Universal asymptotic ergodicity of the unfolded eigenvalues and localization centers of random Hamiltonians in the localized regime

F. Klopp

Université Paris 13

March 7, 2011

Outline

The prototypical example

On $\ell^2(\mathbb{Z}^d)$, consider the random Anderson model

$$H_\omega = -\Delta + \lambda V_\omega$$

where, for $u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$,

- $-\Delta$ is the free discrete Laplace operator $(-\Delta u)_n = \sum_{|m-n|=1} u_m$;
- V_ω is the random potential $(V_\omega u)_n = \omega_n u_n$;
  the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are i.i.d. with compactly supported bounded density, say g.

Let N be the integrated density of states of H_ω; it is a probability distribution on the essential spectrum, say, Σ of H_ω.
For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L = [0,L]^d$ be a large box and $H_\omega(\Lambda)$ be the operator H_ω restricted to Λ with periodic boundary conditions.

It is well known that, for λ large, the whole spectrum of H_ω is localized.

The talk is devoted to the description of various statistics of the eigenvalues and localization centers of H_ω.

UNIVERSITÉ PARIS 13

Let us denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega,\Lambda) \le E_2(\omega,\Lambda) \le \cdots \le E_{|\Lambda|}(\omega,\Lambda)$.
Unfolded eigenvalues: $N(E_1(\omega,\Lambda)) \le N(E_2(\omega,\Lambda)) \le \cdots \le N(E_{|\Lambda|}(\omega,\Lambda)) \le \cdots$

For a fixed ω, consider the point process $\Xi(\omega,t,\Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{|\Lambda|[N(E_n(\omega,\Lambda))-t]}$ under the uniform distribution in $t \in [0,1]$.

Theorem (Kl. 10)

For sufficiently large coupling constant λ, ω-almost surely, when $|\Lambda| \to +\infty$, the probability law of the point process $\Xi(\omega,\cdot,\Lambda)$ under the uniform distribution $\mathbf{1}_{[0,1]}(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.

This result was conjectured by Minami (09): asymptotic ergodicity.
In dimension 1, Minami proved a weaker version, namely L^2-convergence, using decorrelation estimates (Kl. 10). Rem.: $\lambda \neq 0$ can be arbitrary.

The universal joint statistics:

To each eigenvalue $E_n(\omega,\Lambda)$, one can associate a localization center, say, $x_n(\omega,\Lambda)$.

UNIVERSITÉ PARIS 13

Fix an increasing sequence of scales $\ell = (\ell_\Lambda)_\Lambda$ and $\Lambda = \Lambda_L$ such that

$$\ell_\Lambda \leq L \quad \text{and} \quad \log L = o(\ell_\Lambda).$$

such that the following limit exists $\lim_{L\to+\infty} L^{-1}\ell_\Lambda =: c \in [0,1]$.

Let $g_E : [0,1] \to \mathbb{R}^+$ and $g_X : [0,1]^d \to \mathbb{R}^+$ be two probability densities.
For a fixed configuration ω, consider the point process

$$\Xi^2_\Lambda(e,x;\ell,\omega) = \sum_{j=1}^{N} \delta_{\ell^d_\Lambda[N(E_j(\omega,\Lambda))-e]} \otimes \delta_{(x_j(\omega,\Lambda)-Lx)/\ell_\Lambda}.$$

under the distribution of density $g_E \otimes g_X$ over $\Lambda \times [0,1]^{d+1}$.

Our main result on the Anderson model is

Theorem

For sufficiently large coupling constant λ, ω-almost surely, the probability law of the point process $\Xi^2_\Lambda(\cdot,\cdot;\ell,\omega)$ under the distribution $g_E \otimes g_X$ converges to the law of the Poisson point process with intensity 1 on $\mathbb{R} \times [-1/c, 1/c]^d$.

Implies the almost sure convergence of many statistics constructed from the eigenvalues and centers (Minami (09)), for example, the level spacings statistics.

The random model

Consider H_ω, a $\mathbb{Z}^d$-ergodic random operator on $\mathscr{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$ i.e.

- for Ω, a probability space, $\omega \in \Omega \mapsto H_\omega$ is a weakly measurable family of self-adjoint operator on $\mathscr{H}$,
- there exists $(T_\gamma)_{\gamma\in\mathbb{Z}^d}$, an ergodic group of probability preserving transformation on Ω s.t.

$$\tau_\gamma H_\omega \tau^*_\gamma = H_{T_\gamma\omega}$$

  where $(\tau_\gamma u)(x) = u(x-\gamma)$ for $\gamma \in \mathbb{Z}^d$.

"Spectral" objects almost surely constant e.g. spectrum, a.c., s.c., p.p. spectra.

Two standard examples:

- The discrete Anderson model: on $\ell^2(\mathbb{Z}^d)$, $H_\omega = -\Delta + V_\omega$
  - $-\Delta$ discrete Laplacian,
  - V_ω diagonal matrix with i.i.d. entries with nice distribution.
- The continuous Anderson model: on $L^2(\mathbb{R}^d)$, $H_\omega = -\Delta + W + V_\omega$
  - $-\Delta$ Laplacian on $\mathbb{R}^d$ and $W : \mathbb{R}^d \to \mathbb{R}$ $\mathbb{Z}^d$-periodic potential,
  - $V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma u(\cdot - \gamma)$
    - ★ $(\omega_\gamma)_\gamma$ i.i.d. random variables with nice distribution,
    - ★ u bounded with compact support and fixed sign.

Basic assumptions on the random model

Let Σ be the almost sure spectrum of H_ω.

Assume that H_ω admits an integrated density of states i.e.

$$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ less than E}\}}{|\Lambda|}$$

where $H_\omega(\Lambda)$ is the operator H_ω restricted to Λ (periodic BC).
We assume
(IAD) $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are independent if $\mathrm{d}(\Lambda,\Lambda') > R$.

Fix $I \subset \mathbb{R}$ a compact interval.
In I, we assume that H_ω satisfies a Wegner estimate i.e. for $J \subset I$,

$$\text{(W)} \qquad \mathbb{P}(\{\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \geq 1\}) \leq C|J|\,|\Lambda|$$

where $\mathbb{P}(\Omega)$ denotes the probability of the event Ω.
In I, $N(E)$ dist. funct. of a.c. measure with bounded density $\nu(E)$.
Known to hold for many models in particular for the Anderson models under mild regularity conditions on the random variables.

In I, we assume that H_ω satisfies a Minami estimate i.e. for some $\rho \in (0,1]$ and any $J \subset I$,

$$\text{(M)} \qquad \mathbb{P}(\{\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \geq 2\}) \leq C(|J|\,|\Lambda|)^{1+\rho}.$$

This is known to hold for

- the discrete Anderson model under mild regularity assumptions on the r.v. (Minami, Bellissard et al., Graf et al., Combes-Germinet-Klein);
- the continuous Anderson model in the "Lifshitz tails" region (CGK);
- in dimension 1 in the localized regime once a Wegner estimate is known (Kl.).

The localized regime: Basic result in theory of RSO: there exists regions in the spectrum, typically the edges of the spectrum, where spectrum is p.p. and the eigenfunctions are exp. decaying.

We assume:

- for $\xi \in (0,1)$, for any $p > 0$, there exists $q > 0$ such that, for $L \geq 1$, with probability larger than $1 - L^{-p}$, if
  - $\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to $E_{n,\omega} \in I$,
  - $x_n(\omega) \in \Lambda_L$ is a maximum of $x \mapsto \|\varphi_{n,\omega}\|_{x+C}$ on Λ_L

  then, for $x \in \Lambda_L$, one has $\|\varphi_{n,\omega}\|_{x+C} \leq L^q e^{-|x-x_n(\omega)|^\xi}$.

FMM provides $\xi = 1$, MSA ξ arbitrarily close to 1.

The point process:

Fix $J=[a,b]$ be a compact interval such that $N(b)-N(a)=:N(J)>0$. Set $N_J(\cdot):=N(J)^{-1}[N(\cdot)-N(a)]$. Consider eigenvalues in J.

Unfolded eigenvalues: $N_J(E_n(\omega,\Lambda))\leq N_J(E_{n+1}(\omega,\Lambda))\leq\cdots\leq N_J(E_m(\omega,\Lambda))\leq\cdots$
The localization centers: $x_n(\omega,\Lambda),\quad x_{n+1}(\omega,\Lambda),\quad \cdots,\quad x_m(\omega,\Lambda),\quad \cdots$

Fix $\alpha>1$ and an increasing sequence of scales $\ell=(\ell_\Lambda)_\Lambda$ (recall $\Lambda=\Lambda_L$) such that

- $(\log L)^\alpha\leq\ell_\Lambda\leq L$,
- the following limit exists $\lim_{L\to+\infty} L^{-1}\ell_\Lambda=:c\in[0,1]$.

Let $g_E:[0,1]\to\mathbb{R}^+$ and $g_X:[0,1]^d\to\mathbb{R}^+$ be two probability densities.
For a fixed configuration ω, consider the point process

$$\Xi^2_\Lambda(e,x;\ell,\omega)=\sum_{j=1}^N\delta_{N(J)\ell^d_\Lambda[N_J(E_j(\omega,\Lambda))-e]}\otimes\delta_{(x_j(\omega,\Lambda)-Lx)/\ell_\Lambda}.$$

under the distribution of density $g_E\otimes g_X$ over $\Lambda\times[0,1]^{d+1}$.

Universal asymptotic ergodicity:

Theorem

Assume (IAD), (W), (M) and (Loc) hold. Assume that $J\subset I$, the localization region, that $N(J)>0$.
Then, ω-almost surely, the probability law of the point process $\Xi^2_\Lambda(\cdot,\cdot;\ell,\omega)$ under the distribution $g_E\otimes g_X$ converges to the law of the Poisson point process with intensity 1 on $\mathbb{R}\times[-1/c,1/c]^d$.

Statistics of the eigenvalues and localization centers:

Theorem

Assume (IAD), (W), (M) and (Loc) hold. Assume that $J=[a,b]\subset I$ is a compact interval in the localization region satisfying $N(J)>0$.
Define

- *the probability density $\nu_J:=\frac{1}{N(J)}\nu(E)\mathbf{1}_J(E)$ where $\nu=\frac{dN}{dE}$;*
- *the point process $\tilde\Xi^2_J(E,x;\omega,\ell,\Lambda)=\sum_{E_n(\omega,\Lambda)\in J}\delta_{\nu(E)\ell^d_\Lambda[E_n(\omega,\Lambda)-E]}\otimes\delta_{(x_j(\omega,\Lambda)-Lx)/\ell_\Lambda}$.*

Then, ω-almost surely, the law of the point process $\tilde\Xi^2_J(\cdot,\cdot;\omega,\ell,\Lambda)$ under $\nu_J\otimes g_X$ converges to the law of the Poisson point process with intensity 1 on $\mathbb{R}\times[-1/c,1/c]^d$.

Previous results

Minami (09): L^2 convergence in dimension 1.

Local level statistics: Renormalized local levels near E_0:

$$\xi_j(E_0,\omega,\Lambda) = |\Lambda|\nu(E_0)(E_j(\omega,\Lambda)-E_0).$$

Processes:

- $\Xi_{loc}(\xi,E_0,\omega,\Lambda) = \sum \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi)$,
- $\Xi^2_{loc}(\xi,x,E_0,\omega,\Lambda) = \sum \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi)\otimes\delta_{x_j(E_0,\omega,\Lambda)/L}(x)$.

Results: weak convergence to Poisson.
Assume $\nu(E_0)>0$: Molchanov (81), Minami (96), Combes-Germinet-Klein (09), Nakano (07), Nakano-Killip (07).
Unfolded levels (allows for $\nu(E_0)=0$): Germinet-Kl. (10).

Level spacings statistics: Let $(E_j(\Lambda,\omega))_j$ be eigenvalues ordered increasingly; renormalized eigenvalue spacings

$$\delta N_j(\Lambda,\omega) = |\Lambda|(N(E_{j+1}(\Lambda,\omega)) - N(E_j(\Lambda,\omega))) \geq 0.$$

Renormalized eigenvalue spacings distribution:

$$DLS(x;\Lambda,\omega) = \frac{\#\{j;\ \delta N_j(\Lambda,\omega)\geq x\}}{N}.$$

Results: convergence to $x\mapsto e^{-x}$ (Germinet-Kl. (10)).

Another point of view:

Consider the random operator H_ω on the whole space.

Pick J an interval where the Hamiltonian H_ω satisfies (Loc).

Implies that, ω-as, the spectrum in J is made only of eigenvalues associated to eigenfunctions that fall off exponentially.

Assume $\Sigma\cap J = J$.

Proposition (Ge-Kl:10)

Assume (IAD), (M), (W) and (Loc).
With probability 1, if $E\in J$ and φ is a normalized eigenfunction associated to E then, for some $x(E,\omega)\in\mathbb{R}^d$ (or $\mathbb{Z}^d$), a maximum of $x\mapsto\|\varphi\|_x$, for some $C_\omega>0$, one has, for $x\in\mathbb{R}^d$,

$$\|\varphi\|_x \leq C_\omega(1+|x(E,\omega)|^2)^{q/2}e^{-\gamma|x-x(E,\omega)|^\xi}.$$

Moreover, if $N(J,L)$ denotes the number of eigenvalues of H_ω having a center of localization in Λ_L, then

$$N(J,L) = N(J)\,|\Lambda_L|\,(1+o(1)).$$

Enumerate the finitely many eigenvalues of H_ω in J with localization center in Λ_L: $E_1(\omega,\Lambda) \le E_2(\omega,\Lambda) \le \cdots \le E_n(\omega,\Lambda)$.

Pick the scale $(\ell_\Lambda)_\Lambda$ as above and define

$$\Xi^2_\Lambda(e,x;\ell,\omega) = \sum \delta_{N(J)\ell^d_\Lambda[N_J(E_j(\omega,\Lambda))-e]} \otimes \delta_{(x_j(\omega,\Lambda)-Lx)/\ell_\Lambda}.$$

Theorem

Assume (IAD), (W), (M) and (Loc) hold on J s.t. $N(J) > 0$.
Then, ω-almost surely, the law of the point process $\Xi^2_J(\cdot,\cdot;\omega,\ell,\Lambda)$ under the law $g_E \otimes g_X$ converges to the law of the Poisson point process with intensity 1 on $\mathbb{R} \times [-1/c,1/c]^d$.

Results that are local in energy:

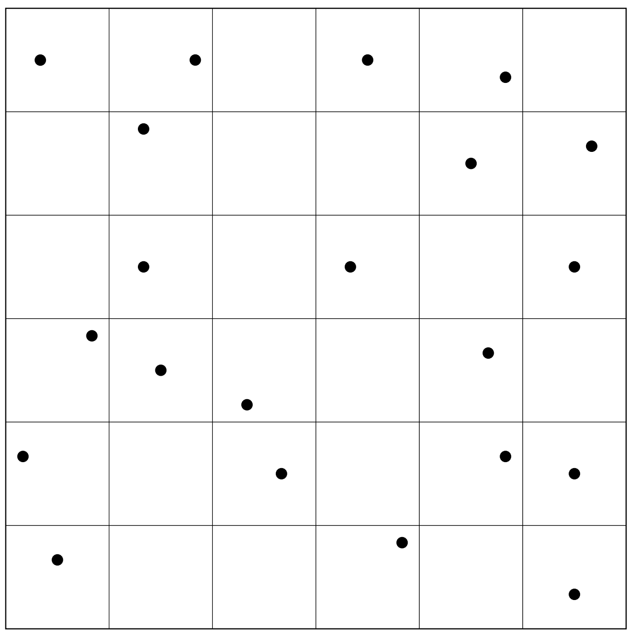
One can prove analogues of these results if J is replaced with J_Λ such that, for some $\tilde\rho > 0$ small enough,

$$(1)\ |J_\Lambda| \to 0, \quad (2)\ N(J_\Lambda)|\Lambda| \to +\infty, \quad (3)\ N(J_\Lambda)|J_\Lambda|^{-1-\tilde\rho} \to +\infty.$$

- (1) describes the local character;
- (2) guarantees that the interval contains asymptotically infinitely many eigenvalues;
- (3) is imposed by the Minami estimate.

Reduction to i.i.d. random variables

Localized regime ⇒ e.v. depend only on local picture of potential ⇒ explicit description of the eigenvalues in I in terms of eigenvalues of smaller independent cubes.



- pick cube of size L
- the localization centers
- Cut large cube into smaller cubes of size ℓ
- Problems:
  - multiple centers in small cubes probability is small due to Minami's estimate: $\ell^{2d}|I|^2(L/\ell)^d = \ell^d L^d |I|^2$
  - centers not localized well in cube probability is small due to Wegner's estimate: $l \cdot L^{d-1} \cdot |I|$
- With good probability, this does not occur.

So with good probability, in I, e.v. of big cube given by e.v. of small cubes.

Theorem (Ge-Kl:10)

Pick $\alpha = (\alpha_N)_N$ s.t. $\lim_N \alpha_N \searrow 0$. Set $\tilde{N} = n/\alpha_N$, $n = n'/\alpha_N$, $n' = (R\log N)^d$, with R large. Set $I_N^\alpha = [a_n, b_n] \subset I$ with $b_N - a_N = n^{1/d} \geq c(\log N)^{-d}\alpha_N^2$.

For any $p > 0$ and N large enough, there exists a set of configurations $\mathscr{Z}_N$ so that

- $\mathbb{P}(\mathscr{Z}_N) \geq 1 - N^{-p}$,
- *for $\omega \in \mathscr{Z}_N$, there exists at least $\dfrac{N}{n}(1 - o(1))$ disjoint boxes $\Lambda_\ell(\gamma_j)$ satisfying the properties:*

  1. *the Hamiltonian $H_\omega(\Lambda_\ell(\gamma_j))$ has at most one eigenvalue in I_N^α, say, $E_j(\omega, \Lambda_\ell(\gamma_j))$;*
  2. *$\Lambda_\ell(\gamma_j)$ contains at most one center of localization, say $x_{k_j}(\omega, L)$, of an eigenvalue of $H_{\omega,L}$ in I_N^α, say $E_{k_j}(\omega, L)$;*
  3. *$\Lambda_\ell(\gamma_j)$ contains a center $x_{k_j}(\omega, L)$ if and only if $\sigma(H_\omega(\Lambda_\ell(\gamma_j))) \cap I_N^\alpha \neq \emptyset$;*

  then, $|E_{k_j}(\omega, L) - E_j(\omega, \Lambda_\ell(\gamma_j))| = O(L^{-\infty})$ and $dist(x_{k_j}(\omega, L), \Lambda_L \setminus \Lambda_\ell(\gamma_j)) \geq L^p$.

Pick $1 \ll \ell' \ll \ell$. Consider $\Lambda = \Lambda_\ell$. Pick an interval $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$
Consider the following random variables:

- $X = X(\Lambda, I_\Lambda)$ is the Bernoulli random variable

$$X = \mathbf{1}_{H_\omega(\Lambda) \text{ has exactly one eigenvalue in } I_\Lambda \text{ with localization center in } \Lambda_{\ell-\ell'}}$$

- $\tilde{E} = \tilde{E}(\Lambda, I_\Lambda)$ is this eigenvalue conditioned on $X = 1$.

Let $\tilde{\Xi}$ be the distribution function of $\tilde{E}$.

Lemma (Ge-Kl:10)

Assume (W), (M) and (Loc) hold. For $\nu \in (0, 1/d)$, one has

$$\begin{aligned}&|\mathbb{P}(X=1) - N(I_\Lambda)|\Lambda|| \\ &\qquad \leq C\left(|\Lambda|^{1+\rho}|I_\Lambda|^{1+\rho} + |I_\Lambda|^{-C}e^{-\ell^\nu/C} + N(I_\Lambda)|\Lambda|\ell'\ell^{-1} + \ell^d e^{-(\ell')^\nu}\right)\end{aligned}$$

Set $N(x,y) := (N(a_\Lambda + x|I_\Lambda|) - N(a_\Lambda + y|I_\Lambda|))|\Lambda|$. For $(x,y) \in [0,1]^2$, one has

$$\begin{aligned}&\left|(\tilde{\Xi}(x) - \tilde{\Xi}(y))P(X=1) - N(x,y,\Lambda)\right| \\ &\leq C\left(|x-y|^{1+\rho}|\Lambda|^{1+\rho}|I_\Lambda|^{1+\rho} + (|I_\Lambda||x-y|)^{-C}e^{-\ell^\nu/C} + N(x,y,\Lambda)\ell'\ell^{-1} + \ell^d e^{-(\ell')^\nu}\right).\end{aligned}$$

Define the random numbers

$$N(I_\Lambda,\Lambda,\omega) := \#\{j;\ E_j(\omega,\Lambda)\in I_\Lambda\}.$$

Write $I_\Lambda = [a_\Lambda, b_\Lambda]$ and recall that $N(I_\Lambda) = N(b_\Lambda) - N(a_\Lambda)$ where N is the integrated density of states.

Theorem (Ge-Kl:10)

Assume (IAD), (W), (M) and (Loc) hold. Assume that $(I_\Lambda)_\Lambda$ is a sequence of compact intervals in the localization region I such that

- *for some $\nu\in(0,1)$, one has $N(I_\Lambda)|\Lambda|^{1-\nu}\to+\infty$,*
- *$N(I_\Lambda)|I_\Lambda|^{-1-\rho}\to+\infty$ as $|\Lambda|\to+\infty$ where ρ is defined in Assumption (M).*

Then, there exists $\delta>0$ such that, for $|\Lambda|$ sufficiently large, one has

$$\mathbb{P}\left(N(I_\Lambda,\Lambda,\omega)-N(I_\Lambda)|\Lambda| = o(N(I_\Lambda)|\Lambda|)\right)\geq 1-e^{-(N(I_\Lambda)|\Lambda|)^\delta}.$$

UNIVERSITÉ PARIS 13

The proof of the asymptotic ergodicity

It suffices to prove that, for $\varphi:\ \mathbb{R}\to\mathbb{R}^+$ continuous and compactly supported,if one sets

$$\mathscr{L}_{\omega,\Lambda}(\varphi) := \mathscr{L}_{\omega,J,\Lambda} := \int_0^1 e^{-\langle\Xi(\omega,t,\Lambda),\varphi\rangle}dt$$

and

$$\langle\Xi(\omega,t,\Lambda),\varphi\rangle := \sum_{E_n(\omega,\Lambda)\in J}\varphi(|\Lambda|[N(E_n(\omega,\Lambda))-t)])$$

then, ω-almost surely,

$$\mathscr{L}_{\omega,\Lambda}(\varphi) \underset{|\Lambda|\to+\infty}{\to} \exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right).$$

Basic idea: split $[0,1]=\cup_n N(I_{n,\Lambda})$ where $I_{n,\Lambda}$ are intervals satisfying

- $I_{n,\Lambda}$ are small enough so that one can apply the reduction theorem;
- $I_{n,\Lambda}$ are large enough so that there are not too many of them.

One computes

$$\int_0^1 e^{-\langle\Xi(\omega,t,\Lambda),\varphi\rangle}dt = \sum_n N(I_{n,\Lambda})\int_0^1 e^{-\langle\Xi_{I_{n,\Lambda}}(\omega,t,\Lambda),\varphi\rangle}dt.$$

UNIVERSITÉ PARIS 13

We are left with analyzing $\int_0^1 e^{-\langle \Xi_{I_{m,\Lambda}}(\omega,t,\Lambda),\varphi\rangle} dt$ where

$$\langle \Xi_{I_{m,\Lambda}}(\omega,t,\Lambda),\varphi\rangle = \sum_{E_n(\omega,\Lambda)\in I_{m,\Lambda}} \varphi(N(I_{m,\Lambda})|\Lambda|[N_{I_{m,\Lambda}}(E_n(\omega,\Lambda))-t]).$$

Want to replace $E_n(\omega,\Lambda)$ with "local" eigenvalues.

- Pb.: this does not control all eigenvalues;
- Sol.: integral in t enables cutting off ngbhd of uncontrolled eigenvalues.
- Pb.: $N(I_{m,\Lambda})$ is not the number of eigenvalues in $I_{m,\Lambda}$ of the "local" random operator;
- Sol.: close by large deviation estimate.
- Pb.: $N_{I_{m,\Lambda}}$ is not the distribution function of the "local" eigenvalues;
- Sol.: close by lemma on their distribution.

Happy 65th, Jean!!!