Lifshitz tails for some random Schrödinger operators or an aspect of random walk in random traps

F. Klopp

Université Paris 13
and
Institut Universitaire de France

Analysis and Probability
Nice - 23/06/2008

Outline

The setting and the questions

On $\mathbb{R}^d$, consider a stationnary ergodic random field $x \mapsto V_\omega(x)$.

Spectral theory On $L^2(\mathbb{R}^d)$, consider the random Schrödinger operator

$$H_\omega = -\frac{1}{2}\Delta + V_\omega$$

and the associated evolution equation

$$\begin{cases} i\partial_t \psi_t = H_\omega \psi_t, \\ \psi_{t|t=0} = \psi_0 \end{cases}$$

where

- $-\Delta$ is the Laplace operator on $\mathbb{R}^d$.

Questions:

- the spectral data of H_ω,
- the large time behavior of the semi-group.

Probability theory Consider the brownian motion in this random field i.e. the path measures

$$Q_t = \frac{1}{S_{t,\omega}} \exp\left(-\int_0^t V_\omega(Z_s)ds\right) P_0$$

$$Q_{t,\omega} = \frac{1}{S_t} \exp\left(-\int_0^t V_\omega(Z_s)ds\right) P_0 \otimes \mathbb{P}$$

where

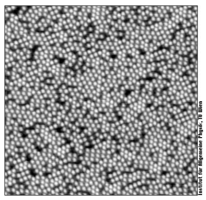
- $\mathbb{P}$ is the law of the random field V_ω,
- Z_s is the standard Brownian motion,
- P_0 is the Wiener measure,
- S_t and $S_{t,\omega}$ are normalizing constants.

Questions:

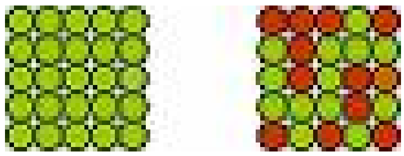
- the large t behavior of the path measures.

Various random potentials

1. The Poisson model: $V_\omega(x) = \sum_{n\in\mathbb{N}} V(x-\xi_n)$ where
   - $V: \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported;
   - $(\xi_n)_n$ is a Poisson point process.

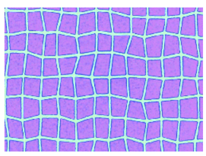


2. The alloy type model: $V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma)$ where
   - $V: \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported;
   - $(\omega_\gamma)_\gamma$ are real valued i.i.d random variables.



3. The displacement model: $V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} V(x-\gamma-\xi_\gamma)$

   where
   - $V: \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported;
   - $(\xi_\gamma)_\gamma$ are $\mathbb{R}^d$-valued i.i.d random variables.



The integrated density of states and the annealed random walk

Define the integrated density of states (IDS) of H_ω as

$$N(E) = \lim_{L\to+\infty} \frac{1}{(2L)^d} \#\{\text{eigenvalues of } H_{\omega|[-L,L]^d} \text{ less that } E\}.$$

Almost surely, the limit exists, is independent of ω and non decreasing.
The Pastur-Shubin formula:

$$N(E) = \begin{cases} \mathbb{E}\left[\mathbf{1}_{(-\infty,E]}(H_\omega)(0,0)\right] & \text{when } V_\omega \text{ is } \mathbb{R}^d\text{-ergodic,} \\ \mathbb{E}\left[\operatorname{tr}(\mathbf{1}_{[0,1]^d}\mathbf{1}_{(-\infty,E]}(H_\omega))\right] & \text{when } V_\omega \text{ is } \mathbb{Z}^d\text{-ergodic.} \end{cases}$$

Related to the heat kernel of H_ω by Laplace transform:

$$L(t) = \int_{\mathbb{R}} e^{-tE} dN(E) = \begin{cases} \mathbb{E}\left[e^{-tH_\omega}(0,0)\right] \text{ when } V_\omega \text{ is } \mathbb{R}^d\text{-ergodic,} \\ \mathbb{E}\left[\operatorname{tr}\left(\mathbf{1}_{[0,1]^d} e^{-tH_\omega}\right)\right] \text{ when } V_\omega \text{ is } \mathbb{Z}^d\text{-ergodic} \end{cases}$$

$$= (2\pi t)^{-d/2} \begin{cases} E^t_{0,0}\left(\mathbb{E}\left[\exp\left(-\int_0^t V_\omega(Z_s)ds\right)\right]\right) \text{ when } V_\omega \text{ is } \mathbb{R}^d\text{-ergodic,} \\ \int_{[0,1]^d} E^t_{x,x}\left(\mathbb{E}\left[\exp\left(-\int_0^t V_\omega(Z_s)ds\right)\right]\right) dx \text{ when } V_\omega \text{ is } \mathbb{Z}^d\text{-ergodic} \end{cases}$$

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Random operators

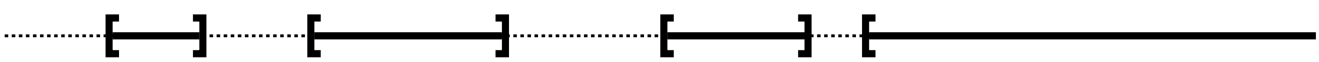
Under our assumptions, $H_\omega = -\Delta + V_\omega$ is essentially self-adjoint on $\mathscr{C}_0^\infty(\mathbb{R}^d)$. It is a metrically transitive family of operators i.e. there exists

- $(U_\alpha)_\alpha$ a family of unitary transform of $L^2(\mathbb{R}^d)$
- $(\tau_\alpha)_\alpha$, an ergodic family of transformation

such that

$$H_{\tau_\alpha \omega} = U_\alpha H_\omega U_\alpha^*.$$

The family $(H_\omega)_\omega$ admits an almost sure spectrum, say Σ such that $\Sigma = \operatorname{supp} dN$.
Typically Σ is a union of bands



One wants to study the behavior of $N(E)$ near spectral edges, in particular near $E_- = \inf(\Sigma)$.
It is known that the behavior of this function is instrumental in the study of the nature spectrum of H_ω (Lifshitz '63).

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The monotonous alloy type model

On $\mathbb{R}^d$, consider the alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma)$$

where

- $V : \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported; assume, moreover, $V \geq 0$;
- $(\omega_\gamma)_\gamma$ are i.i.d random variables distributed in $[0,a]$, $a>0$.

To fix ideas let us assume that $\log|\log \mathbb{P}(\{\omega_0 \leq \varepsilon\})| = o(|\log\varepsilon|)$ when $\varepsilon \to 0^+$.
Then, $\Sigma = [0,+\infty)$, i.e. $E_- = 0$.
Lifshitz tails:

Theorem (Lifshitz,Pastur,Kirsch,Simon,...)

One has

$$\lim_{E\to 0^+} \frac{\ln|\ln(N(E))|}{\ln(E)} = -\frac{d}{2}.$$

Recall for $H_0 = -\Delta$: $N(E) = C_d \max(E,0)^{d/2}$

An idea of the proof:

By Dirichlet-Neumann bracketing,

$$\mathbb{E}\left(\frac{1}{(2L)^d}\#\{n;\ \lambda_n(H^D_{\omega|[-L,L]^d}) \leq E\}\right) \leq N(E) \leq \mathbb{E}\left(\frac{1}{(2L)^d}\#\{n;\ \lambda_n(H^N_{\omega|[-L,L]^d}) \leq E\}\right)$$

One reduces the problem to estimating

$$\mathbb{P}\left(\{H^N_{\omega|[-L,L]^d} \text{ has an eigenvalue less than } \varepsilon\}\right)$$

for $L \sim \varepsilon^{-\alpha}$.
i.e. the probability that there exists $\psi \in H^1([-L,L]^d)$ such that

$$\langle -\Delta\psi,\psi\rangle + \langle V_\omega\psi,\psi\rangle \leq \varepsilon\|\psi\|^2.$$

As $V_\omega \geq 0$ and $-\Delta \geq 0$, this implies

$$\langle -\Delta\psi,\psi\rangle \leq \varepsilon\|\psi\|^2 \quad \text{and} \quad \langle V_\omega\psi,\psi\rangle \leq \varepsilon\|\psi\|^2.$$

So roughly, one has to estimate

$$\varepsilon^{d/2} \sum_{|\gamma|\leq\varepsilon^{-1/2}} \omega_\gamma \leq C\varepsilon,$$

and one concludes by large deviations.

The Poisson potential

On $\mathbb{R}^d$, consider the alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{n\in\mathbb{N}} V(x - x_n)$$

where

- $V : \mathbb{R}^d \to \mathbb{R}$ is continuous, non negative, non identically vanishing, real valued and compactly supported;
- $(x_n)_{n\in\mathbb{N}}$ are the support of a Poissonian cloud of positive density.

Then, $\Sigma = [0, +\infty)$ and $E_- = 0$.

Theorem (Pastur,Sznitman,...)

One has

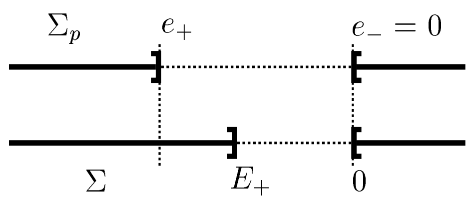
$$\lim_{E\to 0^+} \ln(N(E))E^{d/2} = -C < 0.$$

The result is obtained by probabilistic methods.
Much more precise than the previous result obtained using spectral methods.
But the spectral methods are more flexible.

Internal Lifshitz tails:

Let V_ω be of alloy type.
Lifshitz tails also hold at $\inf(\Sigma)$ when $H_0 = -\Delta$ becomes $H_0 = -\Delta + V_0$ where V_0 is $\mathbb{Z}^d$-periodic.

Let $n(E)$ be the IDS of H_0. Assume that $\Sigma_p = \sigma(H_0)$, the spectrum of H_0 has a gap below energy 0.
Assume that, for $t \in [0,1]$, $\sigma(H_0 + tV_\omega)$ has a gap below 0.



Theorem (K.,K.-Wolff)

Then

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{2} \Longleftrightarrow \lim_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = \frac{d}{2},$$

When $d = 2$, then

$$\limsup_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} < 0.$$

The non monotonous alloy type model:

On $\mathbb{R}^d$, consider the standard continuous alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma)$$

where

- $V: \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported;
- $(\omega_\gamma)_\gamma$ are i.i.d random variables distributed in $[a,b]$, a and b in the support.

One wants to study the spectrum or spectral quantities for H_ω near $E_- = \inf(\Sigma)$.

When V has a fixed sign, it is clear that

- $E_- = \inf(\sigma(-\Delta + V_{\overline{b}}))$ if $V \le 0$;
- $E_- = \inf(\sigma(-\Delta + V_{\overline{a}}))$ if $V \ge 0$.

We want to address the case when V changes sign i.e. we assume

(H1) there exists $x_+ \neq x_-$ such that $V(x_-)\cdot V(x_+) < 0$.

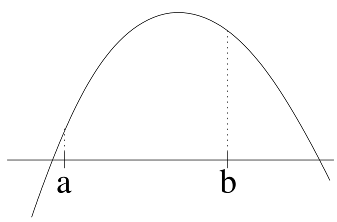
We require one more assumption:

(H2) V is supported in $(-1/2,1/2)^d$ and reflection symmetric i.e. for any $\sigma = (\sigma_1,\dots,\sigma_d) \in \{0,1\}^d$ and any $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, one has

$$V(x_1,\dots,x_d) = V((-1)^{\sigma_1}x_1,\dots,(-1)^{\sigma_d}x_d).$$

Determining the bottom of the spectrum:

Consider the operator $H_\lambda^N = -\Delta + \lambda V$ with Neumann b. c. on $[-1/2,1/2]^d$.
Its spectrum is discrete and let $E_-(\lambda)$ be its ground state energy.
It is a simple eigenvalue and $\lambda \mapsto E_-(\lambda)$ is a real analytic concave function.



Proposition (K.-Nakamura)

One has $E_- = \inf(\inf\sigma(H_{\overline{a}}), \inf\sigma(H_{\overline{b}})) = \inf(E_-(a), E_-(b))$.

If a and b sufficiently small, Najar proved proposition assuming $\int_{\mathbb{R}^d} V(x)dx = E_-'(0) \neq 0$ without (H2).

Lifshitz tails : when $E_-(a) \neq E_-(b)$

Denote by $N(E)$ the integrated density of states of H_ω.

Theorem (K.-Nakamura)

Assume $E_-(a) \neq E_-(b)$. Then

$$-\frac{d}{2} - \alpha_- \leq \liminf_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq \limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq -\frac{d}{2} - \alpha_+$$

where $c = a$ if $E_-(a) < E_-(b)$ and $c = b$ if $E_-(a) > E_-(b)$ and

$$\alpha_- = -\liminf_{\varepsilon \to 0} \frac{\log|\log \mathbb{P}(\{|c-\omega_0| \leq \varepsilon\})|}{\log \varepsilon} \geq 0,$$
$$\alpha_+ = -\limsup_{\varepsilon \to 0} \frac{\log|\log \mathbb{P}(\{|c-\omega_0| \leq \varepsilon\})|}{\log \varepsilon} \geq 0.$$

This result is similar to the one obtained in the monotonous case.

Lifshitz tails: when $E_-(a) = E_-(b)$

Theorem (K.-Nakamura)

Assume (H1) and (H2) and $E_- := E_-(a) = E_-(b)$. Then,

1. *If the random variables $(\omega_\gamma)_\gamma$ are not Bernoulli distributed i.e. if $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$, then*

$$-\frac{d}{2} - \alpha_- \leq \liminf_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq \limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq -\frac{1}{2} - \alpha_+. \quad (2.1)$$

2. *If $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1$, there exists potentials V satisfying assumption (H1) and (H2) such that $E_-(a) = E_-(b)$ and, there exists $C > 0$ such that, for $E \geq E_-$,*

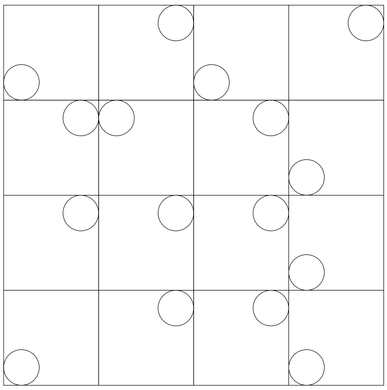
$$\frac{1}{C}(E-E_-)^{d/2} \leq N(E) \leq C(E-E_-)^{d/2}.$$

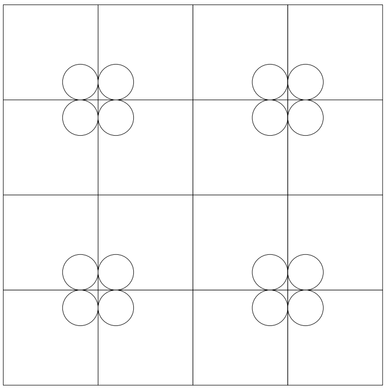
A random displacement model

Consider
$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} V(x-\gamma-\xi_\gamma).$$
where

- $V: \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing and supported in $(-r,r)^d$, $0<r<1/2$ and satisfies (H2);
- $(\xi_\gamma)_\gamma$ are independent identically distributed (i.i.d.) random variables distributed in $\{-1/2+r, 1/2-r\}^d$ such that all these points have a positive probability.

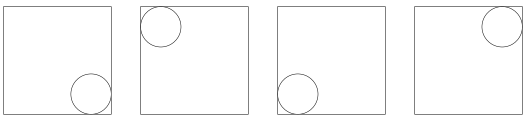
By work of Baker, Loss and Stolz, minimizing configurations given by a symmetric "clusterization".





For $\xi \in \{-1/2+r, 1/2-r\}^d$, we define $H_\xi = -\Delta + V(x-\xi)$ on $[-1/2,1/2]^d$ with Neumann BC.

All the $(H_\xi)_\xi$ have the same ground state energy, say E_-.



H_{ξ_1} and H_{ξ_2} match in the direction e_j if E_- is also the ground state energy of $-\Delta + V(\cdot-\xi_1)+V(\cdot-e_j-\xi_2)$ on $[-1/2,1/2]^d \cup (e_j+[-1/2,1/2]^d)$ (Neumann BC).

Theorem (K.-Nakamura)

Let $N(E)$ denote the IDS of H_ω. Then,

1. *if, at least, two of the $(H_\xi)_\xi$ do not match in, at least, one direction, one has*
$$\limsup_{E\to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} \le -\frac{1}{2};$$
2. *if all the $(H_\xi)_\xi$ match in all directions, then $N(E) \ge c(E-E_-)^{d/2}$.*

Finding the minimum: decoupling due to symmetry

Recall that $E_-(\lambda)$ is the ground state energy of the operator H^N_λ i.e. $-\Delta+\lambda V$ on $[-1/2,1/2]^d$ with Neumann boundary conditions.
To fix ideas, assume $E_-(a)\le E_-(b)$.
Partitioning $\mathbb{R}^d$ into cubes $\gamma+[-1/2,1/2]^d$ for $\gamma\in\mathbb{Z}^d$, we get that

$$H_\omega \ge \bigoplus_{\gamma\in\mathbb{Z}^d} H^N_{\omega_\gamma}.$$

Hence, $H_\omega\ge E_-(a)$.
Consider $H^P_{\omega,L}$, the operator H_ω restricted to the cube $[-L-1/2,L+1/2]^d$ with periodic boundary conditions.
One proves

Lemma

$$\Sigma=\overline{\bigcup_{L\ge 1}\ \bigcup_{\omega\ admissible}\sigma(H^P_{\omega,L})}.$$

The characterization of the infimum of the almost sure spectrum follows from

$$\inf_{\omega\in[a,b]^{C^d_L}}\inf\sigma(H^P_{\omega,L})\le E_-(a)\quad \text{where } C^d_L=\mathbb{Z}^d\cap[-L-1/2,L+1/2]^d.$$

The normalized positive ground state of H^N_a, say ψ, is simple and unique.

The reflection symmetry of the potential V guarantees that ψ is reflection symmetric.

For $\gamma\in\mathbb{Z}^d$ such that $|\gamma|_1=1$, we continue ψ to the $\gamma+[-1/2,1/2]^d$ by reflection symmetry with respect to the common boundary of $[-1/2,1/2]^d$ and $\gamma+[-1/2,1/2]^d$.

As ψ is reflection symmetric, we obtain a continuation of ψ that is $\mathbb{Z}^d$-periodic, positive and reflection symmetric with respect to any plane that is common boundary to two cubes of the form $\gamma+[-1/2,1/2]^d$.

Moreover ψ satisfies, for any $L\ge 0$, $H^P_{\bar a,L}\psi=H^P_{a,0}\psi=H^N_{a,0}\psi=E_-(a)\psi$. This proves that $E_-(a)\ge\inf\sigma(H^P_{\bar a,L})$. □

When the single site potential of fixed sign, $H^P_{\omega,L}$ is increasing/decreasing in any ω_γ $\Longrightarrow$ one can optimize each random variables separately.

With symmetry assumption, also decoupling the dependence on the random variables.

The upper bound in the Lifshitz tails when $E_-(a) \neq E_-(b)$

Theorem (K.-Nakamura)

Suppose assumptions (H1) and (H2) are satisfied, and, that $E_-(a) < E_-(b)$. Then, there exists $C > 0$ such that, for E close to $E_-(a)$, one has $N(E) \leq N_m(C(E - E_-(a)))$ where N_m is the integrated density of states of the random operator

$$H^m_\omega = H_{\overline{a}} - E_-(a) + \sum_{\gamma\in\mathbb{Z}^d} (\omega_\gamma - a)\mathbf{1}_{[-1/2,1/2]^d}(x-\gamma)$$

and $H_{\overline{a}}$ is defined above.

This is a consequence of Neumann BC and the simple

Lemma

Let H_0 be self-adjoint on $\mathscr{H}$ a separable Hilbert space such that $0 = \inf\sigma(H_0)$. Let V_1 be a non trivial closed symmetric operator relatively bounded with respect to H_0 with bound 0. Set $H_1 = H_0 + V_1$ and $E_1 = \inf\sigma(H_1)$. Assume $E_1 > 0$. Then, there exists $C > 0$ such that, for $t \in [0,1]$, one has

$$C(H_0 + tV_1) \geq H_0 + t$$

When $E_-(a) = E_-(b)$: absence of Lifshitz tails.

Let $\varphi \in \mathscr{C}^\infty((-1/2,1/2)^d)$ be positive, reflection symmetric, constant near the boundary of $[-1/2,1/2]^d$ and normalized in the cube.

Let $V = \Delta\varphi/\varphi$. Then, φ is the positive normalized ground state of $-\Delta + V$ on $[-1/2,1/2]^d$ with Neumann boundary conditions.

Let $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ be Bernoulli r.v. with support $\{0,1\}$.

Let φ_L be ground state of $H^N_{\omega,L}$: it is equal to

- in $\gamma + [-1/2,1/2]^d$, $\varphi_L(\cdot) = \varphi(\cdot - \gamma)$ if $\omega_\gamma = 1$;
- in $\gamma + [-1/2,1/2]^d$, $\varphi_L(\cdot) =$ cst if $\omega_\gamma = 0$.

As the ground state is uniformly bounded (in ω and L), a result of [KiSi89] and a calculation imply that, there exists $C_D \geq c_N > 0$, for all ω,

- the second eigenvalue of the Neumann problem is larger than $c_N L^{-2}$;
- the ground state of the Dirichlet problem is smaller than $C_D L^{-2}$.

As

$$\frac{1}{L^d}\mathbb{E}\left(\#\{\text{eigenvalues of } H^D_{\omega,L} \ \leq E\}\right) \leq N(E) \leq \frac{1}{L^d}\mathbb{E}\left(\#\{\text{eigenvalues of } H^N_{\omega,L} \ \leq E\}\right),$$

for $L = cE^{-1/2}$, we get $C^{-1}E^{d/2} \leq N(E) \leq CE^{d/2}$.

The upper bound in the Lifshitz tails when $E_-(a) = E_-(b)$

Assume that $(\omega_\gamma)_\gamma$ are not Bernoulli distributed i.e. $P(\omega_0 = a) + P(\omega_0 = b) < 1$. Pick $\varepsilon > 0$ such that

$$P(\omega_0 \leq a + \varepsilon) + P(\omega_0 \geq b - \varepsilon) < 1.$$

Let $H^N_{\omega,L}$ be the operator H_ω restricted to the cube $[-L-1/2, L+1/2]^d$ with Neumann boundary conditions.
Define

$$N^N_L(E) = (2L+1)^{-d}\mathbb{E}(\#\{\text{eigenvalues of } H^N_{\omega,L} \leq E\}).$$

Well known : the sequence $N^N_L(E)$ is decreasing and converges to $N(E)$ (except possibly at countably many E).

Define $E_{-,L}(\omega) = \inf \sigma(H^N_{\omega,L})$. One has $N^N_L(E) \leq C\mathbb{P}(\{E_{-,L}(\omega) \leq E\})$

Sufficient to prove a suitable upper bound for $\mathbb{P}(\{E_{-,L}(\omega) \leq E\})$ for a well chosen value of L.

Basic property:

Lemma

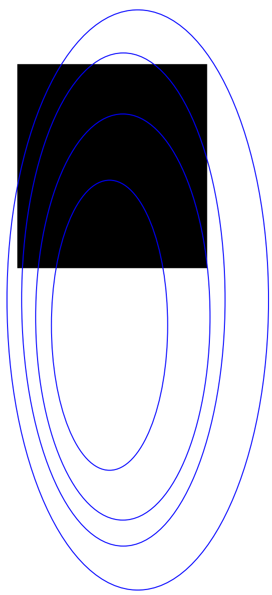
The function $\omega \mapsto E_{-,L}(\omega)$ is real analytic and strictly concave.

The function $\omega \mapsto E_{-,L}(\omega)$ is defined on $\mathbb{R}^{C^d_L}$.

The upper epigraphs of $\omega \mapsto E_{-,L}(\omega)$ i.e. the sets $\Omega_L(E) := \{\omega \in \Omega_L;\ E_{-,L}(\omega) > E\}$ are convex.

On Ω_L, $E_{-,L}(\omega)$ reaches its minimum only at one or more vertices of Ω_L.

One studies what happens at the vertices of Ω_L i.e. at the points of $\{a,b\}^{C^d_L}$.



A local estimate on the ground state energy

Assume $E_- = E_-(a) = E_-(b)$.

Lemma (K.-Nakamura)

Partition the discrete cube C_L^d into strips

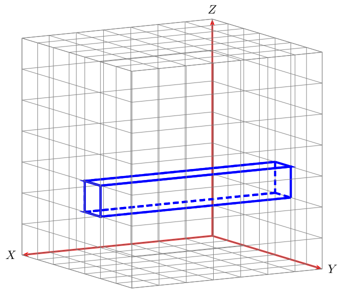
$$C_L^d = \bigcup_{\gamma' \in C_L^{d-1}} S_{L,\gamma'} \text{ where } S_{L,\gamma'} = \{(\gamma_1, \gamma'); \ -L \leq \gamma_1 \leq L\}.$$

There exists $C > 0$ such, for all $L \geq 0$, if $\omega \in \{a, b, a+\varepsilon, b-\varepsilon\}^{C_L^d}$ is such that

(Prop) *for all $\gamma' \in C_L^{d-1}$, there exists $\gamma \in S_{L,\gamma'}$ such that $\omega_\gamma \in \{a+\varepsilon, b-\varepsilon\}$*

then

$$E_{-,L}(\omega) \geq E_- + \frac{1}{CL^2}.$$



The proof of this result relies on Neumann decoupling and on the analysis of the ground state energy of a strip where all but one single site potential are the same.

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Using the concavity of the ground state, one gets

Corollary

There exists $C > 0$, independent of $L \geq 0$ and $\omega \in \Omega_L$, such that if

(Prop (ter)) *for all $\gamma' \in C_L^{d-1}$, there exists $\gamma \in S_{L,\gamma'}$ s.t. $\omega_\gamma \in [a+\varepsilon, b-\varepsilon]$*

then

$$E_{-,L}(\omega) \geq E_- + \frac{1}{CL^2}.$$

with the same constant as in the lemma.

Pick $E > E_-(a) = E_-(b)$, $L = c(E - E_-(a))^{-1/2}$ and $c > 0$ s.t. $Cc^2 < 1$.
Corollary ensures that, if ω satisfies (Prop (ter)), then $E_-(\omega) > E$.
So, the set $\Omega_L(E) := \{\omega \in \Omega_L; \ E_-(\omega) > E\}$ satisfies

$$\Omega_L \setminus \Omega_L(E) \subset \{\omega \in \Omega_L; \ \exists \gamma' \in C_L^{d-1}, \ \forall \gamma \in S_{L,\gamma'}, \ \omega_\gamma \in [a, a+\varepsilon) \cup (b-\varepsilon, b]\}.$$

Hence,

$$\mathbb{P}(\Omega_L \setminus \Omega_L(E)) \leq \sum_{\gamma' \in C_L^{d-1}} \mathbb{P}(\{\forall \gamma \in S_{L,\gamma'}, \ \omega_\gamma \in [a, a+\varepsilon) \cup (b-\varepsilon, b]\})$$
$$= (2L+1)^{d-1} [\mathbb{P}(\omega_0 \in [a, a+\varepsilon)) + \mathbb{P}(\omega_0 \in (b-\varepsilon, b])]^{2L+1}$$

This yields exponential decay.

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