Spectral statistics for random Hamiltonians in the localized regime

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Outline

The random model

Consider H_ω, a $\mathbb{Z}^d$-ergodic random operator on $\mathscr{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$ i.e.

- for Ω, a probability space, $\omega \in \Omega \mapsto H_\omega$ is a weakly measurable family of self-adjoint operator on $\mathscr{H}$,
- there exists $(T_\gamma)_{\gamma\in\mathbb{Z}^d}$, an ergodic group of probability preserving transformation on Ω s.t.

$$\tau_\gamma H_\omega \tau_\gamma^* = H_{T_\gamma \omega}$$

  where $(\tau_\gamma u)(x) = u(x-\gamma)$ for $\gamma \in \mathbb{Z}^d$.

"Spectral" objects almost surely constant e.g. spectrum, a.c., s.c., p.p. spectra.

Two standard examples:

- The discrete Anderson model: on $\ell^2(\mathbb{Z}^d)$, $H_\omega = -\Delta + V_\omega$
  - $-\Delta$ discrete Laplacian,
  - V_ω diagonal matrix with i.i.d. entries with nice distribution.
- The continuous Anderson model: on $L^2(\mathbb{R}^d)$, $H_\omega = -\Delta + W + V_\omega$
  - $-\Delta$ Laplacian on $\mathbb{R}^d$ and $W : \mathbb{R}^d \to \mathbb{R}$ $\mathbb{Z}^d$-periodic potential,
  - $V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma u(\cdot - \gamma)$
    - ★ $(\omega_\gamma)_\gamma$ i.i.d. random variables with nice distribution,
    - ★ u bounded with compact support and fixed sign.

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Basic assumptions on the random model

Let σ be the almost sure spectrum of H_ω.

Assume that H_ω admits an integrated density of states i.e.

$$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ less than E}\}}{|\Lambda|}$$

where $H_\omega(\Lambda)$ is the operator H_ω restricted to Λ (periodic BC).
In I, $N(E)$ dist. funct. of a.c. measure with bounded density $\nu(E)$.

Fix $I \subset \mathbb{R}$ a compact interval.

In I, we assume that H_ω satisfies a Wegner estimate i.e. for $J \subset I$,

$$\text{(W)} \qquad \mathbb{P}(\{\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \geq 1\}) \leq C|J|\,|\Lambda|$$

where

- $\sigma(H)$ is the spectrum of the operator H,
- $\mathbb{P}(\Omega)$ denotes the probability of the event Ω.

Known to hold for many models in particular for the Anderson models under mild regularity conditions on the random variables.

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In I, we assume that H_ω satisfies a Minami estimate i.e. for $J \subset I$,

$$\text{(M)} \qquad \mathbb{P}(\{\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \geq 2\}) \leq C(|J|\,|\Lambda|)^2.$$

This is known to hold for

- the discrete Anderson model under mild regularity assumptions on the r.v. (Minami, Bellissard et al., Graf et al., Combes-Germinet-Klein);
- the continuous Anderson model in the "Lifshitz tails" region (CGK).

The localized regime

Basic result in theory of RSO: there exists regions in the spectrum, typically the edges of the spectrum, where spectrum is p.p. and the eigenfunctions are exp. decaying.

We assume:

- for some $\xi \in (0,1]$ and $\gamma > 0$, for any $p > 0$, there exists $q > 0$ such that, for $L \geq 1$, with probability larger than $1 - L^{-p}$, if
  - $\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to $E_{n,\omega} \in I$,
  - $x_n(\omega) \in \Lambda_L$ is a maximum of $x \mapsto \|\varphi_{n,\omega}\|_{x+C}$ on Λ_L

  then, for $x \in \Lambda_L$, one has $\|\varphi_{n,\omega}\|_{x+C} \leq L^q e^{-\gamma|x-x_n(\omega)|^\xi}$.

FMM provides $\xi = 1$, MSA ξ arbitrarily close to 1.

The questions

Local level statistics: Fix $E_0 \in I$ s.t. $\nu(E_0) := N'(E_0) > 0$.
Renormalized local levels near E_0:

$$\xi_j(E_0,\omega,\Lambda) = |\Lambda|\,\nu(E_0)\,(E_j(\omega,\Lambda) - E_0).$$

Distribution function: $\Xi^l(\xi, E_0, \omega, \Lambda) = \sum_{j=1}^{N} \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi).$

Localization center statistics: Let $\varphi_{n,\omega}$ normalized eigenvector associated to $E_{n,\omega} \in I$.
Localization center for $E_{n,\omega}$ is a maximum of $x \mapsto \|\varphi_{n,\omega}\|_{x+C}$.
A priori not unique!
Localization centers contained in ball of radius $\asymp (\log L)^{1/\xi}$.
Distribution function: $\Xi^c(\xi, x; E_0, \Lambda_L) = \sum_{j=1}^{N} \delta_{x_j(\omega)/L}(x).$

Joint statistics: distribution function:

$$\Xi^2(\xi, x; E_0, \Lambda_L) = \sum_{j=1}^{N} \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi) \otimes \delta_{x_j(\omega)/L}(x).$$

Joint statistics: one can change the scaling. Fix scaling function $\Lambda \mapsto \ell_\Lambda$ s.t.

- $\ell_\Lambda \to +\infty$ when $|\Lambda| \to +\infty$,
- ℓ_Λ not too large, not too small,

Distribution function of covariantly scaled joint statistics:

$$\Xi^2_\Lambda(\xi, x; E_0, \ell) = \sum_{j=1}^{N} \delta_{\nu(E_0)(E_j(\omega,\Lambda)-E_0)|\Lambda_{\ell_\Lambda}|}(\xi) \otimes \delta_{x_j(\omega)/\ell_\Lambda}(x).$$

Level spacings statistics: Let $(E_j(\Lambda,\omega))_{1\le j\le N}$ be eigenvalues ordered increasingly; $N = N(\omega)$ random number.

Renormalized eigenvalue spacings

$$\delta E_j(\Lambda,\omega) = |\Lambda|(E_{j+1}(\Lambda,\omega) - E_j(\Lambda,\omega)) \ge 0.$$

Renormalized eigenvalue spacings distribution:

$$DLS(x;\Lambda,\omega) = \frac{\#\{j;\ \delta E_j(\Lambda,\omega) \ge x\}}{N}.$$

Another point of view: The spectrum of H_ω is I is p.p. with exp. dec. eigenfcts. Localization centers well defined.
For typical ω, consider eigenvalues in I with localization center in Λ_L.
Ask same questions as above.

Local eigenvalue statistics:

Theorem (Molchanov, Minami, Combes-Germinet-Klein, G.-Kl.)

Under the assumptions above, as $|\Lambda| \to +\infty$, $\Xi^l(\xi, E_0, \omega, \Lambda)$ converges weakly to a Poisson process on $\mathbb{R}$ with intensity measure the Lebesgue measure.

Correlation of local statistics:
Consider the limits of $\Xi^l(\xi, E_0, \omega, \Lambda)$ et $\Xi^l(\xi, E_0', \omega, \Lambda)$ of $E_0 \ne E_0'$.

Q: Are they independent ?

Generalized Minami estimate : for $J \subset K \subset I$,

$$\text{(GM)} \qquad \mathbb{P}(\{\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \ge 1 \ \text{ and } \ \mathrm{tr}(\mathbf{1}_K(H_\omega(\Lambda))) \ge 2\}) \le C|J||K|\,|\Lambda|^2.$$

Known for discrete and certain continuous Anderson models [CGK]
Decorrelation estimates (D): for $\alpha \in (0,1)$ and $\{E_0, E_0'\} \subset I$ t.q. $E_0 \ne E_0'$, when $L \to +\infty$ and $\ell \asymp L^\alpha$,

$$\text{(D)} \qquad \mathbb{P}\left[\mathrm{tr}(\mathbf{1}_{I_L}(H_\omega(\Lambda_\ell))) \ge 1 \ \text{ and } \ \mathrm{tr}(\mathbf{1}_{I_L'}(H_\omega(\Lambda_\ell))) \ge 1\right] = o\left((\ell/L)^d\right)$$

where $I_L = E_0 + L^{-d}[-1,1]$, $I_L' = E_0' + L^{-d}[-1,1]$.

Known for discrete Anderson in dim. 1 at all energies and for arbitrary d if $|E_0 - E_0'| > 2d$ [Kl].

Theorem

Assume (W), (M), (Loc), (GM) and (D). Pick $E_0 \neq E_0'$ s.t. $\nu(E_0), \nu(E_0') > 0$. When $|\Lambda| \to +\infty$, $\Xi(E_0, \omega, \Lambda)$ and $\Xi(E_0', \omega, \Lambda)$ converge to two independent Poisson processes i.e. for $U_+ \subset \mathbb{R}$ and $U_- \subset \mathbb{R}$ compact intervals and $\{k_+, k_-\} \in \mathbb{N} \times \mathbb{N}$, one has

$$\mathbb{P}\left(\left\{\omega; \begin{array}{l} \#\{j; \xi_j(E_0, \omega, \Lambda) \in U_+\} = k_+ \\ \#\{j; \xi_j(E_0', \omega, \Lambda) \in U_-\} = k_- \end{array}\right\}\right) \underset{\Lambda \to \mathbb{Z}^d}{\to} e^{-|U_+|} \frac{|U_+|^{k_+}}{k_+!} \cdot e^{-|U_-|} \frac{|U_-|^{k_-}}{k_-!}.$$

Question: minimal distance between E_0 and E_0' to keep independence?

Theorem

Assume (W), (M), (Loc), (GM). Pick E_0 s.t. $\nu(E_0) > 0$ and ν cont. near E_0. If $E_\Lambda \in I$ and $E_\Lambda' \in I$ such that

- *$E_\Lambda \to E_0 \leftarrow E_\Lambda'$ when $|\Lambda| \to +\infty$,*
- *$|\Lambda| \cdot |E_\Lambda - E_\Lambda'| \to +\infty$ when $|\Lambda| \to +\infty$,*

then, as $|\Lambda| \to +\infty$, $\Xi^l(\xi, E_\Lambda, \omega, \Lambda)$ and $\Xi^l(\xi, E_\Lambda', \omega, \Lambda)$ converge to two independent Poisson processes on $\mathbb{R}$.

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Local localization center statistics

Theorem (Nakano, Nakano-Killip)

Under the assumptions above, as $|\Lambda| \to +\infty$, $\Xi^2(\xi, E_0, \omega, \Lambda_L)$ converges weakly to a Poisson process on $\mathbb{R} \times [-1,1]^d$ with intensity measure the Lebesgue measure.

Fix sequence of scales $\ell = (\ell_\Lambda)_\Lambda$ such that

$$\frac{(\ell_\Lambda)^\xi}{\log|\Lambda|} \underset{|\Lambda| \to +\infty}{\to} +\infty \quad \text{and} \quad 2\ell_\Lambda \leq |\Lambda|^{1/d}.$$

Pick $E_0 \in I$ so that $\nu(E_0) > 0$ and recall covariantly scaled joint local distribution

$$\Xi_\Lambda^2(\xi, x; E_0, \ell) = \sum_{j=1}^N \delta_{\nu(E_0)(E_j(\omega,\Lambda) - E_0)|\Lambda_{\ell_\Lambda}|}(\xi) \otimes \delta_{x_j(\omega)/\ell_\Lambda}(x).$$

The process valued in $\mathbb{R} \times \mathbb{R}^d$. Define $c_\ell = \lim_{|\Lambda| \to +\infty} |\Lambda|^{1/d} (2\ell_\Lambda)^{-1} \in [1, +\infty]$.

Theorem

The point process $\Xi_\Lambda^2(\xi, x; E_0, \ell)$ converges weakly to a Poisson process on $\mathbb{R} \times (-c_\ell, c_\ell)^d$ with intensity measure the Lebesgue measure.

For non covariant scales: consider scales, say $\ell = (\ell_\Lambda)_\Lambda$ and $\ell' = (\ell'_\Lambda)_\Lambda$ as above.
Distribution function:

$$\Xi^2_\Lambda(\xi, x; E_0, \ell, \ell') = \sum_{j=1}^{N} \delta_{\nu(E_0)(E_j(\omega,\Lambda)-E_0)|\Lambda_{\ell_\Lambda}|}(\xi) \otimes \delta_{x_j(\omega)/\ell'_\Lambda}(x).$$

Theorem

Let J and X be bounded open sets respectively in $\mathbb{R}$ and $(-c_{\ell'}, c_{\ell'})^d \subset \mathbb{R}^d$. One has

- *if $\ell'_\Lambda/\ell_\Lambda \asymp |\Lambda|^{-\rho}$ then, ω-almost surely, for $|\Lambda|$ sufficiently large,*

$$\int_{J\times X} \Xi^2_\Lambda(\xi, x; E_0, \ell, \ell') d\xi dx = 0;$$

- *if $\ell'_\Lambda/\ell_\Lambda \asymp |\Lambda|^{\rho}$ then, ω-almost surely,*

$$\left(\frac{\ell_\Lambda}{\ell'_\Lambda}\right)^d \int_{J\times X} \Xi^2_\Lambda(\xi, x; E_0, \ell, \ell') d\xi dx \underset{|\Lambda|\to+\infty}{\to} |J|\cdot|X|.$$

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Level spacing distribution

Let $E_0 \subset I_\Lambda$ compact interval s.t. $\lim_{x,y\to E_0}(x-y)^{-1}(N(x)-N(y)) = \nu(E_0) > 0$.
For statistics, I_Λ needs to contain asymptotically infinitely many energy levels of $H_\omega(\Lambda)$ i.e. assume, for some $\delta > 0$, one has

$$|\Lambda|^{1-\delta}\cdot|I_\Lambda| \underset{|\Lambda|\to+\infty}{\to} +\infty \quad \text{and} \quad \text{if } \ell_L = o(L) \text{ then } \frac{|I_{\Lambda_{L+\ell_L}}|}{|I_{\Lambda_L}|} \underset{L\to+\infty}{\to} 1. \tag{2.1}$$

Let $(E_j(\omega,\Lambda))_{1\le j\le N}$ e.v. in I_Λ ordered increasingly: $E_j(\omega,\Lambda) \le E_{j+1}(\omega,\Lambda)$.
Their number N is random of size $\nu(E_0)|\Lambda|\cdot|I_\Lambda|$ (by existence of IDS).
Consider the renormalized eigenvalue spacings

$$\delta E_j(\omega,\Lambda) = |\Lambda|\nu(E_0)(E_{j+1}(\omega,\Lambda) - E_j(\omega,\Lambda)) \ge 0.$$

Empirical distribution :

$$DLS(x;\omega,\Lambda) = \frac{\#\{j;\ \delta E_j(\omega,\Lambda) \ge x\}}{\nu(E_0)|\Lambda|\cdot|I_\Lambda|} \text{ for } x > 0.$$

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Theorem

ω-almost surely, the empirical distribution of level spacings $DLS(x;\omega,\Lambda)$ converges uniformly to the distribution $x \mapsto e^{-x}$ as $|\Lambda| \to +\infty$, that is

$$\sup_{x\geq 0}\left|DLS(x;\omega,\Lambda) - e^{-x}\right| \underset{|\Lambda|\to+\infty}{\to} 0, \quad \omega\text{-a.s.}$$

If one omits one of conditions (2.1), then a.s. convergence becomes convergence in probability.
Macroscopic energy intervals:

Let $J \subset I$ compact s.t. $E \mapsto \nu(E)$ be continuous on J and let $N(J) := \int_J \nu(E)dE > 0$.
Renorm. spacings: for $1 \leq j \leq N$, $\delta_J E_j(\omega,\Lambda) = |\Lambda| N(J)(E_{j+1}(\omega,\Lambda) - E_j(\omega,\Lambda))$.
Empirical distribution denoted by $DLS'(x;I_\Lambda,\omega,\Lambda)$.

Theorem

ω-almost surely, as $|\Lambda| \to +\infty$, the empirical distribution of level spacings $DLS'(x;I_\Lambda,\omega,\Lambda)$ converges uniformly to the distribution $x \mapsto g_{\nu,J}(x)$ defined by

$$g_{\nu,J}(x) = \int_J \mathrm{e}^{-\nu_J(\lambda)x}\nu_J(\lambda)d\lambda \quad \text{and} \quad \nu_J = \frac{1}{N(J)}\nu.$$

Localization center spacings distribution

In Λ, number of centers corresponding to energies in I_Λ roughly $\nu(E_0)\,|I_\Lambda|\,|\Lambda|$.

Reference mean spacing: $(|\Lambda|/[\nu(E_0)\,|I_\Lambda|\,|\Lambda|])^{1/d} = (\nu(E_0)\cdot|I_\Lambda|)^{-1/d}$.

Empirical distribution of center spacing:

$$DCS(s;\Lambda,\omega) = \frac{\#\{j;\ (\nu(E_0)|I_\Lambda|)^{1/d}\min_{i\neq j}|x_j(\omega) - x_i(\omega)| \geq s\}}{\nu(E_0)\cdot|I_\Lambda|\cdot|\Lambda|}$$

Theorem

ω-almost surely, the empirical distribution of localization center spacings $DCS(s;\Lambda,\omega)$ converges uniformly to the distribution $s \mapsto e^{-s^d}$ as $|\Lambda| \to +\infty$.

The other point of view:

Recall that N is the IDS of H_ω.

Proposition

Fix $q > 2d$ and $\xi \in (0,1]$. Then, there exists $\gamma > 0$ such that, ω-almost surely, there exists $C_\omega > 1$ such that

1. *if $E \in I \cap \sigma(H_\omega)$ and φ is a normalized eigenfunction associated to E then, for $x(E) \in \mathbb{R}^d$ a maximum of $x \mapsto \|\varphi\|_x$, one has, for $x \in \mathbb{R}^d$,*
$$\|\varphi\|_x \leq C_\omega \langle x(E)\rangle^q e^{-\gamma|x-x(E)|^\xi}.$$
2. *if $x(E)$ and $x'(E)$ are two centers of localization for $E \in I$, then*
$$|x(E) - x'(E)| \leq \gamma^{-2}(\log\langle x(E)\rangle + \log C_\omega)^{1/\xi}.$$
3. *for $L \geq 1$, pick $I_L \subset I$ such that $|\Lambda_L| N(I_L) \to +\infty$; if $N(I_L, L)$ denotes the number of eigenvalues of H_ω having a center of localization in Λ_L, then*
$$N(I_L, L) = N(I_L)\,|\Lambda_L|\,(1 + o(1)).$$

For $L \geq 1$, pick $I_L \subset I$ such that $L^d \nu(I_L) \to +\infty$.
Consider the eigenvalues of H_ω having a localization center in Λ_L, say,
$$E_1(\omega, L) \leq E_2(\omega, L) \leq \cdots \leq E_N(\omega, L).$$
Consider the renormalized eigenvalue spacings, for $1 \leq j \leq N$,
$$\delta E_j(\omega, L) = |\Lambda_L|(E_{j+1}(\omega, L) - E_j(\omega, L)) \geq 0.$$
Empirical distribution of spacing: for $x \geq 0$
$$DLS(x; I_L, \omega, L) = \frac{\#\{j;\ E_j(\omega, L) \in I_L,\ \delta E_j(\omega, L) \geq x\}}{N(I_L, L)}$$

Theorem

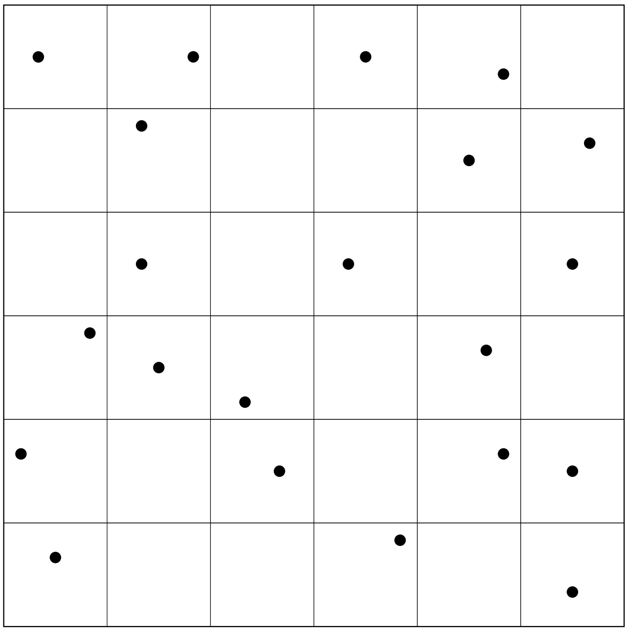
One has

- *if $E_0 \in I_L$ s.t. $\nu(E_0) > 0$ and $|I_L| \to 0$, then, ω-almost surely, $DLS(x; I_L, \omega, L)$ converges uniformly to $e^{-\nu(E_0)x}\mathbf{1}_{x \geq 0}$;*
- *if, for all L large, $I_L = J$ such that $N(J) > 0$ and ν is continuous on J then, ω-almost surely, $DLS(x; I_L, \omega, L)$ converges uniformly to $g_{\nu,J}(N(J)x)$.*

The main ideas of the proofs

Localized regime ⇒ e.v. depend only on local picture of potential ⇒ explicit description of the eigenvalues in I in terms of eigenvalues of smaller independent cubes.

- pick cube of size L
- the localization centers
- Cut large cube into smaller cubes of size ℓ
- Problems:
  - multiple centers in small cubes probability is small due to Minami's estimate: $\ell^{2d}|I|^2(L/\ell)^d = \ell^d L^d |I|^2$
  - centers not localized well in cube probability is small due to Wegner's estimate: $l \cdot L^{d-1} \cdot |I|$
- With good probability, this does not occur.



So with good probability, in I, e.v. of big cube given by e.v. of small cubes.

Following these ideas, we prove

Theorem

Pick α, β s.t. $1 > \alpha$ and $\beta - d/(d+2) = 2(\alpha - (d+1)/(d+2)) > 0$. Set $\ell = L^\beta$ and $\tilde{N} = N^{1-\beta}$. Set $N = L^d$ and $I_N^\alpha = [E_0 - N^{-\alpha}, E_0 + N^{-\alpha}] \subset I$.

There exists $p > 0$ such that, for N large enough, there exists a set of configurations $\mathscr{Z}_N$ s.t.

- $\mathbb{P}(\mathscr{Z}_N) \geq 1 - N^{-p}$,
- *for $\omega \in \mathscr{Z}_N$, each box $\Lambda_\ell(\gamma_j) := \gamma_j + [0, l]^d$ satisfies:*
  1. *the Hamiltonian $H_\omega(\Lambda_\ell(\gamma_j))$ has at most one eigenvalue in I_N^α, say, $E_j(\omega, \Lambda_\ell(\gamma_j))$;*
  2. *$\Lambda_\ell(\gamma_j)$ contains at most one center of localization, say $x_{k_j}(\omega, L)$, of an eigenvalue of $H_{\omega,L}$ in I_N^α, say $E_{k_j}(\omega, L)$;*
  3. *$\Lambda_\ell(\gamma_j)$ contains a center $x_{k_j}(\omega, L)$ if and only if $\sigma(H_\omega(\Lambda_\ell(\gamma_j))) \cap I_N^\alpha \neq \emptyset$;*

  then, $|E_{k_j}(\omega, L) - E_j(\omega, \Lambda_\ell(\gamma_j))| = O(L^{-\infty})$ and $dist(x_{k_j}(\omega, L), \Lambda_L \setminus \Lambda_\ell(\gamma_j)) \geq L^p$.

Problem : for analysis, energy intervals have to be small.

On larger intervals, still possible but cannot control all eigenvalues.

Still enough for level spacings.

Theorem

Pick $\alpha = (\alpha_N)_N$ *s.t.* $\lim_N \alpha_N \searrow 0$. *Set* $\tilde{N} = n/\alpha_N$, $n = n'/\alpha_N$, $n' = (R \log N)^d$, *with* R *large. Set* $I_N^\alpha = [E_0 - i_n, E_0 + i_n] \subset I$ *with* $i_N = n^{1/d} \geq c(\log N)^{-d} \alpha_N^2$.

For any $p > 0$ *and* N *large enough, there exists a set of configurations* $\mathscr{Z}_N$ *so that*

- $\mathbb{P}(\mathscr{Z}_N) \geq 1 - N^{-p}$,
- *for all* $\omega \in \mathscr{Z}_N$, *there exists at least* $\dfrac{N}{n}(1 - o(1))$ *disjoint boxes* $\Lambda_\ell(\gamma_j)$ *satisfying the properties described in the previous theorem.*