Localization for random quantum graphs

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Outline

Quantum graphs

Consider the set of vertices $\mathbb{Z}^d$ equipped with its standard basis $(h_j)_{j\in\{1,\dots,d\}}$.
Two vertices m and m' connected if $|m-m'|=1$.

Edge (m,j): identify edge with $[0,l_j]$ (l_j fixed).

Let $\mathscr{H}_{m,j}=L^2([0,l_j])$ and $\mathscr{H}=\bigoplus_{m\in\mathbb{Z}^d}\bigoplus_{j\in\{1,\dots,d\}}\mathscr{H}_{m,j}$.

We write $f\in\mathscr{H}$ as $f=(f_{m,j})_{m,j}$.

Fix real-valued potentials $U_j\in L^2([0,l_j])$, $1\le j\le d$.
On $\mathscr{H}_{m,j}$, consider the Schrödinger operator

$$H_j=-\frac{d^2}{dt^2}+U_j.$$

Boundary conditions at vertices:

- continuity : $f_{m,j}(0)=f_{m-h_k,k}(l_k)$,$1\le j,k\le d$;
  set $f(m)=f_{m,j}(0)$;
- Kirchoff relations : $f'(m)=\alpha(m)f(m)$ where

$$f'(m):=\sum_{j=1}^{d}f'_{m,j}(0)-\sum_{j=1}^{d}f'_{m-h_j,j}(l_j).$$

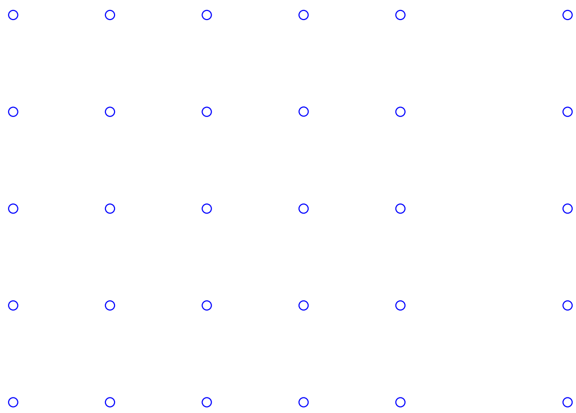

Figure: Quantum graph

The random model

Random parameters: the Kirchoff boundary conditions $(\alpha_\omega(m))_{m\in\mathbb{Z}^d}$

- independent identically distributed random variables
- common distribution with bounded density ρ of support $[\alpha_-,\alpha_+]$

The model is ergodic :

- define τ_m by $(\tau_m\omega)_{m'}=\omega_{m+m'}$, $m,m'\in\mathbb{Z}^d$;
- define U_m on $\mathscr{H}$, $(U_mf)_{m',j'}=f_{m+m',j'}$, $m,m'\in\mathbb{Z}^d$, $j'\in\{1,\dots,d\}$.

Shifts are a measure preserving ergodic family on Ω and

$$\forall m\in\mathbb{Z}^d,\quad H_{\tau_m\omega}=U_m^*H_\omega U_m$$

Theorem

For $\bullet\in\{pp,ac,sc\}$, there exists a closed subset $\Sigma_\bullet\subset\mathbb{R}$ such that, ω-almost surely, $\sigma_\bullet(H_\omega)=\Sigma_\bullet$ for any $\omega\in\Omega'$.

Let $\Sigma=\Sigma_{\mathrm{pp}}\cup\Sigma_{\mathrm{ac}}\cup\Sigma_{\mathrm{sc}}$ be the almost sure spectrum of H_ω.

Rough description of Σ

For the sake of the discussion, set $\alpha_\omega(m) := \lambda\alpha_\omega(m)$, $\lambda > 0$ where $\alpha_\omega(m) \in [\alpha_-, \alpha_+]$.

The Dirichlet operator: let H^0 be the operator on $\mathscr{H}$ with the boundary conditions $f(m) = 0$, $\forall m \in \mathbb{Z}^d$.

- the spectrum of H^0 is a discrete set of point;
- it is of infinite multiplicity.

The almost sure spectrum:

- Σ is a union of intervals (not reduced to a point);
- if $\alpha_- < 0$, for large λ, Σ has spectrum close to $-\infty$;
- except at the bottom of Σ, the edges of Σ are at distance of order λ^{-1} of $\sigma(H^0)$;
- if $0 \in [\alpha_-, \alpha_+]$, $\sigma(H_0) \subset \Sigma$ for any λ (H_0 is H_ω when $\alpha_\omega(m) = 0$, $\forall m$).

Localization results: large coupling and band edge regimes

Theorem

Assume that $\alpha_- < 0$. Then, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$, the spectrum of $H_{\lambda,\omega}$ in $(-\infty, \inf\sigma(H_0) - \varepsilon)$ is dense pure point.

Theorem

Let $0 \in [\alpha_-, \alpha_+]$. Then, for any $E_0 > \inf\sigma H^0$ and any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that, for $\lambda > \lambda_0$, the spectrum of $H_{\lambda,\omega}$ in $(-\infty, E_0) \setminus (\sigma(H^0) + [-\varepsilon, \varepsilon])$ is dense pure point.

Theorem

Let $E_0 \notin \sigma(H^0)$ be an edge of the spectrum of H_ω. Then, the spectrum of H_ω in some neighborhood of E_0 is almost surely pure point.

Actually in all case one obtains strong dynamical localization i.e. exponentially decaying estimates for bounded functions of the operator supported in the localized region.

Reduction to a discrete model

We use the theory of self-adjoint extensions.

Recall

- $\mathscr{H}_{m,j} = L^2([0,l_j])$ and $\mathscr{H} = \bigoplus_{m\in\mathbb{Z}^d} \bigoplus_{j\in\{1,\dots,d\}} \mathscr{H}_{m,j}$;
- we consider the functions $f = (f_{m,j}) \in \bigoplus_{m,j} H^2([0,l_j])$ satisfying boundary conditions:

$$f_{m,j}(0) = f_{m-h_k,k}(l_k) =: f(m), \quad j,k = 1,\dots,d$$

- on $\mathscr{H}$, we consider the "Schrödinger" operator acting as

$$H_j f_{m,j} = -\frac{d^2 f_{m,j}}{dt^2} + U_j f_{m,j};$$

Let S be the operator thus defined. On the domain of S, define

$$f \mapsto \Gamma f := \big(f(m)\big)_{m\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d), \quad f \mapsto \Gamma' f := \big(f'(m)\big)_{m\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$$

where we recall $f'(m) = \sum_{j=1}^{d} f'_{m,j}(0) - \sum_{j=1}^{d} f'_{m-h_j,j}(l_j)$.

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Let H^0 be the restriction of S to $\ker\Gamma$ i.e. H^0 is the direct sum (over all edges) of the Dirichlet restrictions of $-\frac{d^2}{dt^2} + U_j$ on the segments $[0,l_j]$.

Pick $E \notin \sigma(H^0)$. For $\xi \in l^2(\mathbb{Z}^d)$, let $\gamma(E)\xi$ be the unique solution to $(S-E)f = 0$ with $\Gamma f = \xi$. $\gamma(E)$ defines an isomorphism between $l^2(\mathbb{Z}^d)$ and $\ker(S-E)$.

Moreover, define the operator $M(E) : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ by $M(E) := \Gamma'\gamma(E)$.

Let H_A be the "quantum graph" where A diagonal matrix with entries $(\alpha(m))_{m\in\mathbb{Z}^d}$

Proposition

The mappings $E \mapsto \gamma(E)$ and $E \mapsto M(E)$ are analytic on $\mathbb{C}\setminus\sigma(H^0)$. For $E \notin \sigma(H^0)$, M satisfies

$$\frac{dM(E)}{dE} = \gamma^*(E)\gamma(E) \quad \text{and} \quad \frac{\Im M(E)}{\Im E} > 0 \quad \text{for} \quad \Im E \neq 0.$$

For $E \notin \sigma(H^0)\cup\sigma(H_A)$, one has

$$(H_A - E)^{-1} = (H^0 - E)^{-1} - \gamma(E)\big(M(E) - A\big)^{-1}\gamma^*(\bar{E}).$$

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The structure of $M(E)$

Consequences:

1. $\sigma(H_A) \setminus \sigma(H^0)$ coincides with $\{E \notin \sigma(H^0) : 0 \in \sigma\big(M(E) - A\big)\}$;
2. $E \notin \sigma(H^0)$ is an eigenvalue if and only if 0 if an eigenvalue of $M(E) - A$ and $\gamma(E)$ is then an isomorphism between the corresponding eigenspaces.

Let φ_j and ϑ_j be the solutions to $-y'' + U_j y = Ey$ satisfying $\varphi_j(0;E) = \vartheta_j'(0;E) = 0$ and $\varphi_j'(0;E) = \vartheta_j(0;E) = 1$.

Define the functions $a(E) := \sum_{j=1}^{d} \frac{\vartheta_j(l_j;E) + \varphi_j'(l_j;E)}{\varphi_j(l_j;E)}, \quad b_j(E) := \frac{1}{\varphi_j(l_j;E)}$.

For $E \notin \sigma(H^0)$, one computes

$$M(E)\xi(m) = \sum_{j=1}^{d} b_j(E)\big(\xi(m-h_j) + \xi(m+h_j)\big) - a(E)\xi(m).$$

For $\alpha(m) \in [\alpha_-, \alpha_+]$, $E \in \sigma(H_\omega)$ if and only if $0 \in \Sigma_M(E) := \sigma\big(M(E) - A\big)$; hence,

$$E \in \Sigma \setminus \sigma(H^0) \text{ if and only if } (b(E) - a(E) - \alpha_-)\cdot(b(E) + a(E) + \alpha_+) \geq 0$$

where $b(E) = 2\sum_{j=1}^{d} \big|b_j(E)\big|$.

Finite volume criteria

The finite volume criteria is obtained in 2 steps (follows ideas of finite volume criteria for operators on $\mathbb{Z}^d$ (Aizenman et al.)):

1. obtain pure point spectrum from asymptotic estimates of the upper spectral measures;
2. deduce these estimates from finite volume estimates of the reduced operator.

Localization then as usual proved by finding regimes where the finite volume estimates hold.

Localization conditions in terms upper spectral measure

Recall that, for μ a complex valued regular Borel measure and F a Borel set, one defines

$$|\mu|(F) = \sup_{f \in \mathscr{C}_0(\mathbb{R}),\ |f|_\infty \leq 1} \left| \int_F f(E)d\mu(E) \right|.$$

Let $f, g \in \mathscr{H}$. Let $\mu^{f,g}$ denote the spectral measure for H_A associated with H_A and $|\mu^{f,g}|$ denote its absolute value. For any measurable set F and any two edges (m,j), (m',j'), we define the *upper spectral measure*

$$\mu^{(m,j),(m',j')}(F) := \sup_{\substack{f = P_{m,j}f,\ \|f\|=\|g\|=1, \\ g = P_{m',j'}g,\ \|g\|=1}} |\mu^{f,g}|(F).$$

Theorem

Let $F \subset \mathbb{R}$. Assume that, for any (m,j), one has $\sum_{m' \in \mathbb{Z}^d} \sum_{j'=1}^{d} \mu^{(m,j),(m',j')}(F) < \infty$, *then H_A has only pure point spectrum in F.*

The proof: based on a RAGE type characterization of the point spectrum.

Corollary

Let $F \subset \mathbb{R}$. Assume that, for any edge (m,j), one has

$$\mathbb{E}\Big(\sum_{m' \in \mathbb{Z}^d} \sum_{j'=1}^{d} \mu_\omega^{(m,j),(m',j')}(F) \Big) < \infty,$$

then, almost surely, H_ω has only pure point spectrum in F.

Finite volume estimates

Let $\Lambda \subset \mathbb{Z}^d$ be finite. Let H_A^Λ be the H_A with the boundary conditions

$$f'(m) = \alpha(m) f(m) \text{ on } \Lambda \quad \text{and} \quad f(m) = 0 \text{ outside } \Lambda.$$

Let Π_Λ the orthogonal projection from $l^2(\mathbb{Z}^d)$ to $l^2(\Lambda)$. Restrict $M(E) - A$ to Λ and let $\gamma_\Lambda(E) = \gamma(E)\Pi_\Lambda$. Then, for $E \notin \sigma(H^0) \cup \sigma(H_A^\Lambda)$, one has

$$(H_A^\Lambda - E)^{-1} = (H^0 - E)^{-1} - \gamma_\Lambda(E)\big(M_\Lambda(E) - A_\Lambda\big)^{-1} \gamma_\Lambda^*(\bar{E}).$$

Proposition

Let $F \subset \mathbb{R}$ be a segment disjoint from $\sigma(H^0)$. Assume that there exists $A, a > 0$ and $s \in (0,1)$ such that, for all finite $\Lambda \subset \mathbb{Z}^d$ and all $E \in F$,

$$\mathbb{E}\left| \left(M_\Lambda(E) - A_{\Lambda,\omega}\right)^{-1}(m,m') \right|^s \le A e^{-a|m-m'|}.$$

Then, there exists $B, c > 0$ such that, for any two edges (m,j) and (m',j'), one has

$$\mathbb{E}\big(\mu_\omega^{(m,j),(m',j')}(F)\big) \le B e^{-c|m-m'|}.$$

Proof of the main technical result

The proof follows the ideas introduced by Aizenman for the discrete case.
Let $F \subset \mathbb{R} \setminus \sigma(H^0)$. Hence, the spectrum of $H^{\Lambda}_{A,\omega}$ is discrete. and almost surely simple.
Let E_k be the points of F and $\xi_k \neq 0$ vectors such that $(M_\Lambda(E_k) - A_{\Lambda,\omega})\xi_k = 0$; then

$$\mu^{f,g}_{\Lambda,\omega}(F) = \sum_{E_k \in \sigma(H^\Lambda_\omega)\cap F} \frac{\langle f, \gamma_\Lambda(E_k)\xi_k\rangle \langle \gamma_\Lambda(E_k)\xi_k, g\rangle}{\|\gamma_\Lambda(E_k)\xi_k\|^2},$$

Assume $f = P_{m,j}f$ and $g = P_{m',j'}g$. Let $\hat{A}_\omega := A_\omega + (\hat{v} - \alpha(m'))\Pi_{m'}$ where

- $\Pi_{m'}$ is the projection onto $\delta_{m'}$,
- $\hat{v}$ is distributed identically to $\alpha_\omega(m')$.

For almost every $\hat{v}$, if $0 \in \sigma(M_\Lambda(E) - A_{\Lambda,\omega})$, then $M_\Lambda(E) - \hat{A}_{\Lambda,\omega}$ is invertible.

Let $\hat{\varphi}_E := \dfrac{\left(M_\Lambda(E) - A_{\Lambda,\omega}\right)^{-1}\delta_{m'}}{\langle \delta_{m'}, \left(M_\Lambda(E) - A_{\Lambda,\omega}\right)^{-1}\delta_{m'}\rangle} = \dfrac{\left(M_\Lambda(E) - \hat{A}_{\Lambda,\omega}\right)^{-1}\delta_{m'}}{\langle \delta_{m'}, \left(M_\Lambda(E) - \hat{A}_{\Lambda,\omega}\right)^{-1}\delta_{m'}\rangle}$.

Let ξ be an eigenvector of $M_\Lambda(E) - A_{\Lambda,\omega}$ associated to 0. Then,

$$0 = (M_\Lambda(E) - A_{\Lambda,\omega})\xi = (M_\Lambda(E) - \hat{A}_{\Lambda,\omega})\xi + \left(\hat{v} - \alpha_\omega(m')\right)\Pi_{m'}\xi.$$

Thus, $\xi = \left(\alpha_\omega(m') - \hat{v}\right)\langle \delta_{m'}, \xi\rangle (M_\Lambda(E) - \hat{A}_{\Lambda,\omega})^{-1}\delta_{m'}$ i.e. $\xi = C\hat{\varphi}_E$.

One computes $\left(M_\Lambda(E) - A_{\Lambda,\omega}\right)\hat{\varphi}_E = (\alpha_\omega(m') - \hat{v} - \hat{\Gamma}(E))\delta_{m'}$ where

$$\hat{\Gamma}(E) = -\frac{1}{\langle \delta_{m'}, \left(M_\Lambda(E) - \hat{A}_{\Lambda,\omega}\right)^{-1}\delta_{m'}\rangle}.$$

So $E \in \sigma(H^{\Lambda_n}_{A,\omega} \cap F$ if and only if $\alpha_\omega(m') - \tilde{v} = \hat{\Gamma}(E)$. This yields

$$\mu^{f,g}_{\Lambda,\omega}(dE) = \delta(\alpha_\omega(m') - \hat{v} - \hat{\Gamma}(E))\langle f, \gamma_\Lambda(E)\hat{\varphi}_E\rangle\langle \gamma_\Lambda(E)\hat{\varphi}_E, g\rangle dE$$

One has $\mu^{f,g}_{\Lambda,\omega}(dE) = \Psi^{f,g}(E)\mu^{f,f}_{\Lambda,\omega}(dE)$, where $\Psi^{f,g}$ is a measurable function satisfying

$$\int_{\mathbb{R}} |\Psi^{f,g}(E)|^2 \mu^{f,f}_{\Lambda,\omega}(dE) \leq \|g\|^2\|f\|^2.$$

Hence, $\displaystyle\int_{\mathbb{R}} \left|\langle \gamma_\Lambda(E)\hat{\varphi}_E, g\rangle\right|^2 \delta\left(\alpha_\omega(m') - \hat{v} - \hat{\Gamma}(E)\right) dE \leq \|g\|^2.$

One estimates

$$\mathbb{E}\left(\sup_{\|f\|=\|g\|=1} |\mu^{f,g}_{\Lambda,\omega}|(F)\right) \leq C\,\mathbb{E}\left(\sup_{\|f\|=1} \int_F |\langle f, \gamma_\Lambda(E)\hat{\varphi}_E\rangle|\, \delta(\alpha_\omega(m') - \hat{v} - \hat{\Gamma}(E)) dE\right).$$

Recall that, by the definition of $\hat{\varphi}_E$,

$$\langle f, \gamma_\Lambda(E)\hat{\varphi}_E\rangle = \langle f, \gamma_\Lambda(E)(M_\Lambda(E) - \hat{A}_\Lambda)^{-1}\delta_{m'}\rangle.$$

Completing the proof of localization

In the large coupling regime

We use the following result due to Aizenman et al; it is a consequence of general finite volume localization criteria

Theorem

Let $F \subset \mathbb{R}$ be an interval disjoint from $\sigma(H^0)$. Assume that there exists $s \in (0,1/4)$ such that, for all $E \in F$, one has

$$c(E)\left(1 + c(E)\frac{\tilde{C}_s}{\lambda^s}\right)\int_{\alpha_-}^{\alpha_+} \frac{1}{\left|a(E)+\lambda V\right|^s}\rho(dV) < 1, \quad c(E) := 2\sum_{j=1}^{d}\left|b_j(E)\right|^s.$$

Then, there exists $B, c > 0$ such that, for any finite $\Lambda \subset \mathbb{Z}^d$, for any $m, m' \in \Lambda$, and any $E \in F$ there holds

$$\mathbb{E}\left(\left|\left(M_\Lambda(E) - \lambda A_{\Lambda,\omega}\right)^{-1}(m,m')\right|^s\right) \le Be^{-c|m-m'|}.$$

At the band edges

Proposition

Pick $E_0 \in \sigma(H_\omega) \setminus \sigma(H^0)$. If $E_0 \in \partial\sigma(H_\omega)$, then $0 \in \partial\sigma\left(M(E_0) - A_\omega\right)$.

Assume $E_0 \in \partial\Sigma$, then $0 \in \partial\sigma(M(E_0) - A_\omega)$; more precisely, assume $\inf \sigma(M(E_0) - A_\omega) = 0$.

Localization will follow if one proves Lifshitz tail behavior near E_0:

there exists $\delta > 0$ such that for ε sufficiently small and $\#W$ not too large with respect to ε^{-1}, one has

$$\mathbb{P}\left\{\omega \in \Omega : \exists E \in [E_0 - \varepsilon, E_0 + \varepsilon],\ \inf \sigma\left(M_W(E) - A_{W,\omega}\right) \le \varepsilon/\delta\right\} \le e^{-\varepsilon^{-\delta}}.$$

This is a consequence of

- standard Lifshitz tails near 0 for the integrated density of states of the random operator $M(E_0) - A_\omega$;
- analyticity of $E \mapsto M(E)$ which yields a representation of the form $M(E) = M(E_0) + (E - E_0)B(E)$.