Resonances for “large” ergodic systems

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Outline

The general picture

On $\ell^2(\mathbb{Z}^d)$, consider V a bounded ergodic potential and the operator

$$H = -\Delta + V.$$

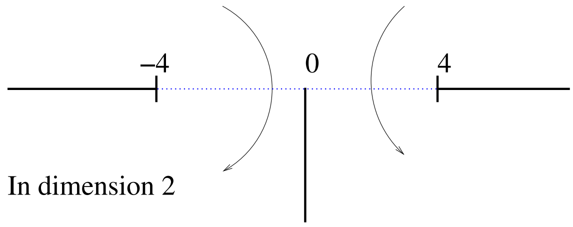
By ergodic potential, we mainly think of:

- V periodic;
- $V = V_\omega$ random e.g. Anderson model;
- V quasi-periodic operator.

Large "ergodic" system: let $L \in \mathbb{N}$, $L \gg 1$ and set $H_L = -\Delta + V\mathbf{1}_{|x|\leq L}$.

Compactly supported perturbation of $-\Delta$:

- $\sigma_{\mathrm{ess}}(H_L) = \sigma(-\Delta) = [-2d, 2d]$;
- outside $\sigma(-\Delta)$, H_L has only discrete eigenvalues.



Theorem

The operator valued function $z \in \mathbb{C}^+ \mapsto (z - H_L)^{-1}$ admits a meromorphic continuation from $\mathbb{C}$ to $\mathbb{C} \setminus \left((-\infty, 2d] \cup [2d, +\infty) \cup \cup_{1\leq k\leq d-1}(4k - 2d + i\mathbb{R}^-)\right)$ with values in the operators from l^2_{comp} to l^2_{loc}.

The poles of the analytic continuation are the resonances of H_L.
They are associated with finite dimensional resonant subspaces.
Pole width: it is the imaginary part of the pole.
Well known: the resonance width plays an important role in the large time behavior of e^{-itH_L},especially the smallest width that gives the leading order contribution.

Goal: "compute" the resonances; relate them (their distribution, the distribution of their width) to the spectral characteristics of $H = -\Delta + V$.

A very simple model:

On $\ell^2(\mathbb{N})$, consider $V = (V_n)_{n\geq 0}$ and the eigenvalue problem

$$u_{n+1} + u_{n-1} + V_n u_n = E u_n \text{ if } n \geq 0 \text{ and } u_{-1} = 0$$

associated to the operator

$$H = -\Delta + V = \begin{pmatrix} V_0 & 1 & 0 & 0 & \cdots \\ 1 & V_1 & 1 & 0 & \ddots \\ 0 & 1 & V_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

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So, for $L > 0$, one has

$$H_L = -\Delta + V\mathbf{1}_{n\leq L} = \begin{pmatrix} V_0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 1 & V_1 & 1 & 0 & & & & & \\ 0 & 1 & V_2 & 1 & 0 & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ \vdots & & 0 & 1 & V_L & 1 & 0 & & \\ \vdots & & & 0 & 1 & 0 & 1 & 0 & \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Model was studied:

- in physics: Titov - Fyodorov, Texier - Combet, etc
- in mathematics: Kunz - Shapiro when $\ell^2(\mathbb{Z})$ instead of $\ell^2(\mathbb{N})$ and $L = +\infty$ (i.e. half-axis filled with potential); they studied the resonances far away from the real axis.

We study the resonances close to spectrum of $-\Delta$ i.e. those that asymptotically tend to $[-2,2]$ when $L \to +\infty$.

The periodic case

We assume, for some $p > 0$, $V_{n+p} = V_n$ for all $n \geq 0$:

- let Σ' be the spectrum of H and Σ_0 be the spectrum of $-\Delta + V$ acting on $\ell^2(\mathbb{Z})$; then $\Sigma' = \Sigma_0 \cup \{v_j; 1 \leq j \leq n\}$ and $\Sigma_0 = \cup_{j=1}^{p} [a_j^-, a_j^+]$;
- Σ_0 is the a.c. spectrum of H;
- the $(v_j)_{0\leq j\leq n}$ are isolated simple eigenvalues associated to exponentially decaying eigenfunctions.

Resonance free regions:

Theorem

Let I be a compact interval in $(-2,2)$. Then,

- *if $I \subset \mathbb{R} \setminus \Sigma'$, then, there exists $C > 0$ such that, for L sufficiently large, there are no resonances in $\{\mathrm{Re}\, z \in I,\ \mathrm{Im}\, z \geq -1/C\}$;*
- *if $I \subset \Sigma_0$, then, there exists $C > 0$ such that, for L sufficiently large, there are no resonances in $\{\mathrm{Re}\, z \in I,\ \mathrm{Im}\, z \geq -1/(CL)\}$;*
- *if $\{v_j\} = \overset{\circ}{I} \cap \Sigma' = I \cap \Sigma'$ and $I \cap \Sigma_0 = \emptyset$, then, for L sufficiently large, there exists a unique resonance in $\{\mathrm{Re}\, z \in I,\ \mathrm{Im}\, z \geq -1/C\}$; moreover, this resonance, say z_j, satisfies $\mathrm{Im}\, z_j \asymp -e^{-\rho_j L}$ and $|z_j - v_j| \asymp e^{-\rho_j L}$.*

Description of the resonances near the real axis

Let I be a compact interval in $(-2,2)\cap\overset{\circ}{\Sigma}_0$.
Let n be the density of states of $-\Delta+V$ and, for $E\in\overset{\circ}{\Sigma}_0$, define

$$S(E)=\text{p.v}\left(\int_{\mathbb{R}}\frac{1}{\lambda-E}dn(\lambda)\right).$$

Let $(\lambda_j)_j=(\lambda_j^L)_j$ be the Dirichlet eigenvalues of $(-\Delta+V)_{|[0,L]}$ in increasing order.

Theorem

There exists $C_0>0$ such that, for $C>C_0$, there exists $L_0>0$ such that for $L>L_0$, for $\lambda_j\in I$ such that $\lambda_{j+1}\in I$, there exists a unique resonance in $[\lambda_j,\lambda_{j+1}]+i[-CL^{-1},0]$, say z_j. It satisfies

$$z_j=\lambda_j+\frac{f(\lambda_j)}{L}\cot^{-1}\left(\left[e^{-i\arccos(\lambda_j/2)}+S(\lambda_j)\right]g(\lambda_j)\right)+o\left(\frac{1}{L}\right)$$

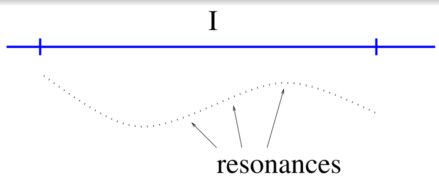
where f and g are real analytic functions defined by the Floquet theory of H on $\mathbb{Z}$.

Corollary

In $I+i[-C/L,0]$, for L sufficiently large,

- *the resonances when rescaled to have imaginary parts of order 1 accumulate on a real analytic curve;*
- *the local (linear) density of resonances is given by the density of states of H.*

Picture of the resonances after rescaling their width by L:



Description of the resonances away from the real axis

Theorem

There exists $C_0>0$ such that, for $C>C_0$, there exists $L_0>0$ such that for $L>L_0$,in $I+i(-\infty,-C/L]$, a resonance, say z, satisfies

$$\int_{\mathbb{R}}\frac{dn(\lambda)}{\lambda-z}+e^{-i\arccos(z/2)}=O\left(\frac{1}{C}\right)$$

Recovers the result of Kunz-Shapiro ($C\to+\infty$).

The random case

Let $V = V_\omega$ where $(V_\omega(n))_n \geq 0$ are bounded i.i.d. random variables with a nice distribution and set $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{N})$.

- let $\sigma(H_\omega)$ be the spectrum of H_ω and Σ be the almost sure spectrum of $-\Delta + V_\omega$ acting on $\ell^2(\mathbb{Z})$ ($\Sigma = [-2,2]+\text{supp}V_\omega(0)$);
- ω-almost surely, $\sigma(H_\omega) = \Sigma \cup K_\omega$;
- Σ is the essential spectrum of H_ω; it consists of simple eigenvalues associated to exponentially decaying eigenfunctions (Anderson localization);
- The set K_ω is the discrete spectrum of H_ω; it may be empty.

Let $\rho(E)$ denote the Lyapunov exponent of H_ω at energy E.

Resonance free regions:

Theorem

Let I be a compact interval in $(-2,2)$. Then, ω-a.s., one has

- *if $I \subset \mathbb{R} \setminus \sigma(H_\omega)$, then, there exists $C > 0$ such that, for L sufficiently large, there are no resonances of $H_{\omega,L}$ in $\{Re\, z \in I,\ Im\, z \geq -1/C\}$;*
- *if $I \subset \overset{\circ}{\Sigma}$, then, there exists $C > 0$ such that, for L sufficiently large, there are no resonances of $H_{\omega,L}$ in $\{Re\, z \in I,\ Im\, z \geq -e^{-2\rho L(1+o(1))})\}$ where ρ is the maximum of the Lyapunov exponent $\rho(E)$ on I.*

Description of the resonances closest to the real axis

Let $n(E)$ denote the density of states of H_ω at energy E.

Theorem

Let I be a compact interval in $(-2,2) \cap \overset{\circ}{\Sigma}$. Then, ω-a.s.,

- *for any $\kappa \in (0,1)$, one has*

$$\frac{1}{L}\#\left\{z \text{ resonance of } H_{\omega,L} \text{ s.t. } Re\, z \in I,\ Im\, z \geq -e^{-L^\kappa}\right\} \to \int_I dn(E);$$

- *fix $E \in I$ such that $n(E) > 0$; then, for $\delta > 0$, there exits $\varepsilon > 0$ such that*

$$\liminf_{L\to+\infty} \frac{1}{L}\#\left\{\text{resonances } z \text{ s.t. } Re\, z \in [E-\varepsilon, E+\varepsilon],\ Im\, z \geq -e^{-2(\rho(E)-\delta)L}\right\} > 0.$$

The resonances are much closer to the real axis than in the periodic case; the lifetime of these resonances is much larger.

Consequence of localization.

The local behavior of the resonances

Let I be a compact interval in $(-2,2)\cap\overset{\circ}{\Sigma}$ and $\kappa\in(0,1)$.
Fix $E_0\in I$ such that $n(E_0)>0$.
Let $(z_i^L(\omega))_i$ be the resonances of $H_{\omega,L}$ in $K_L:=[E_0-\varepsilon,E_0+\varepsilon]+i\left[-e^{-L^\kappa},0\right]$.
Rescaling the resonances: define

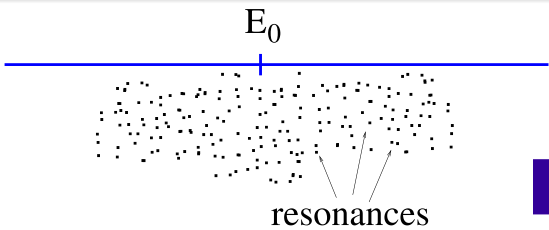
$$x_j=x_j^L(\omega)=(\operatorname{Re} z_j^L(\omega)-E_0)L \quad\text{and}\quad y_j=y_j^L(\omega)=-\frac{1}{2L}\log|\operatorname{Im} z_j^L(\omega)|.$$

Consider now the two-dimensional point process $\xi_L(\omega)=\sum_{z_j^L\in K_L}\delta_{(x_j,y_j)}$.

Theorem

The point process ξ_L converges weakly to a Poisson process in $\mathbb{R}\times[0,1]$ with density $n(E_0)\rho(E_0)dxdy$.

Picture of the resonances after rescaling:

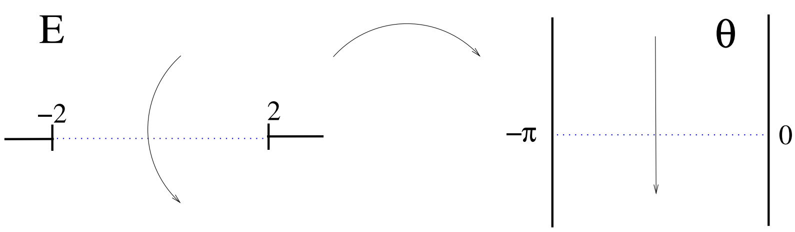


Characterizing resonances

Pick E such that $\operatorname{Im}E>0$ and set $E=2\cos(\theta)$ ($\operatorname{Im}\theta>0$, $\operatorname{Re}\theta\in(-\pi,0)$);

$$\begin{cases} u_{n+1}+u_{n-1}+V_nu_n=E\,u_n, & \forall n\geq 0\\ u_{-1}=0 \end{cases}$$

where $V_n=0$ if $n\geq L+1$.
For $n\geq L+1$, $u_n^+=u_n^+(\theta)=\beta e^{in\theta}$ (exp. decay at $+\infty$).



Hence, we solve, for $\operatorname{Im}\theta>0$ and $\operatorname{Re}\theta\in(-\pi,0)$

$$\begin{pmatrix} V_0 & 1 & 0 & \cdots & 0\\ 1 & V_1 & 1 & 0 & \\ \vdots & \ddots & \ddots & \ddots & \\ & 0 & 1 & V_{L-1} & 1\\ 0 & \cdots & 0 & 1 & V_L+e^{i\theta} \end{pmatrix}\begin{pmatrix} u_0\\ \\ \vdots\\ \\ u_L\end{pmatrix}=E\begin{pmatrix} u_0\\ \\ \vdots\\ \\ u_L\end{pmatrix}\quad\text{where}\quad E=2\cos\theta.$$

By rank one perturbation theory, $E \in \mathbb{C}^-$ is a resonance if and only if

$$\sum_{j=0}^{L} \frac{a_j}{\lambda_j - E} = -e^{-i\theta(E)}, \quad E = 2\cos\theta(E) \tag{3.1}$$

where

- $(\lambda_j)_{0\le j\le L} = (\lambda_j(L))_{0\le j\le L}$ are the Dirichlet eigenvalues of H_L, $\lambda_j < \lambda_{j+1}$;
- $a_j = a_j(L) = |\varphi_j(L)|^2$ where $\varphi_j = (\varphi_j(x))_{0\le x\le L}$ is a normalized eigenvector associated to λ_j.

Resonances near a gap of $\sigma(H)$

For E near gap and $]-2,2[$, $\sin\theta(E) < 0$ (indep of L) and $\operatorname{Im}\left(\sum_{j=0}^{L} \frac{a_j}{\lambda_j - E}\right) \asymp \operatorname{Im} E$.

No solution to (3.1), hence, no resonance, up to distance $O(1)$ to the axis.

Resonances near an isolated eigenvalue of $\sigma(H)$

For $L \gg 1$, there is a unique λ_j near the isolated eigenvalue. (3.1) becomes

$$-e^{-i\theta(E)} = \frac{a_j}{\lambda_j - E} + S_L(E)$$

where

- $a_j \asymp |\varphi(L)|^2$, φ normalized eigenvector associated to isolated eigenvalue;
- $E \mapsto S_L(E)$ nice analytic in E.

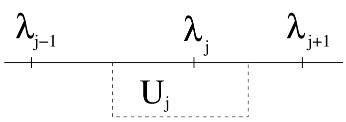
A general result on resonance free regions

define $d_j = \min(\lambda_{j+1} - \lambda_j, \lambda_j - \lambda_{j-1})$ and

$$U_j(C_0) = \left\{ E \in \mathbb{C}^-; \begin{array}{c} E = 2\cos\theta,\ \operatorname{Re} E \in \left[\frac{\lambda_j + \lambda_{j-1}}{2}, \frac{\lambda_j + \lambda_{j+1}}{2}\right] \\ \operatorname{Re}\theta \in (-\pi, 0),\ 0 < -\operatorname{Im}\theta \le a_j d_j^2 |\sin(\operatorname{Re}\theta)|/C_0 \end{array} \right\}$$

Theorem

There exists $C_0 > 0$ such that, for $0 \le j \le$, there are no resonances in $U_j(C_0)$.



Asymptotics in the periodic case

Near E_0 in band:

- $\lambda_j = E_0 + g(j/L) \sim E_0 + \beta(j - j_0)/L$ where $j_0 \sim \rho L$;
- $a_j \sim f(\lambda_j)/L \sim \alpha/L$, f and g nice functions.

So (3.1) becomes

$$\begin{aligned} -e^{-i\theta(E)} &= \sum_{j=-\rho L/2}^{\rho L/2} \frac{\alpha}{j - (E - E_0)L\beta} + S_L(E) \\ &= \alpha\pi\cot(\beta\pi L(E - E_0)) + S(E) + o(1) \end{aligned}$$

In the random case

Localized regime:

- $\mathbb{E}(d_j) \asymp 1/L$ so $d_j(\omega) \asymp 1/L$ fluctuating;
- Minami's estimate: a.s. $d_j(\omega) \geq L^{-3}$;
- $\log a_j(\omega) \asymp -2\rho(\lambda_j)(L - x_j)$ where x_j localization center of φ_j;
- most localization centers are far from L.

If $L - x_{j_0} \geq L^\alpha$ ($\alpha \in (0,1)$), then $a_{j_0} \ll d_{j_0}$. So, for $|E - \lambda_{j_0}| \asymp a_{j_0}$,

$$\sum_{j=0}^{L} \frac{a_j}{\lambda_j - E} = \frac{a_{j_0}}{\lambda_{j_0} - E} + S_L(E)$$

where $E \mapsto S_L(E)$ is well behaved and its imaginary part is of order $a_{j_0}/d_{j_0} \ll 1$.

Solution to (3.1) then of the form

$$E = \lambda_{j_0} + \frac{a_{j_0}}{e^{-i\theta(\lambda_{j_0})} - S_L(\lambda_{j_0})}(1 + o(1))$$