Absolute continuity of the spectrum of a Landau Hamiltonian perturbed by a generic periodic potential

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Outline

The main result:

On $L^2(\mathbb{R}^2)$, consider the Landau Hamiltonian

$$H = (-i\nabla - A)^2, \quad \text{where} \quad A(x_1, x_2) = \frac{B}{2}(-x_2, x_1).$$

The spectrum of H consists of the eigenvalues $\{(2k+1)B;\ k \in \mathbb{N}\}$; each of them is infinitely degenerate.

Let $\Gamma = \oplus_{i=1}^2 \mathbb{Z} e_i$ be a non-degenerate lattice such that $\Phi := \frac{1}{2\pi} B\, e_1 \wedge e_2 \in \mathbb{Q}$.

Define the set of real valued, continuous, Γ-periodic functions

$$C_\Gamma = \{V \in C(\mathbb{R}^2, \mathbb{R});\ \forall x \in \mathbb{R}^2,\ \forall \gamma \in \Gamma,\ V(x+\gamma) = V(x)\}.$$

The space C_Γ is endowed with the uniform topology defined by the norm $\|\cdot\|$.

Question: what is the spectral type of $H(V) := H + V$ for $V \in C_\Gamma$?

Our main result is

Theorem (Kl. Math. Annalen 347 (2010))

There exists a dense G_δ-subset of C_Γ such that, for V in this set, the spectrum of $H(V)$ is purely absolutely continuous.

Some background

Nature of the spectrum for part. diff. operators with periodic coefficients:

Bloch-Floquet theory "implies" absence of singular continuous spectrum.

Absence of point spectrum proved for:

- many Schrödinger operators with or without periodic magnetic fields ([Thomas73], $\cdots$, [Sobolev99], $\cdots$)
- other periodic PDEs ($\cdots$, [Kuchment93], $\cdots$)

Landau Hamiltonian with periodic potential: coefficients are not periodic. Magnetic Bloch-Floquet theory if the magnetic flux Φ is rational. $\Longrightarrow$ absence of s.c. spectrum.

For irrational flux Φ, vastly different situation: spectral theory altogether much more complicated (Hofstadter's butterfly, devil's staircase, etc).

An open question:

Our result in some way optimal. Leads to

Conjecture: For rational flux, the spectrum of $H(V)$ is purely a.c. if V is not constant.

Magnetic Bloch-Floquet theory

For $\alpha \in \mathbb{R}^2$, and $f \in C_0^\infty(\mathbb{R}^2)$, define $U_\alpha^B f(x) := e^{\frac{iB}{2} x\wedge\alpha} f(x+\alpha)$.

One checks $U_\alpha^B U_\beta^B = e^{iB\,\alpha\wedge\beta}\, U_\beta^B U_\alpha^B, \quad [U_\alpha^B, H] = 0 \quad$ and $\quad [U_\alpha^B, V] = 0$.

For (e_1, e_2) a basis of Γ, set $U_j^B := U_{e_j}^B$. The rational flux condition, say, $\Phi = 2\pi p/q$ implies that

$$(U_1^B)^q U_2^B = e^{i2\pi p} U_2^B (U_1^B)^q = U_2^B (U_1^B)^q \quad \text{and} \quad [(U_1^B)^q, H(V)] = 0 = [U_2^B, H(V)].$$

Assume $q = 1$. Define a unitary representation of Γ by

$$W_\gamma^B = \Theta(\gamma) U_\gamma^B \quad \text{where} \quad \Theta(\gamma) = e^{iBe_1\wedge e_2 \gamma_1\gamma_2/2} = e^{i\pi p\gamma_1\gamma_2} \in \{-1, +1\}.$$

As $W_\gamma^B W_\gamma^B = W_{\gamma+\gamma}^B$, the Gelfand-Bloch-Floquet transformation T^B defined by

$$(T^B f)(x, \theta) = \sum_{\gamma\in\Gamma} e^{i\theta\cdot(x+\gamma)} (W_\gamma^B f)(x), \quad \theta \in (\mathbb{R}^2)^*/\Gamma^*, \quad f \in \mathscr{S}(\mathbb{R}^2)$$

satisfies $(W_\gamma^B T^B f)(x,\theta) = (T^B f)(x,\theta)$. Hence, T^B extends to a unitary map form $L^2(\mathbb{R}^2)$ to $L^2((\mathbb{R}^2)^*/\Gamma^*, \mathscr{H}_{B,p})$ where $\mathscr{H}_{B,p} = \{v \in L^2_{loc}(\mathbb{R}^2) \mid W_\gamma^B v = v;\ \forall \gamma \in \Gamma\}$.

Thus, $H(V)$ admits a direct integral $T^B H(V) (T^B)^* = \int_{(\mathbb{R}^2)^*/\Gamma^*}^{\oplus} H(\theta, V)\, d\theta$

where $H(\theta, V) = (i\nabla + A - \theta)^2 + V$.

Reduction to the study of Bloch-Floquet eigenvalues

The spectrum of $H(\theta,V)$ is discrete; its eigenvalues are of finite multiplicity.

Call them $\quad E_1(\theta,V) \le E_2(\theta,V) \le \cdots \le E_n(\theta,V) \le \cdots$.

The function $(\theta,V) \in (\mathbb{R}^2)^*/(\Gamma')^* \times C_\Gamma \mapsto E_n(\theta,V)$ is locally uniformly Lipschitz continuous.

In view of the direct integral decomposition of $H(V)$, our main result is a corollary of

Theorem

There exists a dense G_δ-subset of C_Γ such that, for V in this set, none of the functions $\theta \mapsto E_n(\theta,V)$, $n \ge 1$, is constant.

Definition

$E_n(\theta_0,V_0)$ is an analytically stable eigenvalue of $H(\theta_0,V_0)$ if and only if there exists an orthonormal system of functions, say $((\theta,V) \mapsto \varphi_j(\cdot,\theta,V))_{1\le j\le J}$ s.t.

- for $j \in \{1,\cdots,J\}$, $(\theta,V) \mapsto \varphi_j(\theta,V) \in \mathscr{H}^2_{B,p}$ is analytic near (θ_0,V_0),
- near (θ_0,V_0), $(\varphi_j(\theta,V))_{1\le j\le J}$ spans the eigenspace of $H(\theta,V)$ associated to $E_n(\theta,V)$.

The basic technical lemmas:

Our main result follows from the following two lemmas.

Lemma (1)

Pick $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$ *and* $V_0 \in C_\Gamma$*. Fix* $n \geq 1$*. Then, for any* $\varepsilon > 0$*, there exists* $(\theta_\varepsilon, V_\varepsilon) \in \{\|(\theta, V) - (\theta_0, V_0)\| < \varepsilon\}$ *and* $\delta > 0$ *such that*

- $E_n(\theta, V)$ *is an analytically stable eigenvalue for* $\|(\theta, V) - (\theta_\varepsilon, V_\varepsilon)\| < \delta$*.*

Lemma (2)

Pick $\theta_0 \in (\mathbb{R}^2)^*/(\Gamma')^*$ *and* $V_0 \in C_\Gamma$ *such that* V_0 *is not a constant. Assume that* $E_n(\theta_0, V_0)$ *is an analytically stable eigenvalue of* $H(\theta_0, V_0)$*. Then, for any* $\varepsilon > 0$*, there exists* V *such that* $\|V - V_0\| < \varepsilon$ *and* $\theta \mapsto E_n(\theta, V)$ *is not constant.*

- As the Floquet eigenvalues are locally uniformly Lipschitz continuous in (θ, V), the set of $V \in C_\Gamma$ such that $\theta \mapsto E_n(\theta, V)$ is not constant is open;
- By Lemmas (1) and (2), for any $n \geq 1$, the set of $V \in C_\Gamma$ such that $\theta \mapsto E_n(\theta, V)$ is not constant is dense in C_Γ.

Hence, the set of $V \in C_\Gamma$ for which no Floquet eigenvalue is constant is a dense G_δ-set.

We concentrate on Lemma 2.

The proof of Lemma (2)

Pick $\theta_0 \in (\mathbb{R}^2)^*/\Gamma^*$ and $V_0 \in C_\Gamma$. Assume that $E(\theta_0, V_0)$ is analytically stable.

Assume that Lemma (2) does not hold. Then, for any V close to V_0, the function $\theta \mapsto E(\theta, V)$ is constant and V_0 can be chosen real analytic.

Pick $U \in \mathscr{C}_\Gamma$ such that $\|U\| = 1$ and set $V_t = V_0 + tU$, t complex small.

As $E(\theta_0, V_0)$ is analytically stable, there exists$(\theta, t) \mapsto \varphi(\theta, t)$ analytic such that, for (t, θ) close to $(0, \theta_0)$, one has

- $(H(\theta,t) - E(\theta,t))\varphi(\theta,t) = 0, \quad \|\varphi(\theta,t)\| = 1;$
- $(\theta,t) \mapsto E(\theta,t)$ is real analytic.

Differentiating in t yields $\quad (H(\theta,t) - E(\theta,t))\partial_t\varphi(\theta,t) = [\partial_t E(\theta,t) - U]\varphi(\theta,t).$

Thus, one obtains $\quad \partial_t E(\theta,t) = \langle U\varphi(\theta,t), \varphi(\theta,t)\rangle.$

If $\nabla_\theta E(\theta,t) = 0$, differentiating the expression above in θ, we obtain

$$0 = \partial_t \nabla_\theta E(\theta,t) = 2\mathrm{Re}\left[\langle U\varphi(\theta,t), \nabla_\theta\varphi(\theta,t)\rangle\right].$$

Thus, at $t = 0$, one has

$$0 = \mathrm{Re}\left[\langle U\varphi(\theta,0), \nabla_\theta\varphi(\theta,0)\rangle\right] = \int_{\mathbb{R}^2/\Gamma} U(x)\,\mathrm{Re}\left(\nabla_\theta\varphi(x;\theta,0)\overline{\varphi(x;\theta,0)}\right)dx$$

So, for θ close to θ_0, one has

$$2\,\mathrm{Re}(\nabla_\theta\varphi(x;\theta,0)\overline{\varphi(x;\theta,0)}) = \nabla_\theta\left(|\varphi(x;\theta,0)|^2\right) \equiv 0.$$

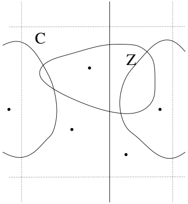
The nodal set 1

The operator $(i\nabla - A - \theta)^2 + V_0$ is elliptic with analytic coefficients; it is analytically hypoelliptic. Hence, $x \mapsto \varphi(x;\theta) := \varphi(x;\theta,V_0)$ is analytic on $\mathbb{R}^2$.

The nodal set $Z = \{x \in \mathbb{R}^2;\ \varphi(x;\theta) = 0\}$ is Γ-periodic and independent of θ.

Let C be the fundamental cell of the lattice Γ. It is compact.

By analytic geometry ([Bierstone-Milman88]), we know that $Z \cap C$ has the following finite decomposition $\quad Z \cap C = \bigcup_{p=1}^{p_0} \mathscr{A}_p \quad$ where the union is disjoint and one has



1. the set $\mathscr{A}_p$ either is reduced to a single point or is a connected real-analytic curve (i.e. a connected real analytic manifold of dimension 1);
2. if $p < p'$ and $\mathscr{A}_p \cap \overline{\mathscr{A}_{p'}} \neq \emptyset$, then
   - $\mathscr{A}_p \subset \overline{\mathscr{A}_{p'}}$, $\mathscr{A}_p$ is reduced to a single point and $\mathscr{A}_{p'}$ is a real analytic curve;
3. assume $\mathscr{A}_p = \{x_0\}$. Then,
   - either x_0 is isolated in $Z \cap C$
   - or, for some $\varepsilon_0 > 0$, $\quad Z \cap C \cap \dot{\overline{D}}(x_0,\varepsilon_0) = \bigcup_{p' \in E} \mathscr{A}_{p'} \cap \dot{\overline{D}}(x_0,\varepsilon_0)$, where E is a non empty, finite set of indices s. t., for $p' \in E$, the set $\mathscr{A}_{p'}$ is a real analytic curve.

The nodal set 2

Let $Z_0 = \bigcup_{\#\mathscr{A}_p=1} \mathscr{A}_p$ be the point components in the above decomposition. We prove

Lemma (3)

Let Z_∇ be the set of points x_0 in C such that $\varphi(x_0;\theta) = 0$ and $\nabla\varphi(x_0;\theta) = 0$. Then, Z_∇ consists of isolated points.

We postpone the proof of Lemma (3).

Consider a line $L_x = x + \mathbb{R} \times \{0\}$ s.t. $L_x \cap (Z_0 \cup Z_\nabla) = \emptyset$. We assume that it intersects these curves transversally in finitely many points.

For $\delta > 0$, define the strip $S_x^\delta = x + \mathbb{R} \times (-\delta, \delta)$. For some small $\delta > 0$, one has

- $\overline{S_x^\delta} \cap (Z_0 \cup Z_\nabla) = \emptyset$,
- S_x^δ intersects Z in C at, at most, finitely many vertical curves, and these curves partition the strip in a finite number of open domains (see figure on next slide).

Define C_k to be the left boundary of D_k. As Z is Γ-periodic, we get that

$$S_x^\delta \setminus Z = \bigcup_{\gamma \in q\mathbb{Z}e_1} \bigcup_{k=1}^{s} \gamma + D_k \quad \text{and} \quad Z \cap S_x^\delta = \bigcup_{\gamma \in q\mathbb{Z}e_1} \bigcup_{k=1}^{s} \gamma + C_k$$

Note that $s = 0$ if $Z = Z_0$.

The nodal set 3

The picture we get for the nodal set is

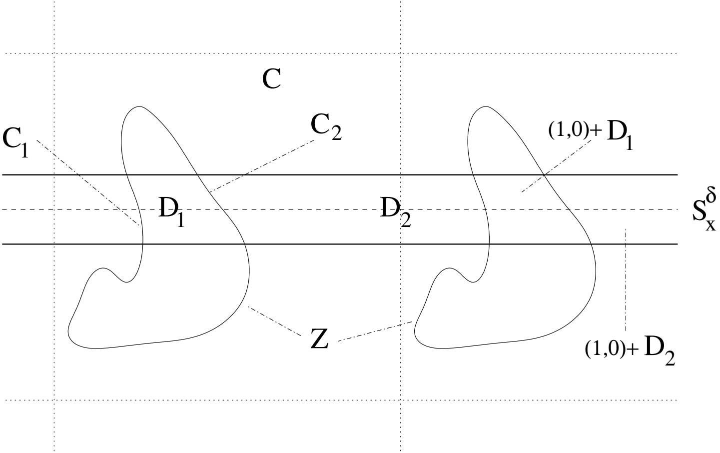


Figure: The strip

The behavior of the phase of the eigenfunction in a horizontal strip

We prove

Lemma (4)

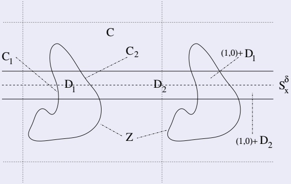
Let D be one of the domains $\gamma+D_k$ for some $1\leq k\leq s$, $\gamma=(\gamma_1,\gamma_2)$ where $\gamma_1\in q\mathbb{Z}$. For $|\theta-\theta_0|<\varepsilon$, there exists two continuous functions $x\in\overline{D}\mapsto g_D(x;\theta)\in\mathbb{R}$ and $x\in\overline{D}\mapsto\psi_D(x)\in\mathbb{R}^+$ such that

$$\forall x\in D,\quad \varphi(x;\theta)=e^{ig_D(x;\theta)}\psi_D(x).$$

and

- *for any $x_0\in D$, $(x,\theta')\mapsto g_D(x;\theta')$ (resp. $x\mapsto\psi_D(x)$) is real analytic in a neighborhood of (x_0,θ) (resp. x_0),*
- *let D' be another domain in the collection $(\gamma+D_k)_{\gamma,k}$; if $\overline{D}\cap\overline{D'}\neq\emptyset$ and D' is to the left of D, then, for $x\in\overline{D}\cap\overline{D'}$, one has*

$$g_D(x;\theta)=g_{D'}(x;\theta)+\pi.$$



We postpone the proof of Lemma (4).

Completing the proof of Lemma (2)

As $\varphi(\theta) \in \mathscr{H}_{B,p}$ one has, for $x \in D_k$ and $\gamma = (\gamma_1, \gamma_2)$, $\gamma_1 \in q\mathbb{Z}$

$$g_{\gamma+D_k}(x+\gamma,\theta) = g_{D_k}(x,\theta) - \frac{B}{2} x \wedge \gamma - \pi\gamma_1\gamma_2 \quad \text{and} \quad \psi_{\gamma+D_k}(x+\gamma) = \psi_{D_k}(x).$$

Plugging the representation of the previous lemma into the eigenvalue equation yields

$$(i\nabla_x + A - \theta - \nabla_x g_D)^2 \psi_D + V_0 \psi_D = E\psi_D.$$

Summing this and its complex conjugate, one obtains that, on D,

$$(A - \theta - \nabla_x g_D)^2 \psi_D = (E - V_0)\psi_D + \Delta\psi_D.$$

There exists $x \in \overline{D} \mapsto h_D(x)$ that is real analytic in D and $\theta \mapsto c_D(\theta)$ also real analytic such that, near θ_0 and for $x \in D$, one has

$$g_D(x,\theta) = -\theta \cdot x + h_D(x) + c_D(\theta).$$

Lemma (4) ensures that, if D' is to the left of D and $\overline{D'} \cap \overline{D} \neq \emptyset$, then we may pick $c_D(\theta) = c_{D'}(\theta) + \pi$.

Thus, we obtain that, near θ_0, for $\gamma = (\gamma_1, \gamma_2)$, $\gamma_1 \in q\mathbb{Z}$ and $x \in D$, one has

$$\theta \cdot \gamma = h_{\gamma+D}(x) - h_D(x) + \frac{B}{2} x \wedge \gamma + \pi\gamma_1\gamma_2 - s\gamma_1\pi.$$

This is absurd. □

The proof of Lemma (3)

The set $Z_\nabla \cap C$ is real analytic; it can be decomposed in the same way as $Z \cap C$.

Assume it contains an analytic curve, say, c. Pick a point $x^0 \in c$. Near $x^0 = (x_1^0, x_2^0)$ assume that the curve is parametrized by $x_1 = c(x_2)$.

The functions $u(x) = \mathrm{Re}(\varphi(x;\theta))$ and $v(x) = \mathrm{Im}(\varphi(x;\theta))$ satisfy

- $-\Delta u + (A-\theta)^2 u + 2A\cdot\nabla v = (E-V)u \;/\; -\Delta v + (A-\theta)^2 v - 2A\cdot\nabla u = (E-V)v$
- on c, one has $0 = u = v = \partial_1 u = \partial_1 v = \partial_2 u = \partial_2 v$.

Prove inductively that, for any $\alpha \in \mathbb{N}^2$, $\partial^\alpha u = \partial^\alpha v = 0$ on c.

Differentiating $\alpha_1 - 1$ in x_1 times equations and $\alpha_2 - 1$ times in x_2 yields that, on c, one has

$$\partial_1^{\alpha_1+1}\partial_2^{\alpha_2-1}u + \partial_1^{\alpha_1-1}\partial_2^{\alpha_2+1}u = \sum_{\beta_1+\beta_2\le N} a_{\beta_1\beta_2}\partial_1^{\beta_1}\partial_2^{\beta_2}u + b_{\beta_1\beta_2}\partial_1^{\beta_1}\partial_2^{\beta_2}v = 0,$$

$$\partial_1^{\alpha_1+1}\partial_2^{\alpha_2-1}v + \partial_1^{\alpha_1-1}\partial_2^{\alpha_2+1}v = \sum_{\beta_1+\beta_2\le N} c_{\beta_1\beta_2}\partial_1^{\beta_1}\partial_2^{\beta_2}u + d_{\beta_1\beta_2}\partial_1^{\beta_1}\partial_2^{\beta_2}v = 0.$$

Differentiating $\partial_1^{\alpha_1}\partial_2^{\alpha_2}u = 0$ along c, we get

$$c'(x_2)\left(\partial_1^{\alpha_1+1}\partial_2^{\alpha_2}u\right)(c(x_2),x_2) + \left(\partial_1^{\alpha_1}\partial_2^{\alpha_2+1}u\right)(c(x_2),x_2) = 0.$$

The end of the proof of Lemma (3)

Using $(\alpha_1,\alpha_2)=(N,0)$ and $(\alpha_1,\alpha_2)=(N-1,1)$ and the first equation in the system for $(\alpha_1,\alpha_2)=(N,1)$, we get

$$\begin{cases} \partial_1^{N+1}u + c'\partial_1^N\partial_2 u & = 0 \\ \partial_1^N\partial_2 u + c'\partial_1^{N-1}\partial_2^2 u & = 0 \\ \partial_1^{N+1}u + \partial_1^{N-1}\partial_2^2 u & = 0 \end{cases}$$

which implies that

$$\partial_1^{N+1}u = \partial_1^N\partial_2 u = \partial_1^{N-1}\partial_2^2 u = 0.$$

Then, using the system inductively, we get that $\partial_1^{N+1-\alpha}\partial_2^{\alpha}u = 0$ for all $0 \le \alpha \le N+1$.

Thus, if $Z_\nabla \cap C$ contains a curve, for all (α_1,α_2), the functions $(\partial_1^{\alpha_1}\partial_2^{\alpha_2})\varphi(\theta)$ vanish identically on this curve.

As $\varphi(\theta)$ is real analytic, this implies that this function vanishes identically which contradicts the assumption that its norm in $\mathscr{H}_{B,p}$ is 1.

The proof of Lemma (3) is complete. □

The proof of Lemma (4)

In the domains $(D_k)_{1\leq k\leq s}$ and their translates, the decomposition in Lemma (4) is the decomposition into argument and modulus of the complex number $\varphi(x;\theta)$.

As $\varphi(x;\theta)$ does not vanish and is analytic, its argument and modulus are real analytic.

So we only need to study what happens at the crossing of one of the curves $(C_k)_{1\leq k\leq s}$. So, we study $x\mapsto\varphi(x;\theta)$ near $x^0\in C_k$.

By Lemma (3), as $S_x^\delta\cap(Z_0\cup Z_\nabla)=\emptyset$, we know that $\nabla\varphi(x^0,\theta)\neq 0$.

As the curve C_k is vertical, we may assume that $\partial_1 u(x^0)\neq 0$.

Rectifying c at x^0 by a real analytic change of variables, in a neighborhood of 0, one obtains

$$u(x_1,x_2)=0\Leftrightarrow x_1=0,\quad \partial_1 u(0,0)\neq 0,\quad v(0,x_2)=0.$$

Write

$$u(x_1,x_2)=\tilde{w}(x_2)+x_1 w(x_1,x_2)\quad\text{and}\quad v(x_1,x_2)=\tilde{t}(x_2)+x_1 t(x_1,x_2).$$

Then, $w(0,0)\neq 0$ and $\tilde{w}(x_2)=\tilde{t}(x_2)=0$ identically. Hence, we obtain that

$$(u+iv)(x_1,x_2)=x_1\,(w+it)(x_1,x_2)\text{ where }|(w+it)(0,0)|\neq 0.$$

The end of the proof of Lemma (4)

Changing back to the initial variables, if $x_2 \mapsto c(x_2)$ is a parametrization of the curve C_k in U a neighborhood of x^0, we can write

$$\varphi(x;\theta) = (x_1 - c(x_2))\,\psi(x) \text{ where } \psi(x^0) \neq 0.$$

Hence, for $x \in D_k \cap U$, one has

$$e^{ig_{D_k}(x;\theta)}\psi_{D_k}(x) = (x_1 - c(x_2))\,\psi(x), \quad x_1 \geq c(x_2)$$

and for $x \in D_{k-1} \cap U$, one has

$$e^{ig_{D_{k-1}}(x;\theta)}\psi_{D_{k-1}}(x) = (x_1 - c(x_2))\,\psi(x) = -(c(x_2) - x_1)\,\psi(x), \quad x_1 \leq c(x_2).$$

This implies that we can continue $g_{D_{k-1}}$ and g_{D_k} continuously up to the boundary C_k and that, on C_k, they satisfy

$$g_{D_k}(x;\theta) = g_{D_{k-1}}(x;\theta) + \pi.$$

This completes the proof of Lemma (4). □