Inverse tunneling estimates and applications to the study of spectral statistics of random operators on the real line

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Two examples

Consider

- the Anderson model

$$H^A_\omega = -\frac{d^2}{dx^2} + W(\cdot) + \sum_{n\in\mathbb{Z}} \omega_n V(\cdot - n)$$

  where
  - $W: \mathbb{R} \to \mathbb{R}$ is a bounded, continuous, $\mathbb{Z}$-periodic function;
  - $V: \mathbb{R} \to \mathbb{R}$ is a bounded, continuous, compactly supported, non negative, not identically vanishing function;
  - $(\omega_n)_{n\in\mathbb{Z}}$ are bounded i.i.d random variables, the common distribution of which admits a continuous density.

- the random displacement model

$$H^D_\omega = -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} V(\cdot - n - \omega_n) \tag{1.1}$$

  where
  - $V: \mathbb{R} \to \mathbb{R}$ is a bounded, odd function that has a fixed sign and is compactly supported in $(-r_0, r_0)$ for some $0 < r_0 < 1/2$;
  - $(\omega_n)_{n\in\mathbb{Z}}$ are bounded i.i.d random variables, the common distribution of which admits a continuously differentiable density supported in $[-r, r] \subset [-1/2 + r_0, 1/2 - r_0]$ and which support contains $\{-r, r\}$.

Let $\bullet \in \{A, D\}$. Consider $H^\bullet_{\omega,L} := (H^\bullet_\omega)_{|[-L,L]}$.

The spectrum of this operator is discrete and accumulates at $+\infty$; denote it by

$$E^\bullet_1(\omega, L) < E^\bullet_2(\omega, L) \leq \cdots \leq E^\bullet_n(\omega, L) \leq \cdots$$

ω almost surely, the *integrated density of states* is defined as

$$N^\bullet(E) = \lim_{L \to +\infty} \frac{\#\{n;\ E^\bullet_n(\omega, L) \leq E\}}{2L}.$$

There exists $\tilde{E}^\bullet \in (\inf \Sigma^\bullet, +\infty]$ such that $N^\bullet$ is Lipschitz continuous on $(-\infty, \tilde{E}^\bullet)$ ([CHKl 07], [KlLNS 10]).

The unfolded levels: Fix E_0. Define the *locally unfolded levels* as

$$\xi^\bullet_n(E_0, \omega, L) = 2L\,[N^\bullet(E^\bullet_n(\omega, L)) - N^\bullet(E_0)].$$

Define the point process

$$\Xi^\bullet(\xi; E_0, \omega, L) = \sum_{n \geq 1} \delta_{\xi^\bullet_n(E_0, \omega, L)}(\xi),$$

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The local level statistics

Theorem

There exists an energy $\inf \Sigma^\bullet < E^\bullet \leq \tilde{E}^\bullet$ *and such that, if* $E_0 \in (-\infty, E^\bullet) \cap \Sigma^\bullet$ *satisfies, for some* $\rho \in [1, 4/3)$*, one has*

$$\forall a > b,\ \exists C > 0,\ \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0),\ |N^\bullet(E_0 + a\varepsilon) - N^\bullet(E_0 + b\varepsilon)| \geq C\varepsilon^\rho \quad (1.2)$$

then, when $L \to +\infty$*, the point process* $\Xi(E_0, \omega, L)$ *converges weakly to a Poisson process on* $\mathbb{R}$ *with intensity the Lebesgue measure.*

If $E \mapsto N(E)$ is differentiable at E_0 and $dN/dE(E_0) > 0$, then (1.2) is satisfied.

What about $E^\bullet$?

$E^\bullet$ such that

- $H^\bullet_\omega$ localized in $(-\infty, E^\bullet]$;
- $H^\bullet_\omega$ satisfies a Wegner estimate in $(-\infty, E^\bullet]$.

Thus, $E^A = +\infty$ and $E^D > \inf \Sigma^D$.

More results on spectral statistics:

For $J = [a,b]$, a compact interval s.t. $N^\bullet(J) := N^\bullet(b) - N^\bullet(a) > 0$ and a fixed configuration ω, consider the point process

$$\Xi_J^\bullet(\omega, t, \Lambda) = \sum_{E_n^\bullet(\omega,\Lambda)\in J} \delta_{N^\bullet(J)|\Lambda|[N_J^\bullet(E_n^\bullet(\omega,\Lambda))-t]}$$

under the uniform distribution in $[0,1]$ in t; here, we have set

$$N_J^\bullet(\cdot) := \frac{N^\bullet(\cdot) - N^\bullet(a)}{N^\bullet(b) - N^\bullet(a)}.$$

The values $(N^\bullet(E_n^\bullet(\omega,L)))_{n\geq 1}$ are called the *unfolded eigenvalues* of the operator $H_{\omega,L}^\bullet$.

Theorem

Fix $J = [a,b] \subset (-\infty, E^\bullet) \cap \Sigma^\bullet$ *a compact interval such that* $N^\bullet(b) - N^\bullet(a) = N^\bullet(J) > 0$. *Then,* ω*-almost surely, the probability law of the point process* $\Xi_J^\bullet(\omega,\cdot,\Lambda)$ *under the uniform distribution* $\mathbf{1}_{[0,1]}(t)dt$ *converges to the law of the Poisson point process on the real line with intensity* 1.

The levelspacing statistics

Define the n-th unfolded eigenvalue spacings

$$\delta N_n^\bullet(\omega,L) = 2LN^\bullet(J)(N^\bullet(E_{n+1}(\omega,L)) - N^\bullet(E_n(\omega,L))) \geq 0.$$

Define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS^\bullet(x;J,\omega,L) = \frac{\#\{n;\ E_n^\bullet(\omega,L) \in J,\ \delta N_n^\bullet(\omega,L) \geq x\}}{N^\bullet(J,\omega,L)}$$

where $N^\bullet(J,\omega,L) := \#\{E_n^\bullet(\omega,L) \in J\}$.

Theorem

Under the assumptions of the previous theorem, ω-almost surely, as $L \to +\infty$, $DLS^\bullet(x;J,\omega,L)$ converges uniformly to the distribution $x \mapsto e^{-x}$.

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A Minami type estimate in the localization regime: the setting

On $L^2(\mathbb{R})$, consider a random Schrödinger operator of the form

$$H_\omega u = -\frac{d^2}{dx^2} u + q_\omega u \tag{2.1}$$

where q_ω is an almost surely bounded $\mathbb{Z}^d$-ergodic random potential.
Let N and Σ be the integrated density of states and almost sure spectrum of H_ω.

Fix $I \subset \Sigma$ an interval. Let $\Lambda := \Lambda_L := [0, L]$ and $H_\omega(\Lambda) = (H_\omega)_{|\Lambda}$.

Assume

- (IAD) There exists $R_0 > 0$ such that for $\mathrm{dist}(\Lambda, \Lambda') > R_0$, the random Hamiltonians $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are independent.
- (W) there exists $C > 0$, $s \in (0, 1]$ and $\rho \geq 1$ such that, for $J \subset I$, and Λ, an interval in $\mathbb{R}$, one has

$$\mathbb{P}(\{\sigma(H_\omega(\Lambda)) \cap J \neq \emptyset\}) \leq \mathbb{E}\left[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))\right] \leq C|J|^s |\Lambda|^\rho.$$

For H^A on the whole axis, [CHKl 07]. For H^D near the bottom of the spectrum, [KlLNS 10].

The second assumption crucial to our study is the existence of a localization region to which I belongs i.e. we assume

(Loc) there exists $\xi > 0$ such that

$$\sup_{\substack{L>0 \\ \mathrm{supp} f \subset I \\ |f| \leq 1}} \mathbb{E}\left(\sum_{n\in\mathbb{Z}} e^{\xi |n|} \| \mathbf{1}_{[-1/2,1/2]} f(H_\omega(\Lambda_L)) \mathbf{1}_{[n-1/2,n+1/2]} \|_2 \right) < +\infty.$$

Many results: many models in 1D ([S], [S]). In higher dimension, mainly Anderson model [GeK 01-11], [A. et al. 06]

For H^A on the whole axis, [DS 10]. For H^D near the bottom of the spec., [KlLNS 10].

Missing ingredient for spectral statistics:

Minami's estimate i.e. estimate on

$$\begin{aligned} \mathbb{P}(\{\#[\sigma(H_\omega(\Lambda)) \cap J] \geq 2\}) &\leq \sum_{k\geq 2} \mathbb{P}(\mathrm{tr}\,[\mathbf{1}_J(H_\omega(\Lambda))] \geq k) \\ &\leq \mathbb{E}\left[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))(\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) - 1)\right]. \end{aligned}$$

Known for special 1D model ([Mol 81]), discrete models (rank one pert. [Mi 96], [BHS 07], [GV 07] [CGK 09]) and continuous model (more assumptions) at bottom of spectrum [CGK 10].

A Minami type estimate in the localization region

Our main technical result is the following Minami type estimates.

Theorem

Assume (W) and (Loc). Fix J compact in I the region of localization. Then, there exists $\beta > 0$, such that, for any $q > 0$, there exists $L_q > 0$ s.t., for $E \in J$, $L \geq L_q$ and $\varepsilon \in \left[L^{-q}, (\log L)^{-1-\beta}\right]$, one has

$$\sum_{k\geq 2} \mathbb{P}\left(\mathrm{tr}\left[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))\right] \geq k\right) \leq \beta\left(\varepsilon^s L(\log L)^\beta\right)^2 e^{\beta\varepsilon^s L(\log L)^\beta} + L^{-q}.$$

This estimate useful when $\varepsilon^s L$ is small (same as (W)).

Stronger standard Minami estimate [Mi 96]: bd on $\mathbb{P}\left(\mathrm{tr}\left[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))\right] \geq 2\right)$.

Weaker than the Minami type estimate in [CGK 09]: bd on

$$\mathbb{E}\left[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))(\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))-1)\right] = \sum_{k\geq 2} k\,\mathbb{P}\left(\mathrm{tr}\left[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))\right] \geq k\right)$$

But nevertheless sufficient to repeat the analysis done in [G-Kl 10] and [Kl 10].

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Universal estimates on the eigenvalue counting function

Assume $q : [0,\ell] \to \mathbb{R}$ is bounded.

On $[0,\ell]$, consider the operator $Hu = -u'' + qu$ with self-adjoint Robin boundary conditions at 0 and ℓ (i.e. $u(0)\cos\alpha + u'(0)\sin\alpha = 0$).

Theorem

Fix J compact. There exists a constant $C > 0$ (depending only on $\|q\|$ and J) such that, for $\ell \geq 1$, if $\varepsilon \in (0,1)$ is such that $|\log\varepsilon| \geq C\ell$, then, for any $E \in J$, the interval $[E-\varepsilon, E+\varepsilon]$ contains at most a single eigenvalue of H.

Theorem

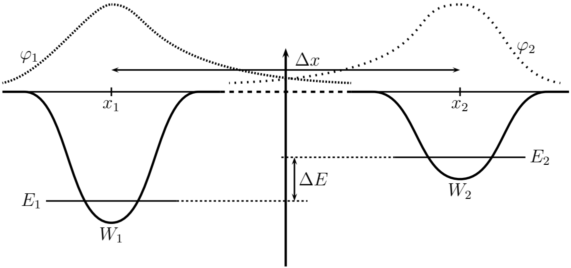
Fix $\nu > 2$ and J compact. There exists $\ell_0 > 1$ and $C > 0$ (depending only on $\|q\|_\infty$ and J) such that, for $\ell \geq \ell_0$, if $\varepsilon \in (0, \ell^{-\nu})$ then, for $E \in J$, the number of eigenvalues of H in the interval $[E-\varepsilon, E+\varepsilon]$ is bounded by $\max(1, C\ell/|\log\varepsilon|)$.

Level repulsion at much smaller scales than levelspacing of random systems.

A heuristic: tunneling and inverse tunneling

Fix $\ell \in \mathbb{R}$ and $q : [0,\ell] \to \mathbb{R}$ a bounded real valued function.
On $[0,\ell]$, consider the Dirichlet eigenvalue problem

$$-\frac{d^2}{dx^2}u(x) + q(x)u(x) = Eu(x), \quad u(0) = u(\ell) = 0.$$



Tunneling occurs if $\Delta E \ll e^{-\Delta x}$.

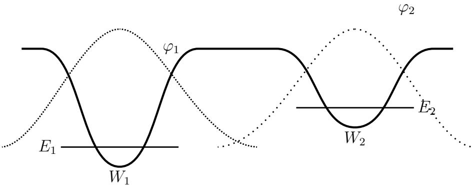


Figure 5: the asymmetric double well.

No tunneling occurs: no overlap between eigenfunctions.

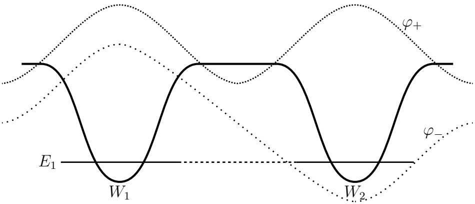


Figure 6: the symmetric double well.

Tunneling occurs: large overlap between eigenfunctions.

Define $r_{\varphi_j} := \sqrt{|\varphi_j|^2 + |\varphi_j'|^2}$ $(j \in \{1, 2\})$.

Two cases:

- if $r_{\varphi_1} \cdot r_{\varphi_2}$ becomes "large" over $[0, \ell]$: the "tunneling case";
- if $r_{\varphi_1} \cdot r_{\varphi_2}$ stays "small" over $[0, \ell]$: the "non tunneling case".

Theorem

Fix $S > 0$ arbitrary and $J \subset \mathbb{R}$ a compact interval. There exists $\varepsilon_0 > 0$ and $\ell_0 > 0$ (depending only on $\|q\|_\infty$, J and S) such that, for $\ell \geq \ell_0$ and $0 < \varepsilon\ell^4 \leq \varepsilon_0$, for $E \in J$, if the operator H defined above has two eigenvalues in $[E-\varepsilon, E+\varepsilon]$, then there exists two points x_+ and x_- in the lattice segment $\varepsilon_0\mathbb{Z} \cap [0,\ell]$ satisfying $S < x_+ - x_- < 2S$ such that, if H_-, resp. H_+, denotes the second order differential operator H defined above and Dirichlet boundary conditions on $[0, x_-]$, resp. on $[x_+, \ell]$, then H_- and H_+ each have an eigenvalue in the interval $[E - \varepsilon\ell^4/\varepsilon_0, E + \varepsilon\ell^4/\varepsilon_0]$.

Assume $E = 0$ and eigenvalues considered in theorem are 0 and $E \in (0, \varepsilon]$.

Let u and v be the eigenfunctions associated respectively to 0 and E.

Goal: find linear combinations of u and v such that

- they vanish at two points, say, x_- and x_+ satisfying the statement of the theorem,
- in $[0, x_-]$ and $[x_+, \ell]$, their masses are of order $\ell^{-\alpha}$ (for $\alpha > 0$ not too large).

Therefore, consider two cases:

1. if $r_u \cdot r_v$ becomes "large" over $[0, \ell]$: the "tunneling case";
2. if $r_u \cdot r_v$ stays "small" over $[0, \ell]$: the "non tunneling case".

In the "no tunneling case", u and v put mass only at different locations locations in $[0, \ell]$.

r_u and r_v are also almost orthogonal.

Using linear combination, one can split $[0, \ell]$ in two by Dirichlet bc.

In the "no tunneling case", u and v put mass at the same locations in $[0, \ell]$

u and v are quite similar except for a phase change.

Analyze this phase difference to show that, with linear combination, one can split $[0, \ell]$ in two by Dirichlet bc.

And localization?

Localization implies $[0, L]$ reduced to $[0, \ell]$ where $\ell \asymp (\log L)^\beta$.