Lyapunov exponents and singular continuous spectrum

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Outline

Quasi-periodic finite difference equations

Consider the finite difference eigenvalue problem

$$(H_\theta \psi)(n) = \psi(n+1) + \psi(n-1) + v(n\omega + \theta)\psi(n) = E\psi(n) \qquad (1.1)$$

where $v : \mathbb{R} \to \mathbb{R}$ is continuous and $v(x+2) = v(x)$, $\omega \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

If $\omega = p/q \in \mathbb{Q}$, $(p,q) \in \mathbb{N} \times \mathbb{N}^*$, then $n \mapsto v(n\omega + \theta)$ is q-periodic.
Hence,

$$\sigma(H_\theta) = \sigma_{ac}(H_\theta)$$

and $\sigma(H_\theta)$ depends on θ.
If $\omega \notin \mathbb{Q}$, then $n \mapsto v(n\omega + \theta)$ is quasi-periodic.
There exist closed sets σ, σ_{pp}, σ_{ac}, σ_{sc}, such that for almost every θ, one has

$$\sigma = \sigma(H_\theta),\ \sigma_{pp} = \sigma_{pp}(H_\theta),\ \sigma_{ac} = \sigma_{ac}(H_\theta),\ \sigma_{sc} = \sigma_{sc}(H_\theta) \quad \text{(Pastur).}$$

Actually, $\sigma = \sigma(H_\theta)$ for every θ.
For analytic v, $\sigma_s(H_\theta) = \sigma_{pp}(H_\theta) \cup \sigma_{sc}(H_\theta)$ independent of θ (Last-Simon).

Study has generated a vast literature:

A. Avila, Y. Avron, J. Bellissard, J. Bourgain, V. Buslaev, V. Chulaevsky, D. Damanik, E. Dinaburg, H. Eliasson, A. F., B. Helffer, M. Hermann, S. Jitomirskaya, F. K., R. Krikorian, Y. Last, J. Puig, M. Shubin, B. Simon, Y. Sinaï, J. Sjöstrand, S. Sorets, T. Spencer, M. Wilkinson, ...

The spectral theory of quasi-periodic operators very rich. Many models exhibit

- Cantorian spectrum;
- spectral nature depending on the “number theoretical” properties of the frequency ω;
- topologically typical singular continuous spectrum (i.e. for a dense G_δ set of parameters).

This has only been shown for a few models (e.g. the almost Mathieu equation $v(x) = 2\lambda\cos(\pi x)$).

The Lyapunov exponent

Equation (1.1) can be rewritten as

$$\begin{pmatrix}\psi(n+1)\\ \psi(n)\end{pmatrix} = M((n-1)\omega+\theta)\begin{pmatrix}\psi(n)\\ \psi(n-1)\end{pmatrix}, \quad M(x) = \begin{pmatrix}E-v(x) & -1\\ 1 & 0\end{pmatrix}$$

The product $M((n-1)\omega+\theta)\cdot M((n-2)\omega+\theta)\cdots M(\theta)$ defines the behavior of the solutions of (1.1). For fixed E, consider

$$\gamma(E,\theta) = \lim_{n\to+\infty}\frac{1}{n}\log\|M((n-1)\omega+\theta)\cdots M(\theta)\|. \tag{1.2}$$

Theorem

For almost every θ, this limit exists and does not depend on θ.

In this case, we call it the Lyapunov exponent $\gamma(E)$.

If the limit (1.2) does not exist or exists but differs from $\gamma(E)$, we say that the Lyapunov exponent does not exist.

When $\omega\in\mathbb{Q}$, $\gamma(E)$ is the imaginary part of the Bloch quasi-momentum.
When $\omega\notin\mathbb{Q}$:

- The absolutely continuous spectrum is the essential closure of the set of energies where the Lyapunov exponent vanishes (Ishii-Pastur-Kotani).
- If $\gamma(E)$ is positive on I, an interval, then $\sigma\cap I\subset\sigma_s$.

One cannot replace σ_s with σ_{pp}.

For $\gamma(E,\theta)>0$, solutions to (1.1) have simple exponential behavior.

For given θ, the limit a priori exists only almost everywhere in E.

In general, solutions to equation don't have a simple behavior.

If $\gamma(E)>0$ on I and if, in I, the spectrum is singular continuous, the spectrum in I is located at the energies where the limit does not exist.

Two examples of singular continuous spectrum

Both examples: the almost Mathieu equation $v(x) = 2\lambda\cos(\pi x)$).

B. Simon's example:

For almost Mathieu, $\gamma(E) = max(0, \log\lambda)$.

If $\lambda > 1$ then $\gamma(E) > 0$.

Let ω be such that, for some sequence $(p_m, q_m) \in \mathbb{N} \times \mathbb{N}^*$,

$$\left|\omega - \frac{p_m}{q_m}\right| \leq m^{-q_m}.$$

Then, no eigenvalues and no absolutely continuous spectrum i.e. the spectrum is purely singular continuous.

Such Liouvillean frequencies are topologically typical but of zero measure.

Simon's result: consequence of a result by A. Gordon:

if potential is exponentially well approximated by periodic potentials, the equation does not admit any decreasing solutions.

In the case of the almost Mathieu equation, one proves

$$\limsup_{m\to\infty} \max\left(\phi(\pm q_m), \phi(\pm 2q_m)\right) \geq \frac{1}{2}\phi(0),$$
$$\phi(n) = (|\psi(n+1)|^2 + |\psi(n)|^2)^{1/2}$$

Theory of subordinate solution (Gilbert-Pearson): for ψ, a generalized eigenfunction of the singular continuous spectrum,

$$\frac{\sum_{n=0}^{N} |\psi(n)|^2}{\sum_{n=0}^{N} |\tilde{\psi}(n)|^2} \to 0 \text{ as } N \to +\infty.$$

for any linearly independent solution $\tilde{\psi}$.

Example of S. Jitomirskaya and B.Simon.

Let θ irrational be such that, for some $B>0$ and infinitely many integers m,

$$\mathrm{dist}(\theta+m\omega,\mathbb{Z}/2)\leq e^{-B|m|}.$$

Then, equation (1.1) does not admit a square summable solution.

The basic idea is that $\psi(k)$ and $\psi(2m-k)$ satisfy almost the same equation and are linearly independent: impossible if ψ decays at infinity.

An almost explicitly solvable model

Consider the equation

$$\psi(n+1)+\psi(n-1)+2\lambda\, e^{i\pi\omega/2}\cos(\pi(n\omega+\theta))\ \psi(n)=0. \qquad (2.1)$$

Our aims:

1. describe when the Lyapunov exponent does not exists.
2. describe solutions of the difference equation whether Lyapunov exponent exists or not.

Three reasons to study (2.1)**:**

1. existence of the Lyapunov exponent and behavior of the solution described quite explicitly.
2. model related to self-adjoint models via cocycle representation e.g.

$$-\psi''(t)+\alpha\sum_{l\geq 0}\delta\left(l(l-1)/2+l\phi_1+\phi_2-t\right)\psi(t)=E\psi(t).$$

3. techniques developed extendable to general real analytic v.

Our tool: an extension of the monodromization method introduced by Buslaev-Fedotov to construct Weyl solutions outside the spectrum. We study the solutions on the spectrum.

Existence of the Lyapunov exponent:

for $L = 0, 1, 2\dots$, define

$$\omega_{L+1} = \frac{1}{\omega_L} \pmod 1, \quad \omega_0 = \omega \quad \text{and} \quad \lambda_{L+1} = \lambda_L^{\frac{1}{\omega_L}}, \quad \lambda_0 = \lambda$$

Theorem

Assume $\lambda_L \omega_L \to \infty$. There exists a function $(k, \omega) \mapsto L(k, \omega) \in \mathbb{N}$ such that the Lyapunov exponent for equation (2.1) exists if and only if there exists a positive sequence $\{c_L\}_{L=1}^{\infty}$ tending to zero and such that, for all positive integers (k, l) satisfying $0 < k - l\omega_0 < 1$, one has:

$$\left|\theta - \frac{1}{2} - (k - l\omega_0)\right| \geq \omega_0 \omega_1 \dots \omega_L e^{-\frac{c_L}{\omega_0 \omega_1 \cdots \omega_L}}, \quad L = L(k, \omega). \tag{2.2}$$

For ω fixed, $k \mapsto L(k, \omega)$ is an increasing sequence of integers such that

$$k \cdot \omega_0 \cdots \omega_{L(k,\omega)-1} \asymp 1 \tag{2.3}$$

Moreover, when the Lyapunov exponent exits, it is equal to $\log \lambda$.

The set of θ satisfying (2.2) is topologically typical and of measure 0.

Comments.

Condition $\lambda_L \omega_L \to \infty$ is satisfied for a set of total measure of frequencies; it contains all the Diophantine frequencies and many more.

Indeed, ω is Diophantine of order β if

$$|\omega - p/q| \geq \frac{C_1(\omega)}{q^{2+\beta}}, \quad \forall q \in \mathbb{Z}_+ \; p \in \mathbb{Z}.$$

Such numbers form a set of full measure.
J.C. Yoccoz has shown that an equivalent condition is

$$\omega_{L+1} \geq C_2(\omega) \omega_L^{1+\beta}.$$

Compare with our requirement

$$\omega_{L+1} \geq \frac{C_L}{\lambda_{L+1}} = C_L e^{-\frac{\ln \lambda}{\omega_0 \omega_1 \dots \omega_L}}$$

for an increasing sequence $\{C_L\}$.
There exists $C > 1$ such, for typical ω, one has $\omega_0 \dots \omega_L \leq C^{-L}$ (Khinchin, Lévy).

What happens when the criterion is not satisfied?

There is a simple result by N. Riedel:

Theorem

Let ω be a Diophantine number. If, for some integers k and l, one has $\theta = 1/2 + k - l\omega$, then

$$\psi(n+1) + \psi(n-1) + 2\lambda\cos(\pi(\theta + n\omega))\psi(n) = 0$$

has a square summable solution.

Our method allows us to construct this solution which has “simple” asymptotics at ∞.

For $\theta = 1/2$, it has a bump at the point $n = 0$ and is exponentially decaying at infinity.

When $\theta = 1/2 + k - l\omega$, the solution has various bumps.

Local behavior

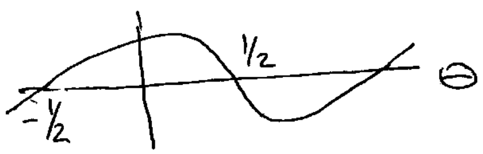
We solve $\quad \psi(k+1) + \psi(k-1) + v(k\omega + \theta)\psi(k) = E\psi(k).$

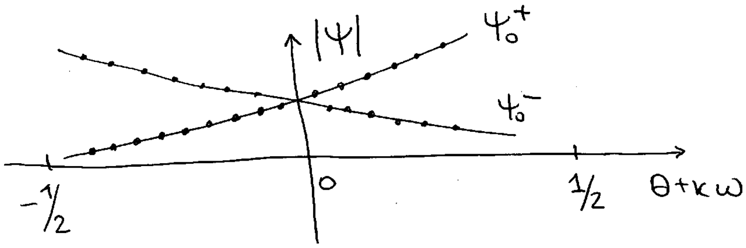
Fix $0 < \varepsilon < 1$. For large λ, for $\theta + k\omega$ in $[-1/2 + \varepsilon\omega, 1/2 - \varepsilon\omega]$, solutions of the form

$$\psi_0^{\pm}(k) = \varphi_0^{\pm}(k)(1 + o(1))$$

where $\varphi_0^{\pm}$ are solutions to

$$\varphi_0^{\pm}(k \pm 1) + \lambda v(\theta + k\omega)\varphi_0^{\pm}(k) = 0.$$

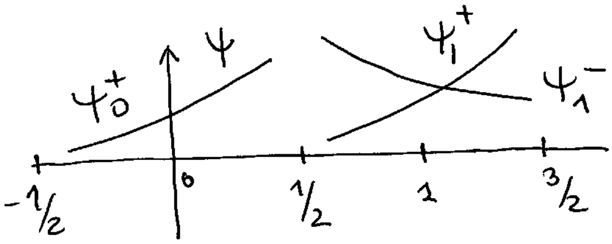




ψ_0^+ is exponentially increasing and ψ_0^+ is exponentially decreasing.

Near $\pm 1/2$, oscillation zones. Complicated asymptotics.

Behavior of ψ_0^+ to the right of $1/2$:



$$\psi_0^+ = b(\theta)\psi_1^+ + a(\theta)\psi_1^-.$$

If $b(\theta) = 0$:

it is a quantization condition.

In general, a and b computed asymptotically for large λ.

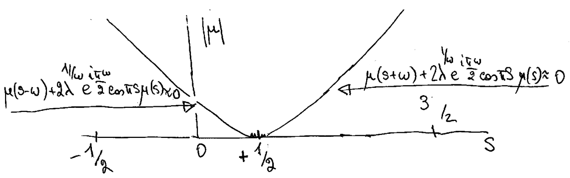
For the simple model

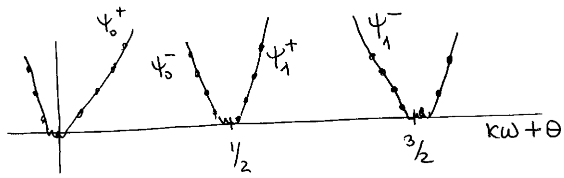
For $v(\theta) = 2\lambda^{1/\omega} e^{i\pi\omega/2}\cos(\pi\theta)$, exact computation of a and b:
one constructs solution of the form $\psi(k,\theta) = \mu(\theta + k\omega)$ where μ satisfies

$$\mu(s+\omega) + \mu(s-\omega) + 2\lambda^{1/\omega} e^{i\pi\omega/2}\cos(\pi s)\mu(s) = 0,$$

$$e^{-i\pi/\omega}\mu(s+1) - \mu(s-1) - 2i\lambda^{1/\omega}\sin(\pi(s-1/2)/\omega)\mu(s) = 0. \quad (3.1)$$

For λ large, one has





Compute asymptotics of solutions for large k: the base $(\psi_0^{\pm})$ given by

$$\psi_0^+(k) = \mu(\theta + 1 + k\omega)(-1)^k \quad \text{and} \quad \psi_0^-(k) = \mu(\theta + k\omega).$$

Then, $\psi_1^-(k) = \mu(\theta - 1 + k\omega)(-1)^k$ and $\psi_1^+(k) = \mu(\theta + k\omega)$.

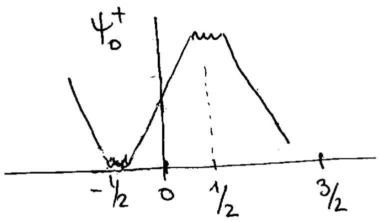
(3.1) rewrites as

$$e^{-i\pi/\omega}\psi_0^+(k) = 2i\lambda^{1/\omega}\sin(\pi(\theta - 1/2)/\omega)\psi_1^+(k) + \psi_1^-(k).$$

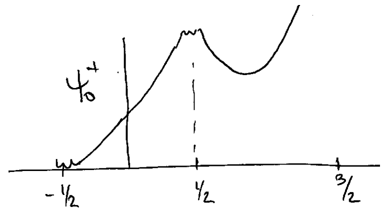
For the simple model, the quantization condition is

$$\sin\left(\frac{\pi}{\omega}(\theta - 1/2)\right) = 0$$

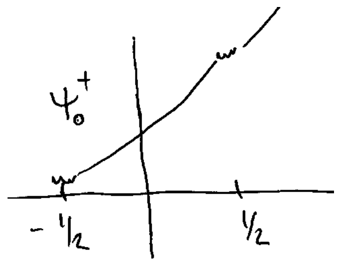
if $\theta - 1/2 \in \omega\mathbb{Z}$



if $\theta - 1/2 \notin \omega\mathbb{Z}$ but $\theta - 1/2$ close to $\omega\mathbb{Z}_-^*$

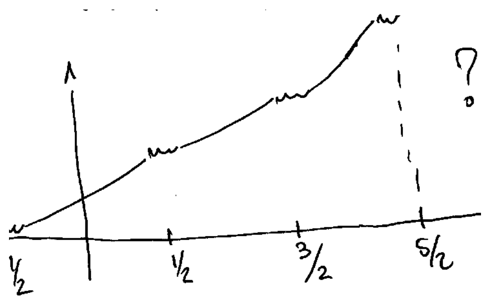


if $\theta - 1/2$ far from $\omega\mathbb{Z}$



When $\theta - 1/2 \notin \omega\mathbb{Z}$ and $\theta - 3/2 \notin \omega\mathbb{Z}$, ...

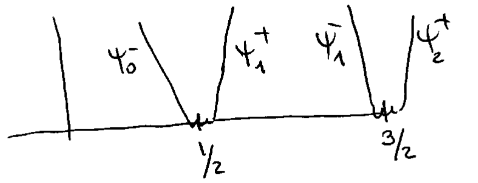
If $\theta - 1/2 \notin \mathbb{N} + \omega\mathbb{Z}_-^*$ for any k, expect ψ_0^+ exponentially increasing



Transition matrices

Pb: to go to ∞, one has to repeat the procedure indefinitely. But, asymptotics only known on compact.
Use transition matrices between $(\psi_0^\pm)$, $(\psi_1^\pm)$, $(\psi_1^\pm)$, ... and study the product.

$$\psi_0^-(k,\theta) = \psi_1^+(k,\theta) = \mu(\theta + k\omega),$$
$$\psi_1^-(k,\theta) = \psi_2^+(k,\theta) = \mu(\theta - 1 + k\omega).$$



Set $\psi_0(k,\theta) = \psi_0^-(k,\theta) = \psi_1^+(k,\theta)$ and $\psi_1(k,\theta) = \psi_1^-(k,\theta) = \psi_2^+(k,\theta)$, and so on ...
The transition matrices:

$$\begin{pmatrix}\psi_{l+1}(k,\theta)\\ \psi_l(k,\theta)\end{pmatrix} = T_l(\theta)\begin{pmatrix}\psi_l(k,\theta)\\ \psi_{l-1}(k,\theta)\end{pmatrix}, \quad T_l(j) = \begin{pmatrix} t_l(\theta) & s_l(\theta)\\ 1 & 0\end{pmatrix}$$

The renormalization

Relation (3.1) gives

$$T_l(\theta) = \begin{pmatrix} 2i\lambda^{1/\omega} e^{i\pi/\omega} \sin(\pi(\theta - 1/2 + l)/\omega) & e^{i\pi/\omega} \\ 1 & 0 \end{pmatrix}$$

Up to conjugation and multiplication by complex unit,

$$\tilde{T}_l(\theta) = \begin{pmatrix} 2\lambda^{1/\omega} \sin(\pi(\theta + l)/\omega) & e^{-i\pi(\theta+l)/\omega} \\ e^{i\pi(\theta+l)/\omega} & 0 \end{pmatrix}$$

$$\tilde{T}_l(\theta) \cdot \tilde{T}_{l-1}(\theta) \cdots \tilde{T}_0(\theta) = \pm \tilde{M}(\theta_1 + l\omega_1) \cdot \tilde{M}(\theta_1 + (l-1)\omega_1) \cdots \tilde{M}(\theta_1)$$

where

$$\theta_1 = \frac{\theta}{\omega} \bmod 1, \quad \omega_1 = \frac{1}{\omega} \bmod 1, \quad \tilde{M}(\theta_1) = \begin{pmatrix} 2\lambda^{1/\omega} \sin(\pi\theta_1) & -e^{-i\pi\theta_1} \\ e^{i\pi\theta_1} & 0 \end{pmatrix}$$

Introduction The results Some ideas from the proof
Monodromization

Studying the Lyapunov exponent for

$$\psi(n+1) + \psi(n-1) + 2\lambda\, e^{i\pi\omega/2} \cos(\pi(n\omega + \theta))\, \psi(n) = 0.$$

comes up to studying the product $M_0(\theta + k\omega) \cdots M_0(\theta + \omega) M_0(\theta)$ for

$$M_0(\theta) = \begin{pmatrix} -2\lambda^{1/\omega} e^{i\pi\omega/2} \cos(\pi(k\omega + \theta)) & -1 \\ 1 & 0 \end{pmatrix}$$

Theorem

For $\theta_1 = \theta/\omega$ mod 1, $\omega_1 = 1/\omega$ mod 1 and $k_1 = [k\omega + \theta]$, one has

$$\begin{aligned} &M_0(\theta + k\omega) \cdots M_0(\theta + \omega) M_0(\theta) \\ &\quad = \Psi_0(\{\theta + k\omega\}) M_1^{-1}(\theta_1 - k_1\omega_1) \cdot M_1^{-1}(\theta_1 - (k_1 - 1)\omega_1) \cdots \\ &\qquad \cdots M_1^{-1}(\theta_1 - \omega_1) \Psi_0^{-1}(\{\theta\}) \end{aligned}$$

where M_1 same form as M_0 and Ψ_0 is constructed from μ.

Completion of the proof of Theorem 2.1

Repeating this

$$\begin{aligned} M_0(\theta+k\omega)\cdots M(\theta+\omega)M_0(\theta) \\ = \Psi_0(\{\theta+k\omega\})\Psi_1(\{\theta_1-k\omega\})\cdots\Psi_l(\{\theta_l\pm k_l\omega_l\})\cdot \\ M_{l+1}(\tilde{\theta}_l)\cdot\Psi_l^{-1}(\{\theta_l\})\cdots\Psi_1^{-1}(\{\theta\})\cdot\Psi_0^{-1}(\{\theta_0\}) \end{aligned}$$

Two gains:

- number of terms in product.
- asymptotic form of each term: $\lambda_l = \lambda_{l-1}^{1/\omega_{l-1}}$.

Under assumption of criterion, the products

$$\Pi_l = \Psi_l M_{l+1}\Psi_l^{-1},\ \Pi_{l-1} = \Psi_{l-1}\Pi_l\Psi_{l-1}^{-1},\ \Pi_{l-2} = \Psi_{l-2}\Pi_{l-1}\Psi_{l-2}^{-1},\dots$$

admit quasi-scalar asymptotics

$$\Pi_k = \tau_k \begin{pmatrix} 1 & o(1) \\ o(1) & o(1) \end{pmatrix}.$$