Complex WKB method for difference equations with meromorphic coefficients

A.Fedotov
Saint Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034, Russia;
e-mail: a.fedotov@spbu.ru

F. Klopp
Sorbonne Universités, UPMC Univ. Paris 06, UMR 7586, IMJ-PRG, F-75005, Paris, France; Univ. Paris Diderot, Sorbonne Paris Cité, UMR 7586, IMJ-PRG, F-75205 Paris, France; CNRS, UMR 7586, IMJ-PRG, F-75005, Paris, France; e-mail: frederic.klopp@imj-prg.fr

In this talk, we discuss the one-dimensional difference Schrödinger equation in the complex plane. To study its solutions in the complex plane in the quasiclassical limit, V.Buslaev, A.Fedotov and E.Shchetka developed an analog of the complex WKB method. We assume that the potential has a simple pole and study the behavior of solutions in its neighborhood.

1 Introduction

Consider the difference equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0, \qquad (1)$$

where z is the complex variable, $z\in\mathbb{C}$, v is a given analytic or meromorphic function, and $h>0$ is a constant *translation* parameter.

One encounters such equations, for example, in solid state physics when studying, say, an electron in a crystal in a weak magnetic field. The translation parameter h is proportional to the magnetic flux through the periodicity cell. In solid state physics, one has $v(z)=w(z)-E$, where the function w is called the potential and the parameter E is called the spectral parameter. When $v(z)=\cos(z)-E$, The equation becomes the famous Harper equation, see, e.g., [1]; when $v(z)=\tan(z)-E$, it is a close relative of the well-known Maryland equation introduced by D.Grempel, S.Fishman and R.Prange in [2].

Difference equations with a small translation parameter arise also in the study of the scattering of waves on wedge-shaped domains in the framework of the Sommerfeld-Malyuzhinets method. In this case, the translation parameter appears to be proportional to the angle of the wedge, see [6].

We study the asymptotics of solutions to (1) as $h\to 0$. Since formally $\exp\left(h\frac{d}{dz}\right)\Psi(z)=\Psi(z+h)$, the parameter h in (1) can be regarded as a small parameter in front of the derivative and, thus, appears to be a standard quasiclassical parameter.

To study the one-dimensional differential Schrödinger equations in the quasiclassical limit, one uses the classical complex WKB method, see [3]. In [4, 5] the authors developed an analog of the complex WKB method to study one-dimensional difference Schrödinger equations with analytic coefficients.

In this paper, we consider the case of meromorphic v. To be more precise, we assume that B_0 is a neighborhood of $z=0$ (here and below a neighborhood of a point is an open disc centered at this point), and that v is analytic in $B_0\setminus\{0\}$ and has a simple pole at zero. Let ψ be a solution to (1) analytic in B_0 to the left of zero, i.e., when $\operatorname{Re} z<0$. Equation (1) implies that $\psi(z)=-\psi(z-2h)-v(z-h)\psi(z-h)$. Therefore, ψ can be meromorphic in B_0 and can have poles at the points $z=h,2h,3h\dots$.

When h is small, these points become close one to another. We describe the quasiclassical asymptotics in B_0 of solutions to (1) having poles at $z=h,2h,3h\dots$.

2 A brief introduction to the complex WKB method

Let us briefly describe the main construction of the complex WKB method for the difference equation (1) with an analytic coefficient v. We note that formally this equation can be written in the form

$$(2\cos\hat{p}+v(z))\psi(z)=0, \qquad (2)$$
$$\hat{p}=-ihd/dz.$$

We define *the complex momentum* p by the formula

$$2\cos p(z)+v(z)=0.$$

It is an analytic multivalued function. Its branch points satisfy the relations $\pm 2 + v(z) = 0$. We call a subset D of the domain of analyticity of v *regular*, if $v(z) \neq \pm 2$ in D.

Let D be a simply connected regular domain, and let p be a branch of the complex momentum analytic in D. All the other branches of p that are analytic in D are of the form $\pm p(z) + 2\pi m$, $m \in \mathbb{Z}$.

The complex momentum is the main analytic object of the complex WKB method. In terms of the complex momentum, one defines *canonical domain* that are the main geometric objects of the method. The precise definition of a canonical domain can be found in [4, 5]. Here, we note that canonical domains are regular and simply connected, and that one has

Theorem 1 *Any regular point is contained in a canonical domain.*

This statement is an analog of Lemma 5.3 from [7]. The principle result of the method is

Theorem 2 ([5]) *Let $K \subset \mathbb{C}$ be a canonical domain, and let p be a branch of the complex momentum analytic in K. For sufficiently small h, there exist ψ, a solution to* (1) *analytic in K and such that in K as $h \to 0$*

$$\psi(z) = \frac{1}{\sqrt{\sin(p(z))}}\, e^{\frac{i}{h}\int^z p(z)\,dz}. \qquad (3)$$

The asymptotic is locally uniform in z.

In the case of meromorphic potentials, when saying that a set D is regular, we additionally assume that the potential is analytic in D.

3 A continuation principle

We now continue to discuss the case of analytic v. Let z_0 be a regular point, and let V_0 be its regular neighborhood. Assume that there exists ψ, a solution to (1) that, in V_0, is analytic and admits the uniform asymptotic representation (3).

There exist general statements (Continuation principles) allowing to describe the asymptotics of ψ outside of V_0. One of them is

Theorem 3 (The rectangle lemma) *Consider the straight line $L = \{z \in \mathbb{C} : \operatorname{Im} z = \operatorname{Im} z_0\}$. Let z_1 be a point of L such that (1) $\operatorname{Re} z_1 > \operatorname{Re} z_0$; (2) the segment $[z_0, z_1] = \{z \in L : \operatorname{Re} z_0 \leq \operatorname{Re} z \leq \operatorname{Re} z_1\}$ is regular. If $\operatorname{Im} p(z) < 0$ along $[z_0, z_1]$, then the asymptotic representation* (3) *is valid and uniform in an independent of h regular neighborhood of $[z_0, z_1]$.*

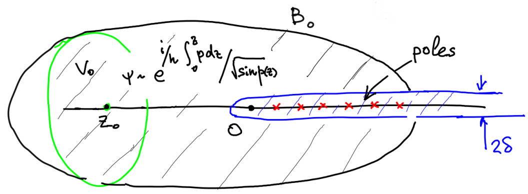

Figure 1: B_0, V_0, the δ-neighborhood of $\mathbb{R}_+$

This theorem is an analog of Lemma 5.1 from [7]. It roughly says that the asymptotic of a solution stays valid along a horizontal line as long as its leading term grows.

4 Typical quasiclassical formulation of the problem

For the sake of simplicity, we additionally assume that $v(z) \in \mathbb{R}$ for $z \in \mathbb{R}$.

Note that if B_0 is sufficiently small then
(1) in B_0 $v(z) \in \mathbb{R}$ only for $z \in \mathbb{R}$;
(2) the set $B_0 \setminus \{0\}$ is regular.
Below, we assume that B_0 possesses these two properties.

We denote by B_0' the domain B_0 cut along $\mathbb{R}_+$, the positive part of the real line. Let p be a branch of the complex momentum analytic in B_0'. For B_0 that we consider, the imaginary part of $p(z)$ does not vanish in B_0'. For the sake of definiteness, we assume that it is negative.

In B_0 we pick $z_0 < 0$, see Fig. 1. In view of Theorem 1, there is a regular neighborhood $V_0 \subset B_0$ of z_0 such that there exists a solution ψ to equation (1) that, in V_0, is analytic and satisfies (3).

Let $\delta > 0$ be sufficiently small. Let B_δ' be the domain B_0 without the δ-neighborhood of $\mathbb{R}_+$. As $\operatorname{Im} p(z) < 0$ in B_0', by Theorem 3 the asymptotic representation (3) for ψ is valid and uniform in B_δ' to the right from V_0. The problem is to describe the asymptotics of ψ in the δ-neighborhood of $\mathbb{R}_+$.

5 The main result

It can be easily checked that the complex momentum p has a logarithmic branch point at zero. More precisely, one has

Lemma 1 *In B_0' fix an analytic branch of* $\ln$. *The function $z \mapsto p(z) + i \ln z$ is analytic in B_0. The* *function $z \mapsto z \sin p(z)$ is analytic and does not vanish in B_0.*

For $z \in B_0$ we set

$$U_0(z) = \sqrt{\frac{h}{-2\pi z \sin p(z)}} \times \exp\left(\frac{z}{h}\ln\frac{1}{h} + \frac{i}{h}\int_0^z (p(z) - i\ln(-z))\,dz\right),$$

where p and $\sqrt{p}$ are the functions used in (3) to describe ψ, $z \mapsto \ln(-z)$ and $z \mapsto \sqrt{-z}$ denote branches analytic in B_0' satisfying $\ln(-z)|_{z=-1} = 0$ and $\sqrt{-z}|_{z=-1} = 1$. By Lemma 1, U_0 is analytic in B_0. Our main result is

Theorem 4 *Let $\delta > 0$ be sufficiently small. In the δ-neighborhood of $\mathbb{R}_+$, the solution ψ_+ admits the following uniform asymptotic representation*

$$\psi(z) = \Gamma\left(1 - \frac{z}{h}\right) U_0(z)\,(1 + o(1)), \quad h \to 0. \quad (4)$$

So, the special function describing the asymptotic behavior of ψ_+ near the small poles generated by the pole of the potential at $z = 0$ is the Euler Γ-function.

Close to the point $z = 0$ the formula (4) can not be simplified. For large values of $|z/h|$ the Γ-function in (4) can be replaced with its asymptotics. To be more precise, let us pick $\epsilon > 0$. By means of the asymptotic formula

$$\Gamma(1+\zeta) = \sqrt{2\pi\zeta}\, e^{\zeta(\ln\zeta - 1) + o(1)}, \ |\zeta| \to \infty \quad (5)$$

that is uniform in the sector $|\arg\zeta| \le \pi - \epsilon$, one easily checks that for $|\arg z - \pi| \le \pi - \epsilon$ and, say, $|z| \ge \delta/2$ representation (4) turns into (3).

Now let us discuss the case where $|z| \ge \delta/2$ and $|\arg z| \le \epsilon$. In this case, to simplify (4), first we use the relation $\Gamma(1-\zeta) = \frac{\pi}{\sin(\pi\zeta)}\frac{1}{\Gamma(\zeta)}$ and, next, the asymptotic representation (5). This leads to the uniform asymptotic representation

$$\psi(z) = \frac{1}{1 - e^{-2\pi i z/h}} \frac{e^{\frac{i}{h}\int^z p(z)\,dz + o(1)}}{\sqrt{\sin(p(z))}}, \quad h \to 0,$$

where p and $\sqrt{\sin(p)}$ are the branches obtained by analytic continuation (in the anticlockwise direction) from the sector $|\arg z - \pi| \le \pi - \epsilon$ to the sector under consideration.

5.1 *The ideas of the proof*

Let us note that the function $z \mapsto f(z) = \psi(z)/(\Gamma(1 - z/h)U_0(z))$ is analytic in z in a neighborhood of zero independent of h, e.g., a disc of radius $\delta > 0$. Therefore, to prove (4), by the maximum principle, it suffices to check that $f(z) = 1 + o(1)$ for, say, $|z| = \delta/2$. This is done by means of a rather standard asymptotic computation made using the complex WKB method for difference equations.

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References

[1] Wilkinson, M., 1984, Critical properties of electron eigenstates in incommensurate systems, *Proc. Roy. Soc. London Ser. A,*, Vol. **391**, pp. 305–350.

[2] D. Grempel, D., Fishman, S., Prange R., 1982, Localization in an incommensurate potential: An exactly solvable model, *Physical Review Letters*, Vol. **49**, pp. 833–836.

[3] Fedoryuk, M. V., 2009, *Asymptotic Analysis. Linear Ordinary Differential Equations*, Springer-Verlag, Berlin-Heidelberg GmbH.

[4] Buslaev, V., Fedotov, A., 1995, The complex WKB method for Harpers equation, *St. Petersburg Math. J.*, Vol. **6**, pp. 495-517.

[5] Fedotov, A., Shchetka, E., 2017, Complex WKB method for the difference Schrödinger equations with the potentials being trigonometric polynomials. *Algebra and Analyz (In Russian; to be translated into English in St.Petersburg Math.J.)*, Vol. **29**, pp. 188-214.

[6] Babich, V., Lyalinov, M. and Grikurov, V., 2008, *Diffraction theory: the Sommerfeld-Malyuzhinets technique*, Alpha Science, Oxford.

[7] Fedotov, A., Klopp, F., 2005, On the absolutely continuous spectrum of one dimensional quasi-periodic Schrödinger operators in the adiabatic limit, *Transactions of the AMS*, Vol. 357, pp. 4481–4516.