Elliptic birational transformations of even complex projective spaces

From 1974 and for 10 years, with Laurent Gruson, we enjoyed dicussing space curves and particularly the relations between their secant lines and their syzygies. As we are gathered to celebrate Laurent's memory I would like to comment on a slight extension of our favorite subject: secant linear spaces to normal elliptic curves. This is a report on part of a work in progress with Laurent Koelblen.

Let E be a complex elliptic curve equipped with an odd degree (say $2n+1$) line bundle $O_E(h)$, and put $V = H^0(O_E(h))$. Consider the rank-$(2n+1)$ vector space $U^\vee \subset S^n V$ of degree n polynomials having multiplicity $\geq n-1$ along E. This vector space generates in $S^{n-1}N^\vee_{E/\mathbb{P}(V)}(nh)$ (the symmetric power of the conormal bundle of E in $\mathbb{P}(V)$) a rank $(2n-1)$ bundle K on E, with $c_1(K) = h$. This induces an exact sequence $0 \to K^\vee \to U \otimes O_E \to Q \to 0$ (where Q is a rank-2 bundle on E, with $c_1(Q) = h$) and embeddings $E \subset G(2, U)$ and $\mathbb{P}(V) \subset \mathbb{P}(\Lambda^2 U)$. For $0 \leq k \leq n-1$, let $S_k(E)$ be the union of all $\mathbb{P}^k \subset \mathbb{P}(V)$ cuting a divisor of degree $k+1$ in E. On the one hand we know, by a theorem of T. Fisher, that $S_k(E) = \mathbb{P}(V) \cap S_k(G(2, U)$ and that its saturated graded ideal is generated by the images of the $(k+2)$-pfaffians. On the other hand, we know, by a theorem of Bothmer and Hulek, that $S_k(E)$ has a Gorenstein projective cone and that its dualizing sheaf is trivial. From the construction we deduce natural embeddings $S^k E \subset G(2k, U) \simeq G(2(n-k)+1, U^\vee)$ for all $k \leq n$. Furthermore the embedding $S^n E \subset G(1, U^\vee) = \mathbb{P}(U^\vee)$ extends to a birational $(n, 2n-1)$ transformation between $\mathbb{P}(V)$ and $\mathbb{P}(U^\vee)$. The graph of this transformation is the blowing up of $\mathbb{P}(V)$ along $S_{n-2}(E)$ (cut out by the n-pfaffians) as well as the blowing up of $\mathbb{P}(U^\vee)$ along the double locus of the projective dual of the ruled surface $\mathbb{P}_E(Q) \subset \mathbb{P}(U)$ (cut out by $(2n+1)$ degree $(2n-1)$ hypersurfaces in $\mathbb{P}(U^\vee)$). For $n = 2$ this is the well known quadro-cubic transformation of $\mathbb{P}^4$. For $n \geq 3$, the graph of this transformation of $\mathbb{P}^{2n}$ is locally complete intersection, has singularities in codimension 3 and a small resolution of singularities obtained by blowing up $\mathbb{P}(V)$ successively along E and all its secant varieties. This construction gives a well defined (by Fitting ideals) description of the multiple loci of the projective dual of the ruled surface $\mathbb{P}_E(Q)$ and relates them to the secant varieties $S_k(E)$ as well as the symmetric powers $S^{k+1}E$. To conclude, we have not been able to describe (by equations) the dimension $(2n+1)^2$ subvariety of $G(2n+1, \Lambda^2 U)$ of all linear subspaces $\mathbb{P}^{2n} \subset \mathbb{P}(\Lambda^2 U)$ intersecting $G(2, U)$ and its secant varieties in a degree $(2n+1)$ elliptic curve and its secant varieties.