \in \mathcal{E}_M(G)$ has M as a Levi subgroup, so that $s^{G'}_M$ is defined.

The conjecture of the weighted fundamental lemma is then that for all G, M, M' as above, we have

$$J^G_{M,M'}(\ell') = \sum_{G'} \iota_{M'}(G, G') s^{G'}_{M'}(\ell'). \tag{9.1.2}$$

for all G-regular elements ℓ' in $\mathbf{c}_{M'}$. The sum on the right runs over $\mathcal{E}_{M'}(G)$.

9.2. Constructibility. By our preceding discussion, we see that the integrand (Equation 8.6.4) of $J^G_{M,M'}(\ell')$ comes as specialization of a constructible function on the definable subassignment

$$Z = \tilde{\mathbf{c}}_H \times_{\tilde{\mathbf{c}}_G} \tilde{\mathbf{g}}_{D,\theta}. \tag{9.2.1}$$

This constructible function depends on parameters a, τ, $\gamma_H \in \tilde{\mathbf{c}}_H$, and $\gamma \in g_{D,\theta,a,\tau}$. If we interpret this p-adically, as we vary the parameter a (under the restriction that it is a unit), the situation specializes to isomorphic groups and Lie algebras. In particular, the fundamental lemma holds for one specialization of a if and only if it holds for all specializations of a. As we vary the generator τ of the Galois group of the unramified field extension $F_{a,r}/F$, we may obtain non-isomorphic data. Different choices of τ correspond to the fundamental lemma for various Lie algebras

$$\mathbf{g}_{D,\theta'} \tag{9.2.2}$$

where θ' and θ generate the same group $\langle\theta'\rangle = \langle\theta\rangle$ of automorphisms of the root data. In particular, for each τ, the constructible version specializes to a version of the p-adic fundamental lemma for Lie algebras.

9.3. The main theorem. We state the transfer principle for the fundamental lemma as two theorems, once in the unweighted case and again in the weighted case. In fact, there is no needed for us to treat these two cases separately; they are both a consequence of the general transfer principle for the motivic integrals of constructible functions given in Theorem 2.8.3. We state them as separate theorems, only because of preprint of Ngô [30], which applies directly to the unweighted case of the fundamental lemma.

Theorem 9.3.1 (Transfer Principle for the Fundamental Lemma)**.** *Let (D,θ) be given. Suppose that the fundamental lemma holds for all p-adic fields of sufficiently large positive characteristic for the endoscopic groups attached to (D,θ'), as θ' ranges over automorphisms of the root data such that $\langle\theta'\rangle = \langle\theta\rangle$. Then, the fundamental lemma holds for all p-adic fields of characteristic zero with sufficiently large residual characteristic p (in the same context of all endoscopic groups attached to (D,θ')).*

Theorem 9.3.2 (Transfer Principle for the weighted Fundamental Lemma)**.** *Let (D,θ) be given. Suppose that the weighted fundamental lemma holds for all p-adic fields of sufficiently large positive characteristic for the endoscopic groups attached to (D,θ'), as θ' ranges over automorphisms of the root data such that $\langle\theta'\rangle = \langle\theta\rangle$. Then, the weighted fundamental lemma holds for all p-adic fields of characteristic zero with sufficiently large residual characteristic p (in the same context of all endoscopic groups attached to (D,θ')).*

Proof. We have successfully represented all the data entering into the fundamental lemma within the general framework of identities of motivic integrals of constructible functions. By the transfer principle given in Theorem 2.8.3, the fundamental lemma holds for all p-adic fields of characteristic zero, for sufficiently large primes p. □

By the main result of [18], the unweighted fundamental lemma holds for all elements of the Hecke algebra for all p, once it holds for all sufficiently large p (for a collection of endoscopic data obtained by descent from the original data (D,θ)). Thus, in the unweighted situation, we can derive the fundamental lemma for all local fields of characteristic zero, without restriction on p, once the fundamental lemma is known for a suitable collection of cases in positive characteristic.

10. Additive characters and the relative fundamental lemma

10.1. Adding exponentials. It is also possible to enlarge $\mathscr{C}(X)$ to a ring $\mathscr{C}(X)^{\exp}$ also containing motivic analogues of exponential functions and to construct a natural extension of the previous theory to $\mathscr{C}^{\exp}$.

This is performed as follows in [10] [11]. Let X be in Def_k. We consider the category $\mathrm{RDef}_X^{\exp}$ whose objects are triples $(Y \to X, \xi, g)$ with Y in RDef_X and $\xi : Y \to h[0,1,0]$ and $g : Y \to h[1,0,0]$ morphisms in Def_k. A morphism $(Y' \to X, \xi', g') \to (Y \to X, \xi, g)$ in $\mathrm{RDef}_X^{\exp}$ is a morphism $h : Y' \to Y$ in Def_X such that $\xi' = \xi \circ h$ and $g' = g \circ h$. The functor sending Y in RDef_X to $(Y, 0, 0)$, with 0 denoting the constant morphism with value 0 in $h[0,1,0]$, resp. $h[1,0,0]$ being fully faithful, we may consider RDef_X as a full subcategory of $\mathrm{RDef}_X^{\exp}$. To the category $\mathrm{RDef}_X^{\exp}$ one assigns a Grothendieck ring $K_0(\mathrm{RDef}_X^{\exp})$ defined as follows. As an abelian group it is the quotient of the free abelian group over symbols $[Y \to X, \xi, g]$ with $(Y \to X, \xi, g)$ in $\mathrm{RDef}_X^{\exp}$ by the following four relations

$$[Y \to X, \xi, g] = [Y' \to X, \xi', g'] \tag{10.1.1}$$

for $(Y \to X, \xi, g)$ isomorphic to $(Y' \to X, \xi', g')$,

$$\begin{aligned}&[(Y \cup Y') \to X, \xi, g] + [(Y \cap Y') \to X, \xi_{|Y\cap Y'}, g_{|Y\cap Y'}]\\ &\qquad = [Y \to X, \xi_{|Y}, g_{|Y}] + [Y' \to X, \xi_{|Y'}, g_{|Y'}]\end{aligned} \tag{10.1.2}$$

for Y and Y' definable subassignments of some W in RDef_X and ξ, g defined on $Y \cup Y'$,

$$[Y \to X, \xi, g + h] = [Y \to X, \xi + \overline{h}, g] \tag{10.1.3}$$

for $h : Y \to h[1,0,0]$ a definable morphism with $\mathrm{ord}(h(y)) \geq 0$ for all y in Y and $\overline{h}$ the reduction of h modulo t, and

$$[Y[0,1,0] \to X, \xi + p, g] = 0 \tag{10.1.4}$$

when $p : Y[0,1,0] \to h[0,1,0]$ is the projection and when $Y[0,1,0] \to X$, g, and ξ factorize through the projection $Y[0,1,0] \to Y$. Fiber product endows $K_0(\mathrm{RDef}_X^{\exp})$ with a ring structure.

Finally, one defines the ring $\mathscr{C}(X)^{\exp}$ of exponential constructible functions as $\mathscr{C}(X)^{\exp} := \mathscr{C}(X) \otimes_{K_0(\mathrm{RDef}_X)} K_0(\mathrm{RDef}_X^{\exp})$. One defines similarly $C(X)^{\exp}$ and $\mathrm{I}_S C(X)^{\exp}$.

In [10] [11], given S in Def_k, we construct for a morphism $f : X \to Y$ in Def_S a push-forward $f_! : \mathrm{I}_S C(X)^{\exp} \to \mathrm{I}_S C(Y)^{\exp}$ extending $f_! : \mathrm{I}_S C(X) \to \mathrm{I}_S C(Y)$

and characterized by certain natural axioms. In particular, the construction of the measure μ and its relative version μ_Λ extend to the exponential setting.

10.2. Specialization and transfer principle. We denote by $\mathscr{C}(S, \mathcal{L}_O)^{\mathrm{exp}}$ the ring of constructible functions on X definable in $\mathcal{L}_O$. We denote by $\mathcal{D}_F$ the set of additive characters $\psi : F \to \mathbb{C}^\times$ such that $\psi(x) = \exp((2\pi i/p)\mathrm{Tr}_{k_F}(\bar{x}))$ for $x \in R_F$, with p the characteristic of k_F, Tr_{k_F} the trace of k_F relatively to its prime field and $\bar{x}$ the class of x in k_F.

The construction of specialization explained in 2.8 extends as follows to the exponential case. Let φ be in $K_0(\mathrm{RDef}_X(\mathcal{L}_O))^{\mathrm{exp}}$ of the form $[W, g, \xi]$. For ψ_F in $\mathcal{D}_F$, one specializes φ into the function $\varphi_{F,\psi_F} : X_F \to \mathbb{C}$ given by

$$x \mapsto \sum_{y \in \pi_F^{-1}(x)} \psi_F(g_F(y)) \exp((2\pi i/p)\mathrm{Tr}_{k_F}(\xi_F(y)))$$

for F in $\mathscr{C}_{O,N}$ with $N \gg 0$. One defines the specialization $\varphi \mapsto \varphi_{F,\psi_F}$ for φ in $\mathscr{C}(X, \mathcal{L}_O)^{\mathrm{exp}}$ by tensor product.

One can show that if φ is relatively integrable, then, for $N \gg 0$ and every F in $\mathscr{C}_{O,N}$, for every λ in Λ_F and every ψ_F in $\mathcal{D}_F$, the restriction $\varphi_{F,\psi_F,\lambda}$ of φ_{F,ψ_F} to $f_F^{-1}(\lambda)$ is integrable.

We denote by $\mu_{\Lambda_F}(\varphi_{F,\psi_F})$ the function on Λ_F defined by

$$\lambda \longmapsto \mu(\varphi_{F,,\psi_F,\lambda}). \tag{10.2.1}$$

The results in 2.8 generalize as follows to the exponential setting:

Theorem 10.2.1 (Exponential specialization, Cluckers-Loeser [10] [11])**.** *Let $f : S \to \Lambda$ be a morphism in* $\mathrm{Def}(\mathcal{L}_O)$*. Let φ be in $\mathscr{C}(S, \mathcal{L}_O)^{\mathrm{exp}}$ relatively integrable with respect to f. For $N \gg 0$, for every F in $\mathscr{C}_{O,N}$ and every ψ_F in $\mathcal{D}_F$, we have*

$$(\mu_\Lambda(\varphi))_{F,\psi_F} = \mu_{\Lambda_F}(\varphi_{F,\psi_F}). \tag{10.2.2}$$

Theorem 10.2.2 (Exponential abstract transfer principle, Cluckers-Loeser [10] [11])**.** *Let φ be in $\mathscr{C}(\Lambda, \mathcal{L}_O)^{\mathrm{exp}}$. There exists N such that for every F_1, F_2 in $\mathscr{C}_{O,N}$ with $k_{F_1} \simeq k_{F_2}$,*

(10.2.3)

$$\varphi_{F_1,\psi_{F_1}} = 0 \quad \textit{for all} \quad \psi_{F_1} \in \mathcal{D}_{F_1} \quad \textit{if and only if} \quad \varphi_{F_2,\psi_{F_2}} = 0 \quad \textit{for all} \quad \psi_{F_2} \in \mathcal{D}_{F_2}.$$

Theorem 10.2.3 (Exponential transfer principle for integrals with parameters, Cluckers-Loeser [10] [11])**.** *Let $S \to \Lambda$ and $S' \to \Lambda$ be morphisms in* $\mathrm{Def}(\mathcal{L}_O)$*. Let φ and φ' be relatively integrable functions in $\mathscr{C}(S, \mathcal{L}_O)^{\mathrm{exp}}$ and $\mathscr{C}(S', \mathcal{L}_O)^{\mathrm{exp}}$, respectively. There exists N such that for every F_1, F_2 in $\mathscr{C}_{O,N}$ with $k_{F_1} \simeq k_{F_2}$,*

$$\mu_{\Lambda_{F_1}}(\varphi_{F_1,\psi_{F_1}}) = \mu_{\Lambda_{F_1}}(\varphi'_{F_1,\psi_{F_1}}) \quad \textit{for all} \quad \psi_{F_1} \in \mathcal{D}_{F_1}$$
$$\textit{if and only if}$$
$$\mu_{\Lambda_{F_2}}(\varphi_{F_2,\psi_{F_2}}) = \mu_{\Lambda_{F_2}}(\varphi'_{F_2,\psi_{F_2}}) \quad \textit{for all} \quad \psi_{F_2} \in \mathcal{D}_{F_2}.$$

10.3. Jacquet-Ye integrals and the relative fundamental lemma. A specific situation where Theorem 2.8.3 applies is that of Jacquet-Ye integrals. Let E/F be a unramified degree two extension of non archimedean local fields of residue characteristic $\neq 2$ and let ψ be a non trivial additive character of F of conductor $\mathcal{O}_F$. Let N_n be the group of upper triangular matrices with 1's on the diagonal and consider the character $\theta : N_n(F) \to \mathbb{C}^\times$ given by

$$\theta(u) := \psi(\sum_i u_{i,i+1}). \tag{10.3.1}$$

For a the diagonal matrix $(a_1, \cdots, a_n)$ with a_i in $F^\times$, we consider the integral

$$I(a) := \int_{N_n(F)\times N_n(F)} \mathbf{1}_{M_n(\mathcal{O}_F)}({}^t u_1 a u_2)\, \theta(u_1 u_2)\, du_1 du_2. \tag{10.3.2}$$

Here du denote the Haar measure on $N_n(F)$ with the normalization $\int_{N_n(\mathcal{O}_F)} du = 1$.

Similarly, one defines

$$J(a) := \int_{N_n(E)} \mathbf{1}_{M_n(\mathcal{O}_E)\cap H_n}({}^t \bar{u} a u)\, \theta(u\bar{u})\, du, \tag{10.3.3}$$

with H_n the set of Hermitian matrices.

The Jacquet-Ye Conjecture [20], proved by Ngô [29] over function fields and by Jacquet [21] in general, asserts that

$$I(a) = \gamma(a)\, J(a) \tag{10.3.4}$$

with

$$\gamma(a) := \prod_{1\leq i\leq n-1} \eta(a_1 \cdots a_i),$$

and η the unramified multiplicative character of order 2 on $F^\times$.

It should be clear by now to the reader that the exponential version of the Transfer Theorem 10.2.3 applies to (10.3.4) using the proxies for field extensions explained in section 3 and viewing a as a parameter. Note that the discrepancy between the conditions on conductors in 10.2 and 10.3 is handled by performing an homothety of ratio t. Also it is most likely that Theorem 10.2.3 can be used for other versions of the relative fundamental lemma.

References

[1] J. Arthur, *The local behaviour of weighted orbital integrals*, Duke Math. J., **56** (1988), 223–293.

[2] J. Arthur, *Canonical normalization of weighted characters and a transfer conjecture*, C. R. Math. Acad. Sci. Soc. R. Can. **20** (1998), 33-52.

[3] J. Arthur, *The trace formula in invariant form*, Annals of Math. **114** (1981), 1–74.

[4] J. Arthur, *A Stable Trace Formula. I. General Expansions*, J. Inst. Math. Jussieu **1** (2002), 175-277.

[5] J. Ax and S. Kochen, *Diophantine problems over local fields. I*, Amer. J. Math. **87**3 (1965), 605–630.

[6] J. Ax and S. Kochen, *Diophantine problems over local fields. II.: A complete set of axioms for p-adic number theory*, Amer. J. Math. **87** (1965), 631–648.

[7] H. Chaudouard, G. Laumon, *Sur l'homologie des fibres de Springer affines tronquées*, Duke Math. J. **145** (2008), 443–535.

[8] R. Cluckers, F. Loeser, *Constructible motivic functions and motivic integration*, Invent. Math. **173** (2008), 23–121.
[9] R. Cluckers, F. Loeser, Ax-Kochen-Ersov Theorems for p-adic integrals and motivic integration, in *Geometric methods in algebra and number theory, edited by F. Bogomolov and Y. Tschinkel*, Progress in Mathematics 235, 109-137 (2005), Birkhäuser.
[10] R. Cluckers, F. Loeser, *Fonctions constructibles exponentielles, transformation de Fourier motivique et principe de transfert*, C. R. Math. Acad. Sci. Paris, **341** (2005), 741–746.
[11] R. Cluckers, F. Loeser, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Ann. of Math. **171** (2010), 1011–1065.
[12] C. Cunningham, T. Hales, *Good Orbital Integrals*, Represent. Theory **8** (2004), 414– 457.
[13] J. Denef, F. Loeser, *Definable sets, motives and p-adic integrals*, J. Amer. Math. Soc. **14** (2001), 429–469.
[14] J. Eršov, *On the elementary theory of maximal normed fields*, Dokl. Akad. Nauk SSSR **165** (1965), 21–23.
[15] T. Hales, Unipotent representations and unipotent classes in $SL(n)$, Amer. J. Math. **115**6 (1993), 1347–1383.
[16] T. Hales, A simple definition of transfer factors for unramified groups, in *Representation theory of groups and algebras*, 109–134, Contemp. Math., 145, Amer. Math. Soc., Providence, RI, 1993.
[17] T. Hales, A statement of the fundamental lemma, in *Harmonic analysis, the trace formula, and Shimura varieties*, 643–658, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.
[18] T. Hales, *On the fundamental lemma for standard endoscopy: reduction to unit elements*, Canad. J. Math. **47**(1995), no. 5, 974–994.
[19] J. Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics, 14. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000.
[20] H. Jacquet, Y. Ye, *Relative Kloosterman integrals for* GL(3), Bull. Soc. Math. France **120** (1992), 263–295.
[21] H. Jacquet, *Kloosterman identities over a quadratic extension*, Ann. of Math. **160** (2004), 755–779.
[22] B. Kostant, *Lie group representations on polynomial rings*, Am. J. Math. **85** (1963), 327-404.
[23] R. Kottwitz, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), 365–399.
[24] R. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), 611–650.
[25] R. Kottwitz, D. Shelstad, *Foundations of Twisted Endoscopy*, Astérisque **255**, 1999.
[26] R. Kottwitz, Harmonic analysis on reductive p-adic groups and Lie algebras, *Harmonic analysis, the trace formula, and Shimura varieties*, 393–522, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.
[27] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), 219-271.
[28] G. Laumon, B. C. Ngô, *Le lemme fondamental pour les groupes unitaires*, Ann. of Math. **168** (2008), 477–573.
[29] B. C. Ngô, *Faisceaux pervers, homomorphisme de changement de base et lemme fondamental de Jacquet et Ye*, Ann. Sci. École Norm. Sup. (4) **32** (1999), 619–679.
[30] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. **11** (2010), 1–169.
[31] J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. **399** (1989), 137–172.
[32] J.-P. Serre, *Corps Locaux*, Hermann, Paris, 1968.
[33] D. Shelstad, *A formula for regular unipotent germs*, Astérisque **171-172** (1989).
[34] J.-L. Waldspurger, *Une formule des traces locale pour les algèbres de Lie p-adiques*, J. Reine Angew. Math. **465** (1995), 41-99.
[35] J.-L. Waldspurger, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), 423–525.

[36] J.-L. Waldspurger, *Endoscopie et changement de caractéristique: intégrales orbitales pondérées*, Ann. Inst. Fourier **59** (2009), 1753 – 1818.
[37] D. Whitehouse, *The twisted weighted fundamental lemma for the transfer of automorphic forms from GSp(4) to GL(4)*, Astérisque **302** (2005), 291–436.

(Raf Cluckers) Université Lille 1, Laboratoire Painlevé, CNRS - UMR 8524, Cité Scientifique, 59655 Villeneuve d'Ascq Cédex, France, and, Katholieke Universiteit Leuven, Departement wiskunde, Celestijnenlaan 200B, B-3001 Leuven, Belgium.
E-mail address: Raf.Cluckers@math.univ-lille1.fr
URL: http://wis.kuleuven.be/algebra/Raf/

(Thomas Hales) Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15217
E-mail address: hales@pitt.edu
URL: www.math.pitt.edu/~thales/

(François Loeser) École Normale Supérieure, Département de mathématiques et applications, 45 rue d'Ulm, 75230 Paris Cedex 05, France (UMR 8553 du CNRS)
E-mail address: Francois.Loeser@ens.fr
URL: www.dma.ens.fr/~loeser/