Non monotonous random Schrödinger operators

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Outline

Introduction: the Anderson model

On $\ell^2(\mathbb{Z}^d)$, consider the standard Anderson model

$$H_\omega = -\Delta + V_\omega \quad \text{where} \quad V_\omega = \sum_{\gamma\in\mathbb{Z}} \omega_\gamma \Pi_\gamma \text{ and } \Pi_\gamma = |\delta_\gamma\rangle\langle\delta_\gamma|.$$

Basic feature: for any γ, the map $\omega_\gamma \mapsto H_\omega$ is monotonous.
Instrumental in the study of Anderson model:

- determine the almost sure spectrum;
- obtain a Wegner estimate estimate i.e. study the local fluctuations of the eigenvalues for a finite volume restriction;
- establish Lifshitz tails i.e. study the band edge behavior of integrated density of states or the probability of "extremal" energies.

Consider now e.g. the alloy type model for $d \geq 2$

$$V_\omega = \sum_{\gamma\in 2\mathbb{Z}^d} \omega_\gamma(\Pi_\gamma - \Pi_{\gamma+e_1}).$$

Many natural models are non monotonous.

The alloy type model

On $\mathbb{R}^d$, consider the standard continuous alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega \quad \text{where} \quad V_\omega(x) = W(x) + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma)$$

where

- $W:\ \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and $\mathbb{Z}^d$-periodic;
- $V:\ \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported;
- $(\omega_\gamma)_\gamma$ are i.i.d. random variables distributed in $[a,b]$, a and b in the support.

H_ω is self-adjoint on $H^2(\mathbb{R}^d)$. It is a metrically transitive family of operators.

Hence, it admits an almost sure spectrum, say, Σ and an integrated density of states, say, $E \mapsto N(E)$. Let $E_- = \inf(\Sigma)$.

The model is non monotonous if V changes sign i.e. we assume

H1 there exists $x_+ \neq x_-$ such that $V(x_-)\cdot V(x_+) < 0$.

The Wegner estimate

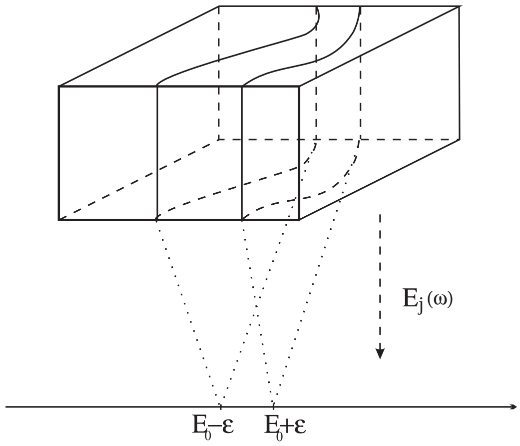
Consider H^L_ω the $(2L+1)\mathbb{Z}^d$-periodic operator

$$H^L_\omega = -\Delta + W + \sum_{\beta\in(2L+1)\mathbb{Z}^d}\ \sum_{\gamma\in\mathbb{Z}^d/(2L+1)\mathbb{Z}^d} \omega_\gamma V(\cdot-\gamma-\beta)$$

on the cube $\mathscr{C}_L = [-L-1/2, L+1/2]^d$ with periodic boundary conditions.

A Wegner estimate is an estimate of the type

$$\mathbb{P}(\{H^L_\omega \text{ has an eigenvalue in } [E_0-\varepsilon, E_0+\varepsilon]\}) \le CL^a\varepsilon^b.$$



The basic idea: such an estimate measures the fluctuation of the eigenvalues of H^L_ω as functions of ω.

Much simpler if this dependence is monotonous.

Theorem (Kl. 1995, Hislop-Kl.2002)

Assume that the distribution of ω_0 conditioned on $(\omega_\gamma)_{\gamma\neq 0}$ is sufficiently regular. Then, for $E_0 < \inf\sigma(-\Delta+W))$ and $\nu\in(0,1)$, there exists $C>0$ such that

$$\mathbb{P}(\{H^L_\omega \textit{ has an eigenvalue in } [E_0-\varepsilon, E_0+\varepsilon]\}) \le CL^d\varepsilon^\nu. \tag{2.1}$$

The case of small perturbations in gaps of $\sigma(-\Delta+W)$:

Let now $H_\omega = -\Delta + W + \lambda V_\omega$.

Theorem (Hislop-Kl.2002)

Assume that the distribution of ω_0 conditioned on $(\omega_\gamma)_{\gamma\neq 0}$ is sufficiently regular. Let (a,b) be a connected component of $\mathbb{R}\setminus\sigma(-\Delta+W)$. Then, for any $\eta>0$, for λ sufficiently small, there exists $C>0$ such that (2.1) holds for $E_0\in[a+\eta\lambda, b-\eta\lambda]$.

Special structure of the single site potential : reduce the problem to a monotonous problem but with dependent random variables (see e.g. Veselic 2002):

$$V(\cdot) = \sum_\gamma \alpha_\gamma v(\cdot-\gamma) \Rightarrow V_\omega(\cdot) = \sum_\beta \tilde\omega_\beta v(\cdot-\beta),\ \tilde\omega_\beta = \sum_\gamma \alpha_\gamma\omega_{\gamma-\beta}.$$

Open problems:

- what happens at high energies?
- what happens for perturbations that are not small?

Lifshitz tails:

Let $\mathscr{C}_0 = [-1/2, 1/2]^d$. We need one more assumption:

H2 V is supported in $\mathscr{C}_0$ and reflection symmetric i.e. for any $\sigma = (\sigma_1, \dots, \sigma_d) \in \{0,1\}^d$ and any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, one has

$$V(x_1, \dots, x_d) = V((-1)^{\sigma_1} x_1, \dots, (-1)^{\sigma_d} x_d).$$

The bottom of the spectrum:

Consider the operator $H_\lambda^N = -\Delta + \lambda V$ with Neumann b.c. on $\mathscr{C}_0$.

Its spectrum is discrete and let $E_-(\lambda)$ be its ground state energy.

It is a simple eigenvalue and $\lambda \mapsto E_-(\lambda)$ is a real analytic concave function.

Proposition

One has $E_- = \inf(\inf \sigma(H_{\overline{a}}), \inf \sigma(H_{\overline{b}})) = \inf(E_-(a), E_-(b))$.

If a and b sufficiently small, Najar proved proposition without H2 but assuming $\int_{\mathbb{R}^d} V(x)dx = E_-'(0) \neq 0$.

Lifshitz tails : when $E_-(a) \neq E_-(b)$

Theorem (Kl.-Nakamura 2008)

Assume $E_-(a) \neq E_-(b)$. *Then* $\limsup_{E \to E_-^+} \dfrac{\log|\log N(E)|}{\log(E - E_-)} \leq -\dfrac{d}{2}$.

Lifshitz tails : when $E_-(a) = E_-(b)$

Theorem (Kl.-Nakamura 2009)

Assume H1 and H2 and $E_-(a) = E_-(b)$. *Then,*

1. *If* $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$, *then* $\limsup_{E \to E_-^+} \dfrac{\log|\log N(E)|}{\log(E - E_-)} \leq -\dfrac{1}{2}$.
2. *If* $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1$, *there exists potentials V satisfying assumption H1 and H2 such that* $E_-(a) = E_-(b)$ *and* $\dfrac{1}{C} \leq N(E)(E - E_-)^{-d/2} \leq C$.

Open problems:

- What happens without the symmetry assumptions H2?

Conjecture: Lifshitz tails (i.e. exponential decay of the IDS at the bottom of the spectrum) hold if random variables are not Bernoulli distributed.

Generalized alloy type models

On $L^2(\mathbb{R}^d)$, consider $H_\omega = -\Delta + W + V_\omega$ where $V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} v_{\omega(\gamma)}(x-\gamma)$ and $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are i.i.d. random variables with values in $\{1,\dots,M\}$.
On $L^2(\mathscr{C}_0)$, define $H_k^N = -\Delta + W + v_k$ with Neumann b.c.
We assume

- A1 W is symmetric about the plane $\{x_d = 0\}$.
- A2 There exists $m \in \{1,\dots,M\}$ such that $\inf\sigma(H_k^N) = 0$ for $k = 1,\dots,m$, and $\inf\sigma(H_k^N) > 0$ for $k > m$.
- A3 Moreover, for $k = 1,\dots,m$, $v_k(x)$ is symmetric about $\{x_d = 0\}$.

Theorem (Kl.-Nakamura 2009)

Suppose Assumption A with $m < M$. Then, $\limsup_{E\to+0} \frac{\log|\log N(E)|}{\log E} \le -\frac{1}{2}$.

Let $(e_1,\cdots,e_d)$ be the standard basis of $\mathbb{R}^d$ and define $U_j = \mathscr{C}_0 \cup (e_j + \mathscr{C}_0)$.
Define

- B In addition to satisfying Assumption A, W and v_k are symmetric about $\{x_j = 0\}$ for all $j = 1,\dots,d$, and $k = 1,\dots,m = M$.

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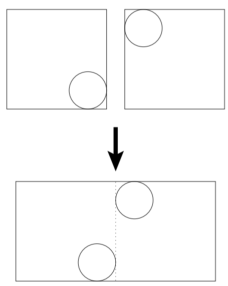
On U_j, with Neumann b.c., define the operator

$$H^N_{k\ell(j)} = \begin{cases} -\Delta + W(x) + v_k(x) & \text{on } \mathscr{C}_0 \\ -\Delta + W(x) + v_\ell(x-e_j) & \text{on } e_j + \mathscr{C}_0 \end{cases}$$



We define $v_k \underset{j}{\sim} v_\ell$ if and only if $\inf\sigma(H^N_{k\ell(j)}) = 0$.

Theorem (Kl.-Nakamura 2009)

Suppose Assumption A with $m = M$. Suppose moreover that $v_k \underset{d}{\not\sim} v_\ell$ for some $k \neq \ell$.

Then, $\limsup_{E\to+0} \frac{\log|\log N(E)|}{\log E} \le -\frac{1}{2}$.

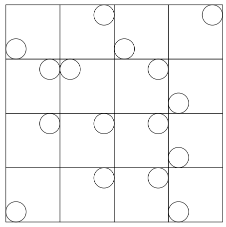
Theorem (Kl.-Nakamura 2009)

Suppose Assumption B. Then

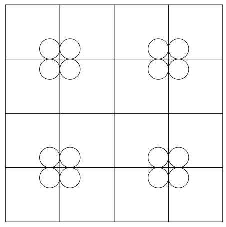
1. *If $v_k \underset{j}{\not\sim} v_\ell$ for some j and $k \neq \ell$, then* $\limsup_{E\to+0} \frac{\log|\log N(E)|}{\log E} \le -\frac{1}{2}$.
2. *If $v_k \underset{j}{\sim} v_\ell$ for all j and k,ℓ, then* $\frac{1}{C} \le N(E)E^{-d/2} \le C$ *for some* $C > 0$.

The random displacement model

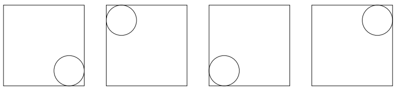
Consider $H_\omega = -\Delta + V_\omega$ where $V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} V(x-\gamma-\xi_\gamma)$ and



- $V: \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing and supported in $(-r,r)^d$, $0<r<1/2$ and satisfies H2;
- $(\xi_\gamma)_\gamma$ are independent identically distributed (i.i.d.) random variables distributed in $\{-1/2+r, 1/2-r\}^d$ such that all these points have a positive probability;



For $\xi \in \{-1/2+r, 1/2-r\}^d$, we define $H_\xi = -\Delta + V(x-\xi)$ on $\mathscr{C}_0$ with Neumann b.c. All the $(H_\xi)_\xi$ have the same ground state energy, say 0.



Theorem (Baker-Loss-Stolz 2008, 2009)

If there exists $\xi_{1,2}$ and j such that $H_{\xi_1} \underset{j}{\not\sim} H_{\xi_2}$, then the minimizing configurations are given by a symmetric "clusterization", and they are the only ones.
If, for all ξ_1, ξ_2 and j, one has $H_{\xi_1} \underset{j}{\sim} H_{\xi_2}$, then all the configurationhave the same ground state energy 0.

Theorem (Baker-Loss-Stolz 2009)

If $d=1$, then, for some $c>0$, when $E\to+0$, one has $N(E)\geq c\log^{-2}(E)$.

Theorem (Kl.-Nakamura 2009)

Let $d\geq 2$ and $N(E)$ denote the IDS of H_ω.

1. *If there exists $\xi_{1,2}$ and j such that $H_{\xi_1} \underset{j}{\not\sim} H_{\xi_2}$ then,* $\limsup_{E\to+0} \frac{\log|\log N(E)|}{\log(E)} \leq -\frac{1}{2}$;
2. *If, for all ξ_1, ξ_2 and j, one has $H_{\xi_1} \underset{j}{\sim} H_{\xi_2}$, then $N(E)\geq cE^{d/2}$.*

Wegner estimates: No results for this model!
Consider $H_{\omega,\lambda} = -\Delta + \lambda V_\omega$ for $V_\omega(x)$ displacement model as above where V need not satisfy the symmetry assumption, is assumed to be smooth, non positive and to have a single global minimum that is non degenerate.

Theorem (Kl. 1993)

$H_{\omega,\lambda}$ admit a Wegner estimate near the bottom of its spectrum.

Obtained through analysis of the tunnel effect.

Small random displacements:

Consider $H_{\lambda,\xi} = -\Delta + W + V_\xi$ where $V_\xi(x) = \sum_{\gamma\in\mathbb{Z}^d} V(x-\gamma-\lambda\xi_\gamma)$ where

- the potential W is a real valued, $\mathbb{Z}^d$-periodic function;
- $V:\ \mathbb{R}^d \to \mathbb{R}$ is C^2, non identically vanishing and compactly supported;
- $(\xi_\gamma)_\gamma$ is a collection of non trivial, independent, identically distributed, bounded random variables; let K be the support of their common distribution.
- λ is a small positive coupling constant.

Define $H_{\lambda,\xi,n} = -\Delta + W(\cdot) + \sum_{\beta\in(2n+1)\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d} V(\cdot-\beta-\gamma-\lambda\xi_\gamma)$.

Let $H^P_{\lambda,\xi,n}$ be the restriction of $H_{\lambda,\xi,n}$ to the cube $\mathscr{C}_n = [-n-1/2, n+1/2]^d$ with periodic boundary conditions.

Define $H_\zeta = H_{\lambda,\overline{\zeta}} = -\Delta + W(\cdot) + \sum_{\gamma\in\mathbb{Z}^d} V(\cdot-\gamma-\lambda\zeta)$.

Assume that

H1 there exits $\lambda_0 > 0$ such that, for $\lambda \in (0,\lambda_0)$, there exists a unique point in K, say, $\zeta(\lambda)$, so that $E(\lambda,\zeta(\lambda)) = \min_{\zeta\in K} E(\lambda,\zeta)$;

H2 there exists $\alpha_0 > 0$ such that, for $\lambda \in (0,\lambda_0)$ and $\zeta \in K$, one has

$$\nabla_\zeta E(\lambda,\zeta(\lambda)) \cdot (\zeta-\zeta(\lambda)) \geq \alpha_0 \lambda |\zeta-\zeta(\lambda)|^2.$$

Theorem (Ghribi-Kl. 2009)

Under assumption H1 and H2, there exists $\lambda_0 > 0$ such that, for any $n \geq 0$, for $\lambda \in (0,\lambda_0]$, on $K^{(2n+1)^d}$, the function $\xi \mapsto E^n_0(\lambda\xi)$ reaches its infimum $E(\lambda,\zeta(\lambda))$ at a single point, the point $\xi = (\zeta(\lambda))_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d}$.

So $E_\lambda = \inf(\sigma(H_{\lambda,\omega})) = E(\lambda,\zeta(\lambda))$.

The minimum is reached at a single point like in the monotonous case.

Lifshitz tails for small displacements:

Theorem

Under assumptions H1 and H2, there exists $\lambda_0 > 0$ such that, for all $\lambda \in]0,\lambda_0]$,

$$\lim_{E\to E_\lambda} \frac{\log|\log(N_\lambda(E)-N_\lambda(E_\lambda)|}{\log(E-E_\lambda)} \leq -\frac{d}{2}$$

Wegner estimates for small displacements:

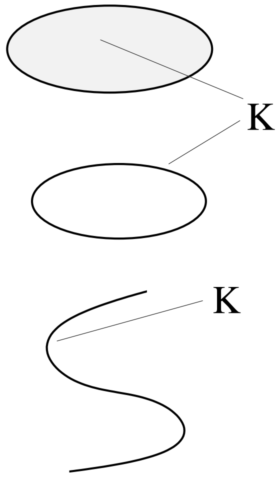
Assume

- **H3** There exists $C > 0$ such that, for λ sufficiently small, one has $E_\lambda \leq E_0 - \lambda/C$.

Clearly, the first theorem on the small displacement shows that this assumption is a consequence of assumption H1.
Assume

- **H4** for almost all $\sigma \in \mathbb{S}^{d-1}$, the distribution of $r_\sigma(\xi_0)$ admits a density with respect to the Lebesgue measure, say, h_σ that itself is absolutely continuous; moreover, one has

$$\operatorname*{ess-sup}_{\sigma \in \mathbb{S}^{d-1}} \|h'_\sigma\|_\infty < +\infty.$$



Theorem (Hislop-Kl. 2002, Ghribi-Kl. 2009)

Fix $\nu \in (0,1)$. Under assumptions H3 and H4, there exists $\lambda_0 > O$ such that, for $\lambda \in (0, \lambda_0]$, there exists $C_\lambda > 0$ such that, for all $E \in [E_\lambda, E_\lambda + \lambda/C]$ and $\varepsilon > 0$,

$$\mathbb{P}(dist(\sigma(H^P_{\lambda,\xi,n}), E) \leq \varepsilon) \leq C_\lambda \varepsilon^\nu n^d.$$

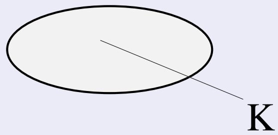
The validity of assumption H1 and H2:

Let E_0 be the infimum of $\sigma(H_{\lambda,0})$ and φ_0 be the positive normalized ground state of the periodic b.c. operator $H_{\lambda,0}$ considered on $\mathscr{C}_0$.

Proposition (Ghribi-Kl. 2009)

Assume that K is a strictly convex set with C^2-boundary and V be such that $v(V) := -\int_{\mathbb{R}^d} \nabla V(x)|\varphi_0(x)|^2 dx \neq 0$.

Then, assumption H1 is satisfied.

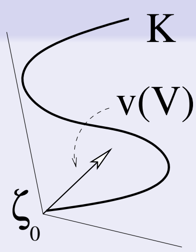


Non vanishing condition satisfied for small generic V.

Proposition (Ghribi-Kl. 2009)

Assume that $v(V) \neq 0$ and that there exists $\varepsilon > 0$ and $\zeta_0 \in K$, such that, for all $\zeta \in K$ and $|v - v(V)| < \varepsilon$, one has $v \cdot (\zeta - \zeta_0) \geq 0$.

Then, assumption H1 is satisfied, and, for λ small, $\zeta(\lambda) = \zeta_0$.



Random magnetic fields:

Let $x \in \mathbb{R}^d \mapsto A_\omega(x)$ be a random vector field and, on $L^2(\mathbb{R}^d)$, consider

$$H_\omega = (i\nabla - A_\omega)^2.$$

Assume that

- $B_\omega = dA_\omega$ is a $\mathbb{R}^d$-ergodic closed 2-form;
- B_ω is almost surely bounded;
- $B_\omega(x)\mathbf{1}_{x\in\Lambda}$ and $B_\omega(x)\mathbf{1}_{x\in\Lambda'}$ satisfy a decorrelation condition when $d(\Lambda,\Lambda')$ grows.

Theorem (Nakamura 2000)

Then, the integrated density of states exhibits Lifshitz tails at 0.

No Wegner estimate known for such operators.

Also results by Ueki, Leschke, Warzel, Weichlein (continuous) and, in the discrete case, by Nakamura and Kl.-Nakamura-Nakano-Nomura.

Small random magnetic potential:

On $\mathbb{R}^d$, consider $H_{\lambda,\omega} = (i\nabla + A_0 + \lambda A_\omega)^2 + V_0$ where

- A_0 and V_0 are smooth and $\mathbb{Z}^d$-periodic,
- $A_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma a(x-\gamma)$,
- the single-site vector potential a is a real, smooth, vector-valued function of compact support,
- the $(\omega_\gamma)_\gamma$ are bounded i.i.d. random variables with a smooth density,
- λ is a small real parameter.

We assume that $H_0 = H_{0,\omega}$ has an internal gap that is

$$\sigma(H_0)\cap(E_{-1},E_1) = (E_{-1},E_-]\cup[E_+,E_1), \quad E_{-1} < E_- < E_+ < E_1.$$

Assume

H1 The edge of the spectrum E_+ is *simple* i.e. it is attained by a single Floquet eigenvalue $E_{n_0}(\theta)$.

H2 The IDS $N_0(E)$ of H_0 at E_+ is *non degenerate* that is

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1}\log(N_0(E_+ + \varepsilon) - N_0(E_+)) = \frac{d}{2}$$

An assumption to control local behavior of eigenvalues:

H3 The matrix $M \equiv (M_{kk'})_{1\le k,k'\le m}$, with matrix elements given by

$$M_{kk'} = \int_{\mathscr{C}_0} [(a(x)\cdot i\nabla + i\nabla\cdot a(x) + 2a(x)\cdot A_0)\phi_{0,n_0}](\theta_k,x)\overline{\phi_{0,n_0}(\theta_{k'},x)}\,dx$$

is either positive or negative definite.

Let $(E_-,E_-(\lambda))\cup(E_+(\lambda),E_+) = \sigma(H_{\lambda,\omega})\cap(E_-,E_+)$.

Theorem (Ghribi 2007)

Under the assumptions given above, there exists $\lambda_0 > 0$ and $C > 0$ so that, for all $\lambda\in[0,\lambda_0]$, one has $E_+ - C\lambda \le E_+(\lambda) \le E_+ - \lambda/C$ and

$$\lim_{E\to E_+(\lambda)^+} \frac{\log|\log(N_\lambda(E)-N_\lambda(E_+(\lambda)))|}{\log(E-E_+(\lambda))} = -\frac{d}{2}.$$

A similar statement holds at the lower band edge $E_-(\lambda)$.

For $G=(E_-,E_+)$, a gap in $\sigma(H_0)$ and $\eta_0>0$, let $G_{\eta_0}(\lambda) = (E_-+\eta_0\lambda, E_+-\eta_0\lambda)$

Theorem (Hislop-Kl. 2002, Ghribi-Hislop-Kl. 2007)

Fix $q>1$. Under the assumptions stated above, there exists $\lambda_0>0$, $\eta_0>0$ and $C_0>0$ such that, for all $|\lambda|<\lambda_0$, $\eta\in(0,\eta_0)$, $E_0\in G_{2\eta}(\lambda)$, $n\ge 1$ and $\delta\in(0,\eta\lambda)$, one has

$$\mathbb{P}\{\,dist\,(\sigma(H^P_{\lambda,\omega,n}),E_0)\le\delta\}\ \le C_\eta\lambda^{-1}\delta^{1/q}n^d$$

where $H^P_{\lambda,\omega,n}$ is the operator $(i\nabla + A_0 + \lambda A^n_\omega)^2 + V_0$ restricted to $\mathscr{C}_n$ with periodic b.c. and

$$A^n_\omega(\cdot) = \sum_{\beta\in(2n+1)\mathbb{Z}^d}\ \sum_{\gamma\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d} \omega_\gamma a(\cdot-\gamma-\beta).$$

One then obtains localization at the edge of the spectrum.

Related results by Ueki (2008).

Open problems:

- What happens when the random magnetic potentials are not small?
- What happens for non zero flux random magnetic operators?