Gaussian exponential sums

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Outline

Exponential sums

We consider sums of the form $S(N,P) = \sum_{0\le n\le N-1} e(P(n))$ where

- $N = 1,2,3,\cdots$,
- $e(z) = e^{2\pi iz}$,
- $P \in \mathbb{R}[X], \quad \deg P = k > 1.$

Such sums play an important role in many areas of mathematics:

- ergodic theory: equidistribution of numbers mod 1,
- number theory: Diophantine equations, Waring's problem, Hardy-Littlewood circle method,
- harmonic analysis: lacunary series,
- analysis of PDEs: linear, nonlinear, on manifolds.

Question: behavior of $S(N,P)$ when $N \to +\infty$.

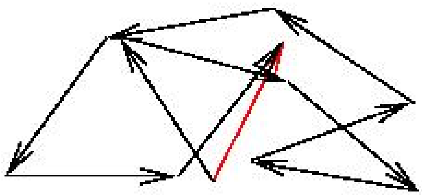
We concentrate on Gaussian sums i.e. $k = 2$ or

$$S(N,a,b) = \sum_{0\le n\le N-1} e\left(-\frac{an^2}{2} + nb\right)$$

The graph of a Gaussian sums

We now restrict to the case $b = 0$ i.e. to sums of the form

$$S(N,a,0) = \sum_{0\le n\le N-1} e\left(-\frac{an^2}{2}\right), \quad N = 1,2,3\ldots,$$



where $0 < a < 1$.

The graphs :

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A renormalition formula: we shall see that

$$S(N,a,0) \sim \frac{e(-1/8)}{\sqrt{a}} S(N_1,a_1,2), \quad N_1 = [aN], \quad a_1 = -\frac{1}{a}.$$

If a is small, there are less terms in the right hand side. Moreover, the right hand side varies more slowly with N. One erases details (Hardy - Littlewood, Mendès-France, Berry - Goldberg).

The Poisson formula

Let $f \in \mathscr{S}(\mathbb{R})$. Then, one has $\sum\limits_{n^* \in 2\pi\mathbb{Z}} \hat{f}(n^*) = \sum\limits_{n \in \mathbb{Z}} f(n)$.

Renormalizing the sums:

One can apply this to the exponential sum taking $f(x) = e(iax^2/2)\mathbf{1}_{[0,N)}(x)$.

Thus,

$$\hat{f}(\xi) = \frac{e(-1/8)}{\sqrt{a}} \exp\left(-\frac{i\xi^2}{4\pi a}\right) \mathbf{1}_{[0,aN)}(\xi) + g(\xi)$$

where g is a remainder term.

This yields

$$S(N,a,0) \sim \frac{e(-1/8)}{\sqrt{a}} S(N_1,a_1,2), \quad N_1 = [aN], \quad a_1 = -\frac{1}{a};$$

The basic formula:

$$S(N,a,0) = \frac{e(-1/8)}{\sqrt{a}} S(N_1,a_1,2) + \text{error terms}.$$

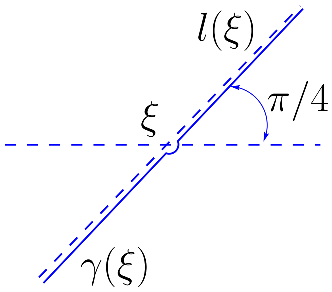
Three questions:

- describe the “error terms”?
- are the “error terms” really smaller than the main term?
- how do these terms behave?

A special function:

Consider the function $\mathscr{F} : \mathbb{C} \to \mathbb{C}$ defined by the formula:

$$\mathscr{F}(\xi,a) = \int_{\gamma(\xi)} \frac{e\left(\frac{p^2}{2a}\right) dp}{e(p-\xi)-1},$$



An exact renormalization formula (A. Fedotov, F.K.) :

Lemma

For $a > 0$, the function $\xi \mapsto \mathscr{F}(\xi,a)$ is entire and satisfies, for $\xi \in \mathbb{C}$,

$$\mathscr{F}(\xi,a) - \mathscr{F}(\xi-1,a) = e\left(\frac{\xi^2}{2a}\right) \quad \textit{and} \quad \mathscr{G}(\xi+a,a) - \mathscr{G}(\xi,a) = e\left(-\frac{\xi^2}{2a}\right)$$

where $\mathscr{G}(\xi,a) = c(a)\,e\left(-\frac{\xi^2}{2a}\right)\mathscr{F}(\xi,a)$ *and* $c(a) = e(-1/8)\,a^{-1/2}$.

Moreover, one has $\mathscr{F}(-\xi,a) + \mathscr{F}(\xi,a) = e\left(\frac{\xi^2}{2a}\right) - \frac{1}{c(a)}$.

Proof. The first relation follows from the residue theorem.
The second relation is obvious after the change of variable $z = p - \xi$ in the integral defining $\mathscr{F}$.
To get the third relation, change variable $p \to -p$ in the integral $\mathscr{F}(-\xi,a)$ and use the residue theorem to get $\mathscr{F}(-\xi,a) = e\left(\frac{\xi^2}{2a}\right) - \int_{\gamma(\xi)} \frac{e\left(\frac{p^2}{2a}\right)e(p-\xi)\,dp}{e(p-\xi)-1}$.

This leads to the renormalization formula

Theorem

Fix $N \in \mathbb{N}$ and $(a,b) \in (0,1) \times (-1/2,1/2]$. Let

$$\xi = \{aN\}, \quad N_1 = [aN], \quad a_1 = \left\{\frac{1}{a}\right\}, \quad b_1 \equiv \left\{-\frac{b}{a} + \frac{1}{2}\left[\frac{1}{a}\right]\right\}_0,$$

where $\{x\}$ and $[x]$ denote the fractional and the integer parts of the real number x, and $\{x\}_0 = x \bmod 1$ and $-1/2 < \{x\}_0 \le 1/2$.

Then,

$$S(N,a,b) = c(a)\left[e\left(\frac{b^2}{2a}\right)\overline{S(N_1,a_1,b_1)} + e\left(-\frac{aN^2}{2} + Nb\right)\mathscr{F}(\xi-b,a) - \mathscr{F}(-b,a)\right].$$

Analogous formulae were already known (Hardy-Littlewood, Van der Corput, Mordell), but the error terms were not known.

Proof

The second equation of Lemma 1 yields we get

$$\mathscr{G}(Na-b,b) = \sum_{k=0}^{N-1} e\left(-\frac{(ka-b)^2}{2a}\right) + \mathscr{G}(-b,a) = e\left(-\frac{b^2}{2a}\right) S(N,a,b) + \mathscr{G}(-b,a).$$

Thus, by the definition of $\mathscr{G}$,

$$S(N,a,b) = c(a)\left[e\left(-\frac{N^2a}{2} + Nb\right)\mathscr{F}(Na-b,a) - \mathscr{F}(-b,a)\right]$$

On the other hand, using the first equation of Lemma 1, we obtain

$$\mathscr{F}(Na-b,b) - \mathscr{F}(\xi-b,a) = e\left(\frac{(Na-b)^2}{2a}\right)\sum_{k=0}^{N_1-1} e\left(\frac{k^2}{2a} - \frac{k(Na-b)}{a}\right).$$

As $e(l) = 1$ for all $l \in \mathbb{Z}$, and as, modulo 1, one has

$$\frac{k^2}{2a} + \frac{b}{a}k = \frac{k(k+1)}{2}\frac{1}{a} + \left(\frac{b}{a} - \frac{1}{2a}\right)k = \frac{k(k+1)}{2}a_1 - k\left(b_1 + \frac{a_1}{2}\right) = \frac{k^2}{2}a_1 - kb_1,$$

we finally get $\mathscr{F}(Na-b,b) = e\left(\frac{(Na-b)^2}{2a}\right)\overline{S(N_1,a_1,b_1)} + \mathscr{F}(\xi-b,a)$.

Expressing $\mathscr{G}$ in terms of $\mathscr{F}$ completes the proof. □

Analysis of the curlicues

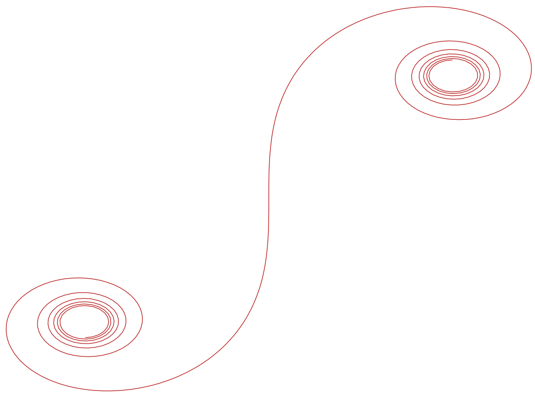
The asymptotics of $\mathscr{F}$:

Proposition

Let $-1/2 \le \xi \le 1/2$ *and* $0 < a \le 1$. *When* $a \to 0$, $\mathscr{F}$ *satisfies*

$$\mathscr{F}(\xi,a) \sim e(1/8)\,e(a^{-1}\xi^2/2) f(a^{-1/2}\xi), \quad f(t) = \int_{-\infty}^{t} e(-\tau^2/2)\,d\tau.$$

The graph of f is



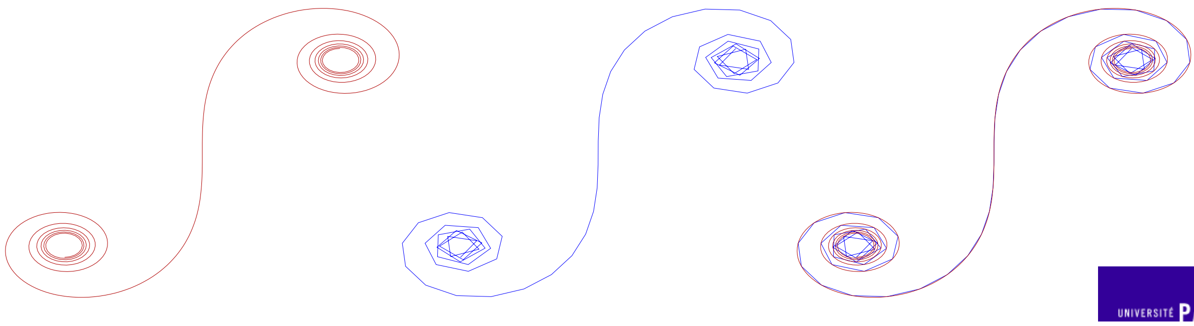
To simplify, set $b = 0$. Recall that, for $\xi = \{aN\}$, $\quad N_1 = [aN]$, $\quad a_1 \equiv \frac{1}{a} \bmod 1$, one has

$$S(N,a,0) = c(a)\left[\overline{S(N_1,a_1,0)} + e\left(-\frac{aN^2}{2}\right)\mathscr{F}(\xi,a) - \mathscr{F}(0,a)\right].$$

Let a be small. Then, $N_1 = [aN]$ stays constant over an interval of size roughly $1/a$. On this interval, $\{aN\}$ varies by increments of a. Hence, in this interval, one has $\{a(N+k)\} - \{aN\} = ak$.

Therefore,

$$e((N+k)^2/2a)\mathscr{F}(\{aN+ak\},a) = e((N+k)^2/2a)\mathscr{F}(\{aN\}+ka,a)$$
$$\sim e(1/8)\,e(N_1^2/2a)f(a^{-1/2}\{aN\}+a^{1/2}k).$$



Define the sequence $a_{n+1} \equiv a_n^{-1} \pmod 1$, $a_0 = a$.

In terms of the continued fraction expansion of a

$$a = \cfrac{1}{n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots}}},$$

we get

$$a_j = \cfrac{1}{n_j + \cfrac{1}{n_{j+1} + \cfrac{1}{n_{j+2} + \cdots}}} \text{ for } j \geq 0.$$

Thus, a_j is small if and only if n_j is large.

Some examples:

$a = [30, 60, 120, 1000, \ldots]$ $\qquad$ $a = [1,1,1,1,1,1,1,1,1,60,1,1,1,1,1,1,1,1,1,\ldots]$

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A short preliminary:

For $b \in \mathbb{R}$, one computes

$$\int_0^1 |S(N,a,b)|^2 da = N + \sum_{0\le m<n\le N} \int_0^1 \cos(\pi a(n^2-m^2) - 2\pi b(n-m))da = N.$$

Thus, it is reasonable to expect that, for many a, $|S(N,a,b)|/\sqrt{N}$ not be too large.

Two theorems on the growth of $S(N,a,b)$: first, consider the case of typical (a,b).

Theorem

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function.
For almost every $(a,b) \in (0,1) \times (-1/2,1/2]$, one has

$$\limsup_{N\to+\infty} \left(g(\ln N) \frac{|S(N,a,b)|}{\sqrt{N}} \right) < \infty \iff \sum_{l\ge1} g^6(l) < \infty.$$

Let $\varphi(N) = (\ln N)^{1/4}$.

For typical a and b, the ratio $\dfrac{S(N,a,b)}{\sqrt{N}}$ grows slower than $(\varphi(N))^{2/3+0}$.

Define the set

$$B_a = \left\{ \{\frac{1}{2}(ma+n)\}_0;\ (m,n) \in \mathbb{Z}^2 \setminus (2\mathbb{Z}+1)^2 \right\}$$

where, for $x \in \mathbb{R}$, $\{x\}_0 = x \bmod 1$ and $-1/2 < \{x\}_0 \le 1/2$.

For every irrational a, the set B_a is dense in $(-1/2,1/2]$ as $\{ma+n;\ (m,n) \in \mathbb{Z}^2\}$ is dense in $\mathbb{R}$.

Theorem

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function. Then, for almost all $a \in (0,1)$, there exists a dense G_δ, say $\tilde{B}_a$, such that $B_a \subset \tilde{B}_a$ and, for $b \in \tilde{B}_a$, one has

$$\limsup_{N\to+\infty} \left(g(\ln N) \frac{|S(N,a,b)|}{\sqrt{N}} \right) < \infty \iff \sum_{l\ge1} g^4(l) < \infty.$$

Let $\varphi(N) = (\ln N)^{1/4}$. For a typical a and for $b \in \tilde{B}_a$, the ratio $S(N,a,b)/\sqrt{N}$ grows faster than $\varphi(N)$.

For $b = 0$, the result is known (see Fiedler-Jurkat-Körner).

Multiple renormalizations

The renormalization formula expresses $S(N,a,b)$ in terms $S(N_1,a_1,b_1)$ containing a smaller number of terms. One can apply it inductively.

Assume a irrational. For $l \geq 0$, we let $a_{l+1} = \left\{\frac{1}{a_l}\right\}$, $a_0 = a$, $N_{l+1} = [a_l N_l]$, $N_0 = N$, $b_{l+1} \equiv \left\{-\frac{b_l}{a_l} + \frac{1}{2}\left[\frac{1}{a_l}\right]\right\}_0$, $b_0 = b$.

The sequence $\{N_l\}$ is strictly decreasing until it reaches the value zero and then becomes constant.

Denote by $L(N)$ the unique natural number such that $N_{L(N)+1} = 0$ and $N_{L(N)} \geq 1$.

Corollary

One has $S(N,a,b) = \sum_{l=0}^{L(N)} \frac{e(\theta_l)}{(a_0 a_1 \ldots a_l)^{1/2}} \Delta\mathscr{F}_l^{*l}$ *where*

- $\Delta\mathscr{F}_l = e(-a_l N_l^2/2 + N_l b_l)\mathscr{F}(\xi_l - b_l, a_l) - \mathscr{F}(-b_l, a_l)$,
- $*l$ *denotes the complex conjugation applied* l *times,*
- $\xi_l = \{a_l N_l\}$ *and* $\theta_{l+1} = \theta_l + (-1)^l \left(\frac{1}{8} + \frac{b_l^2}{2a_l}\right)$ *where* $\theta_0 = -1/8$.

Define $N^-(L) = \min\{N;\ L(N) = L\}$ and $N^+(L) = \max\{N;\ L(N) = L\}$. Clearly, $N^+(L-1) = N^-(L) - 1$. One has $\frac{1}{a_0 a_1 \ldots a_{L-1}} < N^-(L) < \frac{1}{a_0 a_1 \ldots a_{L-1}}(1 + 4a_{L-1})$.

Theorem

Let a *be irrational and* L *be a positive integer. Assume that* $N^-(L) \leq N \leq N^+(L)$. *Let* $\xi_L(N) = a_L N_L(N)$. *Then, for* $\xi_L(N) - b_L \leq 1/2$, *one has*

$$S(N,a,b) = \frac{e(\theta_{L+1})}{\sqrt{a_0 a_1 \ldots a_L}} \left(\int_{-\frac{b_L}{\sqrt{a_L}}}^{\frac{\xi_L(N)-b_L}{\sqrt{a_L}}} e(-\tau^2/2)\, d\tau + O(\sqrt{a_L}) \right)^{*L}$$

and, for $\xi_L(N) - b_L \geq 1/2$, *one has*

$$S(N,a,b) = \frac{e(\theta_{L+1})}{\sqrt{a_0 a_1 \ldots a_L}} \left(\int_{-\frac{b_L}{\sqrt{a_L}}}^{\infty} e(-\tau^2/2)\, d\tau + O(\sqrt{a_L}) + \right.$$

$$\left. + e\left(\frac{b_L - \xi_L(N) + 1/2}{2a_L}\right) \int_{\frac{1-(\xi_L(N)-b_L)}{\sqrt{a_L}}}^{\infty} e(-\tau^2/2)\, d\tau \right)^{*L}$$

where $*L$ *and* θ_l *are defined in previous slide.*

Estimates on Gaussian sums

Using previous theorem, we now estimate $S(N,a,b)$ in terms of $(a_l)_l$ and $(b_l)_l$.

For $L \in \mathbb{N}$, define $M(L,a,b) = \max_{N^-(L) \le N \le N^+(L)} \left| \frac{S(N,a,b)}{\sqrt{N}} \right|$.

Proposition

There exist $c > 0$ and $C > 0$ independent of a, and b such that, for $L \in \mathbb{N}$,

$$M(L,a,b) \le C \frac{1}{\sqrt{|b_L| + \sqrt[4]{a_L}}}$$

and

$$\text{if } \sqrt{|b_l|} + \sqrt[4]{a_L} \le c, \text{ then } \frac{1}{C} \frac{1}{\sqrt{|b_L| + \sqrt[4]{a_L}}} \le M(L,a,b).$$

So studying the growth of the sum reduces to studying the dynamical system defined by $(a_l, b_l) \mapsto (a_{l+1}, b_{l+1})$ defined by $a_{l+1} = \left\{ \frac{1}{a_l} \right\}$, $b_{l+1} \equiv \left\{ -\frac{b_l}{a_l} + \frac{1}{2} \left[\frac{1}{a_l} \right] \right\}_0$ and $(a_0, b_0) = (a,b)$.

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A special dynamical system

For $(a,b) \in (0,1) \times (-1/2, 1/2)$, for $l \ge 0$, define

$$a_{l+1} = \left\{ \frac{1}{a_l} \right\}, \; b_{l+1} \equiv \left\{ -\frac{b_l}{a_l} + \frac{1}{2} \left[\frac{1}{a_l} \right] \right\}_0 \text{ where } (a_0, b_0) = (a,b).$$

Note that the sequence $(a_l)_l$ is defined by the Gauss map, thus given by the continued fraction expansion of a. It is well studied.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function.

Let $\gamma(a,b)$ be the trajectory of the dynamical system.

Let $\mathfrak{N}(L, \varphi, a, b)$ be the number of the conditions

$$\text{“ } \sqrt[4]{a_l} \le \varphi(l) \quad \text{and} \quad \sqrt{|b_l|} \le \varphi(l) \text{ ”}$$

that are satisfied along $\gamma(a,b)$ while $0 \le l \le L$.

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On $K := (0,1) \times (-1/2, 1/2)$, let m be the probability measure of density $\frac{1}{\ln 2} \frac{da\,db}{1+a}$.

Let $\|\mathfrak{N}(L,\varphi)\|_1$ and $\|\mathfrak{N}(L,\varphi)\|_2$ denote the $L^1(K,m)$ and $L^2(K,m)$ norms of the function $(a,b) \to \mathfrak{N}(L,a,b)$.

Lemma

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function such that, for all $l \in \mathbb{N}$, $\varphi(l) \leq 1/2$.

Then, $\|\mathfrak{N}(L,\varphi)\|_1 \leq C \quad \forall L \in \mathbb{N} \quad \Longleftrightarrow \quad \sum_{N \geq 1} \varphi^6(N) < \infty.$

Lemma

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be as above. If $\sum_{N \geq 1} \varphi^6(N)$ diverges then,

$$\|\mathfrak{N}(L,\varphi)\|_2 \underset{L \to \infty}{\sim} \|\mathfrak{N}(L,\varphi)\|_1 .$$

Note that, if $\|\mathfrak{N}(L,\varphi)\|_2 = \|\mathfrak{N}(L,\varphi)\|_1$ then $\mathfrak{N}(L,\varphi,\cdot,\cdot) = \|\mathfrak{N}(L,\varphi)\|_1$ a.s.