Chapter 3

Introduction to the Theory of Distributions

3.1 Test Functions and Distributions

3.1.1 Smooth compactly supported functions

Let Ω be an open subset of $\mathbb{R}^n$; we define $C_c^\infty(\Omega)$ as the vector space of complex-valued compactly supported functions defined on Ω. Even in the case $n = 1$ and $\Omega = \mathbb{R}$, it is not completely obvious that this space is not reduced to $\{0\}$. We leave to the reader as an exercise to check that the function

$$\rho_0(t) = \begin{cases} e^{-t^{-1}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \tag{3.1.1}$$

is a C^∞ function on $\mathbb{R}$. Starting with ρ_0, we may define a function ρ on $\mathbb{R}^n$ by

$$\rho(x) = \rho_0(1 - \|x\|^2) \tag{3.1.2}$$

and we see right away that $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \rho = \bar{B}(0,1)$. Here we have defined the support of ρ as the closure of the set $\{x \in \mathbb{R}^n, \rho(x) \neq 0\}$. Although that definition is fine when we deal with a continuous function, it will produce strange results if we want to define the support of a function in $L^1(\mathbb{R})$: for instance the characteristic function of $\mathbb{Q}$ is 0 a.e. and thus 0 as a function of $L^1(\mathbb{R})$, nevertheless the above set is $\mathbb{R}$. It is better to use the following definition, say for a function in $u \in L^1_{\text{loc}}(\Omega)$, Ω open subset of $\mathbb{R}^n$:

$$\operatorname{supp} u = \{x \in \Omega, \nexists U \text{open} \in \mathscr{V}_x, u_{|U} = 0\}, \quad (\operatorname{supp} u)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x, u_{|U} = 0\}. \tag{3.1.3}$$

The above definition makes sense for an L^1_{loc} function with $u_{|U} = 0$ meaning $u = 0$ a.e. in U. The smooth compactly supported functions are very useful as mollifiers, as shown by the next proposition.

Proposition 3.1.1. *Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $\epsilon > 0$, we define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x\epsilon^{-1})$. Then, if $f \in C_c^m(\mathbb{R}^n)$, $\lim_{\epsilon \to 0_+} \phi_\epsilon * f = f$ (convergence in $C_c^m(\mathbb{R}^n)$) and if $f \in L^p(\mathbb{R}^n)$ with $1 \le p < +\infty$, $\lim_{\epsilon \to 0_+} \phi_\epsilon * f = f$ (convergence in $L^p(\mathbb{R}^n)$). In both cases the function $\phi_\epsilon * f$ is C^∞.*

Proof. We write

$$(\phi_\epsilon * f)(x) - f(x) = \int \phi_\epsilon(x-y) f(y) dy - f(x) = \int \phi(y)\big(f(x-\epsilon y) - f(x)\big) dy,$$

so that, if $\operatorname{supp}\phi \subset \bar{B}(0, R_0)$,

$$|(\phi_\epsilon * f)(x) - f(x)| \le \int |\phi(y)| dy \sup_{|x_1 - x_2| \le \epsilon R_0} |f(x_1) - f(x_2)|.$$

The function f is continuous and compactly supported, so is uniformly continuous on $\mathbb{R}^n$ (an easy consequence of the Heine theorem 1.5.10), thus

$$\lim_{\epsilon \to 0_+} \big(\sup_{x \in \mathbb{R}^n} |(\phi_\epsilon * f)(x) - f(x)|\big) = 0,$$

yielding the uniform convergence of $\phi_\epsilon * f$ towards f. If f is C_c^m, a simple differentiation under the integral sign (see e.g. the *Théorème 3.3.2.* in [9]) gives as well the uniform convergence of the derivatives, up to order m. The smoothness of $\phi_\epsilon * f$ for $\epsilon > 0$ is due to the same theorem when $f \in C_c^m(\mathbb{R}^n)$, since we have $(\phi_\epsilon * f)(x) = \int \phi_\epsilon(x-y) f(y) dy$.

Remark 3.1.2. *We have not defined a topology on the vector space $C_c^m(\mathbb{R}^n)$, but at the moment it will be enough for us to say that a sequence $(u_k)_{k \in \mathbb{N}}$ of functions in $C_c^m(\mathbb{R}^n)$ is converging if it converges in $C^m(\mathbb{R}^n)$ and if there exists a compact set K such that, for all $k \in \mathbb{N}$, $\operatorname{supp} u_k \subset K$.*

We note in particular that these conditions are satisfied by the "sequences" $(\phi_\epsilon * f)_{\epsilon > 0}$ since for $\epsilon \le 1$, $\operatorname{supp}(\phi_\epsilon * f) \subset \operatorname{supp} f + \operatorname{supp} \phi_\epsilon \subset \operatorname{supp} f + \operatorname{supp} \phi$.

Let us now take $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. With $\psi \in C_c^0(\mathbb{R}^n)$, we have

$$f * \phi_\epsilon - f = (f - \psi) * \phi_\epsilon + \psi * \phi_\epsilon - \psi + \psi - f,$$

so that

$$\begin{aligned} \|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} &\le (1 + \|\phi\|_{L^1}) \|f - \psi\|_{L^p(\mathbb{R}^n)} + \|\psi * \phi_\epsilon - \psi\|_{L^p(\mathbb{R}^n)} \\ &\le (1 + \|\phi\|_{L^1}) \|f - \psi\|_{L^p(\mathbb{R}^n)} + \underbrace{|\operatorname{supp}\phi + \epsilon|}_{\text{Lebesgue measure}}{}^{1/p} \|\psi * \phi_\epsilon - \psi\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Since $\psi \in C_c^\infty(\mathbb{R}^n)$, the previous convergence argument implies the inequality

$$\limsup_{\epsilon \to 0_+} \|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} \le (1 + \|\phi\|_{L^1}) \|f - \psi\|_{L^p(\mathbb{R}^n)}, \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n).$$

The density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ (see e.g. the *Théorème 3.4.1* in [9]) yields the result. For $\epsilon > 0, R > 0$, all the functions

$$\psi_{R,\epsilon}(y) = \sup_{|x| \le R} |(\partial_x^\alpha \phi_\epsilon)(x-y) f(y)|$$

belong to $L^1(\mathbb{R}^n_y)$ since

$$\int \psi_{R,\epsilon}(y) dy \le \|f\|_{L^p(\mathbb{R}^n)} \Big(\int \sup_{|x| \le R} |(\partial_x^\alpha \phi_\epsilon)(x-y)|^{p'} dy\Big)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and $\operatorname{supp}\phi \subset \bar{B}_{R_0}$ gives that $|x - y| \le \epsilon R_0, |x| \le R$ imply $|y| \le \epsilon R_0 + R$, and the finiteness of the integral above, proving the smoothness of $\phi_\epsilon * f$ for $\epsilon > 0$. □

N.B. The result of the proposition does not extend to the case $p = \infty$, since the uniform convergence of the continuous function $f * \phi_\epsilon$ would imply the continuity of the limit.

It will be also useful to use the compactly supported functions to construct some partitions of unity and, to begin with, to find C_c^∞ functions identically equal to 1 near a compact set.

Lemma 3.1.3. *Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. Then there exists a function $\varphi \in C_c^\infty(\Omega; [0,1])$ such that $\varphi = 1$ on a neighborhood of K.*

Proof. We claim that there exists $\epsilon_0 > 0$ such that $K + \epsilon_0 B_1 \subset \Omega$, ($B_1$ is the open unit ball). First we note that

$$d(K, \Omega^c) = \inf_{x \in K, y \in \Omega^c} |x - y| > 0, \tag{3.1.4}$$

otherwise, we could find sequences $(x_k)_{k\ge1}$ in K, $(y_k)_{k\ge1}$ in Ω^c such that $\lim_k |x_k - y_k| = 0$, and since K is compact, we may suppose that (x_k) converges with limit $x \in K$, implying $\Omega^c \ni \lim_k y_k = x$, which is impossible since $K \subset \Omega$. As a result, we have with $\epsilon_0 = d(K, \Omega^c)$

$$K + \epsilon_0 B_1 \subset \Omega,$$

otherwise, we could find $|t| < 1, x \in K$ such that $x + \epsilon_0 t = y \in \Omega^c$, implying $|x - y| < \epsilon_0 = d(K, \Omega^c)$, which is impossible. With the function ρ defined in 3.1.2, we define with $0 < \epsilon \le \frac{\epsilon_1}{2} < \frac{\epsilon_0}{4}$,

$$\varphi(x) = \int \mathbf{1}_{K+\epsilon_1 \bar{B}_1}(y) \rho\big((x-y)\epsilon^{-1}\big)\epsilon^{-n} dy \Big(\int \rho(t) dt\Big)^{-1}.$$

The function φ is C^∞ and such that

$$\operatorname{supp} \varphi \subset K + \epsilon_1 \bar{B}_1 + \epsilon \bar{B}_1 \subset K + \frac{3}{2}\epsilon_1 \bar{B}_1 \subset \underbrace{K + \frac{3}{4}\epsilon_0 \bar{B}_1}_{\text{compact}} \subset K + \epsilon_0 B_1 \subset \Omega.$$

Moreover $\varphi = 1$ on $K + \frac{\epsilon_1}{2}\bar{B}_1$ (which is a neighborhood of K), since if $x \in K + \frac{\epsilon_1}{2}\bar{B}_1$, we have, for y satisfying $|x - y| \le \epsilon$, that $y \in K + \frac{\epsilon_1}{2}\bar{B}_1 + \epsilon \bar{B}_1 \subset K + \epsilon_1 \bar{B}_1$. As a result, with $\tilde{\rho} = \rho\big(\int \rho(t) dt\big)^{-1}$, for $x \in K + \frac{\epsilon_1}{2}\bar{B}_1$, we have

$$1 = \int \tilde{\rho}((x-y)\epsilon^{-1})\epsilon^{-n} dy = \int \tilde{\rho}((x-y)\epsilon^{-1})\epsilon^{-n} \mathbf{1}_{K+\epsilon_1 \bar{B}_1}(y) dy = \varphi(x).$$

We note also that, since $\tilde{\rho} \ge 0$ with integral 1, $\mathbf{1}_L(y) \in [0,1]$, we have, for all $x \in \mathbb{R}^n$, $0 \le \varphi(x) \le 1$. The proof of the lemma is complete. □

3.1.2 Distributions

Definition 3.1.4. *Let Ω be an open set of $\mathbb{R}^n$ and let $T : C_c^\infty(\Omega) \longrightarrow \mathbb{C}$ be a linear form with the following continuity property,*

$$\forall K \, compact \subset \Omega, \exists C_K > 0, \exists N_K \in \mathbb{N}, \forall \varphi \in C_K^\infty(\Omega), \ |\langle T, \varphi\rangle| \le C_K \sup_{\substack{|\alpha|\le N_K \\ x\in\mathbb{R}^n}} |(\partial_x^\alpha \varphi)(x)|, \tag{3.1.5}$$

where $C_K^\infty(\Omega) = \{\varphi \in C_c^\infty(\Omega), \operatorname{supp}\varphi \subset K\}$.

N.B. We shall use also the notation $\mathscr{D}(\Omega)$ for the space of test functions $C_c^\infty(\Omega)$ and $\mathscr{D}'(\Omega)$ for the space of distributions on Ω. We have not introduced a topology on $\mathscr{D}(\Omega)$ but we have defined a notion of converging sequence with the remark 3.1.2. It would have been certainly more elegant to start with the display of the natural topological structure on $\mathscr{D}(\Omega)$, at the (heavy) cost of having to deal with a non-metrizable locally convex topology defined by an uncountable family of semi-norms. The study of inductive limits of increasing sequences of Fréchet spaces is outlined in the appendix 3.7.2. Anyhow, one should think of $\mathscr{D}'(\Omega)$ as the topological dual of $\mathscr{D}(\Omega)$, a view supported by the next lemmas and remarks.

Remark 3.1.5. With $\mathscr{D}_K(\Omega) = C_K^\infty(\Omega)$, we have, using the sequence of compact sets $(K_j)_{j\ge1}$ of the lemma 2.3.1

$$\mathscr{D}(\Omega) = \cup_{j\ge1}\mathscr{D}_{K_j}(\Omega)$$

and it is not difficult to see that each $\mathscr{D}_{K_j}(\Omega)$ is a Fréchet space with the natural countable family of semi-norms given by $p_{K_j,m}(u) = \sup_{\substack{|\alpha|\le m \\ x\in K_j}} |(\partial_x^\alpha u)(x)|$. If we want to use the countable family $p_{K_j,m}$, we end-up with the topology on the Fréchet space $C^\infty(\Omega)$ as described in the subsection 2.3.3; the actual topology on $\mathscr{D}(\Omega)$ is finer and it is important to understand that, with ρ defined in (3.1.2) (say with $n = 1$), the sequence $(u_k)_{k\in\mathbb{N}}$, given by

$$u_k(x) = \rho(x - k)$$

does converge to 0 in the Fréchet space $C^\infty(\mathbb{R})$ but is *not* convergent in $C_c^\infty(\mathbb{R})$, since the second condition of the remark 3.1.2 is not satisfied: there is no compact subset K of $\mathbb{R}$ such that $\forall k \in \mathbb{N}, \operatorname{supp} u_k \subset K$.

Remark 3.1.6. Note that a linear form T on $C_c^\infty(\Omega)$ is a distribution if and only if, for all compact subsets K of Ω, its restriction to the Fréchet space $\mathscr{D}_K(\Omega)$ is continuous.

A L^1_{loc} function is a distribution: for Ω open subset of $\mathbb{R}^n$, for $f \in L^1_{\text{loc}}(\Omega)$, we define for $\varphi \in \mathscr{D}(\Omega)$

$$\langle T, \varphi\rangle = \int f(x)\varphi(x)dx \Longrightarrow |\langle T, \varphi\rangle| \le \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\operatorname{supp}\varphi} |f(x)|dx, \tag{3.1.6}$$

so that (3.1.5) is satisfied with $C_K = \int_K |f(x)|dx, N_K = 0$. Moreover the canonical mapping from $L^1_{\text{loc}}(\Omega)$ into $\mathscr{D}'(\Omega)$ is injective, as shown by the next lemma.

Lemma 3.1.7. *Let Ω be an open subset of $\mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ such that, for all $\varphi \in \mathscr{D}(\Omega)$, $\int f(x)\varphi(x)dx = 0$. Then we have $f = 0$.*

Proof. Let K be a compact subset of Ω and $\chi \in \mathscr{D}(\Omega)$ equal to 1 on a neighborhood of K as in the lemma 3.1.3. With ϕ as in the proposition 3.1.1, we get that $\lim_{\epsilon\to 0_+} \phi_\epsilon * (\chi f) = \chi f$ in $L^1(\mathbb{R}^n)$. We have

$$\big(\phi_\epsilon * (\chi f)\big)(x) = \int f(y) \underbrace{\chi(y)\phi\big((x-y)\epsilon^{-1}\big)\epsilon^{-n}}_{=\varphi_x(y)} dy, \quad \operatorname{supp}\varphi_x \subset K, \varphi_x \in \mathscr{D}(\Omega),$$

and from the assumption of the lemma, we obtain $\big(\phi_\epsilon * (\chi f)\big)(x) = 0$ for all x, implying $\chi f = 0$ from the convergence result; the conclusion follows. □

We note that it makes sense to restrict a distribution $T \in \mathscr{D}'(\Omega)$ to an open subset $U \subset \Omega$: just define

$$\langle T_{|U}, \varphi\rangle_{\mathscr{D}'(U),\mathscr{D}(U)} = \langle T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \tag{3.1.7}$$

and $T_{|U}$ is obviously a distribution on U. With this in mind, we can define the support of a distribution exactly as in (3.1.8).

Definition 3.1.8. *Let Ω be an open subset of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. We define the support of T as*

$$\operatorname{supp} T = \{x \in \Omega, \forall U \, open \in \mathscr{V}_x, \ T_{|U} \neq 0\}. \tag{3.1.8}$$

We define the C^∞ singular support of T as

$$\operatorname{singsupp} T = \{x \in \Omega, \forall U \, open \in \mathscr{V}_x, \ T_{|U} \notin C^\infty(U)\}. \tag{3.1.9}$$

Note that the support and the singular support are closed subset of Ω since their complements in Ω are open: we have

$$(\operatorname{supp} T)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x, \ T_{|U} = 0\}, \tag{3.1.10}$$

$$(\operatorname{singsupp} T)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x, \ T_{|U} \in C^\infty(U)\}. \tag{3.1.11}$$

A simple consequence of that definition is that, for $T \in \mathscr{D}'(\Omega)$, $\varphi \in \mathscr{D}(\Omega)$,

$$\operatorname{supp}\varphi \subset (\operatorname{supp} T)^c \Longrightarrow \langle T, \varphi\rangle = 0. \tag{3.1.12}$$

3.1.3 First examples of distributions

The Dirac mass

We define for $\varphi \in C^0_c(\mathbb{R}^n)$, $\langle \delta_0, \varphi\rangle = \varphi(0)$; the property (3.1.5) is satisfied with $C_K = 1, N_K = 0$. We have $\operatorname{supp}\delta_0 = \{0\}$. From this, the Dirac mass cannot be an L^1_{loc} function, otherwise, since it is 0 a.e., it would be 0. Let ϕ, ϵ as in the proposition 3.1.1: then we have from that proposition

$$\lim_{\epsilon\to 0_+} \int \phi_\epsilon(x)\varphi(x)dx = \varphi(0),$$

so that the Dirac mass appears as the weak limit of $\epsilon^{-n}\phi(x\epsilon^{-1})$.

The simple layer

We consider in $\mathbb{R}^n$ the hypersurface $\Sigma = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_n = f(x')\}$, where $f \in C^1(\mathbb{R}^{n-1})$. We define for $\varphi \in C_c^0(\mathbb{R}^n)$,

$$\langle \delta_\Sigma, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi\big(x', f(x')\big)\big(1 + |\nabla f(x')|^2\big)^{1/2} dx'.$$

The property (3.1.5) is satisfied with $C_K = area(\Sigma \cap K), N_K = 0, \operatorname{supp} \delta_\Sigma = \Sigma$, and since Σ has Lebesgue measure 0 in $\mathbb{R}^n$, the simple layer potential cannot be an L^1_{loc} function.

The principal value of $1/x$

We define for $\varphi \in C_c^1(\mathbb{R})$,

$$\langle \operatorname{pv} \frac{1}{x}, \varphi \rangle = \lim_{\epsilon \to 0_+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x} dx. \tag{3.1.13}$$

Let us check that this limit exists. We have for parity reasons,

$$\begin{aligned} \int_{|x|\geq\epsilon} \frac{\varphi(x)}{x} dx &= \int_\epsilon^{+\infty} \big(\varphi(x) - \varphi(-x)\big) \frac{dx}{x} \\ &= \big[\ln x \big(\varphi(x) - \varphi(-x)\big)\big]_{x=\epsilon}^{x=+\infty} - \int_\epsilon^{+\infty} \big(\varphi'(x) + \varphi'(-x)\big) \ln x dx \end{aligned}$$

and thus, using that $\lim_{\epsilon \to 0_+} \epsilon \ln \epsilon = 0$, $\ln |x| \in L^1_{\text{loc}}(\mathbb{R})$, we get

$$\langle \operatorname{pv} \frac{1}{x}, \varphi \rangle = -\int_0^{+\infty} \big(\varphi'(x) + \varphi'(-x)\big) \ln x dx = -\int_{\mathbb{R}} \varphi'(x)(\ln |x|) dx,$$

yielding $|\langle \operatorname{pv} \frac{1}{x}, \varphi \rangle| \leq \int_{\operatorname{supp} \varphi'} |\ln |x|| dx \|\varphi'\|_{L^\infty}$.

3.1.4 Continuity properties

Definition 3.1.9. *Let Ω be an open subset of $\mathbb{R}^n$ and let $(\varphi_j)_{j\geq 1}$ be a sequence of functions in $C_c^\infty(\Omega)$. We shall say that $\lim_j \varphi_j = 0$ in $C_c^\infty(\Omega)$ when the two following conditions are satisfied:*
(1) there exists a compact set $K \subset \Omega$, such that $\forall j \geq 1, \operatorname{supp} \varphi_j \subset K$,
(2) $\lim_j \varphi_j = 0$ in the Fréchet space $C_K^\infty(\Omega)$, i.e. $\forall \alpha \in \mathbb{N}^n$, $\lim_j \big(\sup_{x \in K} |(\partial_x^\alpha \varphi_j)(x)|\big) = 0$.

Proposition 3.1.10. *Let Ω be an open subset of $\mathbb{R}^n$ and T be a linear form defined on $C_c^\infty(\Omega)$. The linear form T is a distribution on Ω if and only if it is sequentially continuous.*

Proof. Assuming $|\langle T, \varphi \rangle| \leq C_K \max_{|\alpha| \leq N_K} \|\partial_x^\alpha \varphi\|_{L^\infty}$ for all $\varphi \in C_K^\infty(\Omega)$ and all K compact $\subset \Omega$ implies readily the sequential continuity. Conversely, if T does not satisfy (3.1.5), we have

$$\exists K_0 \text{compact} \subset \Omega, \forall k \geq 1, \forall N \in \mathbb{N}, \exists \varphi_{k,N} \in C_{K_0}^\infty(\Omega), |\langle T, \varphi_{k,N} \rangle| > k \max_{|\alpha| \leq N} \|\partial_x^\alpha \varphi_{k,N}\|_{L^\infty}.$$

From the strict inequality, we infer that the function $\varphi_{k,N}$ is not identically 0, and we may define

$$\psi_k = \frac{\varphi_{k,k}}{k \max_{|\alpha| \le k} \|\partial_x^\alpha \varphi_{k,k}\|_{L^\infty}}, \quad \text{so that } |\langle T, \psi_k\rangle| > 1.$$

But the sequence $(\psi_k)_{k\ge 1}$ converges to 0 since $\operatorname{supp}\psi_k \subset K_0$ and for $|\beta| \le k$, $\|\partial_x^\beta \psi_k\|_{L^\infty} \le 1/k$, implying for each multi-index β that $\lim_k \|\partial_x^\beta \psi_k\|_{L^\infty} = 0$. The sequential continuity is violated since $|\langle T, \psi_k\rangle| > 1$ and the converse is proven. □

Definition 3.1.11. *Let Ω be an open subset of $\mathbb{R}^n$, $T \in \mathscr{D}'(\Omega)$ and $N \in \mathbb{N}$. The distribution T will be said of finite order N if*

$$\exists N \in \mathbb{N}, \forall K \, compact \subset \Omega, \exists C_K > 0, \forall \varphi \in C_K^\infty(\Omega), |\langle T, \varphi\rangle| \le C_K \sup_{\substack{|\alpha|\le N \\ x \in \mathbb{R}^n}} |(\partial_x^\alpha \varphi)(x)|. \tag{3.1.14}$$

The vector space of distributions of order N on Ω will be denoted by ${\mathscr{D}'}^N(\Omega)$. The vector space ${\mathscr{D}'}^0(\Omega)$ is called the space of Radon measures on Ω.

Proposition 3.1.12. *Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$. The vector space ${\mathscr{D}'}^m(\Omega)$ is equal to the sequentially continuous[1] linear forms on $C_c^m(\Omega)$: if $T \in {\mathscr{D}'}^m(\Omega)$, it can be extended to a sequentially continuous linear form on $C_c^m(\Omega)$. If T is a sequentially continuous linear form on $C_c^m(\Omega)$, then $T \in {\mathscr{D}'}^m(\Omega)$.*

Proof. Let us first consider $T \in {\mathscr{D}'}^m(\Omega), \varphi \in C_c^m(\Omega)$. Applying the proposition 3.1.1, we find a sequence $(\varphi_k)_{k\ge 1}$ in $C_c^\infty(\Omega)$, converging in $C_c^m(\Omega)$ with limit φ. Since we may assume that all the functions φ_k and φ are supported in a fixed compact subset K of Ω, we have, according to the estimate (3.1.14),

$$|\langle T, \varphi_k - \varphi_l\rangle| \le C \max_{|\alpha|\le m} \|\partial_x^\alpha(\varphi_k - \varphi_l)\|_{L^\infty} = Cp(\varphi_k - \varphi_l),$$

where p is the norm in the Banach space $C_K^m(\Omega)$. Since the sequence $(\varphi_k)_{k\ge 1}$ converges in $C_K^m(\Omega)$, we get that the sequence $(\langle T, \varphi_k\rangle)_{k\ge 1}$ is a Cauchy sequence in $\mathbb{C}$, thus converges; moreover, if for some compact subset L of Ω, $(\psi_k)_{k\ge 1}$ is another sequence of $C_L^m(\Omega)$ converging to φ, we have

$$|\langle T, \psi_k - \varphi_k\rangle| \le C' \max_{|\alpha|\le m} \|\partial_x^\alpha(\varphi_k - \psi_k)\|_{L^\infty} = C'p(\varphi_k - \psi_k) \le C'p(\varphi_k - \varphi) + C'p(\varphi - \psi_k)$$

and $\lim_k \langle T, \psi_k - \varphi_k\rangle = 0$ so that, we can extend the linear form to $C_c^m(\Omega)$ by defining $\langle T, \varphi\rangle = \lim_k \langle T, \varphi_k\rangle$. We get also immediately that (3.1.14) holds with $N = m$ and $C_K^\infty(\Omega)$ replaced by $C_K^m(\Omega)$, so that T is obviously sequentially continuous.

Let us now consider a sequentially continuous linear form T on $C_c^m(\Omega)$; reproducing the proof of the proposition 3.1.10, we get that the estimate (3.1.14) holds with $N = m$, proving that $T \in {\mathscr{D}'}^m(\Omega)$. The proof of the proposition is complete. □

Remark 3.1.13. We have already proven directly that functions in $L^1_{\text{loc}}(\Omega)$(see (3.1.6)), the Dirac mass and a simple layer (see the section 3.1.3) are distributions of order 0. It is an exercise left to the reader to prove that the distribution $\operatorname{pv}\frac{1}{x}$ defined in (3.1.13) is of order 1 and not of order 0.

[1] The convergence of a sequence in $C_c^m(\Omega)$ is analogous to the convergence given in the definition 3.1.9, except that (2) is required in the Banach space $C_K^m(\Omega)$, i.e. $|\alpha| \le m$.

3.1.5 Partitions of unity and localization

Theorem 3.1.14 (Partition of unity). *Let Ω be an open subset of $\mathbb{R}^n$, K a compact subset of Ω and $\Omega_1, \dots, \Omega_m$ open subsets of Ω such that $K \subset \Omega_1 \cup \dots \cup \Omega_m$. Then for $1 \le j \le m$, there exists $\psi_j \in C_c^\infty(\Omega_j; [0,1])$ and V open such that*

$$\Omega \supset V \supset K, \ \forall x \in V, \sum_{1 \le j \le m} \psi_j(x) = 1,$$

and for all $x \in \Omega$, $\sum_{1 \le j \le m} \psi_j(x) \in [0,1]$.

Proof. The case $m = 1$ of the theorem is proven in the lemma 3.1.3. We consider now $m > 1$ and we note that, since $x \in K$ implies $x \in$ one of the Ω_j,

$$K \subset \cup_{x \in K} B(x, r_x), \quad \bar{B}(x, r_x) \subset \text{one of the } \Omega_j, \quad r_x > 0.$$

From the compactness of K, we get that $K \subset \cup_{1 \le l \le N} B(x_l, r_{x_l})$ and we may assume that

$$\begin{aligned}
&\bar{B}(x_l, r_{x_l}) \subset \Omega_1, \quad \text{for } 1 \le l \le N_1, \\
&\bar{B}(x_l, r_{x_l}) \subset \Omega_2, \quad \text{for } N_1 < l \le N_2, \\
&\dots\dots\dots\dots\dots \\
&\bar{B}(x_l, r_{x_l}) \subset \Omega_m, \quad \text{for } N_{m-1} < l \le N_m = N.
\end{aligned}$$

We define then the compact sets

$$K_1 = \cup_{1 \le l \le N_1} \bar{B}(x_l, r_{x_l}), \quad \dots \quad , K_m = \cup_{N_{m-1} < l \le N_m} \bar{B}(x_l, r_{x_l}),$$

and we have $K \subset \cup_{1 \le j \le m} K_j$, and for each j, $K_j \subset \Omega_j$. Using the lemma 3.1.3, we find $\varphi_j \in C_c^\infty(\Omega_j; [0,1])$ such that $\varphi_j = 1$ on a neighborhood $V_j (\subset \Omega_j)$ of K_j. We define then

$$\begin{aligned}
&\psi_1 = \varphi_1, \\
&\psi_2 = \varphi_2(1 - \varphi_1), \\
&\dots\dots \\
&\psi_j = \varphi_j(1 - \varphi_1) \dots (1 - \varphi_{j-1}),
\end{aligned}$$

so that $\psi_j \in C_c^\infty(\Omega_j; [0,1])$ and we have

$$\sum_{1 \le j \le m} \psi_j = \sum_{1 \le j \le m} \varphi_j \Big(\prod_{1 \le k < j} (1 - \varphi_k) \Big) = 1 - \prod_{1 \le k \le m} (1 - \varphi_k), \tag{3.1.15}$$

since the formula (second equality above) is true for $m = 1$ and inductively,

$$\begin{aligned}
\sum_{1 \le j \le m+1} \varphi_j \Big(\prod_{1 \le k < j} (1 - \varphi_k) \Big) &= 1 - \prod_{1 \le k \le m} (1 - \varphi_k) + \varphi_{m+1} \prod_{1 \le k \le m} (1 - \varphi_k) \\
&= 1 - (1 - \varphi_{m+1}) \prod_{1 \le k \le m} (1 - \varphi_k) = 1 - \prod_{1 \le k \le m+1} (1 - \varphi_k).
\end{aligned}$$

We have thus for $x \in \cup_{1 \le j \le m} V_j$ (which is a neighborhood of K in Ω), using (3.1.15) and $\varphi_j = 1$ on V_j, $\sum_{1 \le j \le m} \psi_j(x) = 1$. On the other hand, (3.1.15) and φ_j valued in $[0,1]$ show that $\sum_{1 \le j \le m} \psi_j(x) \in [0,1]$ for all x. The proof is complete. □

Theorem 3.1.15. *Let Ω be an open set of $\mathbb{R}^n$ and $(\Omega_j)_{j\in J}$ be an open covering of Ω: each Ω_j is open and $\cup_{j\in J}\Omega_j = \Omega$. Let us assume that for each $j \in J$, we are given $T_j \in \mathscr{D}'(\Omega_j)$ in such a way that*

$$T_{j|\Omega_j\cap\Omega_k} = T_{k|\Omega_j\cap\Omega_k}. \tag{3.1.16}$$

Then there exists a unique $T \in \mathscr{D}'(\Omega)$ such that for all $j \in J$, $T_{|\Omega_j} = T_j$.

Proof. Uniqueness: if T, S are such distributions, we get that $(T-S)_{|\Omega_j} = 0$, so that for all $j \in J$, $\Omega_j \subset (\operatorname{supp}(T-S))^c$ and thus $\Omega = \cup_{j\in J}\Omega_j \subset (\operatorname{supp}(T-S))^c$, i.e. $T - S = 0$.
Existence: let $\varphi \in \mathscr{D}(\Omega)$ and let us consider the compact set $K = \operatorname{supp}\varphi$. We have $K \subset \cup_{j\in M}\Omega_j$ with M a finite subset of J. Using the theorem on partitions of unity, we find some function $\psi_j \in C_c^\infty(\Omega_j)$ for $j \in M$ such that $\sum_{j\in M}\psi_j = 1$ on a neighborhood of K. As a consequence, we have $\varphi = \sum_{j\in M}\psi_j\varphi$ and we define

$$\langle T, \varphi\rangle = \sum_{j\in M}\langle T_j, \psi_j\varphi\rangle.$$

The required estimates (3.1.5) are easily checked, but the linearity and the independence with respect to the decomposition deserve some attention. Assume that we have $\varphi = \sum_{k\in N}\phi_k\varphi$, where N is a finite subset of J and $\phi_k \in C_c^\infty(\Omega_k)$: we have

$$\sum_{k\in N}\langle T_k, \phi_k\varphi\rangle = \sum_{j\in M, k\in N}\langle T_k, \phi_k\psi_j\varphi\rangle \underbrace{=}_{\text{from (3.1.16)}} \sum_{j\in M, k\in N}\langle T_j, \phi_k\psi_j\varphi\rangle = \sum_{j\in M}\langle T_j, \psi_j\varphi\rangle,$$

proving that T is defined independently of the decomposition. The linearity follows at once. The proof is complete. □

3.1.6 Weak convergence of distributions

We have not defined a topology on the space of test functions $\mathscr{D}(\Omega)$, although we gave the definition of convergence of a sequence (see the definition 3.1.9); we shall need also a simple notion of weak-dual convergence of a sequence of distributions, which is the $\sigma(\mathscr{D}', \mathscr{D})$ convergence.

Definition 3.1.16. *Let Ω be an open set of $\mathbb{R}^n$, $(T_j)_{j\ge1}$ be a sequence of $\mathscr{D}'(\Omega)$ and $T \in \mathscr{D}'(\Omega)$. We shall say that $\lim_j T_j = T$ in the weak-dual topology if*

$$\forall\varphi \in \mathscr{D}(\Omega), \quad \lim_j\langle T_j, \varphi\rangle = \langle T, \varphi\rangle. \tag{3.1.17}$$

Remark 3.1.17. We have already seen (see the section 3.1.3) that for $\rho \in C_c^\infty(\mathbb{R}^n)$, $\epsilon > 0$, $\rho_\epsilon(x) = \epsilon^{-n}\rho(x\epsilon^{-1})$, $\lim_{\epsilon\to0_+}\rho_\epsilon = \delta_0\int\rho(t)dt$. Moreover, on $\mathscr{D}'(\mathbb{R})$, we have with $T_\lambda(x) = e^{i\lambda x}$, $\lim_{\lambda\to+\infty}T_\lambda = 0$ since for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} e^{i\lambda x}\varphi(x)dx = (i\lambda)^{-1}\int_{\mathbb{R}}\frac{d}{dx}(e^{i\lambda x})\varphi(x)dx = -(i\lambda)^{-1}\int_{\mathbb{R}} e^{i\lambda x}\varphi'(x)dx.$$

Theorem 3.1.18. *Let Ω be an open set of $\mathbb{R}^n$, $(T_j)_{j\ge 1}$ be a sequence of $\mathscr{D}'(\Omega)$ such that, for all $\varphi \in \mathscr{D}(\Omega)$, the (numerical) sequence $(\langle T_j, \varphi\rangle)_{j\ge 1}$ converges. Defining the linear form T on $\mathscr{D}(\Omega)$, by $\langle T, \varphi\rangle = \lim_j \langle T_j, \varphi\rangle$, we obtain that T belongs to $\mathscr{D}'(\Omega)$.*

Proof. This is an important consequence of the Banach-Steinhaus theorem 2.1.8; let us consider a compact subset K of Ω. Then defining $T_{j,K}$ as the restriction of T_j to the Fréchet space $\mathscr{D}_K(\Omega)$, we see that the assumptions of the corollary 2.1.8 are satisfied since $T_{j,K}$ belongs to the topological dual of $\mathscr{D}_K(\Omega)$, according to the remark 3.1.6. As a consequence the restriction of T to $\mathscr{D}_K(\Omega)$ belongs to the topological dual of $\mathscr{D}_K(\Omega)$ and from the same remark 3.1.6, it gives that $T \in \mathscr{D}'(\Omega)$. □

N.B. The reader may note that we have used $E = \mathscr{D}(\Omega) = \cup_{j\in\mathbb{N}} \mathscr{D}_{K_j}(\Omega) = \cup_j E_j$, and that our definition of the topological dual of E as linear forms T on E such that, for all j, $T_{|E_j} \in$ the topological dual of the Fréchet space E_j. This structure allows us to use the Banach-Steinhaus theorem, although we have not defined a topology on E; this observation is a good introduction to the more abstract setting of LF spaces, the so-called inductive limits of Fréchet spaces.

3.2 Differentiation of distributions, multiplication by C^∞ functions

3.2.1 Differentiation

Definition 3.2.1. *Let Ω be an open set of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. We define the distributions $\partial_{x_j} T$ and for a multi-index $\alpha \in \mathbb{N}^n$ (see (2.3.6)), $\partial_x^\alpha T$ by*

$$\langle \partial_{x_j} T, \varphi\rangle = -\langle T, \partial_{x_j}\varphi\rangle, \quad \langle \partial_x^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial_x^\alpha \varphi\rangle. \tag{3.2.1}$$

We note that $\partial_x^\alpha T$ is indeed a distribution on Ω, since the mappings $\varphi \mapsto \partial_x^\alpha \varphi$ are continuous on each Fréchet space $\mathscr{D}_K(\Omega)$.

Remark 3.2.2. If $\lim_j T_j = T$ in the weak-dual topology of $\mathscr{D}'(\Omega)$, then, for all multi-indices α, $\lim_j \partial_x^\alpha T_j = \partial_x^\alpha T$ (in the weak-dual topology): we have, for each $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_x^\alpha T_j, \varphi\rangle = (-1)^{|\alpha|}\langle T_j, \partial_x^\alpha \varphi\rangle \underset{j\to+\infty}{\longrightarrow} (-1)^{|\alpha|}\langle T, \partial_x^\alpha \varphi\rangle = \langle \partial_x^\alpha T, \varphi\rangle.$$

Remark 3.2.3. If $u \in C^1(\Omega)$, its derivative $\partial_{x_j} u$ as a distribution coincides with the distribution defined by the continuous function $\partial u/\partial x_j$: for $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_{x_j} u, \varphi\rangle = -\langle u, \partial_{x_j}\varphi\rangle = -\int u(x)\frac{\partial \varphi}{\partial x_j}(x)dx = \int \frac{\partial u}{\partial x_j}(x)\varphi(x)dx = \langle \frac{\partial u}{\partial x_j}, \varphi\rangle.$$

Also, if $u, v \in C^0(\Omega)$ are such that $\partial_{x_1} u = v$ in $\mathscr{D}'(\Omega)$, then the function u admits v as a partial derivative with respect to x_1. To prove this, we may assume that u, v are both compactly supported in Ω: in fact it is enough to prove that for $\chi \in C_c^\infty(\Omega)$

identically equal to 1 near a point x_0, the function χu (compactly supported) has a partial derivative with respect to x_1 which is $\chi v + u\partial_{x_1}\chi$ (compactly supported) and we know that in $\mathscr{D}'(\Omega)$ we have

$$\langle \partial_{x_1}(\chi u), \varphi\rangle = -\langle u, \chi\partial_{x_1}\varphi\rangle = -\langle u, \partial_{x_1}(\chi\varphi)\rangle + \langle u, \varphi\partial_{x_1}\chi\rangle = \langle \partial_{x_1}u, \chi\varphi\rangle + \langle u\partial_{x_1}\chi, \varphi\rangle$$

which implies a particular case of Leibniz' formula $\partial_{x_1}(\chi u) = \chi\partial_{x_1}u + u\partial_{x_1}\chi = \chi v + u\partial_{x_1}\chi$. Assuming then that u, v are compactly supported, we have from the proposition 3.1.1, $u = \lim_\epsilon(u * \phi_\epsilon)$ in $C_c^0(\Omega)$ and the functions $u * \phi_\epsilon \in C_c^\infty(\Omega)$. Also we have, with the ordinary differentiation,

$$(\partial_{x_1}(u*\phi_\epsilon))(x) = \int u(y)(\partial_{x_1}\phi_\epsilon)(x-y)dy = \langle u(\cdot), -\partial_{y_1}\big(\phi_\epsilon(x-\cdot)\big)\rangle = \int v(y)\phi_\epsilon(x-y)dy,$$

and $\lim_\epsilon(v * \phi_\epsilon) = v$ in $C_c^0(\Omega)$. As a result the sequences $(u * \phi_\epsilon), (\partial_{x_1}(u * \phi_\epsilon))$ are both uniformly converging sequences of (compactly supported) continuous functions with respective limits u, v, and this implies that the continuous function u has v as a partial derivative with respect to x_1.

3.2.2 Examples

Defining the Heaviside function H as $\mathbf{1}_{\mathbb{R}_+}$, we get

$$H' = \delta_0 \tag{3.2.2}$$

since for $\varphi \in \mathscr{D}(\mathbb{R})$, we have $\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^{+\infty}\varphi'(t)dt = \varphi(0)$. Still in one dimension, we have

$$\langle \delta_0^{(k)}, \varphi\rangle = (-1)^k\varphi^{(k)}(0), \tag{3.2.3}$$

since it is true for $k = 0$ and inductively $\langle\delta_0^{(k+1)}, \varphi\rangle = -\langle\delta_0^{(k)}, \varphi'\rangle = -(-1)^k\varphi'^{(k)}(0) = (-1)^{k+1}\varphi^{(k+1)}(0)$. Looking at the definition (3.1.13), we see that we have proven

$$\mathrm{pv}\,(\frac{1}{x}) = \frac{d}{dx}(\ln|x|), \qquad \text{(distribution derivative)}. \tag{3.2.4}$$

Let f be a finitely-piecewise C^1 function defined on $\mathbb{R}$: it means that there is an increasing finite sequence of real numbers $(a_n)_{1\le n\le N}$, so that f is C^1 on all closed intervals $[a_n, a_{n+1}]$ for $1 \le n < N$ and on $]-\infty, a_1]$ and $[a_N, +\infty[$. In particular, the function f has a left-limit $f(a_n^-)$ and a right-limit $f(a_n^+)$ which may be different. Let us compute the distribution derivative of f; for $\varphi \in \mathscr{D}(\mathbb{R})$, since f is locally integrable, we have, setting $a_0 = -\infty, a_{N+1} = +\infty$,

$$\begin{aligned}
\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{\mathbb{R}} f(x)\varphi'(x)dx = -\sum_{0\le n\le N}\int_{a_n}^{a_{n+1}} f(x)\varphi'(x)dx \\
= \sum_{0\le n\le N}\int_{a_n}^{a_{n+1}}\frac{df}{dx}(x)\varphi(x)dx + \sum_{0\le n\le N}\big(f(a_n^+)\varphi(a_n) - f(a_{n+1}^-)\varphi(a_{n+1})\big) \\
= \int\varphi(x)\big(\sum_{0\le n\le N}\frac{df}{dx}(x)\mathbf{1}_{[a_n,a_{n+1}]}(x)\big) + \sum_{1\le n\le N} f(a_n^+)\varphi(a_n) - \sum_{1\le n\le N} f(a_n^-)\varphi(a_n),
\end{aligned}$$

so that we have obtained the so-called *formula of jumps*

$$f' = \sum_{0\le n\le N} \frac{df}{dx}\mathbf{1}_{[a_n,a_{n+1}]} + \sum_{1\le n\le N} \big(f(a_n^+) - f(a_n^-)\big)\delta_{a_n}, \tag{3.2.5}$$

where δ_{a_n} is the Dirac mass at a_n, defined by $\langle \delta_{a_n}, \varphi\rangle = \varphi(a_n)$.

We consider now the following determination of the logarithm given for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\xi}{\xi}, \tag{3.2.6}$$

which makes sense since $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, i.e. the segment $[1, z] \subset \mathbb{C}\backslash\mathbb{R}_-$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. Since the function Log coincides with ln on $\mathbb{R}_+^*$ and is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$, we get by analytic continuation that

$$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C}\backslash\mathbb{R}_-. \tag{3.2.7}$$

Also by analytic continuation, we have for $|\operatorname{Im} z| < \pi, \quad \operatorname{Log}(e^z) = z$. We want now to study the distributions on $\mathbb{R}$,

$$u_y(x) = \operatorname{Log}(x + iy), \quad \text{where } y \neq 0 \text{ is a real parameter.}$$

We leave as an exercise for the reader to prove that

$$\lim_{y\to 0_\pm} \operatorname{Log}(x + iy) = \ln|x| \pm i\pi\big(1 - H(x)\big), \tag{3.2.8}$$

where the limits are taken in the sense of the definition 3.1.16; also the reader can check

$$\frac{1}{x \pm i0} = \operatorname{pv}\Big(\frac{1}{x}\Big) \mp i\pi\delta_0, \tag{3.2.9}$$

where we have defined

$$\langle \frac{1}{x \pm i0}, \varphi\rangle = \lim_{\epsilon\to 0_+} \int \frac{\varphi(x)}{x \pm i\epsilon} dx \tag{3.2.10}$$

(part of the exercise is to prove that these limits exist for $\varphi \in \mathscr{D}(\mathbb{R})$). We conclude that section of examples with a more general lemma on a simple ODE.

Lemma 3.2.4. *Let I be an open interval of $\mathbb{R}$. The solutions in $\mathscr{D}'(I)$ of $u' = 0$ are the constants. The solutions in $\mathscr{D}'(I)$ of $u' = f$ make a one-dimensional affine subspace of $\mathscr{D}'(I)$.*

Proof. We assume first that $f = 0$; if u is a constant, then it is of course a solution. Conversely, let us assume that $u \in \mathscr{D}'(I)$ satisfies $u' = 0$. Let $\chi_0 \in C_c^\infty(I)$ such that $\int_{\mathbb{R}} \chi_0(x)dx = 1$; then we have for any $\varphi \in C_c^\infty(I)$, with $J(\varphi) = \int_{\mathbb{R}} \varphi(x)dx$, $\psi(x) = \int_{-\infty}^x \big(\varphi(t) - J(\varphi)\chi_0(t)\big)dt$, noting that ψ belongs[2] to $C_c^\infty(I)$,

$$\langle u, \varphi - J(\varphi)\chi_0\rangle = \langle u, \psi'\rangle = -\langle u', \psi\rangle = 0,$$

[2]The function ψ is obviously smooth and if φ, χ_0 are both supported in $\{a \le x \le b\}, a, b \in I$, so is ψ, thanks to the condition $\int \chi_0 = 1$.

which gives $\langle u,\varphi\rangle = J(\varphi)\langle u,\chi_0\rangle$, i.e. $u = \langle u,\chi_0\rangle$ proving that u is indeed a constant. We have proven that the solutions $u \in \mathscr{D}'(I)$ of $u' = 0$ are simply the constants. If $f \in \mathscr{D}'(I)$, we need only to construct a solution v_0 of $v_0' = f$ and then use the previous result to obtain that the set of solutions of $u' = f$ is $v_0 + \mathbb{R}$. Let us construct such a solution v_0. For $\varphi \in \mathscr{D}(I)$, we define with the same ψ as above,

$$\langle v_0,\varphi\rangle = -\langle f,\psi\rangle. \tag{3.2.11}$$

It is a distribution since for $\operatorname{supp}\varphi$ compact $\subset I$, we define (the compact set) $K_1 = \operatorname{supp}\varphi \cup \operatorname{supp}\chi_0$, and we have

$$|\langle v_0,\varphi\rangle| = |\langle f,\psi\rangle| \le C_{K_1} \max_{0\le j\le N_{K_1}} \|\psi^{(j)}\|_{L^\infty} \le C \max_{0\le j\le (N_{K_1}-1)_+} \|\varphi^{(j)}\|_{L^\infty}.$$

Moreover the formula (3.2.11) implies the sought result

$$\langle v_0',\varphi\rangle = -\langle v_0,\varphi'\rangle = \langle f,\psi_{\varphi'}\rangle = \langle f,\varphi\rangle,$$

since $\psi_{\varphi'}(x) = \int_{-\infty}^x \big(\varphi'(t) - J(\varphi')\chi_0(t)\big)dt = \varphi(x)$ because $J(\varphi') = 0$. The proof of the lemma is complete. □

3.2.3 Product by smooth functions

We define now the product of a C^∞ (resp. C^N) function by a distribution (resp. of order N).

Definition 3.2.5. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. For $f \in C^\infty(\Omega)$, we define the product $f\cdot u$ as the distribution defined by*

$$\langle f\cdot u,\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle u, f\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}. \tag{3.2.12}$$

If u is of order N and $f \in C^N(\Omega)$, we define the product $f\cdot u$ as the distribution of order N defined by

$$\langle f\cdot u,\varphi\rangle_{\mathscr{D}'^N(\Omega),C_c^N(\Omega)} = \langle u, f\varphi\rangle_{\mathscr{D}'^N(\Omega),C_c^N(\Omega)}. \tag{3.2.13}$$

Remark 3.2.6. Since the multiplication by a $C^\infty(\Omega)$ (resp. $C^N(\Omega)$) function is a continuous linear operator from $C_c^\infty(\Omega)$ (resp. $C_c^N(\Omega)$) into itself, we get that the above formulas actually define the products as distributions on Ω with the right order (see the proposition 3.1.12). Also the product defined in the second part coincides with the first definition whenever $f \in C_c^\infty(\Omega)$ and if $u \in L^1_{\mathrm{loc}}(\Omega), f \in C^0(\Omega)$, the usual product fu coincides with the $f\cdot u$ defined here, thanks to the lemma 3.1.7.

The next theorem is providing an extension to the classical Leibniz' formula for the derivatives of a product.

Theorem 3.2.7. *Let Ω be an open set of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}^n$ be a multi-index (see (2.3.6)). Then we have*

$$\frac{\partial_x^\alpha(fu)}{\alpha!} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \frac{\partial_x^\beta(f)}{\beta!}\frac{\partial_x^\gamma(u)}{\gamma!}. \tag{3.2.14}$$

Proof. We get immediately by induction on $|\alpha|$ the formula

$$\frac{\partial_x^\alpha(fu)}{\alpha!} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma} \frac{\partial_x^\beta(f)}{\beta!}\frac{\partial_x^\gamma(u)}{\gamma!}, \quad \text{with } \sigma_{\beta,\gamma}\in\mathbb{R}_+.$$

To find the $\sigma_{\beta,\gamma}$, we choose $f(x) = e^{x\cdot\xi}, u(x) = e^{x\cdot\eta}$, with $\xi,\eta\in\mathbb{R}^n$. We find then for all $\xi,\eta\in\mathbb{R}^n$, the identity

$$\frac{(\xi+\eta)^\alpha}{\alpha!} = \frac{\partial_x^\alpha(e^{x\cdot(\xi+\eta)})}{\alpha!}_{|x=0} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma} \frac{\partial_x^\beta(e^{x\cdot\xi})}{\beta!}\frac{\partial_x^\gamma(e^{x\cdot\eta})}{\gamma!}_{|x=0} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\xi^\beta}{\beta!}\frac{\eta^\gamma}{\gamma!},$$

and the formula (2.3.7) shows that for β,γ such that $\beta+\gamma=\alpha$

$$\sigma_{\beta,\gamma} = \partial_\xi^\beta\partial_\eta^\gamma\Big(\frac{(\xi+\eta)^\alpha}{\alpha!}\Big)_{|\xi=\eta=0} = 1,$$

completing the proof of the theorem. □

Examples. Let f be a continuous function on $\mathbb{R}$ and δ_0 be the Dirac mass at 0. The product $f\cdot\delta_0$ is equal to $f(0)\delta_0$: since δ_0 is a distribution of order 0, we can multiply it by a continuous function and if $\varphi\in C_c^0(\mathbb{R})$, we have

$$\langle f\cdot\delta_0,\varphi\rangle = \langle\delta_0, f\varphi\rangle = f(0)\varphi(0) = \langle f(0)\delta_0,\varphi\rangle \Longrightarrow f\cdot\delta_0 = f(0)\delta_0. \tag{3.2.15}$$

On the other hand if $f\in C^1(\mathbb{R})$ we have

$$f\cdot\delta_0' = f(0)\delta_0' - f'(0)\delta_0, \tag{3.2.16}$$

since the Leibniz' formula (3.2.14) gives $f(0)\delta_0' = (f\cdot\delta_0)' = f'\cdot\delta_0 + f\cdot\delta_0' = f'(0)\delta_0 + f\cdot\delta_0'$. In particular $x\delta_0' = -\delta_0$.

3.2.4 Division of distribution on $\mathbb{R}$ by x^m

We want now to address the question of division of a function (or a distribution) by a polynomial; a typical example is the division of 1 by the linear function x expressed by the identity

$$x\,\mathrm{pv}(1/x) = 1 \tag{3.2.17}$$

which is an immediate consequence of (3.1.13). We note also from the previous examples that, for any constant c, we have $x\big(\mathrm{pv}(1/x)+c\delta_0\big) = 1$. The next theorem shows that $T = \mathrm{pv}(1/x) + c\delta_0$ are the only distributions solutions of the equation $xT = 1$.

Theorem 3.2.8. *Let $m\ge 1$ be an integer.*
(1) If $u\in\mathscr{D}'(\mathbb{R})$ is such that $x^m u = 0$, then $u = \sum_{0\le j<m} c_j\delta_0^{(j)}$.
(2) Let $v\in\mathscr{D}'(\mathbb{R})$; there exists $u\in\mathscr{D}'(\mathbb{R})$ such that $v = x^m u$.

Proof. Let us first prove (1). For $\varphi, \chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0 = 1$ near 0, we have

$$\varphi(x) = \underbrace{\sum_{0\le j<m} \frac{\varphi^{(j)}(0)}{j!} x^j}_{p_{\varphi,m}(x)} + \underbrace{\int_0^1 \frac{(1-t)^{m-1}}{(m-1)!} \varphi^{(m)}(tx)dt\, x^m}_{\psi_{m,\varphi}(x)}, \quad \psi_{m,\varphi} \in C^\infty(\mathbb{R}),$$

and thus, since $x^m u = 0$,

$$\begin{aligned}\langle u, \varphi\rangle = \overbrace{\langle x^m u, x^{-m}(1-\chi_0)\varphi\rangle}^{=0} + \langle u, \chi_0\varphi\rangle = \langle u, \chi_0 p_{m,\varphi}\rangle + \overbrace{\langle x^m u, \chi_0 \psi_{\varphi,m}\rangle}^{=0} \\ = \sum_{0\le j<m} \frac{\varphi^{(j)}(0)}{j!} \langle u, \chi_0\rangle = \sum_{0\le j<m} \langle c_j \delta_0^{(j)}, \varphi\rangle,\end{aligned}$$

which the sought result. To obtain (2), for $\varphi \in C_c^\infty(\mathbb{R})$, and a given $v_0 \in \mathscr{D}'(\mathbb{R})$, we define, using the above notations,

$$\langle u, \varphi\rangle = \langle v_0, \chi_0 \psi_{m,\varphi}\rangle + \langle v_0, x^{-m}(1-\chi_0)\varphi\rangle.$$

This defines obviously a distribution on $\mathbb{R}$ and $\langle x^m u, \varphi\rangle = \langle u, x^m\varphi\rangle$; for the function $\phi(x) = x^m\varphi(x)$, we have $p_{\phi,m} = 0, x^m\psi_{m,\phi}(x) = x^m\varphi(x)$, so that the smooth functions $\psi_{m,\phi} = \varphi$,

$$\langle x^m u, \varphi\rangle = \langle v_0, \chi_0\varphi\rangle + \langle v_0, x^{-m}(1-\chi_0)x^m\varphi\rangle = \langle v_0, \varphi\rangle. \qquad \square$$

3.3 Distributions with compact support

3.3.1 Identification with $\mathscr{E}'$

Let Ω be an open subset of $\mathbb{R}^n$. We have already seen that the space $C^\infty(\Omega)$ (also denoted by $\mathscr{E}(\Omega)$) is a Fréchet space. Denoting by $\mathscr{E}'(\Omega)$ the topological dual of $\mathscr{E}(\Omega)$, we can consider $T \in \mathscr{E}'(\Omega)$ as a distribution $\tilde{T}$ on Ω by defining

$$\langle \tilde{T}, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} \quad \text{(this makes sense since } \mathscr{D}(\Omega) \subset \mathscr{E}(\Omega)).$$

The linearity is obvious and the continuity of T as a linear form on the Fréchet space $\mathscr{E}(\Omega)$ implies that there exists $C > 0, N \in \mathbb{N}$, K compact subset of Ω such that

$$\forall \varphi \in \mathscr{E}(\Omega), \quad |\langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)}| \le C \sup_{|\alpha|\le N,\ x\in K} |(\partial_x^\alpha \varphi)(x)|.$$

This estimates also proves that $\tilde{T}$ belongs to $\mathscr{D}'(\Omega)$; moreover, it has compact support in the sense of the definition (3.1.8): we have $\langle \tilde{T}, \varphi\rangle = 0$ for $\varphi \in C_c^\infty(\Omega)$, $\operatorname{supp}\varphi \subset K^c$, so that $\tilde{T}_{|K^c} = 0$ and thus $\operatorname{supp}\tilde{T} \subset K$. The next theorem proves that we can identify the space $\mathscr{E}'(\Omega)$ with the distributions on Ω with compact support, denoted by $\mathscr{D}'_{\text{comp}}(\Omega)$.

Theorem 3.3.1. *Let Ω be an open subset of $\mathbb{R}^n$. The mapping $\iota : \mathscr{E}'(\Omega) \to \mathscr{D}'_{comp}(\Omega)$, defined as above by $\iota(T) = \tilde{T}$ is bijective.*

Proof. The mapping ι is linear and if $\iota(T) = 0$, we know that T vanishes on all functions of $\mathscr{D}(\Omega)$.

Lemma 3.3.2. *Let Ω be an open subset of $\mathbb{R}^n$. The space $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$.*

Proof of the lemma. We consider a sequence $(K_j)_{j\geq 1}$ of compact subsets of Ω such that the lemma 2.3.1 is satisfied. For each $j \geq 1$, we may use the lemma 3.1.3 to construct a function $\chi_j \in \mathscr{D}(\Omega)$ with $\chi_j = 1$ near K_j. For a given $\varphi \in \mathscr{E}(\Omega)$, the sequence $(\varphi\chi_j)_{j\geq 1}$ of functions in $\mathscr{D}(\Omega)$ converges in $\mathscr{E}(\Omega)$ to φ, thanks to the last property of the lemma 2.3.1, proving the lemma. □

Since T is continuous on $\mathscr{E}(\Omega)$, $\langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \lim_j \langle T, \varphi\chi_j\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega),} = 0$ since T vanishes on $\mathscr{D}(\Omega)$. Let us consider now $T \in \mathscr{D}'_{\text{comp}}(\Omega)$ with $\operatorname{supp} T = L$ (compact subset of Ω). Using the lemma 3.1.3, we consider $\chi_0 \in \mathscr{D}(\Omega)$ such that $\chi_0 = 1$ on a neighborhood of L. For $\varphi \in \mathscr{E}(\Omega)$, we define $S \in \mathscr{E}'(\Omega)$ by

$$\langle S, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \langle T, \chi_0\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \quad \text{(note that } |\langle S, \varphi\rangle| \leq C \sup_{|\alpha|\leq N,\ x\in \operatorname{supp}\chi_0} |\partial_x^\alpha \varphi|),$$

We have $\iota(S) = T$ because

$$\langle \iota(S), \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle S, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \langle T, \chi_0\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \chi_0 T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},$$

and since for $\varphi \in \mathscr{D}(\Omega)$, the function $(1 - \chi_0)\varphi$ vanishes on an open neighborhood V of L implying

$$\operatorname{supp}\big((1 - \chi_0)\varphi\big) \subset V^c \subset L^c \Longrightarrow \langle T, (1 - \chi_0)\varphi\rangle = 0,$$

so that $\iota(S) = \chi_0 T = \chi_0 T + \underbrace{(1 - \chi_0)T}_{=0} = T$. The proof of the theorem is complete.

□

Remark 3.3.3. We can then identify $\mathscr{D}'_{\text{comp}}(\Omega)$ with $\mathscr{E}'(\Omega)$, and we may note that for $T \in \mathscr{D}'_{\text{comp}}(\Omega)$ with $\operatorname{supp} T = L$, T is of finite order N, and for all neighborhoods K of L, there exists $C > 0$ such that, for all $\varphi \in \mathscr{E}(\Omega)$,

$$|\langle T, \varphi\rangle| \leq C \sup_{|\alpha|\leq N,\ x\in K} |(\partial_x^\alpha \varphi)(x)|. \tag{3.3.1}$$

In general, it is not possible to take $K = L$ in the above estimate.

3.3.2 Distributions with support at a point

The next theorem characterizes the distributions supported in $\{0\}$.

Theorem 3.3.4. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$ and let $u \in \mathscr{D}'(\Omega)$ such that $\operatorname{supp} u = \{x_0\}$. Then $u = \sum_{|\alpha|\leq N} c_\alpha \delta_{x_0}^{(\alpha)}$, where the c_α are some constants.*

Proof. Let $\varphi \in C^\infty(\Omega)$; we have for $x \in V_0 \subset$ open neighborhood of x_0 (included in Ω), N_0 the order of u,

$$\varphi(x) = \sum_{|\alpha|\le N_0} \frac{(\partial_x^\alpha \varphi)(x_0)}{\alpha!}(x-x_0)^\alpha + \underbrace{\int_0^1 \frac{(1-\theta)^{N_0}}{N_0!}\varphi^{(N_0+1)}(x_0+\theta(x-x_0))d\theta(x-x_0)^{N_0+1}}_{\psi(x),\quad \psi\in C^\infty(V_0)},$$

and thus for $\chi_0 \in C_c^\infty(V_0), \chi_0 = 1$ near x_0,

$$\langle u, \varphi\rangle = \langle u, \chi_0\varphi\rangle = \sum_{|\alpha|\le N_0} \frac{(\partial_x^\alpha \varphi)(x_0)}{\alpha!}\langle u, \chi_0(x)(x-x_0)^\alpha\rangle + \langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1}\rangle. \tag{3.3.2}$$

We have also

$$|\langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1}\rangle| \le C_0 \sup_{|\alpha|\le N_0} |\partial_x^\alpha\big(\chi_0(x)\psi(x)(x-x_0)^{N_0+1}\big)|. \tag{3.3.3}$$

We can take $\chi_0(x) = \rho(\frac{x-x_0}{\epsilon})$, where $\rho \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball B_1, $\rho = 1$ in $\frac{1}{2}B_1$ and $\epsilon > 0$. We have then

$$\chi_0(x)\psi(x)(x-x_0)^{N_0+1} = \epsilon^{N_0+1}\rho(\frac{x-x_0}{\epsilon})\psi\big(x_0+\epsilon\frac{(x-x_0)}{\epsilon}\big)\frac{(x-x_0)^{N_0+1}}{\epsilon^{N_0+1}}$$
$$= \epsilon^{N_0+1}\rho_1(\frac{x-x_0}{\epsilon})$$

with $\rho_1(t) = \rho(t)\psi(x_0+\epsilon t)t^{N_0+1}$, so that $\rho_1 \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball B_1 has all its derivatives bounded independently of ϵ. From (3.3.3), we get for all $\epsilon > 0$,

$$|\langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1}\rangle| \le C_0 \sup_{|\alpha|\le N_0} \epsilon^{N_0+1-|\alpha|}|(\partial_t^\alpha \rho_1)(\frac{x-x_0}{\epsilon})| \le C_1\epsilon,$$

which implies that the left-hand-side of (3.3.3) is zero. On the other hand, for $\chi_1 \in C_c^\infty(V_0), \chi_1 = 1$ near the support of χ_0, we have

$$\langle u, \chi_1(x)(x-x_0)^\alpha\rangle = \langle u, \underbrace{\chi_1(x)\chi_0(x)}_{=\chi_0(x)}(x-x_0)^\alpha\rangle + \langle u, \underbrace{\chi_1(x)(1-\chi_0(x))}_{\text{supported in } (\operatorname{supp} u)^c}(x-x_0)^\alpha\rangle$$
$$= \langle u, \chi_0(x)(x-x_0)^\alpha\rangle$$

so that the latter does not depend on ε for ε small enough. The result of the theorem follows from (3.3.2). □

3.4 Tensor products

Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$ and $f \in C_c^\infty(X), g \in C_c^\infty(Y)$. The tensor product $f \otimes g$ is defined by $(f\otimes g)(x,y) = f(x)g(y)$ and belongs to $C_c^\infty(X \times Y)$. Now if $T \in \mathscr{D}'(X), S \in \mathscr{D}'(Y)$, we want to define a distribution $T \otimes S \in \mathscr{D}'(X \times Y)$ such that

$$\langle T \otimes S, f \otimes g \rangle = \langle T, f \rangle \langle S, g \rangle.$$

This triggers several questions: is such a construction possible? Is the definition above sufficient to determine unambiguously the distribution $T \otimes S$? We shall answer positively to these questions, but we first address a related question of derivation of an "integral" depending on a parameter.

3.4.1 Differentiation of a duality product

Theorem 3.4.1. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, U an open subset of $\mathbb{R}^m$ and $\phi \in C^\infty(\Omega \times U)$ such that*

$$\forall t \in U, \exists V_t \in \mathscr{V}_t, \exists K_t \text{ compact subset of } \Omega, \quad \forall s \in V_t, \quad \operatorname{supp} \phi(\cdot, s) \subset K_t. \tag{3.4.1}$$

Then the function f defined on U by $f(t) = \langle u, \phi(\cdot, t) \rangle$ makes sense and belongs to $C^\infty(U)$. Moreover we have for all $\alpha \in \mathbb{N}^m$, $(\partial_t^\alpha f)(t) = \langle u, (\partial_t^\alpha \phi)(\cdot, t) \rangle$.

Proof. The function f makes sense since for all $t \in U$, the function $\phi(\cdot, t)$ belongs to $C_c^\infty(\Omega)$. Let $t_0 \in U$ and B_0 be a closed ball with center t_0 and positive radius r_0 included in V_{t_0} given by (3.4.1). For $|h| \le r_0$, we have

$$f(t_0 + h) - f(t_0) = \langle u, \underbrace{\phi(\cdot, t_0 + h) - \phi(\cdot, t_0)}_{\text{supported in } K_{t_0},} \rangle$$

and using Taylor's formula with integral remainder, we get

$$f(t_0 + h) - f(t_0) = \langle u, (\partial_t \phi)(\cdot, t_0) \rangle h + \underbrace{\langle u, \int_0^1 (1 - \theta) \overbrace{\partial_s^2 \phi(\cdot, t_0 + \theta h)}^{\text{support in } K_{t_0}} d\theta \rangle h^2}_{r(t_0, h)}.$$

We have, since $K_{t_0} \times B_0$ is a compact subset of $\Omega \times U$,

$$|r(t_0, h)| \le |h|^2 C_0 \sup_{x \in K_{t_0}, |\alpha| \le N_0} \int_0^1 (1 - \theta) |(\partial_x^\alpha \partial_s^2 \phi) \underbrace{(x, t_0 + \theta_0 h)}_{\in K_{t_0} \times B_0}| d\theta \le C_1 |h|^2,$$

proving the differentiability of f on U along with $df(t) = \langle u, \partial_t \phi(\cdot, t) \rangle$. Inductively, we get that f is smooth and the result of the theorem. □

Corollary 3.4.2. *Let X, Y be open subsets of $\mathbb{R}^n, \mathbb{R}^m$, $\phi \in C^\infty(X \times Y)$ and $u \in \mathscr{D}'(X)$.*
(1) If ϕ is compactly supported in $X \times Y$, the function ψ defined by $\psi(y) = \langle u, \phi(\cdot, y) \rangle$ belongs to $C_c^\infty(Y)$.
(2) If $u \in \mathscr{E}'(X)$, the function ψ defined by $\psi(y) = \langle u, \phi(\cdot, y) \rangle$ belongs to $C^\infty(Y)$.

Proof. To prove (1), we need only to verify (3.4.1): we have indeed for all $y \in Y$

$$\operatorname{supp} \phi(\cdot, y) \subset proj_X(\operatorname{supp} \phi) \quad \text{which is a compact subset of } X,$$

which implies that $\psi \in C^\infty(Y)$; moreover the function $\phi(\cdot, y) = 0$ on the open subset of Y, $\big(proj_Y(\operatorname{supp} \phi)\big)^c$, and thus $\operatorname{supp} \psi \subset proj_Y(\operatorname{supp} \phi)$ which is a compact subset of Y. To obtain (2), we consider $\chi \in C_c^\infty(X)$ equal to 1 near the compact support of u. We have then $u = \chi u$ and consequently,

$$\langle u, \phi(\cdot, y)\rangle = \langle u, \phi(\cdot, y)\chi(\cdot)\rangle.$$

The function $\Phi(x, y) = \phi(x, y)\chi(x)$ is smooth on $X \times Y$ and $\operatorname{supp} \Phi(\cdot, y) \subset \operatorname{supp} \chi$ so that we can apply the theorem 3.4.1 whose assumptions are satisfied. □

3.4.2 Pull-back by the affine group

Let us now recall the definition of the affine group of $\mathbb{R}^n$: it is the group of mappings from $\mathbb{R}^n$ into itself of the form $x \mapsto Ax + t = \theta_{A,t}(x)$ where $A \in Gl(n, \mathbb{R})(n \times n$ invertible matrices) and $t \in \mathbb{R}^n$. When A is the identity, $\Theta_{\mathrm{Id},t}$ is simply the translation of vector t; we have also $\theta_{A,t}^{-1} = \Theta_{A^{-1},-A^{-1}t}$. If u belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Theta_{A,t}$ is in the affine group of $\mathbb{R}^n$, we can define the *pull-back* of u by the map Θ by the identity

$$\Theta_{A,t}^* u = u \circ \Theta_{A,t}, \quad \text{so that } (\Theta_{A,t}^* u)(x) = u(Ax + t). \tag{3.4.2}$$

As a result for $\varphi \in C_c^0(\mathbb{R}^n)$, we find

$$\langle \Theta_{A,t}^* u, \varphi\rangle = \int_{\mathbb{R}^n} u(Ax + t)\varphi(x)dx = \int_{\mathbb{R}^n} u(y)\varphi(A^{-1}y - A^{-1}t)|\det A|^{-1}dy. \tag{3.4.3}$$

We want to use that formula to define the pull-back of a distribution on $\mathbb{R}^n$ by an affine transformation.

Definition 3.4.3. *Let $A \in Gl(n, \mathbb{R}), t \in \mathbb{R}^n$, $\Theta_{A,t}$ the affine transformation defined above and let $u \in \mathscr{D}'(\mathbb{R}^n)$. We define the distribution $\Theta_{A,t}^* u$ by the identity*

$$\langle \Theta_{A,t}^* u, \varphi\rangle = \langle u, \varphi \circ \Theta_{A,t}^{-1}\rangle |\det A|^{-1}. \tag{3.4.4}$$

Remark 3.4.4. (1) Note that this defines a distribution on $\mathbb{R}^n$, since the mapping $\varphi \mapsto \varphi \circ \Theta_{A,t}^{-1}$ is an isomorphism of $\mathscr{D}(\mathbb{R}^n)$. Moreover, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the previous definition ensures that $\Theta_{A,t}^* u = u \circ \Theta_{A,t}$, thanks to the lemma 3.1.7.
(2) The mapping $u \mapsto \Theta_{A,t}^* u$ is sequentially continuous from $\mathscr{D}'(\mathbb{R}^n)$ into itself.
(3) A distribution u on $\mathbb{R}^n$ is even (resp. odd) if $\Theta_{-\mathrm{Id},0}^* u = u$ (resp. $-u$). Using the notation

$$\check{u} = \Theta_{-\mathrm{Id},0}^* u \quad (\text{for a function } u,\ \check{u}(x) = u(-x)), \tag{3.4.5}$$

u is even means $\check{u} = u$, odd means $\check{u} = -u$.

3.4.3 Homogeneous distributions

Definition 3.4.5. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution u is said to be homogeneous with degree λ if for all $t > 0$, $u(t\cdot) = t^\lambda u(\cdot)$ (here $u(t\cdot) = \theta^*_{t\,\mathrm{Id},0} u$).*

Proposition 3.4.6. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution u is homogeneous of degree λ if and only if the Euler equation is satisfied, namely*

$$\sum_{1\le j\le n} x_j \partial_{x_j} u = \lambda u. \tag{3.4.6}$$

Proof. A distribution u on $\mathbb{R}^n$ is homogeneous of degree λ means:

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \forall t > 0, \qquad \langle u(y), \varphi(y/t)t^{-n}\rangle = t^\lambda \langle u(x), \varphi(x)\rangle,$$

which is equivalent to $\forall \varphi \in C_c^\infty(\mathbb{R}^n), \forall s > 0, \langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle$, also equivalent to

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \qquad \frac{d}{ds}\big(\langle u(y), \varphi(sy)s^{n+\lambda}\rangle\big) = 0 \quad \text{on } s > 0. \tag{3.4.7}$$

Note that the differentiability property is due to the theorem 3.4.1 and that

$$\langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle \quad \text{at } s = 1.$$

As a consequence, applying the theorem 3.4.1, we get that the homogeneity of degree λ of u is equivalent to

$$\forall s > 0, \quad \langle u(y), s^{n+\lambda-1}\big((n+\lambda)\varphi(sy) + \sum_{1\le j\le n} (\partial_j\varphi)(sy)sy_j\big)\rangle = 0,$$

also equivalent to $0 = \langle u(y), (n+\lambda+\sum_{1\le j\le n} y_j\partial_j)\big(\varphi(sy)\big)\rangle$ and by the definition of the differentiation of a distribution, it is equivalent to $(n+\lambda)u - \sum_{1\le j\le n} \partial_j(y_j u) = 0$, which is (3.4.6) by the Leibniz rule (3.2.14). □

Remark 3.4.7. (1) The Dirac mass at 0 in $\mathbb{R}^n$ is homogeneous of degree $-n$: we have for $t > 0$

$$\langle \delta_0(tx), \varphi(x)\rangle = \langle \delta_0(y), \varphi(y/t)t^{-n}\rangle = t^{-n}\varphi(0) = t^{-n}\langle \delta_0, \varphi\rangle.$$

(2) If T is an homogeneous distribution of degree λ, then $\partial_x^\alpha T$ is also homogeneous with degree $\lambda - |\alpha|$: taking the derivative of the Euler equation (3.4.6), we get

$$\partial_{x_k} u + \sum_{1\le j\le k} x_j \partial_{x_j}\partial_{x_k} u - \lambda \partial_{x_k} u = 0,$$

proving that $\partial_{x_k} u$ is homogeneous of degree $\lambda - 1$ and the result by iteration.
(3) It follows immediately from the definition (3.1.13) that the distribution $\mathrm{pv}(\frac{1}{x})$ is homogeneous of degree -1. The same is true for the distributions $\frac{1}{x\pm i0}$ as it is clear from (3.2.9)and (3.2.10).

(4) For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -1$ we define the $L^1_{\rm loc}(\mathbb{R})$ functions

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \le 0., \end{cases} \quad \chi_+^\lambda = \frac{x_+^\lambda}{\Gamma(\lambda+1)}. \tag{3.4.8}$$

The distributions χ_+^λ and x_+^λ are homogeneous of degree λ and by an analytic continuation argument, we can prove that χ_+^λ may be defined for any $\lambda \in \mathbb{C}$, is an homogeneous distribution of degree λ and satisfies

$$\chi_+^\lambda = (\frac{d}{dx})^k(\chi_+^{\lambda+k}), \quad \chi_+^{-k} = \delta_0^{(k-1)}, \quad k \in \mathbb{N}^*.$$

Lemma 3.4.8. *Let $(u_j)_{1\le j\le m}$ be non-zero homogeneous distributions on $\mathbb{R}^n$ with distinct degrees $(\lambda_j)_{1\le j\le m}$ ($j \ne k$ implies $\lambda_j \ne \lambda_k$). Then they are independent in the complex vector space $\mathscr{D}'(\mathbb{R}^n)$.*

Proof. We assume that $m \ge 2$ and that there exists some complex numbers $(c_j)_{1\le j\le m}$ such that $\sum_{1\le j\le m} c_j u_j = 0$. Then applying the operator $\mathcal{E} = \sum_{1\le j\le m} x_j \partial_{x_j}$, we get for all $k \in \mathbb{N}$,

$$0 = \sum_{1\le j\le m} c_j \mathcal{E}^k(u_j) = \sum_{1\le j\le m} c_j \lambda_j^k u_j.$$

We consider now the Vandermonde matrix $m \times m$

$$V_m = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \dots. & & & \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{pmatrix}, \quad \det V_m = \prod_{1\le j<k\le m} (\lambda_k - \lambda_j) \ne 0.$$

We note that for $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $X \in \mathbb{C}^m$ given by

$$X = \begin{pmatrix} c_1\langle u_1, \varphi\rangle \\ c_2\langle u_2, \varphi\rangle \\ \dots\dots\dots \\ c_m\langle u_m, \varphi\rangle \end{pmatrix},$$

we have $V_m X = 0$, so that $X = 0$, i.e. $\forall j, \forall \varphi \in C_c^\infty(\mathbb{R}^n), c_j\langle u_j, \varphi\rangle = 0$, i.e. $c_j u_j = 0$ and since u_j is not the zero distribution, we get the sought conclusion $c_j = 0$ for all j. □

3.4.4 Tensor products of distributions

We begin with a lemma.

Lemma 3.4.9. *Let $\phi \in C_c^\infty(]0,1[^n)$; one can find a sequence of functions in*

$$\operatorname{Vect}(\otimes^n C_c^\infty(]0,1[) \quad \textit{(the vector space generated by the tensor products)}$$

converging to ϕ in $C_c^\infty(]0,1[^n)$ in the sense of the definition 3.1.9.

Proof. We define for $k \in \mathbb{Z}^n$, $\hat\phi(k) = \int e^{-2i\pi x\cdot k}\phi(x)dx$, and we note that, with $\Delta = \sum_{1\le j\le n}\partial_{x_j}^2$, $m \in \mathbb{N}$,

$$\hat\phi(k) = (1+|k|^2)^{-m}\int (1-\frac{1}{4\pi^2}\Delta)^m\big(e^{-2i\pi x\cdot k}\big)\phi(x)dx$$
$$= (1+|k|^2)^{-m}\int e^{-2i\pi x\cdot k}\big((1-\frac{1}{4\pi^2}\Delta)^m\phi\big)(x)dx$$

so that

$$|\hat\phi(k)| \le (1+|k|^2)^{-m}C_m \max_{|\alpha|\le 2m}\|\partial_x^\alpha\phi\|_{L^\infty}. \tag{3.4.9}$$

As a result the series $\Phi(x) = \sum_{k\in\mathbb{Z}^n}\hat\phi(k)e^{2i\pi x\cdot k}$ converges and is a smooth function, periodic with periods $\mathbb{Z}^n$: we need only to check that $\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-1} < +\infty$.[3] Moreover,

$$\text{for } x\in[0,1]^n, \qquad \Phi(x) = \phi(x). \tag{3.4.10}$$

We verify this first for $n = 1$. We have in that case

$$\Phi(x) = \lim_{N\to+\infty}\int\sum_{|k|\le N}e^{2i\pi k(x-y)}\phi(y)dy,$$

and since $\displaystyle\sum_{|k|\le N}e^{2i\pi kt} = 1+2\operatorname{Re}\sum_{1\le k\le N}e^{2i\pi kt} = 1+2\operatorname{Re}\big(e^{2i\pi t}\frac{e^{2i\pi Nt}-1}{e^{2i\pi t}-1}\big)$

$$= 1+2\operatorname{Re}\big(e^{i\pi(N+1)t}\frac{\sin(\pi Nt)}{\sin(\pi t)}\big) = \frac{\sin(\pi t(2N+1))}{\sin(\pi t)},$$

we get that, since $\phi\in C_c^\infty(]0,1[)$, and for $x\in]0,1[$,

$$\begin{aligned}\Phi(x) &= \lim_{N\to+\infty}\int\frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\phi(y)dy\\ &= \lim_{N\to+\infty}\Big(\int_0^1\frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\big(\phi(y)-\phi(x)\big)dy+\phi(x)\int_0^1\sum_{|k|\le N}e^{2i\pi k(x-y)}dy\Big)\\ &= \phi(x),\end{aligned}$$

because with $\psi\in C^\infty(\mathbb{R}^2)$, $\theta(s) = \frac{s}{\sin\pi s}$ (which is in $C^\infty(\mathbb{R}\backslash\pi\mathbb{Z}^*)$ and in particular on $]-1,+1[$), we have

$$\int_0^1\frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\big(\phi(y)-\phi(x)\big)dy$$
$$= \int_0^1\sin\big(\pi(x-y)(2N+1)\big)\overbrace{\psi(x,y)\theta(x-y)}^{\substack{\text{smooth of } y \text{ on } [0,1]\\ \text{since } x\in]0,1[}}dy \underset{N\to+\infty}{\longrightarrow} 0,$$

[3] In fact, with $Q_k = k+(0,1)^n$ we have, replacing the Euclidean norm $|k|$ by the (equivalent) sup-norm $\|k\| = \max_{1\le j\le k}|k_j|$, we have for $x\in Q_k$, $k_j < x_j < k_j+1$ and thus

$$\|x\| = \max|x_j| \le 1+\|k\| \Longrightarrow 1+\|x\|\le 2+\|k\|$$

and $\sum_{k\in\mathbb{Z}^n}(2+\|k\|)^{-n-1}\le\int\sum_{k\in\mathbb{Z}^n}\mathbf{1}_{Q_k}(x)(1+\|x\|)^{-n-1}dx = \int(1+\|x\|)^{-n-1}dx < +\infty$.

since with $\omega \in C^\infty([0,1])$, we have $\int_0^1 \sin\big(\pi(x-y)(2N+1)\big)\omega(y)dy =$

$$\Big[\frac{\cos\big(\pi(x-y)(2N+1)\big)}{\pi(2N+1)}\omega(y)\Big]_{y=0}^{y=1} - \int_0^1 \frac{\cos\big(\pi(x-y)(2N+1)\big)}{\pi(2N+1)}\omega'(y)dy.$$

We have proven (3.4.10) for $n = 1$ and $x \in]0,1[$. Since Φ, ϕ are both smooth on $[0,1]$ the equality holds as well for $x \in \{0,1\}$.

N.B. We could have used the Riemann-Lebesgue lemma (see e.g. the lemma 3.4.4 in [9]), but we have preferred a simple self-contained argument with an integration by parts since there was no shortage of regularity for the function ω.

To handle the case $n \geq 2$, we use an induction and in $n+1$ dimensions, we have for $\phi \in C_c^\infty(]0,1[^{n+1})$,

$$\forall x \in [0,1]^n, \quad \Phi(x, x_{n+1}) = \sum_{k\in\mathbb{Z}^n} \int_{(0,1)^n} e^{2i\pi(x-y)\cdot k}\phi(y, x_{n+1})dy = \phi(x, x_{n+1}),$$

and thus $\forall x \in [0,1]^n, \forall x_{n+1} \in [0,1], \Phi(x, x_{n+1}) =$

$$\sum_{k\in\mathbb{Z}^n} \int_{(0,1)^n} e^{2i\pi(x-y)\cdot k}\Big(\sum_{k_{n+1}\in\mathbb{Z}} \int_0^1 e^{2i\pi(x_{n+1}-y_{n+1})k_{n+1}}\phi(y, y_{n+1})dy_{n+1}\Big)dy = \phi(x, x_{n+1}),$$

which is (3.4.10) since the series are uniformly converging. Since $\operatorname{supp}\phi \subset]0,1[^n$, there exists $\epsilon_0 > 0$ such that[4] $\operatorname{supp}\phi \subset [\epsilon_0, 1-\epsilon_0]^n$, and with $\chi \in C_c^\infty(]0,1[)$ equal to 1 on $[\epsilon_0, 1-\epsilon_0]$, we have

$$\chi(x_1)\dots\chi(x_n)\phi(x) = \phi(x) = \sum_{k\in\mathbb{Z}^n} e^{2i\pi x\cdot k}\hat{\phi}(k)\chi(x_1)\dots\chi(x_n). \tag{3.4.11}$$

The series is uniformly converging as well as all its derivatives, thanks to the fast decay of $\hat{\phi}(k)$ expressed by (3.4.9), and the functions

$$\sum_{|k|\leq N} e^{2i\pi x_1 k_1}\dots e^{2i\pi x_n k_n}\hat{\phi}(k)\chi(x_1)\dots\chi(x_n)$$

belong to $\operatorname{Vect}(\otimes^n C_c^\infty(]0,1[)$ with fixed compact support in $]0,1[^n$. The proof of the lemma is complete. □

As a consequence, we get the following result.

Proposition 3.4.10. *Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$. $\operatorname{Vect} C_c^\infty(X) \otimes C_c^\infty(Y)$ is dense in $C_c^\infty(X \times Y)$.*

[4]In fact, each projection $K_j = proj_j(\operatorname{supp}\phi)$ is a compact subset of $]0,1[$, thus $0 < \inf_{t\in K_j} t \leq \sup_{t\in K_j} t < 1$.

Proof. Let K be a compact subset of $X\times Y$. For each point $(x,y)\in K$, we can find some open bounded intervals $I_1,\dots,I_m,J_1,\dots,J_n$ of $\mathbb R$ such that

$$(x,y)\in Q=I_1\times\dots\times I_m\times J_1\times\dots\times J_n\subset X\times Y.$$

As a result, we can cover K with a finite number of open "cubes" $(Q_l)_{1\le l\le N}$ included in $X\times Y$. Using a partition of unity given by the theorem 3.1.14, we can find $\psi_l\in C_c^\infty(Q_l)$ such that $\sum_{1\le l\le N}\psi_l(x)=1$ for $x\in V$ open such that $K\subset V\subset X\times Y$. For $\varphi\in C_c^\infty(X\times Y)$, $\operatorname{supp}\varphi=K$ compact subset of $X\times Y$, we have

$$\varphi=\sum_{1\le l\le N}\varphi\psi_l,\quad \varphi\psi_l\in C_c^\infty(Q_l).$$

We can then apply the lemma 3.4.9 for each $\varphi\psi_l$ (rescaling the cube Q_l to $]0,1[^n$) to obtain the conclusion of the proposition. □

Theorem 3.4.11. *Let X be an open subset of $\mathbb R^m$, Y be an open subset of $\mathbb R^n$, and $u\in\mathscr D'(X),v\in\mathscr D'(Y)$. Then there exists a unique $w\in\mathscr D'(X\times Y)$ such that, $\forall\phi\in\mathscr D(X),\forall\psi\in\mathscr D(Y)$,*

$$\langle w,\phi\otimes\psi\rangle_{\mathscr D'(X\times Y),\mathscr D(X\times Y)}=\langle u,\phi\rangle_{\mathscr D'(X),\mathscr D(X)}\langle v,\psi\rangle_{\mathscr D'(Y),\mathscr D(Y)},\tag{3.4.12}$$

where $(\phi\otimes\psi)(x,y)=\phi(x)\psi(y)$. We shall denote w by $u\otimes v$ and call it the tensor product of u and v.

Proof. The uniqueness follows from the proposition 3.4.10. To find such a w, we define for $\Phi\in C_c^\infty(X\times Y)$, with obvious notations,

$$\langle w,\Phi\rangle=\big\langle v(y),\langle u(x),\Phi(x,y)\rangle\big\rangle.\tag{3.4.13}$$

As a matter of fact, thanks to the corollary 3.4.2 (1), the function $Y\ni y\mapsto\langle u(\cdot),\Phi(\cdot,y)\rangle$ belongs to $C_c^\infty(Y)$ so that (3.4.13) makes sense. Using the theorem 3.4.1, we obtain $\partial_y^\alpha\langle u(\cdot),\Phi(\cdot,y)\rangle=\langle u(\cdot),\partial_y^\alpha\Phi(\cdot,y)\rangle$. If $K=\operatorname{supp}\Phi$ (compact subset of $X\times Y$), both projections $proj_XK,proj_YK$ are compact so that

$$|\langle u(\cdot),\partial_y^\alpha\Phi(\cdot,y)\rangle|\le C_1\sup_{|\beta|\le N_1,\ x\in proj_XK}|(\partial_x^\beta\partial_y^\alpha\Phi)(x,y)|$$

and thus

$$\begin{aligned}|\big\langle v(y),\langle u(x),\Phi(x,y)\rangle\big\rangle|&\le C_2\sup_{\substack{|\alpha|\le N_2\\ y\in proj_YK}}|\partial_y^\alpha\langle u(\cdot),\Phi(\cdot,y)\rangle|\\&\le C_1C_2\sup_{\substack{|\beta|\le N_1,|\alpha|\le N_2\\(x,y)\in K}}|(\partial_x^\beta\partial_y^\alpha\Phi)(x,y)|,\end{aligned}$$

implying that w is indeed a distribution on $X\times Y$. Since the formula (3.4.12) follows from (3.4.13), this concludes the proof of the theorem. □

Remark 3.4.12. (1) The uniqueness ensures that $w = u \otimes v$ is also defined by

$$\langle w, \Phi \rangle = \big\langle u(x), \langle v(y), \Phi(x,y) \rangle \big\rangle, \tag{3.4.14}$$

a formula for which (3.4.12) also holds.
(2) If $u \in L^1_{\text{loc}}(X), v \in L^1_{\text{loc}}(Y)$, then $u \otimes v$ belongs to $L^1_{\text{loc}}(X \times Y)$ and is defined by $u(x)v(y)$, thanks to the lemma 3.1.7 and to the proposition 3.4.10.
(3) For $u \in \mathscr{D}'(X), v \in \mathscr{D}'(Y)$, we have

$$\operatorname{supp}(u \otimes v) = \operatorname{supp} u \times \operatorname{supp} v. \tag{3.4.15}$$

In fact, if $\Phi \in C_c^\infty(X \times Y)$ with $\operatorname{supp} \Phi \subset X \times (\operatorname{supp} v)^c$ or with $\operatorname{supp} \Phi \subset (\operatorname{supp} u)^c \times Y$, it follows from (3.4.14) or (3.4.13) that $\langle u \otimes v, \Phi \rangle = 0$; this holds as well when

$$\operatorname{supp} \Phi \subset (\operatorname{supp} u \times \operatorname{supp} v)^c = \big((\operatorname{supp} u)^c \times Y\big) \cup \big(X \times (\operatorname{supp} v)^c\big),$$

since $\operatorname{supp} \Phi \subset \Omega_1 \cup \Omega_2$ with Ω_j open subset of $X \times Y$ and, thanks to the theorem 3.1.14, the compactly supported $\Phi = \Phi_1 + \Phi_2$, with $\operatorname{supp} \Phi_j \subset \Omega_j$ (it is also a direct consequence of the theorem 3.1.15 since $(u \otimes v)_{|\Omega_j} = 0$). We have proven that $\operatorname{supp}(u \otimes v) \subset \operatorname{supp} u \times \operatorname{supp} v$. Conversely, if $x_0 \in \operatorname{supp} u, y_0 \in \operatorname{supp} v$, and U, V are respective open neighborhoods of x_0, y_0 in X, Y, we can find $\phi_0 \in C_c^\infty(U), \psi_0 \in C_c^\infty(V)$ such that $\langle u, \phi_0 \rangle \neq 0$ and $\langle v, \psi_0 \rangle \neq 0$. As a result $\phi_0 \otimes \psi_0 \in C_c^\infty(U \times V)$ and $\langle u \otimes v, \phi_0 \otimes \psi_0 \rangle = \langle u, \phi_0 \rangle \langle v, \psi_0 \rangle \neq 0$, so that $(u \otimes v)_{|U \times V}$ is not zero, proving that $(x_0, y_0) \in \operatorname{supp}(u \otimes v)$ and the sought result.
(4) With the notations of the previous theorem, we have obviously from the expression (3.4.13) and the theorem 3.4.1 that $\partial_x^\alpha \partial_y^\beta (u \otimes v) = (\partial_x^\alpha u) \otimes (\partial_y^\beta v)$.

Proposition 3.4.13. *Let $n \in \mathbb{N}^*$, U be an open subset of $\mathbb{R}^{n-1}$, I an interval of $\mathbb{R}$. Let $u \in \mathscr{D}'(U \times I)$ such that $\partial_{x_n} u = 0$. Then, there exists $v \in \mathscr{D}'(U)$ such that $u = v \otimes 1$. In other words, the differential equation $\partial_{x_n} u = 0$ has the only solutions $u(x', x_n) = v(x')$.*

Proof. From the remark 3.4.12 (3) above, the tensor products $v(x') \otimes 1$ are indeed solutions of $\partial_{x_n} u = 0$. Conversely the proposition is proven for $n = 1$ by the lemma 3.2.4. Let us assume $n \geq 2$; we consider $\chi_0 \in C_c^\infty(I)$ such that $\int \chi_0(t) dt = 1$ and we define $v \in \mathscr{D}'(U)$ by the identity

$$\langle v, \varphi \rangle_{\mathscr{D}'(U), \mathscr{D}(U)} = \langle u, \varphi \otimes \chi_0 \rangle_{\mathscr{D}'(U \times I), \mathscr{D}(U \times I)}.$$

For $\varphi \in \mathscr{D}(U), \psi \in \mathscr{D}(I)$, we have with $J(\psi) = \int \psi(t) dt$,

$$\langle v \otimes 1, \varphi \otimes \psi \rangle = \langle u, \varphi \otimes \chi_0 \rangle J(\psi).$$

From the proof of the lemma 3.2.4, we see that $\psi - \chi_0 J(\psi) = \theta'$ with $\theta \in C_c^\infty(I)$, and we get $\langle u, \varphi \otimes \big(\chi_0 J(\psi) - \psi\big) \rangle = \langle u, \partial_{x_n}(\varphi \otimes \theta) \rangle = 0$ so that $\langle v \otimes 1, \varphi \otimes \psi \rangle = \langle u, \varphi \otimes \psi \rangle$, which is the sought result. □

3.5 Convolution

We want to define the convolution of two distributions on $\mathbb{R}^n$, provided one of them has compact support. Assuming first that $u \in L^1_{\text{comp}}(\mathbb{R}^n), v \in L^1_{\text{loc}}(\mathbb{R}^n), \phi \in C_c^\infty(\mathbb{R}^n)$ the integral

$$\iint u(x-y)v(y)\phi(x)dxdy = \iint u(x)v(y)\phi(x+y)dxdy, \tag{3.5.1}$$

makes sense since x and $x+y$ are moving in a compact set in the last integral (and so is y). This formula allows us to define

$$(u*v)(x) = \int u(x-y)v(y)dy = \int u(y)v(x-y)dy$$

and can naturally be extended to $u, v \in L^1(\mathbb{R}^n)$ so that $\|u*v\|_{L^1(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}\|v\|_{L^1(\mathbb{R}^n)}$, making $L^1(\mathbb{R}^n)$ a Banach algebra (without unit). The inequality of Young (see e.g. the Théorème 6.2.1 in [9]) is a non-trivial extension of that inequality. Anyhow, at the moment, we want to use the formula (3.5.1) for our general definition.

3.5.1 Convolution $\mathscr{E}'(\mathbb{R}^n) * \mathscr{D}'(\mathbb{R}^n)$

Definition 3.5.1. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We define the convolution $u*v$ by the following bracket of duality*

$$\langle u*v, \phi\rangle_{\mathscr{D}'(\mathbb{R}^n),\mathscr{D}(\mathbb{R}^n)} = \big\langle u(x), \langle v(y), \phi(x+y)\rangle\big\rangle = \big\langle v(y), \langle u(x), \phi(x+y)\rangle\big\rangle. \tag{3.5.2}$$

We note that the theorem 3.4.1 shows that the function $\mathbb{R}^n \ni x \mapsto \langle v(y), \phi(x+y)\rangle$ is C^∞ and thus that the first definition makes sense from the corollary 3.4.2 (2). To check the second equality above, we note that with $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the support of u, we have $\chi u = u$ and thus from the remark 3.4.12(1) and the formula (3.4.13),

$$\big\langle u(x), \langle v(y), \phi(x+y)\rangle\big\rangle = \big\langle u(x), \langle v(y), \chi(x)\phi(x+y)\rangle\big\rangle = \langle u(x)\otimes v(y), \chi(x)\phi(x+y)\rangle,$$

which is also equal to $\big\langle v(y), \langle u(x), \chi(x)\phi(x+y)\rangle\big\rangle = \big\langle v(y), \langle u(x), \phi(x+y)\rangle\big\rangle$. This proves as well that $u*v$ is a distribution on $\mathbb{R}^n$ since the mapping $C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \Phi \in C_c^\infty(\mathbb{R}^{2n})$, with $\Phi(x,y) = \phi(x+y)\chi(x)$ is continuous.

Remark 3.5.2. We note that whatever is $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the support of u, we have for $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$,

$$\langle u*v, \phi\rangle = \langle u(x)\otimes v(y), \chi(x)\phi(x+y)\rangle. \tag{3.5.3}$$

Proposition 3.5.3. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We have*

$$\operatorname{supp}(u*v) \subset \operatorname{supp} u + \operatorname{supp} v. \tag{3.5.4}$$

Proof. Note first that $\operatorname{supp} u + \operatorname{supp} v$ is a closed subset of $\mathbb{R}^n$ as the sum of a compact set and a closed set (exercise). Now if $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset (\operatorname{supp} u + \operatorname{supp} v)^c$, then

$$\operatorname{supp}\big((x,y) \mapsto \phi(x+y)\big) \subset (\operatorname{supp} u \times \operatorname{supp} v)^c. \tag{3.5.5}$$

In fact, if $(x_0, y_0) \in \operatorname{supp} u \times \operatorname{supp} v$, then $x_0 + y_0 \in \operatorname{supp} u + \operatorname{supp} v \subset (\operatorname{supp} \phi)^c$, the latter being open so that there exists U open in $\mathscr{V}_0$ with $\phi(x_0 + U + y_0 + U) = 0$. As a consequence, the open set $(x_0 + U) \times (y_0 + U) \subset \big(\operatorname{supp}((x,y) \mapsto \phi(x+y))\big)^c$ and this implies $(x_0, y_0) \in \big(\operatorname{supp}((x,y) \mapsto \phi(x+y))\big)^c$ and proves (3.5.5), so that (3.5.3), (3.4.15) give the conclusion of the proposition. □

Remark 3.5.4. For u, v both in $\mathscr{E}'(\mathbb{R}^n)$, the formula (3.5.2) ensures that $u * v = v * u$.

3.5.2 Regularization

Proposition 3.5.5. *Let $u \in \mathscr{D}'(\mathbb{R}^n), \rho \in C_c^\infty(\mathbb{R}^n)$. Then $\rho * u$ belongs to $C^\infty(\mathbb{R}^n)$.*

Proof. We have from the definitions, with $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near $\operatorname{supp} \rho$, $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \rho * u, \phi \rangle = \langle \rho(x) \otimes u(y), \chi(x)\phi(x+y) \rangle = \langle u(y), \langle \rho(x), \chi(x)\phi(x+y) \rangle \rangle, \tag{3.5.6}$$

and we note that $\langle \rho(x), \chi(x)\phi(x+y) \rangle = \int \rho(x)\phi(x+y)dx = \int \rho(x-y)\phi(x)dx$. As a result, we have

$$\langle \rho * u, \phi \rangle = \langle u(y), \int \underbrace{\rho(x-y)\phi(x)}_{\in C_c^\infty(\mathbb{R}^{2n})} dx \rangle = \int \phi(x) \langle u(y), \rho(x-y) \rangle dx$$

where the last equality is due to the theorem 3.4.1[5] which gives also that $\psi(x) = \langle u(y), \rho(x-y) \rangle$ is C^∞; we have proven $\rho * u = \psi$ and the result. We note also the formula following from (3.5.6)

$$\langle \rho * u, \phi \rangle = \langle u, \check{\rho} * \phi \rangle. \tag{3.5.7}$$

□

Lemma 3.5.6. *Let Ω be an open subset of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. There exists a sequence $(\psi_j)_{j \ge 1}$ in $\mathscr{D}(\Omega)$ such that $\lim_j \psi_j = T$ in the weak-dual topology sense of the definition 3.1.16.*

Proof. We consider first a sequence $(K_j)_{j \ge 1}$ of compact subsets of Ω as in the lemma 2.3.1 and a sequence $(\chi_j)_{j \ge 1}$ such that $\chi_j \in C_c^\infty(\operatorname{int} K_{j+1}), \chi_j = 1$ near K_j (see the lemma 3.1.3). In the weak-dual topology sense, we have $\lim_j \chi_j T = T$: let $\varphi \in \mathscr{D}(\Omega)$, $K = \operatorname{supp} \varphi$. From the lemma 2.3.1, there exists j such that $\operatorname{supp} \varphi \subset K_j$ and thus $\varphi \chi_j = \varphi$, implying $\langle T\chi_j, \varphi \rangle = \langle T, \chi_j \varphi \rangle = \langle T, \varphi \rangle$. We can also consider the compactly supported distribution $\chi_j T$ and see it as a distribution on $\mathbb{R}^n$. We take now a function $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\int \rho(x)dx = 1$. According to the first example

[5] For $\Phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathscr{D}'(\mathbb{R}^n)$, $\langle 1 \otimes u, \Phi \rangle = \langle u(y), \int \Phi(x,y)dx \rangle = \int \langle u(y), \Phi(x,y) \rangle dx$.

in the section 3.1.3, we define ρ_ϵ (it tends to the Dirac mass at 0 in the weak-dual topology when $\epsilon \to 0_+$). For $\varphi \in \mathscr{D}(\Omega)$, using (3.5.7), we have

$$\langle \rho_\epsilon * (\chi_j T), \varphi \rangle = \langle \chi_j T, \check{\rho}_\epsilon * \varphi \rangle. \tag{3.5.8}$$

Considering now a decreasing sequence of positive numbers (ϵ_j) with limit 0 such that

$$\operatorname{supp} \chi_j + \epsilon_j \operatorname{supp} \rho \subset \operatorname{int}(K_{j+1}) \subset \Omega,$$

and we define $T_j = \rho_{\epsilon_j} * \chi_j T$. We have from the proposition 3.5.3 that $\operatorname{supp} T_j$ is compact included in Ω and also that $T_j \in C^\infty$ (proposition 3.5.5). Going back to (3.5.8), for a fixed φ, we can find j such that $\operatorname{supp} \varphi \subset K_{j-1}$ for $j \geq j_0$, implying that

$$\operatorname{supp}(\check{\rho}_{\epsilon_j} * \varphi) \subset K_{j-1} + \epsilon_j \operatorname{supp} \rho \subset \operatorname{supp} \chi_{j-1} + \epsilon_{j-1} \operatorname{supp} \rho \subset K_j,$$

implying that $\chi_j(\check{\rho}_{\epsilon_j} * \varphi) = \check{\rho}_{\epsilon_j} * \varphi$ and $\langle \rho_{\epsilon_j} * (\chi_j T), \varphi \rangle = \langle T, \check{\rho}_{\epsilon_j} * \varphi \rangle$. The result follows from the proposition 3.1.1 (implying $\lim_j (\check{\rho}_{\epsilon_j} * \varphi) = \varphi$ in $C_c^\infty(\Omega)$) and the (sequential) continuity of the distribution T. □

Proposition 3.5.7. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We have*

$$\operatorname{singsupp}(u * v) \subset \operatorname{singsupp} u + \operatorname{singsupp} v. \tag{3.5.9}$$

Proof. We can choose $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the $\operatorname{singsupp} u$, $\psi \in C^\infty$ equal to 1 near the singular support of v. We have from the proposition 3.5.5

$$u * v = (\chi u) * v + \underbrace{(\overbrace{(1-\chi)u}^{\in C_c^\infty(\mathbb{R}^n)}) * v}_{\in C^\infty(\mathbb{R}^n)} \equiv (\chi u) * (\psi v) + \underbrace{\overbrace{(\chi u)}^{\in \mathscr{E}'(\mathbb{R}^n)} * (\overbrace{(1-\psi)v}^{\in C^\infty(\mathbb{R}^n)})}_{\in C^\infty(\mathbb{R}^n)} \mod C^\infty(\mathbb{R}^n)$$

and thus we get for all $\epsilon > 0$

$$\operatorname{singsupp}(u * v) \subset \operatorname{supp} \psi + \operatorname{supp} \psi \subset \operatorname{singsupp} u + \epsilon \bar{B}_1 + \operatorname{singsupp} v + \epsilon \bar{B}_1,$$

which gives the result. □

3.5.3 Convolution with a proper support condition

Looking at the formula (3.5.1), we see that we can extend it easily for $L^1_{\text{loc}}(\mathbb{R}^n)$ functions u, v so that the mapping

$$\operatorname{supp} u \times \operatorname{supp} v \ni (x, y) \mapsto x + y = \sigma(x, y) \in \mathbb{R}^n \tag{3.5.10}$$

is *proper*, i.e. such that $\sigma^{-1}(K)$ is compact for K compact subset of $\mathbb{R}^n$. In fact if $u, v \in L^1_{\text{loc}}(\mathbb{R}^n)$ are such that the map σ of (3.5.10) is proper, the function $u * v$ defined by

$$(u * v)(x) = \int u(x - y) v(y) dy$$

is also $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, since for K compact subset of $\mathbb{R}^n$, we have

$$\iint |u(x-y)||v(y)|\mathbf{1}_K(x)dydx = \iint |u(x)||v(y)|\mathbf{1}_K(x+y)dxdy$$
$$= \iint_{\sigma^{-1}(K)} |u(x)||v(y)|dxdy < \infty, \quad \text{since } \sigma^{-1}(K) \text{ is compact in } \mathbb{R}^{2n}.$$

We can extend as well the convolution product of distributions u, v, provided σ in (3.5.10) is proper. Before doing so, we prove a simple lemma.

Lemma 3.5.8. *Let $F_1, \dots, F_m$ be closed subsets of $\mathbb{R}^n$ such that the mapping $\sigma : F_1 \times \cdots \times F_m \to \mathbb{R}^n$, defined by $\sigma(x_1, \dots, x_m) = x_1 + \cdots + x_m$ is proper. Defining for $\epsilon > 0$, $F_{j,\epsilon} = \{x \in \mathbb{R}^n, |x - F_j| \le \epsilon\}$, the mapping $\sigma_\epsilon : F_{1,\epsilon} \times \cdots \times F_{m,\epsilon} \to \mathbb{R}^n$, defined by $\sigma_\epsilon(x_1, \dots, x_m) = x_1 + \cdots + x_m$ is also proper.*

Proof. We note first that $F_{j,\epsilon} = F_j + \epsilon \bar{B}_1$ ($\bar{B}_1$ is the closed unit ball of $\mathbb{R}^n$) is closed as the sum of a compact and a closed set. Let K be compact subset of $\mathbb{R}^n$; if $(x_1, \dots, x_m) \in \sigma_\epsilon^{-1}(K)$, then there exists $y_j \in F_j, t_j \in \mathbb{R}^n, |t_j| \le \epsilon$, such that $x_j = y_j + t_j$, $\sum_{1\le j\le m}(y_j + t_j) \in K$ and thus $\sum_{1\le j\le m} y_j \in K + m\epsilon \bar{B}_1$, so that $(y_j)_{1\le j\le m} \in \sigma^{-1}(K + m\epsilon\bar{B}_1)$, a compact subset of $\prod F_j$. As a consequence, $(x_j)_{1\le j\le m} \in \sigma^{-1}(K + m\epsilon\bar{B}_1) + \epsilon \bar{B}_{1,nm}$ ($\bar{B}_{1,nm}$ is the closed unit ball of $\mathbb{R}^{nm}$), which is compact. As a result, $\sigma_\epsilon^{-1}(K)$ is compact as a closed subset of $\prod F_{j,\epsilon}$ (σ_ϵ is continuous) included in a compact set. □

Definition 3.5.9. *Let $u_1, \dots, u_m \in \mathscr{D}'(\mathbb{R}^n)$ such that the mapping σ*

$$\prod_{1\le j\le m} \operatorname{supp} u_j \ni (x_j)_{1\le j\le m} \mapsto \sum_{1\le j\le m} x_j \in \mathbb{R}^n \quad \textit{is proper.} \tag{3.5.11}$$

For $\epsilon > 0$, we take $\chi_{j,\epsilon} \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \chi_{j,\epsilon} \subset \operatorname{supp} u_j + \epsilon \bar{B}_1$ and $\operatorname{supp} \chi_{j,\epsilon}$ is 1 on a neighborhood of $\operatorname{supp} u_j$. We define then

$$\langle u_1 * \cdots * u_m, \phi \rangle_{\mathscr{D}'(\mathbb{R}^n), \mathscr{D}(\mathbb{R}^n)} = \langle u_1 \otimes \cdots \otimes u_m, \tilde{\phi} \rangle_{\mathscr{D}'(\mathbb{R}^{nm}), \mathscr{D}(\mathbb{R}^{nm})} \tag{3.5.12}$$

with $\tilde{\phi}(x_1, \dots, x_m) = \prod_{1\le j\le m} \chi_{j,\epsilon}(x_j)\phi(\sum_{1\le j\le m} x_j)$: we note that $\tilde{\phi}$ is in $\mathscr{D}(\mathbb{R}^{nm})$ since

$$\operatorname{supp} \tilde{\phi} \subset \{(x_j)_{1\le j\le m} \in \prod_{1\le j\le m} \operatorname{supp} \chi_{j,\epsilon} \textit{ with } \sigma((x_j)) \in \operatorname{supp} \phi\}$$

which is compact from the previous lemma and (3.5.11).

It is also easy to prove that this definition does not depend on the choices of the functions $\chi_{j,\epsilon}$ having the properties listed above and that this definition coincides with the definition of convolution in the previous section. In particular, we can prove the associativity of the convolution using the identity (3.5.12), *provided the condition (3.5.11) is satisfied.* As a counterexample we can take $u_1 = 1, u_2 = \delta_0', u_3 = H$ and we have since $1 * \delta_0' = 0, \delta_0' * H = \delta_0$,

$$(u_1 * u_2) * u_3 = 0, \quad u_1 * (u_2 * u_3) = 1 * \delta_0 = 1.$$

Naturally the hypothesis (3.5.11) is violated here since the mapping σ defined on $\mathbb{R} \times \{0\} \times \mathbb{R}_+$ is not proper: $\sigma^{-1}(\{0\}) \supset \{(-N, 0, N)\}_{N\in\mathbb{N}}$. The assumption (3.5.11) is satisfied for $m = 2$ if $\operatorname{supp} u_1$ is compact and also for distributions on $\mathbb{R}$ with support in $\mathbb{R}_+$. We get also that

$$\forall u \in \mathscr{D}'(\mathbb{R}^n), \quad u * \delta = u, \quad \text{since } \langle u(x_1) \otimes \delta(x_2), \phi(x_1 + x_2)\rangle = \langle u, \phi\rangle. \tag{3.5.13}$$

and for $u, v \in \mathscr{D}'(\mathbb{R}^n)$ such that (3.5.11) holds

$$\partial_x^\alpha(u * v) = (\partial_x^\alpha u) * v = u * (\partial_x^\alpha v), \tag{3.5.14}$$

since $\langle \partial_x^\alpha(u * v), \phi\rangle = (-1)^{|\alpha|}\langle u * v, \partial_x^\alpha \phi\rangle = (-1)^{|\alpha|}\langle u(x) \otimes v(y), (\partial^\alpha \phi)(x + y)\rangle = \langle(\partial_x^\alpha u)(x) \otimes v(y), \phi(x + y)\rangle$ and putting inside the brackets the cut-off functions χ_ϵ does not change the outcome of the computation.

3.6 Some fundamental solutions

3.6.1 Definitions

Definition 3.6.1. *We consider a constant coefficients differential operator*

$$P = P(D) = \sum_{|\alpha|\le m} a_\alpha D_x^\alpha, \quad \textit{where } a_\alpha \in \mathbb{C}, D_x^\alpha = \frac{1}{(2i\pi)^{|\alpha|}}\partial_x^\alpha. \tag{3.6.1}$$

A distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution of P when $PE = \delta_0$.

We note that if $f \in \mathscr{E}'(\mathbb{R}^n)$ and E is a fundamental solution of P, we have from (3.5.14), (3.5.13),

$$P(E * f) = PE * f = \delta_0 * f = f,$$

which allows to find a solution of the Partial Differential Equation (PDE for short) $P(D)u = f$, at least when f is a compactly supported distribution.

Examples. We have on the real line already proven (see (3.2.2)) that $\frac{dH}{dt} = \delta_0$, so that the Heaviside function is a fundamental solution of d/dt (note that from the lemma 3.2.4, the other fundamental solutions are $C + H(t)$). This also implies that

$$\partial_{x_1}\big(H(x_1) \otimes \delta_0(x_2) \otimes \cdots \otimes \delta_0(x_n)\big) = \delta_0(x), \quad \text{(the Dirac mass at 0 in } \mathbb{R}^n).$$

Let $N \in \mathbb{N}$. With x_+^λ defined in (3.4.8), we get, since $\partial_{x_1}^{N+1}(x_{1,+}^{N+1}) = H(x_1)(N + 1)!$, that

$$(\partial_{x_1} \dots \partial_{x_n})^{N+2}\Big(\prod_{1\le j\le n}\Big(\frac{x_{j,+}^{N+1}}{(N + 1)!}\Big)\Big) = \delta_0(x).$$

The last example has the following interesting consequence.

Proposition 3.6.2. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and Ω a bounded open set. Then $u_{|\Omega}$ is a derivative of finite order of a continuous function.*

Proof. We consider for $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on Ω the distribution $\chi u \in \mathscr{E}'(\mathbb{R}^n)$ whose restriction to Ω coincides with $u_{|\Omega}$. The distribution χu has finite order N (see the remark 3.3.3). We have with $E(x) = \prod_{1\le j\le n} \frac{x_{j,+}^{N+1}}{(N+1)!}$

$$\chi u = \chi u * \delta_0 = (\partial_{x_1} \dots \partial_{x_n})^{N+2}\big(\chi u * E\big). \tag{3.6.2}$$

Since the function E is C^N with Nth derivatives (Lipschitz) continuous, we may consider the function ψ defined by

$$\psi(x) = \langle \chi(y)u(y), E(x-y)\rangle.$$

Since χu is compactly supported with order N, we have with K compact subset of $\mathbb{R}^n$,

$$|\psi(x+h) - \psi(x)| \le C \sup_{|\alpha|\le N, y\in K} |\partial_y^\alpha\big(E(x+h-y) - E(x-y)\big)|.$$

Since the function E is C^N with Nth derivatives Lipschitz continuous, we find that ψ is Lipschitz continuous. We have from the definitions, with $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle E * \chi u, \phi\rangle = \langle E(x) \otimes (\chi u)(y), \phi(x+y)\rangle = \langle (\chi u)(y), \langle E(x), \phi(x+y)\rangle\rangle,$$

and we note that $\langle E(x), \phi(x+y)\rangle = \int E(x-y)\phi(x)dx$. As a result, we have

$$\langle E * \chi u, \phi\rangle = \langle u(y), \int \underbrace{\chi(y)E(x-y)\phi(x)}_{\in C_c^N(\mathbb{R}^{2n})} dx\rangle = \int \phi(x)\langle (\chi u)(y), E(x-y)\rangle dx$$

where the last equality is due to the theorem 3.4.1[6] and gives also that $\psi = \chi u * E$. The result follows from the continuity of ψ and (3.6.2). □

3.6.2 The Laplace and Cauchy-Riemann equations

We define the Laplace operator Δ in $\mathbb{R}^n$ as

$$\Delta = \sum_{1\le j\le n} \partial_{x_j}^2. \tag{3.6.3}$$

In one dimension, we have from (3.2.2) that $\frac{d^2}{dt^2}(t_+) = \delta_0$ and for $n \ge 2$ the following result describes the fundamental solutions of the Laplace operator. In $\mathbb{R}^2_{x,y}$, we define the operator $\bar\partial$ (a.k.a. the Cauchy-Riemann operator) by

$$\bar\partial = \frac{1}{2}(\partial_x + i\partial_y). \tag{3.6.4}$$

Theorem 3.6.3. *We have $\Delta E = \delta_0$ with $\|\cdot\|$ standing for the Euclidean norm,*

$$E(x) = \frac{1}{2\pi}\ln\|x\|, \quad \text{for } n = 2, \tag{3.6.5}$$

$$E(x) = \|x\|^{2-n}\frac{1}{(2-n)|S^{n-1}|}, \quad \text{for } n \ge 3, \text{ with } |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{3.6.6}$$

$$\bar\partial\Big(\frac{1}{\pi z}\Big) = \delta_0, \quad \text{with } z = x+iy \text{ (equality in } \mathscr{D}'(\mathbb{R}^2_{x,y})). \tag{3.6.7}$$

[6]For $\Phi \in C_c^N(\mathbb{R}^n\times\mathbb{R}^n)$, $v \in \mathscr{D}'(\mathbb{R}^n)$, $\text{order}(v) \le N$ $\langle 1\otimes v, \Phi\rangle = \langle v(y), \int \Phi(x,y)dx\rangle = \int \langle v(y), \Phi(x,y)\rangle dx$.

Proof. We start with $n \ge 3$, noting that the function $\|x\|^{2-n}$ is L^1_{loc} and homogeneous with degree $2-n$, so that $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ (see the remark 3.4.7 (2)). Moreover, the function $\|x\|^{2-n} = f(r^2)$, $r^2 = \|x\|^2$, $f(t) = t_+^{1-\frac{n}{2}}$ is smooth outside 0 and we can compute there

$$\Delta(f(r^2)) = \sum_j \partial_j(f'(r^2)2x_j) = \sum_j f''(r^2)4x_j^2 + 2nf'(r^2) = 4r^2 f''(r^2) + 2nf'(r^2),$$

so that with $t = r^2$,

$$\Delta(f(r^2)) = 4t(1-\frac{n}{2})(-\frac{n}{2})t^{-\frac{n}{2}-1} + 2n(1-\frac{n}{2})t^{-\frac{n}{2}} = t^{-\frac{n}{2}}(1-\frac{n}{2})(-2n+2n) = 0.$$

As a result, $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ and supported in $\{0\}$. From the theorem 3.3.4, we obtain that

$$\underbrace{\Delta\|x\|^{2-n} = c\delta_0}_{\substack{\text{homogeneous}\\ \text{degree } -n}} + \sum_{1\le j\le m} \underbrace{\sum_{|\alpha|=j} c_{j,\alpha}\delta_0^{(\alpha)}}_{\substack{\text{homogeneous}\\ \text{degree } -n-j}}.$$

The lemma 3.4.8 implies that for $1 \le j \le m$, $0 = \sum_{|\alpha|=j} c_{j,\alpha}\delta_0^{(\alpha)}$ and $\Delta\|x\|^{2-n} = c\delta_0$. It remains to determine the constant c. We calculate, using the previous formulas for the computation of $\Delta(f(r^2))$, here with $f(t) = e^{-\pi t}$,

$$\begin{aligned}
c = \langle \Delta\|x\|^{2-n}, e^{-\pi\|x\|^2}\rangle &= \int \|x\|^{2-n} e^{-\pi\|x\|^2}\big(4\|x\|^2\pi^2 - 2n\pi\big)dx \\
&= |S^{n-1}| \int_0^{+\infty} r^{2-n+n-1} e^{-\pi r^2}(4\pi^2 r^2 - 2n\pi)dr \\
&= |S^{n-1}|\Big(\frac{1}{2\pi}[e^{-\pi r^2}(4\pi^2 r^2 - 2n\pi)]^0_{+\infty} + \frac{1}{2\pi}\int_0^{+\infty} e^{-\pi r^2} 8\pi^2 r dr\Big) \\
&= |S^{n-1}|(-n+2),
\end{aligned}$$

giving (3.6.6). For the convenience of the reader, we calculate explicitely $|S^{n-1}|$. We have indeed

$$\begin{aligned}
1 &= \int_{\mathbb{R}^n} e^{-\pi\|x\|^2} dx = |S^{n-1}| \int_0^{+\infty} r^{n-1} e^{-\pi r^2} dr \\
&\underbrace{=}_{r=t^{1/2}\pi^{-1/2}} |S^{n-1}|\pi^{(1-n)/2} \int_0^{+\infty} t^{\frac{n-1}{2}} e^{-t}\frac{1}{2}t^{-1/2} dt \pi^{-1/2} = |S^{n-1}|\pi^{-n/2}2^{-1}\Gamma(n/2).
\end{aligned}$$

Turning now our attention to the Cauchy-Riemann equation, we see that $1/z$ is also $L^1_{\text{loc}}(\mathbb{R}^2)$, homogeneous of degree -1, and satisfies $\bar\partial(z^{-1}) = 0$ on the complement of $\{0\}$, so that the same reasoning as above shows that

$$\bar\partial(\pi^{-1}z^{-1}) = c\delta_0.$$

To check the value of c, we write $c = \langle \bar\partial(\pi^{-1}z^{-1}), e^{-\pi z\bar z}\rangle = \int_{\mathbb{R}^2} e^{-\pi z\bar z}\pi^{-1}z^{-1}\pi z dxdy = 1$, which gives (3.6.7). We are left with the Laplace equation in two dimensions and we note that with $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\partial_x + i\partial_y)$, we have in two dimensions

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = 4\frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}. \tag{3.6.8}$$

Solving the equation $4\frac{\partial E}{\partial z} = \frac{1}{\pi z}$ leads us to try $E = \frac{1}{2\pi}\ln|z|$ and we check directly[7] that $\frac{\partial}{\partial z}\big(\ln(z\bar{z})\big) = z^{-1}$

$$\Delta(\frac{1}{2\pi}\ln|z|) = \pi^{-1}2^{-2}4\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}\big(\ln(z\bar{z})\big) = \pi^{-1}\frac{\partial}{\partial\bar{z}}\big(z^{-1}\big) = \delta_0. \qquad \square$$

3.6.3 Hypoellipticity

Definition 3.6.4. *Let P be a linear operator of type (3.6.1). We shall say that P is hypoelliptic when for all open subsets Ω of $\mathbb{R}^n$ and all $u \in \mathscr{D}'(\Omega)$, we have*

$$\operatorname{singsupp} u = \operatorname{singsupp} Pu. \tag{3.6.9}$$

It is obvious that $\operatorname{singsupp} Pu \subset \operatorname{singsupp} u$, so the hypoellipticity means that $\operatorname{singsupp} u \subset \operatorname{singsupp} Pu$, which is a very interesting piece of information since we can then determine the singularities of our (unknown) solution u, which are located at the same place as the singularities of the source f, which is known when we try to solve the equation $Pu = f$.

Theorem 3.6.5. *Let P be a linear operator of type (3.6.1) such that P has a fundamental solution E satisfying*

$$\operatorname{singsupp} E = \{0\}. \tag{3.6.10}$$

Then P is hypoelliptic. In particular the Laplace and the Cauchy-Riemann operators are hypoelliptic.

N.B. The condition (3.6.10) appears as an iff condition for the hypoellipticity of the operator P since it is also a consequence of the hypoellipticity property.

Proof. Assume that (3.6.10) holds, let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. We consider $f = Pu \in \mathscr{D}'(\Omega)$, $x_0 \notin \operatorname{singsupp} f$, $\chi_0 \in C_c^\infty(\Omega)$, $\chi_0 = 1$ near x_0. We have from the proposition 3.5.5 that

$$\chi u = \chi u * PE = (P\chi u) * E = ([P,\chi]u) * E + \underbrace{\overbrace{(\chi f)}^{\in C_c^\infty(\mathbb{R}^n)} * E}_{\in C^\infty(\mathbb{R}^n)}$$

and thus, using the the proposition 3.5.7 for singular supports, we get

$$\operatorname{singsupp}(\chi u) \subset \operatorname{singsupp}([P,\chi]u) + \operatorname{singsupp} E = \operatorname{singsupp}([P,\chi]u) \subset \operatorname{supp}(u\nabla\chi),$$

and since χ is identically 1 near x_0, we get that $x_0 \notin \operatorname{supp}(u\nabla\chi)$, implying $x_0 \notin \operatorname{singsupp}(\chi u)$, proving that $x_0 \notin \operatorname{singsupp} u$ and the result. $\square$

[7] Noting that $\ln(x^2+y^2)$ and its first derivatives are $L^1_{\text{loc}}(\mathbb{R}^2)$, we have for $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\langle\frac{\partial}{\partial z}(\ln|z|^2), \varphi\rangle =$

$$\frac{1}{2}\iint_{\mathbb{R}^2}(-\partial_x\varphi + i\partial_y\varphi)\ln(x^2+y^2)dxdy = \iint \varphi(x,y)(xr^{-2} - iyr^{-2})dxdy = \iint (x+iy)^{-1}\varphi(x,y)dxdy.$$

3.7 Appendix

3.7.1 The Gamma function

The gamma function Γ is a meromorphic function on $\mathbb{C}$ given for $\operatorname{Re} z > 0$ by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{3.7.1}$$

For $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$; another interesting value is $\Gamma(1/2) = \sqrt{\pi}$. The functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{3.7.2}$$

is easy to prove for $\operatorname{Re} z > 0$ and can be used to extend the Γ function into a meromorphic function with simple poles at $-\mathbb{N}$ and $\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}$. For instance, for $-1 < \operatorname{Re} z \leq 0$ with $z \neq 0$ we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad \text{where we can use (3.7.1) to define } \Gamma(z+1).$$

More generally for $k \in \mathbb{N}$, $-1-k < \operatorname{Re} z \leq -k$, $z \neq -k$, we can define

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\dots(z+k)}.$$

There are manifold references on the Gamma function. One of the most comprehensive is certainly the chapter VII of the Bourbaki volume *Fonctions de variable réelle* [2].

3.7.2 *LF* spaces

3.7.3 The Schwartz kernel theorem

3.7.4 Coordinate transformations and pullbacks

Chapter 4

Introduction to Fourier Analysis

4.1 Fourier Transform of tempered distributions

4.1.1 The Fourier transformation on $\mathscr{S}(\mathbb{R}^n)$

Let $n \geq 1$ be an integer. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined in the section 2.3.5, is a Fréchet space, as the space of C^∞ functions u from $\mathbb{R}^n$ to $\mathbb{C}$ such that, for all multi-indices[1] $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally if A is a symmetric positive definite $n \times n$ matrix the function

$$v_A(x) = e^{-\pi \langle Ax, x \rangle}$$

belongs to the Schwartz class.

Definition 4.1.1. *For $u \in \mathscr{S}(\mathbb{R}^n)$, we define its Fourier transform $\hat{u}$ as*

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x \cdot \xi} u(x) dx. \tag{4.1.1}$$

Lemma 4.1.2. *The Fourier transform sends continuously $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. Just notice that $\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x \xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta| - |\alpha|} (-1)^{|\beta|}$. □

Lemma 4.1.3. *For a symmetric positive definite $n \times n$ matrix A, we have*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi \langle A^{-1} \xi, \xi \rangle}. \tag{4.1.2}$$

[1] Here we use the multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{1 \leq j \leq n} \alpha_j.$$

Proof. In fact, diagonalizing the symmetric matrix A, it is enough to prove the one-dimensional version of (4.1.2), i.e. to check

$$\int e^{-2i\pi x\xi}e^{-\pi x^2}dx = \int e^{-\pi(x+i\xi)^2}dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality is obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2}dx$: we have indeed

$$\frac{d}{d\xi}\Big(\int e^{-\pi(x+i\xi)^2}dx\Big) = \int e^{-\pi(x+i\xi)^2}(-2i\pi)(x+i\xi)dx = -i\int \frac{d}{dx}\big(e^{-\pi(x+i\xi)^2}\big)dx = 0.$$

For $a>0$, we obtain $\int_{\mathbb{R}} e^{-2i\pi x\xi}e^{-\pi a x^2}dx = a^{-1/2}e^{-\pi a^{-1}\xi^2}$, which is the sought result in one dimension. If $n \ge 2$, and A is a positive definite symmetric matrix, there exists an orthogonal $n\times n$ matrix P (i.e. ${}^tPP = \mathrm{Id}$) such that

$$D = {}^tPAP, \quad D = \mathrm{diag}(\lambda_1,\dots,\lambda_n), \text{ all } \lambda_j > 0.$$

As a consequence, we have, since $|\det P| = 1$,

$$\int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi}e^{-\pi\langle Ax,x\rangle}dx = \int_{\mathbb{R}^n} e^{-2i\pi(Py)\cdot\xi}e^{-\pi\langle APy,Py\rangle}dy = \int_{\mathbb{R}^n} e^{-2i\pi y\cdot({}^tP\xi)}e^{-\pi\langle Dy,y\rangle}dy$$
$$(\text{with } \eta = {}^tP\xi) \quad = \prod_{1\le j\le n}\int_{\mathbb{R}} e^{-2i\pi y_j\eta_j}e^{-\pi\lambda_j y_j^2}dy_j = \prod_{1\le j\le n}\lambda_j^{-1/2}e^{-\pi\lambda_j^{-1}\eta_j^2}$$
$$= (\det A)^{-1/2}e^{-\pi\langle D^{-1}\eta,\eta\rangle} = (\det A)^{-1/2}e^{-\pi\langle {}^tPA^{-1}P\ {}^tP\xi,{}^tP\xi\rangle} = (\det A)^{-1/2}e^{-\pi\langle A^{-1}\xi,\xi\rangle}. \quad \square$$

Proposition 4.1.4. *The Fourier transformation is an isomorphism of the Schwartz class and for $u \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$u(x) = \int e^{2i\pi x\xi}\hat u(\xi)d\xi. \tag{4.1.3}$$

Proof. Using (4.1.2) we calculate for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$, dealing with absolutely converging integrals,

$$\begin{aligned} u_\epsilon(x) &= \int e^{2i\pi x\xi}\hat u(\xi)e^{-\pi\epsilon^2|\xi|^2}d\xi \\ &= \iint e^{2i\pi x\xi}e^{-\pi\epsilon^2|\xi|^2}u(y)e^{-2i\pi y\xi}dyd\xi \\ &= \int u(y)e^{-\pi\epsilon^{-2}|x-y|^2}\epsilon^{-n}dy \\ &= \int \underbrace{\big(u(x+\epsilon y) - u(x)\big)}_{\text{with absolute value}\le\epsilon|y|\|u'\|_{L^\infty}} e^{-\pi|y|^2}dy + u(x). \end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

$$u(x) = \int e^{2i\pi x\xi}\hat u(\xi)d\xi. \tag{4.1.4}$$

We have also proven for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\check{u}(x) = u(-x)$

$$u = \check{\hat{\hat{u}}}. \tag{4.1.5}$$

Since $u \mapsto \hat{u}$ and $u \mapsto \check{u}$ are continuous homomorphisms of $\mathscr{S}(\mathbb{R}^n)$, this completes the proof of the proposition. □

Proposition 4.1.5. *Using the notation*

$$D_{x_j} = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad D_x^\alpha = \prod_{j=1}^n D_{x_j}^{\alpha_j} \quad \textit{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \tag{4.1.6}$$

we have, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad (D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha u(x)}(\xi) \tag{4.1.7}$$

Proof. We have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx$ and thus

$$(D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\int e^{-2i\pi x\cdot\xi} x^\alpha u(x)dx,$$
$$\xi^\alpha \hat{u}(\xi) = \int (-2i\pi)^{-|\alpha|}\partial_x^\alpha\big(e^{-2i\pi x\cdot\xi}\big)u(x)dx = \int e^{-2i\pi x\cdot\xi}(2i\pi)^{-|\alpha|}(\partial_x^\alpha u)(x)dx,$$

proving both formulas. □

N.B. The normalization factor $\frac{1}{2i\pi}$ leads to a simplification in the formulas (4.1.7), but the most important aspect of these formulas is certainly that the Fourier transformation exchanges the operation of derivation against the operation of multiplication. For instance if $P(D)$ is given by a formula (3.6.1), we have

$$\widehat{Pu}(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi).$$

Remark 4.1.6. We have the following continuous inclusions[2]

$$\mathscr{D}(\mathbb{R}^n) \hookrightarrow \mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{E}(\mathbb{R}^n), \tag{4.1.8}$$

triggering the (continuous) inclusions of topological duals,

$$\mathscr{E}'(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n). \tag{4.1.9}$$

The space $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$ and is called the space of *tempered distributions on* $\mathbb{R}^n$. We shall sometimes omit the "$\mathbb{R}^n$" in $\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n)$, at least when it is clear that the dimension is fixed equal to n.

The Fourier transformation can be extended to $\mathscr{S}'(\mathbb{R}^n)$.

[2]The first inclusion is certainly sequentially continuous according to the definition 3.1.9 and the second is an inclusion of Fréchet spaces: for each semi-norm p on $\mathscr{E}(\mathbb{R}^n)$, there exists a semi-norm q on $\mathscr{S}(\mathbb{R}^n)$ such that for all $u \in \mathscr{S}(\mathbb{R}^n)$, $p(u) \le q(u)$.

4.1.2 The Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$

Definition 4.1.7. *Let T be a tempered distribution ; the Fourier transform $\hat{T}$ of T is the tempered distibution defined by the formula*

$$\langle \hat{T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{4.1.10}$$

The linear form $\hat{T}$ is obviously a tempered distribution since the Fourier transformation is continuous on $\mathscr{S}$. Thanks to the lemma 3.1.7, if $T \in \mathscr{S}$, the present definition of $\hat{T}$ and (4.1.1) coincide.

Note that for $T, \varphi \in \mathscr{S}$, we have $\langle \hat{T}, \varphi\rangle = \iint T(x)e^{-2i\pi x\cdot\xi}\varphi(\xi)dxd\xi = \langle T, \hat{\varphi}\rangle$. This definition gives that

$$\widehat{\delta_0} = 1, \tag{4.1.11}$$

since $\langle \widehat{\delta_0}, \varphi\rangle = \langle \delta_0, \widehat{\varphi}\rangle = \widehat{\varphi}(0) = \int \varphi(x)dx = \langle 1, \varphi\rangle$.

Theorem 4.1.8. *The Fourier transformation is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$. Let T be a tempered distribution. Then we have*[3]

$$T = \check{\hat{\hat{T}}}. \tag{4.1.12}$$

With obvious notations, we have the following extensions of (4.1.7),

$$\widehat{D_x^\alpha T}(\xi) = \xi^\alpha \hat{T}(\xi), \qquad (D_\xi^\alpha \hat{T})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha T(x)}(\xi). \tag{4.1.13}$$

Proof. Using the notation $(\check{\varphi})(x) = \varphi(-x)$ for $\varphi \in \mathscr{S}$, we define $\check{S}$ for $S \in \mathscr{S}'$ by (see the remark 3.4.4), $\langle \check{S}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle S, \check{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}$ and we obtain for $T \in \mathscr{S}'$

$$\langle \check{\hat{\hat{T}}}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{\hat{T}}, \check{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \hat{\check{\varphi}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\hat{\check{\varphi}}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

where the last equality is due to the fact that $\varphi \mapsto \check{\varphi}$ commutes[4] with the Fourier transform and (4.1.4) means $\check{\hat{\hat{\varphi}}} = \varphi$, a formula also proven true on $\mathscr{S}'$ by the previous line of equality. The formula (4.1.7) is true as well for $T \in \mathscr{S}'$ since, with $\varphi \in \mathscr{S}$ and $\varphi_\alpha(\xi) = \xi^\alpha\varphi(\xi)$, we have

$$\langle \widehat{D^\alpha T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, (-1)^{|\alpha|}D^\alpha\hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \widehat{\varphi_\alpha}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \varphi_\alpha\rangle_{\mathscr{S}',\mathscr{S}},$$

and the other part is proven the same way. □

The following lemma will be useful.

Lemma 4.1.9. *Let $T \in \mathscr{S}'(\mathbb{R}^n)$ be a homogeneous distribution of degree m. Then its Fourier transform is a homogeneous distribution of degree $-m-n$*

Proof. We check

$$(\xi\cdot D_\xi)\hat{T} = -\xi\cdot\widehat{xT} = -(\widehat{D_x\cdot xT}) = -\frac{n}{2i\pi}\hat{T} - \frac{1}{2i\pi}(\widehat{x\cdot\partial_x T}) = -\frac{(n+m)}{2i\pi}\hat{T},$$

so that the Euler equation (3.4.6) $\xi\dot{\partial}_\xi\hat{T} = -(n+m)\hat{T}$ is satisfied. □

[3] According to the remark 3.4.4, $\check{T}$ is the distribution defined by $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$ and if $T \in \mathscr{S}'$, $\check{T}$ is also a tempered distribution since $\varphi \mapsto \check{\varphi}$ is an involutive isomorphism of $\mathscr{S}$.

[4] If $\varphi \in \mathscr{S}$, we have $\widehat{\check{\varphi}}(\xi) = \int e^{-2i\pi x\cdot\xi}\varphi(-x)dx = \int e^{2i\pi x\cdot\xi}\varphi(x)dx = \hat{\varphi}(-\xi) = \check{\hat{\varphi}}(\xi)$.

4.1.3 The Fourier transformation on $L^1(\mathbb{R}^n)$

Theorem 4.1.10. *The Fourier transformation is linear continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$, we have*

$$\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx, \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{4.1.14}$$

Proof. The formula (4.1.1) can be used to define directly the Fourier transform of a function in $L^1(\mathbb{R}^n)$ and this gives an $L^\infty(\mathbb{R}^n)$ function which coincides with the Fourier transform: for a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and $u \in L^1(\mathbb{R}^n)$, we have by the definition (4.1.10) above and the Fubini theorem

$$\langle \hat{u}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int u(x)\hat{\varphi}(x)dx = \iint u(x)\varphi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int \widetilde{u}(\xi)\varphi(\xi)d\xi$$

with $\widetilde{u}(\xi) = \int e^{-2i\pi x\cdot\xi}u(x)dx$ which is thus the Fourier transform of u. □

4.1.4 The Fourier transformation on $L^2(\mathbb{R}^n)$

We refer the reader to the section 5.3 in Chapter 5.

4.1.5 Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \le 0$; it is obviously a tempered distribution, so that we can compute its Fourier transform. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\text{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\,\text{sign}}(\xi) = \xi\widehat{\text{sign}}\xi$$

so that $\xi\big(\widehat{\text{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\big) = 0$ and from the theorem 3.2.8, we get

$$\widehat{\text{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0,$$

with $c = 0$ since the lhs is odd. We obtain

$$\widehat{\text{sign}}(\xi) = \frac{1}{i\pi}pv\frac{1}{\xi}, \tag{4.1.15}$$

$$\widehat{pv(\frac{1}{\pi x})} = -i\,\text{sign}\,\xi, \tag{4.1.16}$$

$$\hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi}pv(\frac{1}{\xi}) = \frac{1}{(x-i0)}\frac{1}{2i\pi}. \tag{4.1.17}$$

Let us consider now for $0 < \alpha < n$ the $L^1_{\text{loc}}(\mathbb{R}^n)$ function $u_\alpha(x) = |x|^{\alpha-n}$ ($|x|$ is the Euclidean norm of x); since u_α is also bounded for $|x| \ge 1$, it is a tempered distribution. Let us calculate its Fourier transform v_α. Since u_α is homogeneous of degree $\alpha - n$, we get from the lemma 4.1.9 that v_α is a homogeneous distribution of degree $-\alpha$. On the other hand, if $S \in O(\mathbb{R}^n)$ (the orthogonal group), we have in the distribution sense (see the definition 3.4.3), since u_α is a radial function,

$$v_\alpha(S\xi) = v_\alpha(\xi). \tag{4.1.18}$$

The distribution $|\xi|^\alpha v_\alpha(\xi)$ is homogeneous of degree 0 on $\mathbb{R}^n \backslash \{0\}$ and is also "radial", i.e. satisfies (4.1.18). Moreover on $\mathbb{R}^n \backslash \{0\}$, the distribution v_α is a C^1 function which coincides with

$$\int e^{-2i\pi x\cdot\xi} \chi_0(x)|x|^{\alpha-n} dx + |\xi|^{-2N} \int e^{-2i\pi x\cdot\xi} |D_x|^{2N} \big(\chi_1(x)|x|^{\alpha-n}\big) dx,$$

where $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ is 1 near 0 and $\chi_1 = 1 - \chi_0$, $N \in \mathbb{N}, \alpha + 1 < 2N$. As a result $|\xi|^\alpha v_\alpha(\xi) = c_\alpha$ on $\mathbb{R}^n \backslash \{0\}$ and the distribution on $\mathbb{R}^n$ (note that $\alpha < n$)

$$T = v_\alpha(\xi) - c_\alpha |\xi|^{-\alpha}$$

is supported in $\{0\}$ and homogeneous (on $\mathbb{R}^n$) with degree $-\alpha$. From the theorem 3.3.4 and the lemma 3.4.8, the condition $0 < \alpha < n$ gives $v_\alpha = c_\alpha |\xi|^{-\alpha}$. To find c_α, we compute

$$\int |x|^{\alpha-n} e^{-\pi x^2} dx = \langle u_\alpha, e^{-\pi x^2} \rangle = c_\alpha \int |\xi|^{-\alpha} e^{-\pi \xi^2} d\xi$$

which yields

$$2^{-1}\Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} = \int_0^{+\infty} r^{\alpha-1} e^{-\pi r^2} dr = c_\alpha \int_0^{+\infty} r^{n-\alpha-1} e^{-\pi r^2} dr = c_\alpha 2^{-1}\Gamma(\frac{n-\alpha}{2})\pi^{-\frac{n-\alpha}{2}}.$$

We have proven the following lemma.

Lemma 4.1.11. *Let $n \in \mathbb{N}^*$ and $\alpha \in]0, n[$. The function $u_\alpha(x) = |x|^{\alpha-n}$ is $L^1_{loc}(\mathbb{R}^n)$ and also a temperate distribution on $\mathbb{R}^n$. Its Fourier transform v_α is also $L^1_{loc}(\mathbb{R}^n)$ and given by*

$$v_\alpha(\xi) = |\xi|^{-\alpha} \pi^{\frac{n}{2}-\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

4.2 The Poisson summation formula

4.2.1 Wave packets

We define for $x \in \mathbb{R}^n$, $(y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = 2^{n/4} e^{-\pi(x-y-i\eta)^2} e^{-\pi\eta^2}, \tag{4.2.1}$$

$$\text{where for } \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n, \quad \zeta^2 = \sum_{1\le j\le n} \zeta_j^2. \tag{4.2.2}$$

We note that the function $\varphi_{y,\eta}$ is in $\mathscr{S}(\mathbb{R}^n)$ and with L^2 norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the wave packets transform as a Gabor wavelet.

Lemma 4.2.1. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define*

$$(Wu)(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbb{R}^n)} = 2^{n/4}\int u(x)e^{-\pi(x-y)^2}e^{-2i\pi(x-y)\cdot\eta}dx \tag{4.2.3}$$

$$= 2^{n/4}\int u(x)e^{-\pi(y-i\eta-x)^2}dxe^{-\pi\eta^2}. \tag{4.2.4}$$

For $u \in L^2(\mathbb{R}^n)$, the function Tu defined by

$$(Tu)(y+i\eta) = e^{\pi\eta^2}Wu(y,-\eta) = 2^{n/4}\int u(x)e^{-\pi(y+i\eta-x)^2}dx \tag{4.2.5}$$

is an entire function. The mapping $u \mapsto Wu$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} Wu(y,\eta)\varphi_{y,\eta}(x)dyd\eta. \tag{4.2.6}$$

Proof. For u in $\mathcal{S}(\mathbb{R}^n)$, we have

$$Wu(y,\eta) = e^{2i\pi y\eta}\widehat{\Omega}^1(\eta,y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathcal{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x)e^{-\pi(x-y)^2}2^{n/4}$. Thus the function Wu belongs to $\mathcal{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\overline{u}(x_2)e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]}dyd\eta dx_1dx_2. \tag{4.2.7}$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use the Fubini theorem. Integrating with respect to η involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2 + 2(x_2-y)^2 = (x_1+x_2-2y)^2 + (x_1-x_2)^2,$$

integrating with respect to y yields a factor $2^{-n/2}$. We are left with

$$(Wu,Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\,\overline{u}(x_2)e^{-\pi(x_1-x_2)^2/2}\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}dx_1dx_2. \tag{4.2.8}$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+}\int u(s+\epsilon t/2)\,\overline{u}(s-\epsilon t/2)e^{-\pi\epsilon^2t^2/2}e^{-\pi t^2}dtds = \|u\|^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)| \le C(1+|x|)^{-n-1}$ imply, with $v = u/C$,

$$\begin{aligned}|v(s+\epsilon t/2)\,\overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1}\\ &\le (1+|s+\epsilon t/2|+|s-\epsilon t/2|)^{-n-1}\\ &\le (1+2|s|)^{-n-1}.\end{aligned}$$

Eventually, this proves that

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})} = \|u\|^2_{L^2(\mathbb{R}^n)} \tag{4.2.9}$$

i.e.

$$W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}) \quad \text{with} \quad W^*W = \mathrm{id}_{L^2(\mathbb{R}^n)}. \tag{4.2.10}$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $\|Wu\|_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (4.2.9) for $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(u,v)_{L^2(\mathbb{R}^n)} &= (Wu, Wv)_{L^2(\mathbb{R}^{2n})} \\ &= \iint Wu(y,\eta)(\varphi_{y,\eta}, v)_{L^2(\mathbb{R}^n)}dyd\eta \\ &= (\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta, v)_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding the result of the lemma $u = \iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$. □

4.2.2 Poisson's formula

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

Lemma 4.2.2. *For all complex numbers z, the following series are absolutely converging and*

$$\sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi m^2} e^{2i\pi mz}. \tag{4.2.11}$$

Proof. We set $\omega(z) = \sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2}$. The function ω is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$

$$\sup_{|z|\le R} |e^{-\pi(z+m)^2}| \le \sup_{|z|\le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi |m| R} \in l^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding ω in Fourier series[5],

$$\omega(z) = \sum_{k\in\mathbb{Z}} e^{2i\pi kz} \int_0^1 \omega(x) e^{-2i\pi kx} dx.$$

[5] Note that we use this expansion only for a C^∞ 1-periodic function. The proof is simple and requires only to compute $1 + 2\,\mathrm{Re}\sum_{1\le k\le N} e^{2i\pi kx} = \frac{\sin \pi(2N+1)x}{\sin \pi x}$. Then one has to show that for a smooth 1-periodic function ω such that $\omega(0) = 0$,

$$\lim_{\lambda\to+\infty} \int_0^1 \frac{\sin \lambda x}{\sin \pi x} \omega(x) dx = 0,$$

which is obvious since for a smooth ν (here we take $\nu(x) = \omega(x)/\sin \pi x$), $|\int_0^1 \nu(x) \sin \lambda x dx| = O(\lambda^{-1})$ by integration by parts.

We also check, using Fubini's theorem on $L^1(0,1)\times l^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x)e^{-2i\pi kx}dx &= \sum_{m\in\mathbb{Z}}\int_0^1 e^{-\pi(x+m)^2}e^{-2i\pi kx}dx\\ &= \sum_{m\in\mathbb{Z}}\int_m^{m+1} e^{-\pi t^2}e^{-2i\pi kt}dt\\ &= \int_{\mathbb{R}} e^{-\pi t^2}e^{-2i\pi kt} = e^{-\pi k^2}.\end{aligned}$$

So the lemma is proven for real z and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the n-th dimensional version of the previous lemma: for all $z\in\mathbb{C}^n$, using the notation (4.2.2), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2}e^{2i\pi m\cdot z}. \tag{4.2.12}$$

Theorem 4.2.3 (The Poisson summation formula). *Let n be a positive integer and u be a function in $\mathcal{S}(\mathbb{R}^n)$. Then we have*

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{4.2.13}$$

where $\hat{u}$ stands for the Fourier transform of u. In other words the tempered distribution $D_0=\sum_{k\in\mathbb{Z}^n}\delta_k$ is such that $\widehat{D_0}=D_0$.

Proof. We write, according to (4.2.6) and to Fubini's theorem

$$\begin{aligned}\textstyle\sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n}\iint Wu(y,\eta)\varphi_{y,\eta}(k)dyd\eta\\ &= \iint Wu(y,\eta)\sum_{k\in\mathbb{Z}^n}\varphi_{y,\eta}(k)dyd\eta.\end{aligned}$$

Now, (4.2.12), (4.2.1) give $\sum_{k\in\mathbb{Z}^n}\varphi_{y,\eta}(k)=\sum_{k\in\mathbb{Z}^n}\widehat{\varphi}_{y,\eta}(k)$, so that (4.2.6) and Fubini's theorem imply the result. □

4.3 Fourier transformation and convolution

4.3.1 Fourier transformation on $\mathscr{E}'(\mathbb{R}^n)$

Theorem 4.3.1. *Let $u\in\mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u}$ is an entire function on $\mathbb{C}^n$.*

Proof. We have for $\varphi\in\mathscr{D}(\mathbb{R}^n)$, according to the definition (3.4.14),

$$\begin{aligned}\langle\hat{u},\varphi\rangle = \langle u,\hat{\varphi}\rangle = \langle u(x),\int e^{-2i\pi x\cdot\xi}\varphi(\xi)d\xi\rangle &= \langle u(x)\otimes\varphi(\xi), e^{-2i\pi x\cdot\xi}\rangle_{\mathscr{E}'(\mathbb{R}^{2n}),\mathscr{E}(\mathbb{R}^{2n})}\\ &= \langle\varphi(\xi),\underbrace{\langle u(x),e^{-2i\pi x\cdot\xi}\rangle}_{\tilde{u}(\xi)}\rangle,\end{aligned}$$

an identity which implies $\hat{u} = \tilde{u}$ and moreover the function $\tilde{u}$ is indeed entire, since with $\zeta \in \mathbb{C}^n$, and $\tilde{u}(\zeta) = \langle u(x), e^{-2i\pi x\cdot\zeta}\rangle$ the function $\tilde{u}$ is $C^\infty(\mathbb{C}^n)$ from the corollary 3.4.2, and we can check that $\bar\partial\tilde{u} = 0$ (a direct computation of $\tilde{u}(\zeta+h) - u(\zeta)$ provides elementarily the holomorphy of $\tilde{u}$). □

Definition 4.3.2. *The space $\mathscr{O}_M(\mathbb{R}^n)$ of multipliers of $\mathscr{S}(\mathbb{R}^n)$ is the subspace of the functions $f \in \mathscr{E}(\mathbb{R}^n)$ such that,*

$$\forall\alpha \in \mathbb{N}^n, \exists C_\alpha > 0, \exists N_\alpha \in \mathbb{N}, \quad \forall x \in \mathbb{R}^n, \quad |(\partial_x^\alpha f)(x)| \le C_\alpha(1+|x|)^{N_\alpha}. \tag{4.3.1}$$

It is easy to check that, for $f \in \mathscr{O}_M(\mathbb{R}^n)$, the operator $u \mapsto fu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself, and by transposition from $\mathscr{S}'(\mathbb{R}^n)$ into itself: we have for $T \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{O}_M(\mathbb{R}^n)$,

$$\langle fT, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, f\varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

and if p is a semi-norm of $\mathscr{S}$, the continuity on $\mathscr{S}$ of the multiplication by f implies that there exists a semi-norm q on $\mathscr{S}$ such that for all $\varphi \in \mathscr{S}$, $p(f\varphi) \le q(\varphi)$. A typical example of a function in $\mathscr{O}_M(\mathbb{R}^n)$ is $e^{iP(x)}$ where P is a real-valued polynomial: in fact the derivatives of $e^{iP(x)}$ are of type $Q(x)e^{iP(x)}$ where Q is a polynomial so that (4.3.1) holds.

Lemma 4.3.3. *Let $u \in \mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$.*

Proof. We have already seen that $\hat{u}(\xi) = \langle u(x), e^{-2i\pi x\cdot\xi}\rangle$ is a smooth function so that

$$(D_\xi^\alpha u)(\xi) = \langle u(x), e^{-2i\pi x\cdot\xi}x^\alpha\rangle(-1)^{|\alpha|}$$

which implies $|(D_\xi^\alpha u)(\xi)| \le C_0 \sup_{\substack{|\beta|\le N_0\\ x\in K_0}} |\partial_x^\beta(e^{-2i\pi x\cdot\xi}x^\alpha)| \le C_1(1+|\xi|)^{N_0}$, proving the sought result. □

4.3.2 Convolution and Fourier transformation

Theorem 4.3.4. *Let $u \in \mathscr{S}'(\mathbb{R}^n), v \in \mathscr{E}'(\mathbb{R}^n)$. Then the convolution $u * v$ belongs to $\mathscr{S}'(\mathbb{R}^n)$ and*

$$\widehat{u * v} = \hat{u}\hat{v}. \tag{4.3.2}$$

N.B. We note that both sides of the equality (4.3.2) make sense since the lhs is the Fourier transform of $u * v$ which belongs to $\mathscr{S}'$(this has to be proven) and $\hat{v}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$ so that the product of $\hat{u} \in \mathscr{S}'$ with $\hat{v}$ makes sense.

Proof. Let us prove first that $u * v$ belongs to $\mathscr{S}'$. We have for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\chi \in \mathscr{D}(\mathbb{R}^n)$ equal to 1 near the support of v,

$$\langle u * v, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^n),\mathscr{D}(\mathbb{R}^n)} = \langle u(x) \otimes v(y), \varphi(x+y)\chi(y)\rangle_{\mathscr{D}'(\mathbb{R}^{2n}),\mathscr{D}(\mathbb{R}^{2n})}.$$

Now if $\varphi \in \mathscr{S}(\mathbb{R}^n)$ the function $(x, y) \mapsto \varphi(x+y)\chi(y) = \Phi(x, y)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$: it is a smooth function and $x^\alpha y^\beta \partial_x^\gamma \partial_y^\rho \Phi$ is a linear combination of terms of type

$$(x+y)^\omega(\partial^\nu\varphi)(x+y)y^\lambda(\partial^\mu\chi)(y)$$

which are bounded as product of bounded terms. Moreover, if $\Phi \in \mathscr{S}(\mathbb{R}^{2n})$, the function $\psi(x) = \langle v(y), \Phi(x,y)\rangle$ is smooth (see the corollary 3.4.2(2)) and belongs to $\mathscr{S}(\mathbb{R}^n)$ since $x^\alpha(\partial_x^\beta \psi)(x) = \langle v(y), x^\alpha \partial_x^\beta \Phi(x,y)\rangle$ and for some compact subset K_0 of $\mathbb{R}^n$,

$$|x^\alpha(\partial_x^\beta \psi)(x)| = |\langle v(y), x^\alpha \partial_x^\beta \Phi(x,y)\rangle| \le C \sup_{\substack{|\gamma|\le N_0\\ y\in K_0}} |x^\alpha \partial_x^\beta \partial_y^\gamma \Phi(x,y)| = p(\Phi),$$

where p is a semi-norm on $\mathscr{S}(\mathbb{R}^{2n})$. As a result, we can extend $u * v$ to a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$ so that $u * v \in \mathscr{S}'(\mathbb{R}^n)$. Let $w \in \mathscr{S}'$ such[6] that $\hat w = \hat u \hat v$. For $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle w, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat u \hat v, \check{\hat\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat u, \hat v \check{\hat\varphi}\rangle_{\mathscr{S}',\mathscr{S}}.$$

On the other hand, we have

$$\hat v(\xi)\check{\hat\varphi}(\xi) = \langle v(x), e^{-2i\pi x\cdot\xi}\rangle \int \varphi(y) e^{2i\pi y\cdot\xi} dy = \langle v(x)\otimes\varphi(y), e^{2i\pi(y-x)\cdot\xi}\rangle$$
$$= \langle v(x), \langle \varphi(y), e^{2i\pi(y-x)\cdot\xi}\rangle\rangle = \langle v(x), \langle \check\varphi(y), e^{-2i\pi(y+x)\cdot\xi}\rangle\rangle = \widehat{(v * \check\varphi)}(\xi),$$

so that

$$\langle w, \varphi\rangle = \langle \hat u, \widehat{(v*\check\varphi)}\rangle = \langle \check u, v * \check\varphi\rangle = \langle u(-x), \langle v(x-y), \varphi(-y)\rangle\rangle$$
$$= \langle u(x), \langle v(y-x), \varphi(y)\rangle\rangle = \langle (u*v), \varphi\rangle,$$

which gives $w = u * v$ and (4.3.2). □

4.3.3 The Riemann-Lebesgue lemma

Lemma 4.3.5. *Let $u \in L^1(\mathbb{R}^n)$. Then from (4.1.14) $\hat u(\xi) = \int e^{-2i\pi x\cdot\xi} u(x) dx$; moreover $\hat u$ belongs to $C^0_{(0)}(\mathbb{R}^n)$, where $C^0_{(0)}(\mathbb{R}^n)$ stands for the space of continuous functions on $\mathbb{R}^n$ tending to 0 at infinity. In particular $\hat u$ is uniformly continuous.*

Proof. This follows from the Riemann-Lebesgue lemma (see e.g. the lemma 3.4.4 in [9]); moreover,

$$|\hat u(\xi+h) - \hat u(\xi)| = \int |u(x)||e^{-2i\pi x\cdot h} - 1| dx = \sigma_u(h),$$

and the Lebesgue dominated convergence theorem implies that $\lim_{h\to 0}\sigma_u(h) = 0$, implying as well the uniform continuity. □

4.4 Some fundamental solutions

4.4.1 The heat equation

The heat operator is the following constant coefficient differential operator on $\mathbb{R}_t \times \mathbb{R}^n_x$

$$\partial_t - \Delta_x, \tag{4.4.1}$$

where the Laplace operator Δ_x on $\mathbb{R}^n$ is defined by (3.6.3).

[6] Take $w = \check{\widehat{\hat u \hat v}}$.

Theorem 4.4.1. *We define on $\mathbb{R}_t \times \mathbb{R}^n_x$ the L^1_{loc} function*

$$E(t,x) = (4\pi t)^{-n/2} H(t) e^{-\frac{|x|^2}{4t}}. \tag{4.4.2}$$

The function E is C^∞ on the complement of $\{(0,0)\}$ in $\mathbb{R}\times\mathbb{R}^n$. The function E is a fundamental solution of the heat equation, i.e. $\partial_t E - \Delta_x E = \delta_0(t)\otimes\delta_0(x)$.

Proof. To prove that $E \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$, we calculate for $T \geq 0$,

$$\int_0^T \int_0^{+\infty} t^{-n/2} r^{n-1} e^{-\frac{r^2}{4t}} dt dr \underbrace{=}_{r=2t^{1/2}\rho} \int_0^T \int_0^{+\infty} t^{-n/2} 2^{n-1} t^{(n-1)/2} \rho^{n-1} e^{-\rho^2} 2t^{1/2} dt d\rho$$

$$= 2^n T \int_0^{+\infty} \rho^{n-1} e^{-\rho^2} d\rho < +\infty.$$

Moreover, the function E is obviously analytic on the open subset of $\mathbb{R}^{1+n}$ $\{(t,x)\in \mathbb{R}\times\mathbb{R}^n, t\neq 0\}$. Let us prove that E is C^∞ on $\mathbb{R}\times(\mathbb{R}^n\backslash\{0\})$. With ρ_0 defined in (3.1.1), the function ρ_1 defined by $\rho_1(t) = H(t)t^{-n/2}\rho_0(t)$ is also C^∞ on $\mathbb{R}$ and

$$E(t,x) = H(\frac{|x|^2}{4t})(\frac{|x|^2}{4t})^{n/2} e^{-\frac{|x|^2}{4t}} |x|^{-n}\pi^{-n/2} = |x|^{-n}\pi^{-n/2}\rho_1(\frac{4t}{|x|^2}),$$

which is indeed smooth on $\mathbb{R}_t\times(\mathbb{R}^n_x\backslash\{0\})$. We want to solve the equation $\partial_t u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{n+1})$, we can consider its Fourier transform v with respect to x (well-defined by transposition as the Fourier transform in (4.1.10)), and we end-up with the simple ODE with parameters on v,

$$\partial_t v + 4\pi^2|\xi|^2 v = \delta_0(t). \tag{4.4.3}$$

It remains to determine a fundamental solution of that ODE: we have

$$\frac{d}{dt} + \lambda = e^{-t\lambda}\frac{d}{dt}e^{t\lambda}, \quad \big(\frac{d}{dt}+\lambda\big)(e^{-t\lambda}H(t)) = \big(e^{-t\lambda}\frac{d}{dt}e^{t\lambda}\big)(e^{-t\lambda}H(t)) = \delta_0(t), \tag{4.4.4}$$

so that we can take $v = H(t)e^{-4\pi^2 t|\xi|^2}$, which belongs to $\mathscr{S}'(\mathbb{R}_t\times\mathbb{R}^n_\xi)$. Taking the inverse Fourier transform with respect to ξ of both sides of (4.4.3) gives[7] with $u \in \mathscr{S}'(\mathbb{R}_t\times\mathbb{R}^n_\xi)$

$$\partial_t u - \Delta_x u = \delta_0(t)\otimes\delta_0(x). \tag{4.4.5}$$

To compute u, we check with $\varphi\in\mathscr{D}(\mathbb{R}), \psi\in\mathscr{D}(\mathbb{R}^n)$,

$$\langle u, \varphi\otimes\check{\psi}\rangle = \langle \widehat{v}^x, \varphi\otimes\psi\rangle = \langle v, \varphi\otimes\hat{\psi}\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n}\varphi(t)\hat{\psi}(\xi)e^{-4\pi^2 t|\xi|^2} dt d\xi.$$

We can use the Fubini theorem in that absolutely converging integral and use (4.1.2) to get

$$\langle u, \varphi\otimes\check{\psi}\rangle = \int_0^{+\infty}\varphi(t)\left(\int_{\mathbb{R}^n}(4\pi t)^{-n/2}e^{-\pi\frac{|x|^2}{4\pi t}}\psi(x)dx\right)dt = \langle E, \varphi\otimes\check{\psi}\rangle,$$

where the last equality is due to the Fubini theorem and the local integrability of E. We have thus $E = u$ and E satisfies (4.4.5). The proof is complete. □

[7] The Fourier transformation obviously respects the tensor products.

Corollary 4.4.2. *The heat equation is C^∞ hypoelliptic (see the definition 3.6.4) , in particular for $w \in \mathscr{D}'(\mathbb{R}^{1+n})$,*

$$\operatorname{singsupp} w \subset \operatorname{singsupp}(\partial_t w - \Delta_x w),$$

where singsupp *stands for the C^∞ singular support as defined by (3.1.9).*

Proof. It is an immediate consequence of the theorem 3.6.5, since E is C^∞ outside zero from the previous theorem. □

Remark 4.4.3. It is also possible to define the *analytic singular support* of a distribution T in an open subset Ω of $\mathbb{R}^n$: we define

$$\operatorname{singsupp}_{\mathcal{A}} T = \{x \in \Omega, \forall U \text{open} \in \mathscr{V}_x,\ T_{|U} \notin \mathcal{A}(U)\}, \tag{4.4.6}$$

where $\mathcal{A}(U)$ stands for the analytic[8] functions on the open set U. It is a consequence[9] of the proof of theorem 4.4.1 that

$$\operatorname{singsupp}_{\mathcal{A}} E = \{0\} \times \mathbb{R}^n_x. \tag{4.4.7}$$

In particular this implies that the heat equation is *not* analytic-hypoelliptic since

$$\{0\} \times \mathbb{R}^n_x = \operatorname{singsupp}_{\mathcal{A}} E \not\subset \operatorname{singsupp}_{\mathcal{A}}(\partial_t E - \Delta_x E) = \operatorname{singsupp}_{\mathcal{A}} \delta_0 = \{0_{\mathbb{R}^{1+n}}\}.$$

4.4.2 The Schrödinger equation

We move forward now with the Schrödinger equation,

$$\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x \tag{4.4.8}$$

which looks similar to the heat equation, but which is in fact drastically different.

Lemma 4.4.4.

$$\mathscr{D}(\mathbb{R}^{n+1}) \mapsto \int_0^{+\infty} e^{-i(n-2)\frac{\pi}{4}} (4\pi t)^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx\right) dt = \langle E, \Phi\rangle \tag{4.4.9}$$

is a distribution in $\mathbb{R}^{n+1}$ of order $\le n+2$.

---

[8] A function f is said to be analytic on an open subset U of $\mathbb{R}^n$ if it is $C^\infty(U)$, and for each $x_0 \in U$ there exists $r_0 > 0$ such that $\bar{B}(x_0, r_0) \subset U$ and

$$\forall x \in \bar{B}(x_0, r_0), \quad f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_x^\alpha f(x_0)(x - x_0)^\alpha.$$

[9] In fact, in the theorem, we have noted the obvious inclusion $\operatorname{singsupp}_{\mathcal{A}} E \subset \{0\} \times \mathbb{R}^n_x$, but since E is C^∞ in $t \neq 0$, vanishes identically on $t < 0$, is positive (it means > 0) on $t > 0$, it cannot be analytic near any point of $\{0\} \times \mathbb{R}^n_x$.

Proof. Let $\Phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)$; for $t > 0$ we have, using (4.6.7),

$$e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx = i \int_{\mathbb{R}^n} \hat{\Phi}^x(t,\xi) e^{-4i\pi^2 t|\xi|^2} d\xi,$$

so that with $\mathbb{N} \ni \tilde{n}$ even $> n$, using (4.1.7) and (4.1.14),

$$\sup_{t>0} \left| e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx \right| \le \sup_{t>0} \int_{\mathbb{R}^n} |\hat{\Phi}^x(t,\xi)| d\xi$$
$$\le \sup_{t>0} \int (1+|\xi|^2)^{-\tilde{n}/2} | \underbrace{(1+|\xi|^2)^{\tilde{n}/2}}_{\text{polynomial}} \hat{\Phi}(t,\xi)| d\xi \le C_n \max_{|\alpha| \le \tilde{n}} \|\partial_x^\alpha \Phi\|_{L^\infty(\mathbb{R}^{n+1})}.$$

As a result the mapping

$$\mathscr{D}(\mathbb{R}^{n+1}) \mapsto \int_0^{+\infty} e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx \right) dt = \langle E, \Phi \rangle$$

is a distribution of order $\le n+2$. □

Theorem 4.4.5. *The distribution E given by (4.4.9) is a fundamental solution of the Schrödinger equation, i.e. $\frac{1}{i}\partial_t E - \Delta_x E = \delta_0(t) \otimes \delta_0(x)$. Moreover, E is smooth on the open set $\{t \neq 0\}$ and equal there to*

$$e^{-i(n-2)\frac{\pi}{4}} H(t)(4\pi t)^{-n/2} e^{i\frac{|x|^2}{4t}}. \tag{4.4.10}$$

The distribution E is the partial Fourier transform with respect to the variable x of the $L^\infty(\mathbb{R}^{n+1})$ function

$$\tilde{E}(t,\xi) = iH(t) e^{-4i\pi^2 t|\xi|^2}. \tag{4.4.11}$$

Proof. We want to solve the equation $-i\partial_t u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{n+1})$, we can consider its Fourier transform v with respect to x (well-defined by transposition as the Fourier transform in (4.1.10)), and we end-up with the simple ODE with parameters on v,

$$\partial_t v + i4\pi^2 |\xi|^2 v = i\delta_0(t). \tag{4.4.12}$$

Using the identity (4.4.4), we see that we can take $v = iH(t)e^{-i4\pi^2 t|\xi|^2}$, which belongs to $\mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^n_\xi)$. Taking the inverse Fourier transform with respect to ξ of both sides of (4.4.12) gives with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^n_\xi)$

$$\partial_t u - i\Delta_x u = i\delta_0(t) \otimes \delta_0(x) \quad \text{i.e.} \quad \frac{1}{i}\partial_t u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{4.4.13}$$

To compute u, we check with $\varphi \in \mathscr{D}(\mathbb{R}), \psi \in \mathscr{D}(\mathbb{R}^n)$,

$$\langle u, \varphi \otimes \psi \rangle = \langle \widehat{v}^x, \varphi \otimes \check{\psi} \rangle = \langle v, \varphi \otimes \check{\hat{\psi}} \rangle = i \int_0^{+\infty} \varphi(t) \left(\int_{\mathbb{R}^n} \hat{\psi}(\xi) e^{i\pi(-4\pi t)|\xi|^2} d\xi \right) dt. \tag{4.4.14}$$

We note now that, using (4.6.7) and (4.1.10), for $t > 0$,

$$i \int_{\mathbb{R}^n} \hat{\psi}(\xi) e^{i\pi(-4\pi t)|\xi|^2} d\xi = i \int_{\mathbb{R}^n} \psi(x)(4\pi t)^{-n/2} e^{i\frac{|x|^2}{4t}} dx e^{-n\frac{i\pi}{4}}$$
$$= e^{-i(n-2)\frac{\pi}{4}} (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{i\frac{|x|^2}{4t}} \psi(x) dx.$$

As a result, u is a distribution on $\mathbb{R}^{n+1}$ defined by

$$\langle u, \Phi \rangle = e^{-i(n-2)\frac{\pi}{4}} (4\pi)^{-n/2} \int_0^{+\infty} t^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t, x) e^{i\frac{|x|^2}{4t}} dx \right) dt$$

and coincides with E, so that E satisfies (4.4.13). The identity (4.4.14) is proving (4.4.11). The proof of the theorem is complete. □

Remark 4.4.6. The fundamental solution of the Schrödinger equation is unbounded near $t = 0$ and, since E is smooth on $t \neq 0$, its C^∞ singular support is equal to $\{0\} \times \mathbb{R}^n_x$. In particular, the Schrödinger equation is *not* hypoelliptic. We shall see that it looks like a propagation equation with an infinite speed, or more precisely with a speed depending on the frequency of the wave.

4.4.3 The wave equation

Presentation

The wave equation in d dimensions with speed of propagation $c > 0$, is given by the operator on $\mathbb{R}_t \times \mathbb{R}^d_x$

$$\Box_c = c^{-2} \partial_t^2 - \Delta_x. \tag{4.4.15}$$

We want to solve the equation $c^{-2}\partial_t^2 u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{d+1})$, we can consider its Fourier transform v with respect to x, and we end-up with the ODE with parameters on v,

$$c^{-2}\partial_t^2 v + 4\pi^2 |\xi|^2 v = \delta_0(t), \quad \partial_t^2 v + 4\pi^2 c^2 |\xi|^2 v = c^2 \delta_0(t). \tag{4.4.16}$$

Lemma 4.4.7. *Let $\lambda, \mu \in \mathbb{C}$. A fundamental solution of $P_{\lambda,\mu} = (\frac{d}{dt} - \lambda)(\frac{d}{dt} - \mu)$ (on the real line) is*

$$\begin{cases} \left(\dfrac{e^{t\lambda} - e^{t\mu}}{\lambda - \mu}\right) H(t) & \text{for } \lambda \neq \mu, \\ t e^{t\lambda} H(t) & \text{for } \lambda = \mu. \end{cases} \tag{4.4.17}$$

Proof. If $\lambda \neq \mu$, to solve $(\frac{d}{dt} - \lambda)(\frac{d}{dt} - \mu) = \delta_0(t)$, the method of variation of parameters gives a solution $a(t)e^{\lambda t} + b(t)e^{\mu t}$ with

$$\begin{pmatrix} e^{t\lambda} & e^{t\mu} \\ \lambda e^{t\lambda} & \mu e^{t\mu} \end{pmatrix} \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix} \Longrightarrow \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{1}{\lambda - \mu} \begin{pmatrix} \delta \\ -\delta \end{pmatrix} \Longrightarrow \text{(4.4.17) for } \lambda \neq \mu,$$

which gives also the result for $\lambda = \mu$ by differentiation with respect to λ of the identity $P_{\lambda,\mu}(e^{t\lambda} - e^{t\mu}) = (\lambda - \mu)\delta$. □

Going back to the wave equation, we can take v as the temperate distribution[10] given by

$$v(t,\xi) = c^2 H(t)\frac{e^{2i\pi ct|\xi|} - e^{-2i\pi ct|\xi|}}{4i\pi c|\xi|} = c^2 H(t)\frac{\sin\big(2\pi ct|\xi|\big)}{2\pi c|\xi|}. \tag{4.4.18}$$

Taking the inverse Fourier transform with respect to ξ of both sides of (4.4.16) gives with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}^d_\xi)$

$$c^{-2}\partial_t^2 u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{4.4.19}$$

To compute u, we check with $\Phi \in \mathscr{D}(\mathbb{R}^{1+d})$,

$$\langle u, \Phi\rangle = \langle \widehat{v}^x(t,\xi), \Phi(t,-\xi)\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n} \widehat{\Phi}^x(t,\xi) c\frac{\sin\big(2\pi ct|\xi|\big)}{2\pi|\xi|} d\xi dt. \tag{4.4.20}$$

We have found an expression for a fundamental solution of the wave equation in d space dimensions and proven the following proposition.

Proposition 4.4.8. *Let E_+ be the temperate distribution on $\mathbb{R}^{d+1}$ such that*

$$\widehat{E_+}^x(t,\xi) = cH(t)\frac{\sin\big(2\pi ct|\xi|\big)}{2\pi|\xi|}. \tag{4.4.21}$$

Then E_+ is a fundamental solution of the wave equation (4.4.15), i.e. satisfies $\square_c E_+ = \delta_0(t) \otimes \delta_0(x)$.

Remark 4.4.9. Defining the forward-light-cone $\Gamma_{+,c}$ as

$$\Gamma_{+,c} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^d, ct \ge |x|\}, \tag{4.4.22}$$

one can prove more precisely that E_+ is the only fundamental solution with support in $\{t \ge 0\}$ and that

$$\operatorname{supp} E_+ = \Gamma_+, \text{ when } d = 1 \text{ and } d \ge 2 \text{ is even}, \tag{4.4.23}$$
$$\operatorname{supp} E_+ = \partial\Gamma_+, \text{ when } d \ge 3 \text{ is odd}, \tag{4.4.24}$$
$$\operatorname{singsupp} E_+ = \partial\Gamma_+, \text{ in any dimension}. \tag{4.4.25}$$

Lemma 4.4.10. *Let E_1, E_2 be fundamental solutions of the wave equation such that $\operatorname{supp} E_1 \subset \Gamma_{+,c}, \operatorname{supp} E_2 \subset \{t \ge 0\}$. Then $E_1 = E_2$.*

Proof. Defining $u = E_1 - E_2$, we have $\operatorname{supp} u \subset \{t \ge 0\}$ and the mapping

$$\{t \ge 0\} \times \Gamma_{+,c} \ni \big((t,x),(s,y)\big) \mapsto (t+s, x+y) \in \mathbb{R}^{d+1}$$

is proper since

$$t,s \ge 0, cs \ge |y|, |t+s| \le T, |x+y| \le R \Longrightarrow t,s \in [0,T], |x| \le R + cT, |y| \le cT,$$

so that the section 3.5.3 allows to perform the following calculations

$$u = u * \delta_0 = u * \square_c E_1 = \square_c u * E_1 = 0.$$ □

[10] The function $\mathbb{R} \ni s \mapsto \frac{\sin s}{s} = \sum_{k\ge 0}(-1)^k \frac{s^{2k}}{(2k+1)!} = S(s^2)$ is a smooth bounded function of s^2, so that $v(t,\xi) = c^2 H(t) t S(4\pi^2 c^2 t^2 |\xi|^2)$ is continuous and such that $|v(t,\xi)| \le CtH(t)$, thus a tempered distribution.

The wave equation in one space dimension

Theorem 4.4.11. *On $\mathbb{R}_t \times \mathbb{R}_x$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{c}{2} H(ct - |x|). \tag{4.4.26}$$

where E_+ is defined in (4.4.21). That fundamental solution is bounded and the properties (4.4.23), (4.4.25) are satisfied.

Proof. We have $c^{-2}\partial_t^2 - \partial_x^2 = (c^{-1}\partial_t - \partial_x)(c^{-1}\partial_t + \partial_x)$ and changing (linearly) the variables with $x_1 = ct + x, x_2 = ct - x$, we have $t = \frac{1}{2c}(x_1 + x_2), x = \frac{1}{2}(x_1 - x_2)$, using the notation

$$(x_1, x_2) \mapsto (t, x) \mapsto u(t,x) = v(x_1, x_2),$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x_1} c + \frac{\partial v}{\partial x_2} c, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x_1} - \frac{\partial v}{\partial x_2}, \quad c^{-1}\partial_t - \partial_x = 2\partial_{x_2}, c^{-1}\partial_t + \partial_x = 2\partial_{x_1},$$

and thus $\square_c = 4\frac{\partial^2}{\partial x_1 \partial x_2}$, so that a fundamental solution is $v = \frac{1}{4} H(x_1) H(x_2)$. We have now to pull-back this distribution by the linear mapping $(t,x) \mapsto (x_1, x_2)$: we have the formula

$$\varphi(0,0) = \langle 4 \frac{\partial^2 v}{\partial x_1 \partial x_2}(x_1, x_2), \varphi(x_1, x_2) \rangle = \langle (\square_c u)(t,x), \varphi(ct + x, ct - x) \rangle 2c$$

which gives the fundamental solution $\frac{2c}{4} H(ct+x) H(ct-x) = \frac{c}{2} H(ct - |x|)$. Moreover that fundamental solution is supported in $\Gamma_{+,c}$ and since E_+ is supported in $\{t \geq 0\}$, we can apply the lemma 4.4.10 to get their equality. □

The wave equation in two space dimensions

We consider (4.4.15) with $d = 2$, i.e. $\square_c = c^{-2}\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$.

Theorem 4.4.12. *On $\mathbb{R}_t \times \mathbb{R}_x^2$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{c}{2\pi} H(ct - |x|)(c^2 t^2 - |x|^2)^{-1/2}, \tag{4.4.27}$$

where E_+ is defined in (4.4.21). That fundamental solution is L^1_{loc} and the properties (4.4.23), (4.4.25) are satisfied.

Proof. From the lemma 4.4.10, it is enough to prove that the rhs of (4.4.27) is indeed a fundamental solution. The function $E(t,x) = \frac{c}{2\pi} H(ct - |x|)(c^2t^2 - |x|^2)^{-1/2}$ is locally integrable in $\mathbb{R} \times \mathbb{R}^2$ since

$$\int_0^T \int_0^{ct} (c^2t^2 - r^2)^{-1/2} r dr dt = \int_0^T [(c^2t^2 - r^2)^{1/2}]_{r=ct}^{r=0} dt = cT^2/2 < +\infty.$$

Moreover E is homogeneous of degree -1, so that $\square_c E$ is homogeneous with degree -3 and supported in $\Gamma_{+,c}$. We use now the independently proven three-dimensional case (theorem 4.4.13). We define with $E_{+,3}$ given by (4.4.29), $\varphi \in \mathscr{D}(\mathbb{R}^3_{t,x_1,x_2})$, $\chi \in \mathscr{D}(\mathbb{R})$ with $\chi(0) = 1$,

$$\begin{aligned}
\langle u, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^3),\mathscr{D}(\mathbb{R}^3)} &= \lim_{\epsilon\to 0}\langle E_{+,3}, \varphi(t,x_1,x_2)\otimes\chi(\epsilon x_3)\rangle_{\mathscr{D}'(\mathbb{R}^4),\mathscr{D}(\mathbb{R}^4)}\\
&= \lim_{\epsilon\to 0}\frac{1}{4\pi}\iiint_{\mathbb{R}^3}\frac{\varphi(c^{-1}\sqrt{x_1^2+x_2^2+x_3^2}, x_1, x_2)}{\sqrt{x_1^2+x_2^2+x_3^2}}\chi(\epsilon x_3)dx_1dx_2dx_3\\
&= \frac{1}{4\pi}2\iiint_{\mathbb{R}^2_{x_1,x_2}\times\{x_3\ge 0\}}\frac{\varphi(c^{-1}\sqrt{x_1^2+x_2^2+x_3^2}, x_1, x_2)}{\sqrt{x_1^2+x_2^2+x_3^2}}dx_1dx_2dx_3 \quad (t = c^{-1}\sqrt{x_1^2+x_2^2+x_3^2})\\
&= \frac{1}{2\pi}\iiint_{\mathbb{R}^2_{x_1,x_2}\times\{ct\ge\sqrt{x_1^2+x_2^2}\}}\frac{\varphi(t,x_1,x_2)}{ct}\frac{1}{2}(c^2t^2-x_1^2-x_2^2)^{-1/2}2c^2t\,dx_1dx_2dt\\
&= \frac{c}{2\pi}\iiint_{\mathbb{R}^2_{x_1,x_2}\times\{ct\ge\sqrt{x_1^2+x_2^2}\}}\varphi(t,x_1,x_2)(c^2t^2-x_1^2-x_2^2)^{-1/2}dx_1dx_2dt\\
&= \langle E, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^3),\mathscr{D}(\mathbb{R}^3)}, \qquad \text{so that } E = u.
\end{aligned}$$

With $\square_{c,d}$ standing for the wave operator in d dimensions with speed c, we have, since

$$\square_{c,3}\big(\varphi(t,x_1,x_2)\otimes\chi(\epsilon x_3)\big) = \square_{c,2}\big(\varphi(t,x_1,x_2)\big)\otimes\chi(\epsilon x_3) - \varphi(t,x_1,x_2)\epsilon^2\chi''(\epsilon x_3)$$

$$\begin{aligned}
\langle\square_{c,2}u, \varphi\rangle &= \lim_{\epsilon\to 0}\langle E_{+,3}, (\square_{c,2}\varphi)(t,x_1,x_2)\otimes\chi(\epsilon x_3)\rangle\\
&= \lim_{\epsilon\to 0}\Big(\langle E_{+,3}, \square_{c,3}\big(\varphi(t,x_1,x_2)\otimes\chi(\epsilon x_3)\big)\rangle + \langle E_{+,3}, \varphi(t,x_1,x_2)\epsilon^2\chi''(\epsilon x_3)\rangle\Big)\\
&= \varphi(0,0,0),
\end{aligned}$$

which gives $\square_{c,2}E = \square_{c,2}u = \delta_{0,\mathbb{R}^3}$ and the result. □

The wave equation in three space dimensions

We consider (4.4.15) with $d = 3$, i.e. $\square_c = c^{-2}\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$.

Theorem 4.4.13. *On $\mathbb{R}_t\times\mathbb{R}^3_x$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{1}{4\pi|x|}\delta_{0,\mathbb{R}}(t - c^{-1}|x|), \tag{4.4.28}$$

$$\textit{i.e. for } \Phi\in\mathscr{D}(\mathbb{R}_t\times\mathbb{R}^3_x), \quad \langle E_+, \Phi\rangle = \int_{\mathbb{R}^3}\frac{1}{4\pi|x|}\Phi(c^{-1}|x|, x)dx. \tag{4.4.29}$$

where E_+ is defined in (4.4.21). The properties (4.4.24), (4.4.25) are satisfied.

Proof. The formula (4.4.29) is defining a Radon measure E with support $\partial\Gamma_{+,c}$, so that the last statements of the lemmas are clear. From the lemma 4.4.10, it is enough to prove that (4.4.29) defines indeed a fundamental solution. We check for $\varphi\in\mathscr{D}(\mathbb{R}), \psi\in\mathscr{D}(\mathbb{R}^3)$

$$\begin{aligned}
\langle\square_c E, \varphi(t)\otimes\psi(x)\rangle &= \langle E, \square_c(\varphi\otimes\psi)\rangle\\
&= \frac{1}{4\pi}\int_{\mathbb{R}^3}|x|^{-1}\Big(c^{-2}\varphi''(c^{-1}|x|)\psi(x) - \varphi(c^{-1}|x|)(\Delta\psi)(x)\Big)dx.
\end{aligned}$$

If we assume that $\operatorname{supp}\varphi \subset \mathbb{R}_+^*$, we get

$$\begin{aligned}\int_{\mathbb{R}^3} |x|^{-1}\varphi(c^{-1}|x|)(\Delta\psi)(x)dx &= \int_{\mathbb{R}^3} \Delta\big(|x|^{-1}\varphi(c^{-1}|x|)\big)\psi(x)dx \\ &= \int_{\mathbb{R}^3} \Big(\big(r^{-1}\varphi(c^{-1}r)\big)'' + 2r^{-1}\big(r^{-1}\varphi(c^{-1}r)\big)'\Big)\psi(x)dx \qquad (r = |x|) \\ &= \int \psi(x)\Big(r^{-1}\varphi''(c^{-1}r)c^{-2} + 2(-r^{-2})\varphi'(c^{-1}r)c^{-1} + 2r^{-3}\varphi(c^{-1}r) \\ &\qquad + 2r^{-1}r^{-1}\varphi'(c^{-1}r)c^{-1} + 2r^{-1}(-r^{-2})\varphi(c^{-1}r)\Big)dx,\end{aligned}$$

which gives $\langle \square_c E, \varphi(t) \otimes \psi(x)\rangle = 0$. As a result,

$$\operatorname{supp}(\square_c E) \subset \partial\Gamma_{+,c} \cap \{t \le 0\} = \{(0_{\mathbb{R}}, 0_{\mathbb{R}^3})\},$$

and since E is homogeneous with degree -2, the distribution $\square_c E$ is homogeneous with degree -4 with support at the origin of $\mathbb{R}^4$: the lemma 3.4.8 and the theorem 3.3.4 imply that $\square_c E = \kappa\delta_{0,\mathbb{R}^4}$. To check that $\kappa = 1$, we calculate for $\varphi \in \mathscr{D}(\mathbb{R})$ (noting that $|t| \le C$ and $|x| \le c|t| + 1$ implies $|x| \le cC + 1$)

$$\begin{aligned}\langle \square_c E, \varphi(t) \otimes 1\rangle = \frac{1}{4\pi}\int_0^{+\infty} r^{-1}c^{-2}\varphi''(c^{-1}r)r^2 dr 4\pi &= \int_0^{+\infty} \varphi''(r)rdr \\ &= [\varphi'(r)r]_0^{+\infty} - \int_0^{+\infty}\varphi'(r)dr = \varphi(0),\end{aligned}$$

so that $\kappa = 1$ and the theorem is proven. □

4.5 Periodic distributions

4.5.1 The Dirichlet kernel

For $N \in \mathbb{N}$, the Dirichlet kernel D_N is defined on $\mathbb{R}$ by

$$\begin{aligned}D_N(x) = \sum_{-N\le k\le N} e^{2i\pi kx} = 1 + 2\operatorname{Re}\sum_{1\le k\le N} e^{2i\pi kx} \underbrace{=}_{x\notin\mathbb{Z}} 1 + 2\operatorname{Re}\left(e^{2i\pi x}\frac{e^{2i\pi Nx} - 1}{e^{2i\pi x} - 1}\right) \\ = 1 + 2\operatorname{Re}\big(e^{2i\pi x - i\pi x + i\pi Nx}\big)\frac{\sin(\pi Nx)}{\sin(\pi x)} = 1 + 2\cos(\pi(N+1)x)\frac{\sin(\pi Nx)}{\sin(\pi x)} \\ = 1 + \frac{1}{\sin(\pi x)}\Big(\sin\big(\pi x(2N+1)\big) - \sin(\pi x)\Big) = \frac{\sin\big(\pi x(2N+1)\big)}{\sin(\pi x)},\end{aligned}$$

and extending by continuity at $x \in \mathbb{Z}$ that 1-periodic function, we find that

$$D_N(x) = \frac{\sin\big(\pi x(2N+1)\big)}{\sin(\pi x)}. \tag{4.5.1}$$

Now, for a 1-periodic $v \in C^1(\mathbb{R})$, with

$$(D_N \star u)(x) = \int_0^1 D_N(x-t)u(t)dt, \tag{4.5.2}$$

we have

$$\lim_{N\to+\infty}\int_0^1 D_N(x-t)v(t)dt = v(x) + \lim_{N\to+\infty}\int_0^1 \sin(\pi t(2N+1))\frac{\big(v(x-t)-v(x)\big)}{\sin(\pi t)}dt,$$

and the function θ_x given by $\theta_x(t)=\frac{v(x-t)-v(x)}{\sin(\pi t)}$ is continuous on $[0,1]$, and from the Riemann-Lebesgue lemma 4.3.5, we obtain

$$\lim_{N\to+\infty}\sum_{-N\le k\le N} e^{2i\pi kx}\int_0^1 e^{-2i\pi kt}v(t)dt = \lim_{N\to+\infty}\int_0^1 D_N(x-t)v(t)dt = v(x).$$

On the other hand if v is 1-periodic and C^{1+l}, the Fourier coefficient

$$c_k(v) = \int_0^1 e^{-2i\pi kt}v(t)dt \overbrace{=}^{\text{for } k\neq 0} \frac{1}{2i\pi k}[e^{-2i\pi kt}v(t)]_{t=1}^{t=0} + \int_0^1 \frac{1}{2i\pi k}e^{-2i\pi kt}v'(t)dt, \quad (4.5.3)$$

and iterating the integration by parts, we find $c_k(v) = O(k^{-1-l})$ so that for a 1-periodic C^2 function v, we have

$$\sum_{k\in\mathbb{Z}} e^{2i\pi kx}c_k(v) = v(x). \quad (4.5.4)$$

4.5.2 Pointwise convergence of Fourier series

Lemma 4.5.1. *Let $u:\mathbb{R}\longrightarrow\mathbb{R}$ be a 1-periodic $L^1_{loc}(\mathbb{R})$ function and let $x_0\in[0,1]$. Let us assume that there exists $w_0\in\mathbb{R}$ such that the Dini condition is satisfied, i.e.*

$$\int_0^{1/2}\frac{|u(x_0+t)+u(x_0-t)-2w_0|}{t}dt < +\infty. \quad (4.5.5)$$

Then, $\lim_{N\to+\infty}\sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = w_0$ *with* $c_k(u)=\int_0^1 e^{-2i\pi tk}u(t)dt$.

Proof. Using the calculations of the previous section 4.5.1, we find

$$\sum_{|k|\le N} c_k(u)e^{2i\pi kx_0} = (D_N * u)(x_0) = w_0 + \int_0^1 \frac{\sin\big(\pi t(2N+1)\big)}{\sin(\pi t)}\big(u(x_0-t)-w_0\big)dt,$$

so that, using the periodicity of u and the fact that D_N is an even function , we get

$$(D_N * u)(x_0) - w_0 = \int_0^{1/2}\frac{\sin\big(\pi t(2N+1)\big)}{\sin(\pi t)}\big(u(x_0-t)+u(x_0+t)-2w_0\big)dt.$$

Thanks to the hypothesis (4.5.5), the function $t\mapsto \mathbf{1}_{[0,1]}(t)\frac{u(x_0-t)+u(x_0+t)-2w_0}{\sin(\pi t)}$ belongs to $L^1(\mathbb{R})$ and the Riemann-Lebesgue lemma 4.3.5 gives the conclusion. □

Theorem 4.5.2. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a 1-periodic L^1_{loc} function.*
(1) Let $x_0 \in [0,1], w_0 \in \mathbb{R}$. We define $\omega_{x_0,w_0}(t) = |u(x_0+t)+u(x_0-t)-2w_0|$ and we assume that

$$\int_0^{1/2} \omega_{x_0,w_0}(t)\frac{dt}{t} < +\infty. \tag{4.5.6}$$

Then the Fourier series $(D_N * u)(x_0)$ converges with limit w_0. In particular, if (4.5.6) is satisfied with $w_0 = u(x_0)$, the Fourier series $(D_N * u)(x_0)$ converges with limit $u(x_0)$. If u has a left and right limit at x_0 and is such that (4.5.6) is satisfied with $w_0 = \frac{1}{2}\big(u(x_0+0)+u(x_0-0)\big)$, the Fourier series $(D_N * u)(x_0)$ converges with limit $\frac{1}{2}\big(u(x_0-0)+u(x_0+0)\big)$.
(2) If the function u is Hölder-continuous[11]*, the Fourier series $(D_N * u)(x)$ converges for all $x \in \mathbb{R}$ with limit $u(x)$.*
(3) If u has a left and right limit at each point and a left and right derivative at each point, the Fourier series $(D_N * u)(x)$ converges for all $x \in \mathbb{R}$ with limit $\frac{1}{2}\big(u(x-0)+u(x+0)\big)$.

Proof. (1) follows from the lemma 4.5.1; to obtain (2), we note that for a Hölder continuous function of index $\theta \in]0,1]$, we have for $t \in]0,1/2]$

$$t^{-1}\omega_{x,u(x)}(t) \le Ct^{\theta-1} \in L^1([0,1/2]).$$

If u has a right-derivative at x_0, it means that

$$u(x_0+t) = u(x_0+0) + u_r'(x_0)t + t\epsilon_0(t), \qquad \lim_{t\to 0_+}\epsilon_0(t) = 0.$$

As a consequence, for $t \in]0,1/2]$, $t^{-1}|u(x_0+t)-u(x_0+0)| \le |u_r'(x_0)+\epsilon_0(t)|$. Since $\lim_{t\to 0_+}\epsilon_0(t) = 0$, there exists $T_0 \in]0,1/2]$ such that $|\epsilon_0(t)| \le 1$ for $t \in [0,T_0]$. As a result, we have

$$\int_0^{1/2} t^{-1}|u(x_0+t)-u(x_0+0)|dt$$
$$\le \int_0^{T_0}(|u_r'(x_0)|+1)dt + \int_{T_0}^{1/2}|u(x_0+t)-u(x_0+0)|dtT_0^{-1} < +\infty,$$

since u is also L^1_{loc}. The integral $\int_0^{1/2} t^{-1}|u(x_0-t)-u(x_0-0)|dt$ is also finite and the condition (4.5.6) holds with $w_0 = \frac{1}{2}\big(u(x_0-0)+u(x_0+0)\big)$. The proof of the lemma is complete. □

4.5.3 Periodic distributions

We consider now a distribution u on $\mathbb{R}^n$ which is periodic with periods $\mathbb{Z}^n$. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi = 1$ on $[0,1]^n$. Then the function χ_1 defined by

$$\chi_1(x) = \sum_{k\in\mathbb{Z}^n}\chi(x-k)$$

[11] Hölder-continuity of index $\theta \in]0,1]$ means that $\exists C > 0, \forall t,s,\ |u(t)-u(s)| \le C|t-s|^\theta$.

is C^∞ periodic[12] with periods $\mathbb{Z}^n$. Moreover since $\mathbb{R}^n \ni x \in \prod_{1\le j\le n}[E(x_j), E(x_j)+1[$, the bounded function χ_1 is also bounded from below and such that $1 \le \chi_1(x)$. With $\chi_0 = \chi/\chi_1$, we have

$$\sum_{k\in\mathbb{Z}^n} \chi_0(x-k) = 1, \quad \chi_0 \in C_c^\infty(\mathbb{R}^n).$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have from the periodicity of u

$$\langle u, \varphi\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \varphi(x)\chi_0(x-k)\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \varphi(x+k)\chi_0(x)\rangle,$$

where the sums are finite. Now if $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have, since χ_0 is compactly supported in $|x| \le R_0$,

$$\begin{aligned} |\langle u(x), \varphi(x+k)\chi_0(x)\rangle| &\le C_0 \sup_{|\alpha|\le N_0, |x|\le R_0} |\varphi^{(\alpha)}(x+k)| \\ &\le C_0 \sup_{|\alpha|\le N_0, |x|\le R_0} |(1+R_0+|x+k|)^{n+1}\varphi^{(\alpha)}(x+k)|(1+|k|)^{-n-1} \\ &\le p_0(\varphi)(1+|k|)^{-n-1}, \end{aligned}$$

where p_0 is a semi-norm of φ (independent of k). As a result u is a tempered distribution and we have for $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \langle u(x), \sum_{k\in\mathbb{Z}^n} \underbrace{\varphi(x+k)\chi_0(x)}_{\psi_x(k)}\rangle = \langle u(x), \sum_{k\in\mathbb{Z}^n} \widehat{\psi_x}(k)\rangle.$$

Now we see that $\widehat{\psi_x}(k) = \int_{\mathbb{R}^n} \varphi(x+t)\chi_0(x)e^{-2i\pi kt}dt = \chi_0(x)e^{2i\pi kx}\hat{\varphi}(k)$, so that $\langle u, \varphi\rangle = \sum_{k\in\mathbb{Z}^n} \langle u(x), \chi_0(x)e^{2i\pi kx}\rangle\hat{\varphi}(k)$ which means

$$u(x) = \sum_{k\in\mathbb{Z}^n} \langle u(t), \chi_0(t)e^{2i\pi kt}\rangle e^{-2i\pi kx} = \sum_{k\in\mathbb{Z}^n} \langle u(t), \chi_0(t)e^{-2i\pi kt}\rangle e^{2i\pi kx}.$$

Theorem 4.5.3. *Let u be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. Then u is a tempered distribution and if χ_0 is a $C_c^\infty(\mathbb{R}^n)$ function such that $\sum_{k\in\mathbb{Z}^n} \chi_0(x-k) = 1$, we have*

$$u = \sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \tag{4.5.7}$$

$$\hat{u} = \sum_{k\in\mathbb{Z}^n} c_k(u)\delta_k, \quad \text{with} \quad c_k(u) = \langle u(t), \chi_0(t)e^{-2i\pi kt}\rangle, \tag{4.5.8}$$

and convergence in $\mathscr{S}'(\mathbb{R}^n)$. If u is in $C^m(\mathbb{R}^n)$ with $m > n$, the previous formulas hold with uniform convergence for (4.5.7) and

$$c_k(u) = \int_{[0,1]^n} u(t)e^{-2i\pi kt}dt. \tag{4.5.9}$$

[12] Note that the sum is locally finite since for K compact subset of $\mathbb{R}^n$, $(K-k)\cap \operatorname{supp}\chi_0 = \emptyset$ except for a finite subset of $k \in \mathbb{Z}^n$.

Proof. The first statements are already proven and the calculation of $\hat{u}$ is immediate. If u belongs to L^1_{loc} we can redo the calculations above choosing $\chi_0 = \mathbf{1}_{[0,1]^n}$ and get (4.5.7) with c_k given by (4.5.9). Moreover, if u is in C^m with $m > n$, we get by integration by parts that $c_k(u)$ is $O(|k|^{-m})$ so that the series (4.5.7) is uniformly converging. □

Theorem 4.5.4. *Let u be a periodic distribution on $\mathbb{R}^n$ with periods $\mathbb{Z}^n$. If $u \in L^2_{loc}$ (i.e. $u \in L^2(\mathbb{T}^n)$ with $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$), then*

$$u(x) = \sum_{k\in\mathbb{Z}^n} c_k(u)e^{2i\pi kx}, \quad \textit{with} \quad c_k(u) = \int_{[0,1]^n} u(t)e^{-2i\pi kt}dt, \tag{4.5.10}$$

and convergence in $L^2(\mathbb{T}^n)$. Moreover $\|u\|^2_{L^2(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |c_k(u)|^2$. Conversely, if the coefficients $c_k(u)$ defined by (4.5.8) are in $\ell^2(\mathbb{Z}^n)$, the distribution u is $L^2(\mathbb{T}^n)$

Proof. As said above the formula for the $c_k(u)$ follows from changing the choice of χ_0 to $\mathbf{1}_{[0,1]^n}$ in the discussion preceding the theorem 4.5.3. The formula (4.5.7) gives the convergence in $\mathscr{S}'(\mathbb{R}^n)$ to u. Now, since $\int_{[0,1]^n} e^{2i\pi(k-l)t}dt = \delta_{k,l}$ we see from the theorem 4.5.3 that for $u \in C^{n+1}(\mathbb{T}^n)$, $\langle u, u\rangle_{L^2(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |c_k(u)|^2$. As a consequence the mapping $L^2(\mathbb{T}^n) \ni u \mapsto (c_k(u))_{k\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$ is isometric with a range containing the dense subset $\ell^1(\mathbb{Z}^n)$ (if $(c_k(u))_{k\in\mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n)$, u is a continuous function); since the range is closed, the mapping is onto and is an isometric isomorphism from the open mapping theorem. □

4.6 Appendix

4.6.1 The logarithm of a nonsingular symmetric matrix

The set $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, so that we can define the principal determination of the logarithm for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by the formula

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\zeta}{\zeta}. \tag{4.6.1}$$

The function Log is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$ and we have $\operatorname{Log} z = \ln z$ for $z \in \mathbb{R}^*_+$ and by analytic continuation $e^{\operatorname{Log} z} = z$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. We get also by analytic continuation, that $\operatorname{Log} e^z = z$ for $|\operatorname{Im} z| < \pi$.

Let Υ_+ be the set of symmetric nonsingular $n \times n$ matrices with complex entries and nonnegative real part. The set Υ_+ is star-shaped with respect to the Id: for $A \in \Upsilon_+$, the segment $[1, A] = \big((1-t)\operatorname{Id} + tA\big)_{t\in[0,1]}$ is obviously made with symmetric matrices with nonnegative real part which are invertible[13], since for $0 \le t < 1$, $\operatorname{Re}\big((1-t)\operatorname{Id} + tA\big) \ge (1-t)\operatorname{Id} > 0$ and for $t = 1$, A is assumed to be invertible. We can now define for $A \in \Upsilon_+$

$$\operatorname{Log} A = \int_0^1 (A - I)\big(I + t(A - I)\big)^{-1} dt. \tag{4.6.2}$$

[13]Note that a symmetric matrix B with a positive-definite real part is indeed invertible since for $u \in \mathbb{C}^n$, $Bu = 0$ implies $0 = \operatorname{Re}\langle Bu, \bar{u}\rangle = \langle(\operatorname{Re} B)u, \bar{u}\rangle \ge c_0\|u\|^2$ with $c_0 > 0$ and thus $u = 0$.

We note that A commutes with $(I+sA)$ (and thus with $\operatorname{Log} A$), so that, for $\theta>0$,

$$\frac{d}{d\theta}\operatorname{Log}(A+\theta I)=\int_0^1\bigl(I+t(A+\theta I-I)\bigr)^{-1}dt \\ -\int_0^1(A+\theta I-I)t\bigl(I+t(A+\theta I-I)\bigr)^{-2}dt,$$

and since $\frac{d}{dt}\Bigl\{\bigl(I+t(A+\theta I-I)\bigr)^{-1}\Bigr\}=-\bigl(I+t(A+\theta I-I)\bigr)^{-2}(A+\theta I-I)$, we obtain by integration by parts $\frac{d}{d\theta}\operatorname{Log}(A+\theta I)=(A+\theta I)^{-1}$. As a result, we find that for $\theta>0, A\in\Upsilon_+$, since all the matrices involved are commuting,

$$\frac{d}{d\theta}\left((A+\theta I)^{-1}e^{\operatorname{Log}(A+\theta I)}\right)=0,$$

so that, using the limit $\theta\to+\infty$, we get that $\forall A\in\Upsilon_+,\forall\theta>0,\ e^{\operatorname{Log}(A+\theta I)}=(A+\theta I)$, and by continuity

$$\forall A\in\Upsilon_+,\quad e^{\operatorname{Log} A}=A,\quad\text{which implies}\quad\det A=e^{\operatorname{trace}\operatorname{Log} A}. \tag{4.6.3}$$

Using (4.6.3), we can define for $A\in\Upsilon_+$, using (4.6.2)

$$(\det A)^{-1/2}=e^{-\frac12\operatorname{trace}\operatorname{Log} A}=|\det A|^{-1/2}e^{-\frac{i}{2}\operatorname{Im}(\operatorname{trace}\operatorname{Log} A)}. \tag{4.6.4}$$

- When A is a positive definite matrix, $\operatorname{Log} A$ is real-valued and $(\det A)^{-1/2}=|\det A|^{-1/2}$.
- When $A=-iB$ where B is a real nonsingular symmetric matrix, we note that $B=PD^tP$ with $P\in O(n)$ and D diagonal. We see directly on the formulas (4.6.2),(4.6.1) that

  $$\operatorname{Log} A=\operatorname{Log}(-iB)=P(\operatorname{Log}(-iD))^tP,\quad\operatorname{trace}\operatorname{Log} A=\operatorname{trace}\operatorname{Log}(-iD)$$

  and thus, with (μ_j) the (real) eigenvalues of B, we have $\operatorname{Im}(\operatorname{trace}\operatorname{Log} A)=\operatorname{Im}\sum_{1\le j\le n}\operatorname{Log}(-i\mu_j)$, where the last Log is given by (4.6.1). Finally we get,

  $$\operatorname{Im}(\operatorname{trace}\operatorname{Log} A)=-\frac{\pi}{2}\sum_{1\le j\le n}\operatorname{sign}\mu_j=-\frac{\pi}{2}\operatorname{sign} B$$

  where $\operatorname{sign} B$ is the signature of B. As a result, we have when $A=-iB$, B real symmetric nonsingular matrix

  $$(\det A)^{-1/2}=|\det A|^{-1/2}e^{i\frac{\pi}{4}\operatorname{sign}(iA)}=|\det B|^{-1/2}e^{i\frac{\pi}{4}\operatorname{sign} B}. \tag{4.6.5}$$

4.6.2 Fourier transform of Gaussian functions

Proposition 4.6.1. *Let A be a symmetric nonsingular $n\times n$ matrix with complex entries such that $\operatorname{Re} A\ge0$. We define the Gaussian function v_A on $\mathbb{R}^n$ by $v_A(x)=e^{-\pi\langle Ax,x\rangle}$. The Fourier transform of v_A is*

$$\widehat{v_A}(\xi)=(\det A)^{-1/2}e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{4.6.6}$$

where $(\det A)^{-1/2}$ is defined according to the formula (4.6.4). In particular, when $A=-iB$ with a symmetric real nonsingular matrix B, we get

$$\operatorname{Fourier}(e^{i\pi\langle Bx,x\rangle})(\xi)=\widehat{v_{-iB}}(\xi)=|\det B|^{-1/2}e^{i\frac{\pi}{4}\operatorname{sign} B}e^{-i\pi\langle B^{-1}\xi,\xi\rangle}. \tag{4.6.7}$$

Proof. Let us define Υ_+^* as the set of symmetric $n \times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax = 0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax, \bar{x}\rangle = \langle (\operatorname{Re} A)x, \bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function v_A is in the Schwartz class (and so is its Fourier transform). The set Υ_+^* is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (4.6.6). On the other hand the function $\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi, \xi\rangle}$ is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (4.6.6) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle \widehat{v_A}, \varphi\rangle_{\mathscr{S}', \mathscr{S}} = \int v_A(x)\hat{\varphi}(x)dx$ so that $\Upsilon_+ \ni A \mapsto \langle \widehat{v_A}, \varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is an homeomorphism of Υ_+), using the previous result on Υ_+^*,

$$\langle \widehat{v_A}, \varphi\rangle = \lim_{\epsilon \to 0_+} \langle \widehat{v_{A+\epsilon I}}, \varphi\rangle = \lim_{\epsilon \to 0_+} \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log}(A+\epsilon I)} e^{-\pi\langle (A+\epsilon I)^{-1}\xi, \xi\rangle}\varphi(\xi)d\xi$$

$$\text{(by continuity of Log on } \Upsilon_+ \text{ and domin. cv.)} \qquad = \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi, \xi\rangle}\varphi(\xi)d\xi,$$

which is the sought result. □