Edge statistics in dimension 1

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Outline :

The prototypical example : the Anderson model

On $\mathbb{C}^L$, consider $M_L(\omega) = \begin{pmatrix} \omega_1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & \omega_2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \omega_3 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & \omega_{L-1} & 1 \\ 0 & 0 & \cdots & 0 & 1 & \omega_L \end{pmatrix}$ where $(\omega_j)_{1\leq j\leq L}$ i.i.d.

Consider eigenvalues of $M_L(\omega)$: $E_1(\omega,L) \leq E_2(\omega,L) \leq \cdots \leq E_L(\omega,L)$.

Question : statistics of the eigenvalues of $M_L(\omega)$ when $L \to +\infty$?

Renormalized (unfolded) eigenvalues : define N to be the integrated density of states :

$$N(E) := \lim_{L\to+\infty} \frac{\#\{\text{e.v. of } M_L(\omega) \text{ less than E}\}}{L}.$$

The limit exists a.s., is a.s. constant and defines probability distribution on $\mathbb{R}$.

Renormalized eigenvalues : $N(E_1(\omega,L)) \leq N(E_2(\omega,L)) \leq \cdots \leq N(E_L(\omega,L))$.

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Bulk statistics :

Fix E_0 such that $\frac{dN}{dE}(E_0) > 0$.

Consider the point process $\Xi(E_0,\omega,L) = \sum_{j=1}^{L} \delta_{L\cdot[N(E_j(\omega,L))-N(E_0)]}$.

Theorem (Molchanov, Minami, Germinet-K.)

If the r.v. are "regular", as $L \to +\infty$, $\Xi(E_0,\omega,L)$ converges weakly to the Poisson process on $\mathbb{R}$ with intensity 1.

Edge statistics : consider the point process $\Xi_-(\omega,L) = \sum_{j=1}^{L} \delta_{L\cdot N(E_j(\omega,L))}$.

Theorem (Germinet-K.)

If the r.v. are "regular", as $L \to +\infty$, $\Xi_-(\omega,L)$ converges weakly to the Poisson process on $\mathbb{R}^+$ with intensity 1.

Note that $\Xi_-(\omega,L) = \Xi(E_0,\omega,L)$ if $N(E_0) = 0$.

One can equivalently consider $\Xi_+(\omega,L) = \sum_{j=1}^{L} \delta_{L\cdot[1-N(E_j(\omega,L))]}$.

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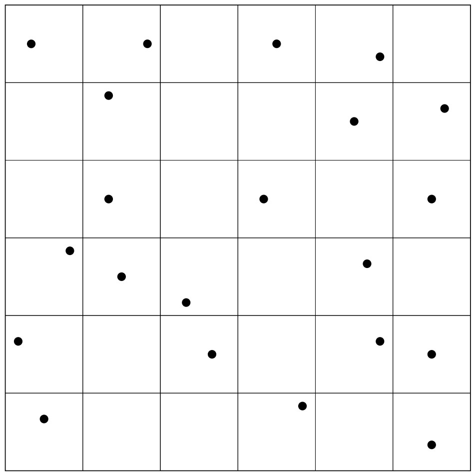
Bulk vs edge : basic analysis of bulk

- Localization : for some $\alpha > 0$, with proba $1 - L^{-q}$,
  if φ e.v. of $M_L(\omega)$ ass. to E, then $\exists n_E$ s.t. $|\varphi(n)| \leq L^q e^{-\alpha|n-n_E|}$;
  [Kunz-Souillard, Fröhlich-Spencer, Aizenman-Molchanov, Germinet-Klein, etc]
- Wegner estimate : $\mathbb{E}(\{\mathrm{tr}(\mathbf{1}_I(M_L(\omega)))\}) \leq C|I|L$; [Wegner, many others]
- Minami estimate : $\mathbb{E}\left(\{\mathrm{tr}(\mathbf{1}_I(M_L(\omega)))\,(\mathrm{tr}(\mathbf{1}_I(M_L(\omega)))-1)\}\right) \leq C(|J|L)^2$.
  [Minami, Bellissard-Hislop-Stolz, Graf-Vaghi, Combes-Germinet-Klein]

The analysis : pick (small) interval I and L s.t. need $1 \ll |N(I)|L \lesssim |I|L$

- pick cube of size L
- find the localization centers
- cut cube into small cubes of size $\ell \ll L$
- possible problems :
  - multiple centers in small cubes
    probability is small due to Minami's estimate : $\ell^2|I|^2(L/\ell) = \ell L|I|^2$
  - centers not localized well in cube
    probability is small due to Wegner's estimate : $l \cdot |I|$



- if $L|N(I)| \gg 1$ and $\ell L|I|^2 \ll 1$, with good prob

So with good probability, e.v. of big cube given by e.v. of small cubes i.e. i.i.d.

Bulk vs edge : basic analysis (continued)

So analysis works if $1 \ll |N(I)|L \lesssim |I|L$ and $\ell L|I|^2 \ll 1$.

Compute distributions of e.v. in small cubes :

- proba. to have e.v. in small cube $\sim |N(I)|\ell + O((|I|\ell)^2)$
- distr. of this (renorm.) e.v. (cond. on its existence) : if $I = [a,b]$

$$\mathbb{P}(\text{e.v. in } [x,y]) \sim \frac{N(a+x|I|) - N(a+y|I|)}{|N(I)|} + O\left(\frac{|I|^2\ell}{|N(I)|}\right).$$

Bulk : $|I| \asymp |N(I)|$ (as typically $\dfrac{dN}{dE}(E_0) > 0$).

Edge : typical Lifshitz tails : $|N(I)| \sim e^{-|I|^{-1/2}} \quad \Rightarrow \quad |N(I)| \ll |I|$.

At edge, standard Wegner and Minami insufficient !

Enhanced Wegner and Minami :

Theorem (Germinet-K.)

Fix $\xi \in (0,1)$. For I compact interval in loc. region s.t. $|N(I)| \geq \exp(-L^\xi/C)$

1. $\mathbb{E}(\mathrm{tr}\,\mathbf{1}_I(M_L(\omega))) \leq 2|N(I)|L$;
2. $\mathbb{E}\left[\mathrm{tr}\,\mathbf{1}_I(M_L(\omega))(\mathrm{tr}\,\mathbf{1}_I(M_L(\omega)) - 1)\right] \leq 2|N(I)||I|L^2$.

The Poisson and the Anderson model

On $L^2(\mathbb{R})$, consider the following random operators $H^\bullet_\omega = -\frac{d^2}{dx^2} + V^\bullet_\omega(x)$:

- $V^P(x) = \int_{\mathbb{R}} v(x-y)d\mu(y,\omega)$ where $\mu(\cdot,\omega)$ is a random Poisson point process i.e.
- $V^A(x) = W(x) + \sum_{n\in\mathbb{Z}} \omega_n v(x-n)$ where W is $\mathbb{Z}$-periodic and $(\omega_n)_{n\in\mathbb{Z}^d}$ are i.i.d. non trivial.

For simplicity $v : \mathbb{R} \to \mathbb{R}^+$ continuous and compact support

For $L > 1$, consider $H^\bullet_{\omega|L}$ to be $H^\bullet_\omega$ restricted to $[-L/2, L/2]$ with, say, Dirichlet b.c. Consider eigenvalues of $H^\bullet_{\omega|L}$: $E_1(\omega,L) \le E_2(\omega,L) \le \cdots \le E_n(\omega,L) \le \cdots$.

Renormalized (unfolded) eigenvalues : define N to be the integrated density of states :

$$N^\bullet(E) := \lim_{L\to+\infty} \frac{\#\{\text{e.v. of } H^\bullet_{\omega|L} \text{ less than E}\}}{L}.$$

The limit exists and is indep. of ω a.s. ; it is continuous and defines distribution of a positive measure on $\mathbb{R}$. Its support is the a.s. spectrum of $H^\bullet_\omega$. Let $\sigma_- = \inf(\sigma(H^\bullet_\omega))$. Assume $\sigma_- = 0$.

Renormalized eigenvalues : $N(E_1(\omega,L)) \le N(E_2(\omega,L)) \le \cdots \le N(E_n(\omega,L)) \le \cdots$.

Lifshitz tails

Theorem

One has $\log N^\bullet(E) = -c^\bullet E^{-1/2}(1+o(1))$ *where*

- $c^P = \pi$ *[Sznitman, Pastur, etc.]*
- $c^A = \pi|\log p|\sqrt{m}$ *for* $(\omega_n)_n$ *Bernoulli s.t.* $p := P(\omega_0 = \inf(\omega_0))$ *and* m *is the effective mass of* $H^A_{\inf(\omega_0)}$ *at* 0.

Joint statistics : Localization centers : for some $\alpha \in (0,1)$, with prob. $1 - L^{-q}$,

$$\text{if } \varphi \text{ e.v. of } H^\bullet_{\omega|L} \text{ ass. to } E, \text{ then } \exists x_E \text{ s.t. } |\varphi(x)| \le L^q e^{-|x-x_E|^\alpha}.$$

This description holds [Bourgain-Kenig, Germinet-Klein-Hislop, Germinet-Klein].

Define $\Xi^\bullet(\omega,L) = \sum_j \delta_{L\cdot N(E_j(\omega,L))}$ and $\Xi^\bullet_2(\omega,L) = \sum_j \delta_{L\cdot N(E_j(\omega,L))} \otimes \delta_{L^{-1}\cdot x_{E_j(\omega,L)}}$.

Theorem

For $\bullet \in \{A, D\}$, *as* $L \to +\infty$, $\Xi^\bullet(\omega,L)$ *(resp* $\Xi^\bullet_2(\omega,L)$*) converges weakly to the Poisson process on* $\mathbb{R}^+$ *(resp.* $\mathbb{R}^+ \times [-1/2, 1/2]$*) with intensity 1.*

Related work : [Grenkova-Molchanov-Sudarev] (sum of δ potentials).

Another point of view

Consider the random operator $H^\bullet_\omega$ on the whole space.

Localization : ω-a.s., the spectrum is pure point and $\exists\alpha\in(0,1)$ s.t. for $q>0$ large,

$$\text{if } E \text{ e.v. assoc. to } \varphi \text{ normalized} \quad |\varphi(n)|\le C_\omega(1+|x_E|^2)^{q/2}e^{-|x-x_E|^\alpha}.$$

Moreover, for $E_0\in\mathbb{R}$, the number of eigenvalues E of $H^\bullet_\omega$ in $(-\infty,E_0]$ s.t. $|x_E|\le L/2$ is bounded by $N(E_0)L(1+o(1))$.

Enumerate the finitely many eigenvalues of $H^\bullet_\omega$ less than 1 with localization center in $[-L/2,L/2]$: $\tilde E_1(\omega,L)\le\tilde E_2(\omega,L)\le\cdots\le\tilde E_n(\omega,L)\le\cdots$.

Define $\tilde\Xi^\bullet_2(\omega,L)=\sum_j \delta_{L\cdot N(\tilde E_j(\omega,L))}\otimes\delta_{L^{-1}\cdot n_{\tilde E_j(\omega,L)}}$ and $\tilde\Xi^\bullet(\omega,L)=\sum_j\delta_{L\cdot N(\tilde E_j(\omega,L))}$.

Theorem

For $\bullet\in\{A,D\}$, *as* $L\to+\infty$,

- $\tilde\Xi^\bullet(\omega,L)$ *converges weakly to the Poisson process on* $\mathbb{R}^+$ *with intensity 1 ;*
- $\tilde\Xi^\bullet_2(\omega,L)$ *converges weakly to the Poisson process on* $\mathbb{R}^+\times[-1/2,1/2]$ *with intensity 1.*

Applications

- Parabolic Anderson model : large time asymptotics of $\begin{cases}\partial_t u_t+H^\bullet_\omega u_t=0,\\ u_{t|t=0}\equiv 1.\end{cases}$

  Intermittency : study $u_t(0)$ (as potential homogeneous) formally

$$u_t(0)=\sum_j e^{-tE_j(\omega)}\varphi_j(0)\int\varphi_j(y)dy\asymp\sum_j e^{-tE_j(\omega)}e^{-|x_{E_j(\omega)}|}$$
$$\asymp\sum_j e^{-tN^{-1}\left(L^{-1}e_j\right)}e^{-L|X_j|}+O(e^{-L/2}).$$

  where (e_j,X_j) are Poisson distributed on $\mathbb{R}^+\times[-1/2,1/2]$.

  Choice of scale L depends on what is to be computed.
- Study of the ground state of fermionic systems in a random potential : consider N copies $H^\bullet_{\omega|L}$ and on $\wedge_{j=1}^N L^2([-L/2,L/2])$, for some interaction potential W,

$$H^{N,\bullet}_{\omega|L}=\sum_{j=1}^N \mathbf{1}\wedge\cdots\wedge H^\bullet_{\omega|L}\wedge\cdots\wedge\mathbf{1}+W.$$

  Study ground state : edge statistics gives description of free ground state (and (not too) excited states).

Only one model dependent parameter : the density of states N.