Decorrelation estimates for the eigenlevels of random operators in the localized regime

F. Klopp

Université Paris 13
and
Institut Universitaire de France

Outline

1. The setting and the results
   - The Anderson model in the localized regime
   - Local renormalized level distribution
   - The independence
   - The decorrelation lemmas
2. Ideas of the proof
   - Basic idea
   - Reduction to localization boxes
   - Analysis on a localization box
   - The fundamental estimate
   - Completing the proof of the decorrelation lemma
   - The proof of the fundamental estimate: case 1
   - The proof of the fundamental estimate: case 2

The Anderson model in the localized regime

On $\ell^2(\mathbb{Z}^d)$, we consider the Anderson model $H_\omega = -\Delta + V_\omega$ where $V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma \pi_\gamma$ and

- $-\Delta$ is the standard discrete Laplacian,
- π_γ is the orthogonal projector on δ_γ,
- the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are non trivial, i.i.d. bounded and admit a bounded density.

$$\begin{pmatrix} \omega_1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & \omega_2 & 1 & & & \vdots \\ 0 & 1 & \omega_3 & 1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & 0 & 1 & \omega_{n-1} & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \omega_n \end{pmatrix}$$

Well known : there exists a set, say $I \subset \mathbb{R}$, such that, in I, the spectrum of H_ω is localized.

Pick $E \in I$ and $L \in \mathbb{N}$. Let $\Lambda = \Lambda_L = [-L,L]^d \cap \mathbb{Z}^d \subset \mathbb{Z}^d$ and $H_\omega(\Lambda) = H_{\omega|\Lambda}$ (per. BC).

Denote its eigenvalues by $E_1(\omega,\Lambda) \le E_2(\omega,\Lambda) \le \cdots \le E_N(\omega,\Lambda)$.

The local level statistics near E is the point process defined by

$$\Xi(\xi,E,\omega,\Lambda) = \sum_{j=1}^{N} \delta_{\xi_j(E,\omega,\Lambda)}(\xi) \quad \text{where} \quad \xi_j(E,\omega,\Lambda) = |\Lambda|\,\nu(E)\,(E_j(\omega,\Lambda) - E).$$

Theorem (Molchanov,Minami)

Assume that $\nu(E) > 0$. When $|\Lambda| \to +\infty$, the point process $\Xi(,\omega,\Lambda)$ converges weakly to a Poisson process on $\mathbb{R}$ with intensity the Lebesgue measure.

Question: pick $E_0 \in I$ and $E_0' \in I$ such that $E_0 \neq E_0'$, $\nu(E_0) > 0$ and $\nu(E_0') > 0$;

Are the point processes $\Xi(E_0,\omega,\Lambda)$ and $\Xi(E_0',\omega,\Lambda)$ asymptotically independent?

Not much known about this question for random Schrödinger operators.

Results for random matrices.

The answer may be model dependent:

$$\begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \omega_{2n} \end{pmatrix} \qquad \begin{pmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_1+1 & 0 & \cdots & 0 \\ \vdots & 0 & \omega_2 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \omega_n+1 \end{pmatrix}$$

The independence

Theorem (Ge-Kl,Kl)

Assume that the dimension $d = 1$.When $|\Lambda| \to +\infty$, the point processes $\Xi(E_0, \omega, \Lambda)$ and $\Xi(E_0', \omega, \Lambda)$ converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure. That is, for $U_+ \subset \mathbb{R}$ and $U_- \subset \mathbb{R}$ compact intervals and $\{k_+, k_-\} \in \mathbb{N} \times \mathbb{N}$, one has

$$\mathbb{P}\left(\left\{\omega; \left\{\begin{array}{c} \#\{j; \xi_j(E_0, \omega, \Lambda) \in U_+\} = k_+ \\ \#\{j; \xi_j(E_0', \omega, \Lambda) \in U_-\} = k_- \end{array}\right\}\right) \underset{\Lambda \to \mathbb{Z}^d}{\to} e^{-|U_+|} \frac{|U_+|^{k_+}}{k_+!} \cdot e^{-|U_-|} \frac{|U_-|^{k_-}}{k_-!}.$$

Theorem (Ge-Kl,Kl)

Pick $E_0 \in I$ and $E_0' \in I$ such that $|E_0 - E_0'| > 2d$, $\nu(E_0) > 0$ and $\nu(E_0') > 0$. When $|\Lambda| \to +\infty$, the point processes $\Xi(E_0, \omega, \Lambda)$ and $\Xi(E_0', \omega, \Lambda)$ converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure.

UNIVERSITÉ PARIS 13

The decorrelation lemmas

Lemma (Kl)

For the discrete Anderson model , fix $\alpha \in (0,1)$, $\beta \in (1/2, 1)$ and $\{E_0, E_0'\} \subset I$ s.t. $|E_0 - E_0'| > 2d$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has

$$\mathbb{P}\left(\left\{\begin{array}{c} \sigma(H_\omega(\Lambda_\ell)) \cap (E_0 + L^{-d}(-1,1)) \neq \emptyset, \\ \sigma(H_\omega(\Lambda_\ell)) \cap (E_0' + L^{-d}(-1,1)) \neq \emptyset \end{array}\right\}\right) \leq C(\ell/L)^{2d} e^{(\log L)^\beta}.$$

Lemma (Kl)

Assume $d = 1$. For the discrete Anderson model, for $\alpha \in (0,1)$ and $\{E_0, E_0'\} \subset I$ s.t. $E_0 \neq E_0'$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, the result of the previous theorem holds.

Another decorrelation estimate: the Minami estimate

Theorem (Min, GV, BHS, CGK)

For $J \subset K$, one has

$$\mathbb{E}\left[tr[\mathbf{1}_J(H_\omega(\Lambda))] \cdot (tr[\mathbf{1}_K(H_\omega(\Lambda))] - 1)\right] \leq C|J|\,|K|\,|\Lambda|^2.$$

Basic idea of the proof of decorrelation lemmas

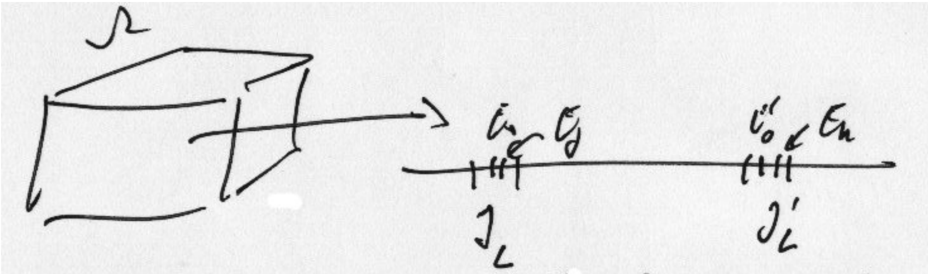
Let $J_L = E_0 + L^{-d}(-1,1)$ and $J'_L = E'_0 + L^{-d}(-1,1)$.
By Minami's estimate

$$\mathbb{P}\left(\#[\sigma(H_\omega(\Lambda_\ell)) \cap J_L] \geq 2 \text{ or } \#[\sigma(H_\omega(\Lambda_\ell)) \cap J'_L] \geq 2\right) \leq C(\ell/L)^{2d}$$

If $\mathbb{P}_0 = \mathbb{P}\left(\#[\sigma(H_\omega(\Lambda_\ell)) \cap J_L] = 1, \#[\sigma(H_\omega(\Lambda_\ell)) \cap J'_L] = 1\right)$, suffices to show that

$$\mathbb{P}_0 \leq C(\ell/L)^{2d} e^{(\log L)^\beta}.$$

Let $E_j(\omega)$ and $E_k(\omega)$ be the eigenvalues resp. in J_L and J'_L.
Need to show that they don't vary "synchronously".

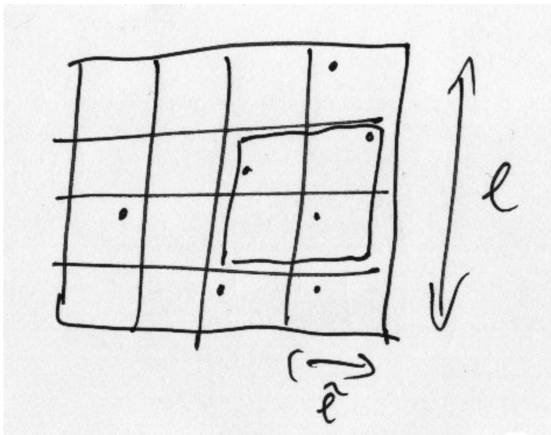


Basic idea: find random variables $(\omega_\gamma, \omega_{\gamma'})$ such that
$\psi : (\omega_\gamma, \omega_{\gamma'}) \mapsto (E_j(\omega), E_k(\omega))$ be a local diffeomorphism.

Problem: even if $|\mathrm{Jac}\,\psi| \asymp 1$, one has

$$\text{Proba} \leq \sum_{j,k} \sum_{\gamma,\gamma'} L^{-2d} \asymp \ell^{4d}/L^{2d}.$$

We need to reduce the volume of the cube Λ_ℓ.



Reduction to localization boxes:

This can be done using localization.

Lemma

There exists $C > 0$ such that for L sufficiently large

$$\mathbb{P}_0 \leq C(\ell/L)^{2d} + C(\ell/\tilde{\ell})^d \, \mathbb{P}_1$$

where

- $\mathbb{P}_1 := \mathbb{P}(\#[\sigma(H_\omega(\Lambda_{\tilde{\ell}})) \cap \tilde{J}_L] = \#[\sigma(H_\omega(\Lambda_{\tilde{\ell}})) \cap \tilde{J}'_L] = 1)$
- $\tilde{\ell} \asymp \log L$, $\tilde{J}_L = J_L + [-L^{-d}, L^{-d}]$ *and* $\tilde{J}'_L = J'_L + [-L^{-d}, L^{-d}]$

Idea of proof: if e.v. distinct loc. centers, use Wegner and spacial independence.

As localization boxes of size $\tilde{\ell}$, remains to estimate $\mathbb{P}_1$.

Analysis on a localization box

Let $\omega \mapsto E(\omega)$ be the e.v of $H_\omega(\Lambda_{\tilde\ell})$ in J_L.

1. $E(\omega)$ being simple, $\omega \mapsto E(\omega)$ and the ass. eigenvect. $\omega \mapsto \varphi(\omega)$ analytic;
2. $\partial_{\omega_\gamma} E(\omega) = \langle \pi_\gamma \varphi(\omega), \varphi(\omega)\rangle \geq 0$; hence $\|\nabla_\omega E(\omega)\|_{\ell^1} = 1$;
3. $\text{Hess}_\omega E(\omega) = ((h_{\gamma\beta}))_{\gamma,\beta}, \quad h_{\gamma,\beta} = -2\text{Re}\langle (H_\omega(\Lambda_{\tilde\ell}) - E(\omega))^{-1}\psi_\gamma(\omega), \psi_\beta(\omega)\rangle$ where
   - $\psi_\gamma = \Pi(\omega)\pi_\gamma\varphi(\omega)$,
   - $\Pi(\omega)$ is the orthogonal projector on the orthogonal to $\varphi(\omega)$.

Lemma

$$\|Hess_\omega(E(\omega))\|_{\ell^\infty \to \ell^1} \leq \frac{C}{\text{dist}\,(E(\omega), \sigma(H_\omega(\Lambda_{\tilde\ell})) \setminus \{E(\omega)\})}.$$

Hence, by Minami's estimate

Lemma

For $\varepsilon \in (4L^{-d}, 1)$, *one has* $\mathbb{P}_1 \leq C\varepsilon \tilde\ell^{2d} L^{-d} + \mathbb{P}_\varepsilon$ *where* $\mathbb{P}_\varepsilon = \mathbb{P}(\Omega_0(\varepsilon))$ *and*

$$\Omega_0(\varepsilon) = \left\{ \omega; \begin{array}{c} \sigma(H_\omega(\Lambda_{\tilde\ell})) \cap \tilde J_L = \{E(\omega)\} = \sigma(H_\omega(\Lambda_{\tilde\ell})) \cap (E - C\varepsilon, E + C\varepsilon), \\ \sigma(H_\omega(\Lambda_{\tilde\ell})) \cap \tilde J'_L = \{E'(\omega)\} = \sigma(H_\omega(\Lambda_{\tilde\ell})) \cap (E' - C\varepsilon, E' + C\varepsilon) \end{array} \right\}$$

To estimate the Jac(ψ), need to show that $\nabla_\omega E(\omega)$ and $\nabla_\omega E'(\omega)$ not colinear as

Lemma

Pick $(u,v) \in (\mathbb{R}^+)^{2n}$ *such that* $\|u\|_1 = \|v\|_1 = 1$. *Then* $\max_{j\neq k} \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix}^2 \geq \frac{1}{2n^3}\|u-v\|_1^2$.

Difficulty : gradient may be colinear e.g. for $\omega = 0$.

The fundamental estimate:

Lemma

1. *In any dimension d: for* $\Delta E > 2d$, *if the random variables* $(\omega_\gamma)_{\gamma\in\Lambda}$ *are bounded by K, for* $E_j(\omega)$ *and* $E_k(\omega)$ *are simple eigenvalues of* $H_\omega(\Lambda_L)$ *such that* $|E_k(\omega) - E_j(\omega)| \geq \Delta E$, *one has* $\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_2 \geq \frac{\Delta E - 2d}{K} L^{-d/2}$;
2. *in dimension 1: fix* $E < E'$ *and* $\beta > 1/2$; *let* $\mathbb{P}$ *denote the probability that there exists* $E_j(\omega)$ *and* $E_k(\omega)$, *simple eigenvalues of* $H_\omega(\Lambda_L)$ *such that* $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq e^{-L^\beta}$ *and such that*

$$\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_1 \leq e^{-L^\beta};$$

then, there exists $c > 0$ *such that* $\qquad \mathbb{P} \leq e^{-cL^{2\beta}}$.

Completing the proof of the decorrelation lemma

One now has $\mathbb{P}_\varepsilon \leq \sum_{\gamma\neq\gamma'} \mathbb{P}(\Omega_{0,\nu}^{\gamma,\gamma'}(\varepsilon)) + \mathbb{P}_r$ where

- $\Omega_{0,\nu}^{\gamma,\gamma'}(\varepsilon) = \Omega_0(\varepsilon) \cap \left\{\omega;\ |J_{\gamma,\gamma'}(E(\omega),E'(\omega))| \geq e^{-\tilde{\ell}^\beta}\right\}$;
- $J_{\gamma,\gamma'}(E(\omega),E'(\omega)) = \begin{vmatrix} \partial_{\omega_\gamma} E(\omega) & \partial_{\omega_{\gamma'}} E(\omega) \\ \partial_{\omega_\gamma} E'(\omega) & \partial_{\omega_{\gamma'}} E'(\omega) \end{vmatrix}$;
- in dimension 1, we have $\mathbb{P}_r \leq Ce^{-c\tilde{\ell}^{2\beta}}$, thus, $\mathbb{P}_r \leq L^{-2d}$;
- in dimension d, as by assumption $\Delta E > 2d$, one has $\mathbb{P}_r = 0$.

The estimate of Jacobian and picking $\varepsilon \asymp L^{-d}\tilde{\ell}^{\nu+1}$ yields

$$\mathbb{P}(\Omega_{0,\nu}^{\gamma,\gamma'}(\varepsilon)) \leq CL^{-2d}e^{2\tilde{\ell}^\beta}.$$

Summing over $(\gamma,\gamma') \in \Lambda_{\tilde{\ell}}^2$, we obtain

$$\mathbb{P}_\varepsilon \leq CL^{-2d}e^{4\tilde{\ell}^\beta}$$

Proof is complete. □

UNIVERSITÉ PARIS 13

The proof of the fundamental estimate: case 1

$E_j(\omega)$ and $E_k(\omega)$ simple evs of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E_j(\omega)| \geq \Delta E > 2d$.

Then, $\omega \mapsto E_j(\omega)$ and $\omega \mapsto E_k(\omega)$ are real analytic functions.

Let $\omega \mapsto \varphi_j(\omega)$ and $\omega \mapsto \varphi_k(\omega)$ be normalized eigenvec. ass. resp. to $E_j(\omega)$ and $E_k(\omega)$.

Differentiating the eigenvalue equation in ω, one computes

$$\begin{aligned}\omega \cdot \nabla_\omega(E_j(\omega) - E_k(\omega)) &= \langle V_\omega \varphi_j(\omega), \varphi_j(\omega)\rangle - \langle V_\omega \varphi_k(\omega), \varphi_k(\omega)\rangle \\ &= E_j(\omega) - E_k(\omega) + \langle -\Delta\varphi_k(\omega), \varphi_k(\omega)\rangle - \langle -\Delta\varphi_j(\omega), \varphi_j(\omega)\rangle.\end{aligned}$$

So

$$\Delta E - 2d \leq |E_j(\omega) - E_k(\omega)| - 2d \leq |\omega \cdot \nabla_\omega(E_j(\omega) - E_k(\omega))|.$$

Hence,

$$\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_2 \geq \frac{\Delta E - 2d}{K} L^{-d/2}.$$

UNIVERSITÉ PARIS 13

The proof of the fundamental estimate: case 2

Let us now assume $d = 1$. We prove a weaker result.

Theorem

Fix $\nu > 8$. For the discrete Anderson model in dimension 1, there exists $\Delta\mathscr{E}$ of total measure such that, for $E - E' \in \Delta\mathscr{E}$, for L sufficiently large, if $E_j(\omega)$ and $E_k(\omega)$ are simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq L^{-\nu}$ then $\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_1 \geq L^{-\nu}$;

Fix $E < E'$. Pick $E_j(\omega)$ and $E_k(\omega)$, simple evs s.t. $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq L^{-\alpha}$. Then,

$$4L^{-2\nu} \geq \|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_2^2 = \sum_{\gamma\in\Lambda_L} |\varphi_\gamma^j(\omega) - \varphi_\gamma^k(\omega)|^2 \cdot |\varphi_\gamma^j(\omega) + \varphi_\gamma^k(\omega)|^2$$

there exists a partition of Λ_L, say $\mathscr{P} \subset \Lambda_L$ and $\mathscr{Q} \subset \Lambda_L$ s.t.

- for $\gamma \in \mathscr{P}$, $|\varphi_\gamma^j(\omega) - \varphi_\gamma^k(\omega)| \leq L^{-\nu}$;
- for $\gamma \in \mathscr{Q}$, $|\varphi_\gamma^j(\omega) + \varphi_\gamma^k(\omega)| \leq L^{-\nu}$.

Introduce the orthogonal projectors P and Q defined by

$$P = \sum_{\gamma\in\mathscr{P}} |\gamma\rangle\langle\gamma| \quad \text{and} \quad Q = \sum_{\gamma\in\mathscr{Q}} |\gamma\rangle\langle\gamma|.$$

UNIVERSITÉ PARIS 13

One has $\|P\varphi^j - P\varphi^k\|_2 \leq L^{-\nu+d/2}$ and $\|Q\varphi^j + Q\varphi^k\|_2 \leq L^{-\nu+d/2}$.

As $\|Pu\|^2 + \|Qu\|^2 = \|u\|^2$ and $\langle\varphi^j, \varphi^k\rangle = 0$, one has

$$\|P\varphi^j\|^2 = \frac{1}{2} + O(L^{-\nu+d/2}) \text{ and } \|Q\varphi^j\|^2 = \frac{1}{2} + O(L^{-\nu+d/2}).$$

This implies that $\mathscr{P} \neq \emptyset$ and $\mathscr{Q} \neq \emptyset$.

To simplify the notation, from now on, we write $u = \varphi_j$. So $\varphi_k = Pu - Qu + O(L^{-\nu})$.

Plugging this into the eigenavalue equations yields

$$\begin{cases} [-(P\Delta Q + Q\Delta P) - \Delta E]u & = O(L^{-\alpha}) \\ [-(P\Delta P + Q\Delta Q) + V_\omega - \overline{E}]u & = O(L^{-\alpha}), \end{cases}$$

where $\Delta E = E' - E$ and $\overline{E} = (E + E')/2$.

So

- ΔE is at a distance at most $L^{-\alpha}$ to the spectrum of $-(P\Delta Q + Q\Delta P)$,
- u is close to being an eigenvector associated to this eigenvalue,
- u is also close to being in the kernel of $-(P\Delta P + Q\Delta Q) + V_\omega - \overline{E}$.

UNIVERSITÉ PARIS 13

The operator $P\Delta Q + Q\Delta P$:

$$-P\Delta Q - Q\Delta P = \sum_{\gamma\in\partial\mathscr{P}} (|\gamma+1\rangle\langle\gamma| + |\gamma\rangle\langle\gamma+1|) + \sum_{\gamma\in\partial\mathscr{Q}} (|\gamma+1\rangle\langle\gamma| + |\gamma\rangle\langle\gamma+1|)$$

where $\partial\mathscr{P} = \{\gamma\in\mathscr{P};\ \gamma+1\in\mathscr{Q}\}\subset\mathscr{P}$ and $\partial\mathscr{Q} = \{\gamma\in\mathscr{Q};\ \gamma+1\in\mathscr{P}\}\subset\mathscr{Q}$.

One checks $\partial\mathscr{P}\neq\emptyset$, and $\partial\mathscr{Q}\neq\emptyset$ and $\partial\mathscr{P}\cap\partial\mathscr{Q}=\emptyset$.

For $\mathscr{A}\subset\Lambda_L$ we define $\mathscr{A}+1=\{p+1;\ p\in\mathscr{A}\}$ to be the shift by one of $\mathscr{A}$.

One clearly has $(\partial\mathscr{P}+1)\subset\mathscr{Q}$ and $(\partial\mathscr{Q}+1)\subset\mathscr{P}$.

Hence, $(\partial\mathscr{P}+1)\cap\partial\mathscr{P}=\emptyset$ and $(\partial\mathscr{Q}+1)\cap\partial\mathscr{Q}=\emptyset$.

Consider the set $\mathscr{C}:=\partial\mathscr{P}\cup\partial\mathscr{Q}$.

Partition it into its “connected components” i.e. $\mathscr{C}$ can be written a a disjoint union of intervals of integers, say $\mathscr{C}=\cup_{l=1}^{l_0}\mathscr{C}_l^c$.

Then, for $l\neq l'$,

$$\mathscr{C}_l^c\cap\mathscr{C}_{l'}^c=\mathscr{C}_l^c\cap(\mathscr{C}_{l'}^c+1)=\emptyset.$$

Define $\mathscr{C}_l=\mathscr{C}_l^c\cup(\mathscr{C}_l^c+1)$. One has, for $l\neq l'$, $\mathscr{C}_l\cap\mathscr{C}_{l'}=\emptyset$.

Note that one may have $\cup_{l=1}^{l_0}\mathscr{C}_l=\Lambda_L$.

UNIVERSITÉ PARIS 13

Then

$$-P\Delta Q - Q\Delta P = -\sum_{l=1}^{l_0} C_l\Delta C_l$$

where C_l is the projector $C_l=\sum_{\gamma\in\mathscr{C}_j}|\gamma\rangle\langle\gamma|$.

The projectors C_l and $C_{l'}$ are orthogonal to each other for $l\neq l'$.

So the spectrum of $-P\Delta Q - Q\Delta P$ is given by the union of the spectra of $(C_l\Delta C_l)_{1\leq j\leq J}$.

Each of these operators : Dirichlet Laplacian on interval of length, the length of C_l.

Its spectral decomposition can be computed explicitly: for segment of length n,

- the eigenvalues are simple and are given by $(2\cos(k\pi/(n+1)))_{1\leq k\leq n}$;
- for $k\in\{1,\cdots,n\}$, the eigenspace associated to $2\cos(k\pi/(n+1))$ is generated by the vector $(\sin(kj\pi/(n+1))_{1\leq j\leq n}$.

Let $\Delta\mathscr{E}_L^c=\cup_{n=0}^{L}\sigma(-C_n\Delta C_n)+[-L^{-\nu},L^{-\nu}]$ then $|\cap_{n\geq1}\cup_{L\geq n}\Delta\mathscr{E}_L^c|=0$.

$\Delta\mathscr{E}=^c(\cap_n\cup_{L\geq n}\Delta\mathscr{E}_L^c)$ is of total measure.

This completes the proof.

UNIVERSITÉ PARIS 13