# Internal Lifshitz tails for the density of surface states of random operators

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SAMP VI 2008, Santiago - 07/01/2008



## Outline

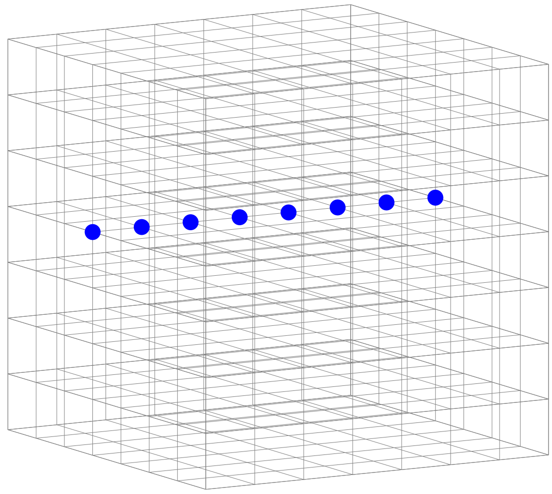
## The main result in a simple setting.

Let $d_1 + d_2 = 3$, $(d_1, d_2) \in \{1,2\}^2$. On $\mathbb{R}^3$,

$$H_\omega = H + V_\omega = H + \sum_{\gamma_1 \in \mathbb{Z}^{d_1} \times \{0\}} \omega_{\gamma_1} V(\cdot - \gamma_1),$$

where

- $H = -\Delta + W$ where $W$ is a $\mathbb{Z}^3$-periodic, bounded, real valued potential;
- $V$ is a non-negative, strictly positive on some open set, bounded, real valued, compactly supported potential;
- $\omega = (\omega_{\gamma_1})_{\gamma_1 \in \mathbb{Z}^{d_1}}$ is a collection of independent identically uniformly distributed, say on $[a,b]$ ($a < b$ real), random variables.



Let $\Sigma_0$ be the spectrum of $H$ and $\Sigma = \sigma(H_\omega)$ be the almost sure spectrum of $H_\omega$. One has $\Sigma_0 \subset \Sigma$.



## The integrated density of surface states (IDSS):

For $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$

$$(\varphi, dN_s) = \mathbb{E}(\mathrm{tr}\,(\mathbf{1}_{|x_1| \leq 1/2}[\varphi(H_\omega) - \varphi(H)]))$$

Outside $\Sigma_0$, this density of states is a positive measure.
In the setting described above, let $E_0 \in \partial\Sigma \setminus \partial\Sigma_0$. Let

$$H_t = H + t \sum_{\gamma_1 \in \mathbb{Z}^{d_1} \times \{0\}} V(\cdot - \gamma_1), \quad t \in [a,b].$$

Then, $E_0 \in \partial\sigma(H_a)$ or $E_0 \in \partial\sigma(H_b)$.

### Theorem

*Assume the spectrum of $H_a$ (or $H_b$) is absolutely continuous near $E_0$. Then, one has*

$$\limsup_{\substack{E \to E_0 \\ E \in \Sigma}} \frac{\ln|\ln(N_s(E) - N_s(E_0))|}{\ln|E - E_0|} < 0$$

Exponential decay of the density of surface states.
This estimate can then be used to obtain localization.

At the bottom of the spectrum, results of Kirsch and Warzel (2006). In the discrete case, results by Kirsch and Klopp (2006).

## The general setting:

Let $d = d_1 + d_2$, $(d_1, d_2) \in (\mathbb{N}^*)^2$.
In $\mathbb{R}^d$, let $\Gamma$ be a degenerate lattice i.e. $\operatorname{Span}\Gamma \subsetneq \mathbb{R}^d$.
Let $\mathscr{E}$ =Span$\{e_j,\ 1 \le j \le d_1\}$ where $(e_j)_{1\le j\le d_1}$ is a basis of $\Gamma$.
Define $\mathscr{C}_0 = \{x = \sum_{j=1}^{d_1} x_j e_j;\ |x_j| \le 1/2\} \subset \mathscr{E}$, the Brillouin zone of $\Gamma$ in $\mathscr{E}$.
Let $(f_j)_{1\le j\le d_2}$ be a basis of $\mathscr{E}^\perp$, the orthogonal of $\mathscr{E}$.

Consider

$$H_\omega = H + V_\omega = H + \sum_{\gamma\in\Gamma} \omega_\gamma V(\cdot - \gamma),$$

where

- $H = -\Delta + W$ and $W$ is a $\Gamma$-periodic (i.e. $\forall\gamma \in \Gamma$, $W(x+\gamma) = W(x)$), bounded potential;
- $V$ is a non-negative, strictly positive on some open set, bounded, real valued, compactly supported potential;
- $\omega = (\omega_\gamma)_{\gamma\in\Gamma}$ is a collection of independent identically distributed bounded random variables.

$H_\omega$ is self-adjoint on $H^2(\mathbb{R}^d)$. Let $\Sigma = \sigma(H_\omega)$ be the almost sure spectrum of $H_\omega$.



## The Floquet decomposition of partially periodic operators

Define

- $\Gamma^*$ be the dual lattice to $\Gamma$ i.e. $\Gamma^* = \{\gamma^* \in \mathscr{E};\ \forall\gamma \in \Gamma,\ \gamma\cdot\gamma^* \in 2\pi\mathbb{Z}\}$;
- $\mathbb{T}^* = \mathscr{E}/\Gamma^*$ and $S_{1/2} = \{x = \sum_{j=1}^{d_1} x_j e_j + \sum_{j=1}^{d_2} y_j f_j;\ |x_j| \le 1/2\} \subset \mathbb{R}^d$.

Define

$$(Uu)(x,\theta) = \sum_{\gamma\in\Gamma} e^{i\gamma\cdot\theta} u(x-\gamma).$$

Define

$$\mathscr{H} = \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^d) \otimes L^2(\mathbb{T}^*);\ \begin{array}{c} \theta \text{ a.e, } \int_{S_{1/2}} |u(x,\theta)|^2 dx < +\infty \text{ and} \\ \forall(x,\theta,\gamma) \in \mathbb{R}^d \times \mathbb{T}^* \times \Gamma,\ u(x+\gamma,\theta) = e^{i\gamma\cdot\theta} u(x,\theta) \end{array} \right\}$$

$U$ is a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathscr{H}$ and $(U^{-1}v)(x) = \dfrac{1}{Vol(\mathbb{T}^*)} \displaystyle\int_{\mathbb{T}^*} v(x,\theta)d\theta$.

$H$ admits the Floquet decomposition $UHU^* = \displaystyle\int_{\mathbb{T}^*}^{\oplus} H_\theta d\theta$ and $\Sigma_0 = \displaystyle\bigcup_{\theta\in\mathbb{T}^*} \sigma(H_\theta)$ where

$\Sigma_0$ denotes the spectrum of $H$.

**Main difference with the totally periodic case:** as $S_{1/2}$ is not compact, the operators $H_\theta$ are not necessarily of compact resolvent.

## The density of surface states for partially periodic operators

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a potential that is $\Gamma$-periodic such that

$$\forall x \in \mathscr{F}, \quad \sup_{y \in \mathscr{E}} |V(x+y)| \leq \frac{C}{(1+|x|)^{\alpha_2}} \text{ where } \alpha_2 > d_2.$$

Clearly $V_\theta$ is a relatively compact perturbation of $H_\theta$.
The spectrum of $H_\theta + V_\theta$ in the gaps of $\Sigma_0$ is discrete.
One defines $n_s(E)$, the IDSS of the operator $H+V$ as follows

$$\int_{\mathbb{R}} \varphi(E) dn_s(E) = \frac{1}{(2\pi)^{d_1}} \int_{\mathbb{T}^*} \mathrm{tr}_\theta \left(\varphi(H_\theta + V_\theta) - \varphi(H_\theta)\right) d\theta, \quad \forall \varphi \in \mathscr{C}_0^\infty(\mathbb{R}).$$

### Lemma

*For* $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(E) dn_s(E) = \frac{1}{Vol(\mathscr{C}_0)} \mathrm{tr}\left(\mathbf{1}_{S_{1/2}}[\varphi(H+V) - \varphi(H)]\right).$



## The integrated density of surface states of the random model

Consider $V_{\omega,L} = \sum_{\beta \in (2L+1)\Gamma} \sum_{\gamma \in \Gamma/(2L+1)\Gamma} \omega_\gamma V(\cdot - \gamma - \beta).$
Define $H_{\omega,L} = H + V_{\omega,L}$ and $dn_s(\omega, L)$, the density of surface states of $H_{\omega,L}$ and $H$.

### Theorem

*When* $L \to +\infty$, $\mathbb{E}(dn_s(\omega,L))$ *converges to* $dN_s$, *the integrated density of surface states. It satisfies, for* $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$,

$$\langle \varphi, dN_s \rangle = \frac{1}{Vol(\mathscr{C}_0)} \mathbb{E}(\mathrm{tr}(\mathbf{1}_{S_{1/2}}[\varphi(H_\omega) - \varphi(H)]))$$

*where* $Vol(\mathscr{C}_0)$ *is the volume of* $\mathscr{C}_0$ *(measured with respect to the restriction of the Lebesgue measure to* $\mathscr{E}$*).*
*Moreover, for* $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ *such that* $\mathrm{supp}\varphi \subset \mathbb{R} \setminus \Sigma_0$, *almost surely,*

$$\langle \varphi, dN_s \rangle = \lim_{L \to +\infty} \langle \varphi, dn_s(\omega, L) \rangle$$

*Moreover,* $dN_s$ *defines a positive measure in the gaps of* $\Sigma_0$.

## Fluctuating edges

The edge $E_0 \in \partial\Sigma$ is *fluctuating* for $H_\omega$ if, for all $\varepsilon > 0$ sufficiently small, $E_0$ is in a gap of the spectrum of either $H_\omega + \varepsilon \sum_{\gamma\in\Gamma} V(\cdot - \gamma)$ or $H_\omega - \varepsilon \sum_{\gamma\in\Gamma} V(\cdot - \gamma)$.

### Theorem

*Let $E_0$ be an edge of the spectrum of $H_\omega$. $E_0$ is fluctuating if and only if it does not belong to the union of the essential spectra of $H_\theta$ when $\theta$ runs over $\mathbb{T}^*$.*

## Low dimensional surfaces

### Theorem

*Assume $d_1 \in \{1,2\}$ and that the common support of the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ is the interval $[a,b]$.*
*Assume that $E_0$ is a fluctuating edge and that if $E_0 \in \sigma(H_a)$ (resp. $E_0 \in \sigma(H_b)$) then $H_a$ (resp. $H_b$) has only a.c. spectrum near $E_0$.*
*Then, one has*

$$\limsup_{\substack{E\to E_0 \\ E\in\Sigma}} \frac{\ln|\ln|N_s(E) - N_s(E_0)||}{\ln|E - E_0|} < 0$$



## Non degenerate band edges

Assume that

(H2) for some $\delta > 0$, $(E_0 - \delta, E_0) \cap \Sigma = \emptyset$ and $E_0 \in \Sigma$.

Consider the operator

$$H_a = H + a \sum_{\gamma\in\Gamma} V(\cdot - \gamma)$$

where the support of $\omega_{\gamma_1}$ is $[a,b]$. This operator is $\Gamma$-periodic.
Under our assumptions of the random variables, $E_0$ is also an edge of $\sigma(H_a)$.
Let $n_s$ be the integrated density of surface states of $H_a$.
We say that $E_0$ is a *non degenerate edge* if

$$\limsup_{E\to E_0^+} \frac{\log(n_s(E) - n_s(E_0))}{\log|E - E_0|} = \frac{d_1}{2}.$$

Assume that $\displaystyle \frac{\log \mathbb{P}(\omega_\gamma - a \le \varepsilon)}{\log \varepsilon} \underset{\varepsilon\to 0^+}{\to} 0$

We prove

## Theorem

*Let $E_0 \in \partial\Sigma$ be a fluctuating edge and that the spectrum of $H_a$ at $E_0$ is absolutely continuous.*
*Then*

1. *one has*
$$\liminf_{E \to E_0^+} \frac{\ln|\ln(N_s(E) - N_s(E_0))|}{\ln(E - E_0)} \geq -\frac{d_1}{2};$$

2. *$E_0$ is a non degenerate edge of the spectrum of $H_a$ if and only if*
$$\limsup_{E \to E_0^+} \frac{\ln|\ln(N_s(E) - N_s(E_0))|}{\ln(E - E_0)} = -\frac{d_1}{2};$$