SEMICLASSICAL ASYMPTOTICS OF MEROMORPHIC SOLUTIONS OF DIFFERENCE EQUATIONS

ALEXANDER FEDOTOV AND FRÉDÉRIC KLOPP

ABSTRACT. We consider the difference Schrödinger equation $\psi(z+h)+\psi(z-h)+v(z)\psi(z)=E\psi(z)$ where z is a complex variable, E is a spectral parameter, and h is a small positive parameter. If the potential v is an analytic function, then, for h sufficiently small, the analytic solutions to this equation have standard semi-classical behavior that can be described by means of an analog of the complex WKB method for differential equations. In the present paper, we assume that v has a simple pole and, in its neighborhood, we study the asymptotics of meromorphic solutions to the difference Schrödinger equation.

1. Introduction

We study the difference Schrödinger equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=E\psi, \tag{1.1}$$

where z is a complex variable, v is a given meromorphic function called *potential*, E is a *spectral* parameter, and h is a small positive *shift* parameter [1].

Instead of (1.1), one often considers equations of the form

$$\phi_{k+1}+\phi_{k-1}+v(kh+\theta)\phi_k=E\phi_k, \tag{1.2}$$

where $k\in\mathbb{Z}$ is an integer variable, and $\theta\in\mathbb{R}$ is a parameter. There is a simple relation between (1.1) and (1.2): if ψ is a solution to (1.1), then the formula $\phi_k=\psi(kh+\theta)$ yields a solution to (1.2). We note that, when h is small, the coefficient in front of ϕ_k in (1.2) varies slowly in k.

Formally, $\psi(z+h)=\sum_{l=0}^{\infty}\frac{h^l}{l!}\frac{d^l\psi}{dz^l}(z)=e^{h\frac{d}{dz}}\psi(z)$; thus, in (1.1) h is a small parameter in front of the derivative. So, h is a standard semiclassical parameter.

The semi-classical asymptotics of solutions to ordinary differential equations, e.g., the Schrödinger equation

$$-h^2\frac{d^2\psi}{dz^2}(z)+v(z)\psi(z)=E\psi(z), \tag{1.3}$$

are described by means of the well-known WKB method (called so after G. Wentzel, H. Kramers and L. Brillouin). There is a huge literature devoted to this method and its applications. If v in (1.3) is analytic, one uses the variant often called the complex WKB method (see, e.g., [20, 6]). This powerful and classical asymptotic method is used to study solutions to (1.3) on the complex plane. Even when studying this equation on the real line, the complex WKB method is used to compute exponentially small quantities (such as the overbarier tunneling coefficient or the exponentially small lengths of spectral gaps in the case of a periodic v) or to simplify the asymptotic analysis (e.g. by going to the complex plane to avoid turning points), see, e.g., [6]. The case of meromorphic coefficients is a classical topic in

Key words and phrases. Difference equations, meromorphic coefficients, semiclassical asymptotics, complex WKB method.

The work was supported by PRC Russie CNRS, France, and the Russian Foundation for Basic Research under the grant No 17-51-150008.

[1] The results were announced in the conference proceedings [10]

the complex WKB theory. The analog of the complex WKB method for difference equations is being developed in [3, 14, 11] and in the present paper, where we turn to meromorphic solutions to equation (1.1).

Difference equations (1.1) on $\mathbb{R}$ or on $\mathbb{C}$ and (1.2) on $\mathbb{Z}$ with a small h arise in many fields of mathematics and physics. In quantum physics, for example, one encounters such equations when studying in various asymptotic situations an electron in a two-dimensional crystal submitted to a constant magnetic field (see, e.g. [21] and references therein). The electron is described by a magnetic Schrödinger operator with a periodic electric potential. And, for example, in the semi-classical limit, in certain cases its analysis asymptotically reduces to analyzing an h-pseudo-differential operator with the symbol $H(x,p) = 2\cos p + 2\cos x$ (see [18]). Its eigenfunctions satisfy equation (1.1) with $v(z) = 2\cos z$. The parameter h is proportional to the magnetic flux through the periodicity cell, and the case when h is small is a natural one. The reader can find more references and examples in [17]. We add only that equation (1.2) for $v(z) = \lambda\cos(2\pi z)$, λ being a coupling constant, is the famous almost Mathieu equation (see, e.g., [1]), and, for $v(z) = \lambda\cot(\pi z)$, it is the well known Maryland equation (Maryland model) introduced by specialists in solid state physics in [15].

Difference equations in the complex plane (with analytic or meromorphic coefficients) arise in many other fields of mathematics and physics, in particular, in the study of the diffraction of acoustic waves by wedges (see, e.g., [2]) or in the theory of differential quasi-periodic equations (see, e.g., [8]). Small shift parameters arise in the problem of diffraction by thin wedges (the shift parameter is proportional to the angle of the wedge (see [2])) and for quasi-periodic equations with two periods of small ratio (the shift parameter is proportional to this ratio (see, e.g., [8])). The semi-classical analysis of difference equations is also used to study the asymptotics of orthogonal polynomials (see, e.g., [16, 4, 22]).

Even when studying (1.2) on $\mathbb{Z}$, it is quite natural to pass to the analysis of (1.1) on $\mathbb{R}$ or on $\mathbb{C}$ as for this equation one can fruitfully use numerous analytic ideas developed in the theory of differential equations, e.g., tools of the theory of pseudo-differential operators and of the complex WKB method. If the coefficient v is periodic, for equation (1.1) one can use ideas of the Floquet theory for differential equations with periodic coefficients, which leads to a natural renormalization method (see [7]).

B. Helffer and J. Sjöstrand (e.g., in [18]) and V. Buslaev and A. Fedotov (see, e.g. [7]) studied the cantorian geometrical structure of the spectrum of the Harper operator in the semiclassical approximation. Therefore, V. Buslaev and A. Fedotov began to develop the complex WKB method for difference equations in [3]. We are going to use the results of the present paper to study in the semiclassical approximation the multiscale structure of the (generalized) eigenfunctions of the Maryland operator (by means of the renormalization method described in [12]). In the "anti-semiclassical" case, for the almost Mathieu operator such a problem was solved in [19].

In this paper, for small h, we describe uniform asymptotics of meromorphic solutions to equation (1.1) near a simple pole of v. In the case of a differential equation, say, equation (1.3) with a meromorphic v, the solutions may have singularities (branch points or isolated singular points) only at poles of v. In the case of equation (1.1), the behavior of its solutions is completely different.

Let $d_x > 0$, $d_y > 0$, and $S = \{z \in \mathbb{C} : |\mathrm{Re}\, z| < d_x, |\mathrm{Im}\, z| < d_y\}$. We assume that v is analytic in $S \setminus \{0\}$ and has a simple pole at zero. Let ψ be a solution to equation (1.1) that is analytic in $\{z \in S : \mathrm{Re}\, z < 0\}$. Equation (1.1) implies that $\psi(z) = -\psi(z-2h) - v(z-h)\psi(z-h)$. Therefore, for sufficiently small h, ψ

can be meromorphically continued into S. It may have poles at the points of $h\mathbb{N}$. When h becomes small, these points become close one to another. We describe the semi-classical asymptotics of such meromorphic solutions in S.
Below, unless stated otherwise, the estimates of the error terms in the asymptotic formulas are locally uniform for z in the domain that we consider (i.e., uniform on any given compact subset of such a domain).
Instead of saying that an asymptotic representation is valid for sufficiently small h, we write that it is valid as $h \to 0$.
In the sequel, we shall not distinguish between a meromorphic function and its meromorphic continuation to a larger domain.
We also use the notations $\mathbb{R}_\pm = \{z \in \mathbb{C} : \operatorname{Im} z = 0,\ \pm\operatorname{Re} z \geq 0\}$, $\mathbb{R}_\pm^* = \mathbb{R}_\pm \setminus 0$ and $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm\operatorname{Im} z > 0\}$.

2. Main results

2.1. The complex WKB method in a nutshell. Formally equation (1.1) can be written in the form

$$(2\cos\hat{p} + v(z))\psi(z) = 0, \quad \hat{p} = -ih\frac{d}{dz}\ . \tag{2.1}$$

One of the main objects of the complex WKB method is *the complex momentum* p defined by the formula

$$2\cos\, p(z) + v(z) = 0. \tag{2.2}$$

It is an analytic multivalued function. Its branch points are solutions to $v(z) = \pm 2$. The points where $v(z) = \pm 2$ are called *turning points*. A subset D of the domain of analyticity of v is *regular* if it contains no turning points.
Let D be a regular simply connected domain, and p_0 be a branch of the complex momentum analytic in D. Any other branch of the complex momentum that is analytic in D is of the form $sp_0 + 2\pi m$ for some $s \in \{\pm 1\}$ and $m \in \mathbb{Z}$.
In terms of the complex momentum, one defines *canonical* domains. The precise definition of a canonical domain can be found in section 3.2. Here, we note only that the canonical domains are regular and simply connected and that any regular point is contained in a canonical domain (independent of h).
One of the basic results of the complex WKB method is

Theorem 2.1 ([13])**.** *Let $K \subset \mathbb{C}$ be a bounded canonical domain; let p be a branch of the complex momentum analytic in K and pick $z_0 \in K$. For sufficiently small h, there exists ψ, a solution to equation (1.1) analytic in K and such that, in K, one has*

$$\psi(z) = \frac{1}{\sqrt{\sin p(z)}}\, e^{\frac{i}{h}\int_{z_0}^{z} p(\zeta)\, d\zeta + o(1)}, \qquad h \to 0, \tag{2.3}$$

where $z \mapsto \sqrt{\sin(p(z))}$ is analytic in K.

Remark 2.1. The function $\sin p$ does not vanish in regular domains; indeed, $\sin p$ only vanishes at the points where $v(z) = -2\cos p(z) \in \{\pm 2\}$, i.e., at the turning points.

In the case of the Harper equation (for unbounded canonical domains), the analog of Theorem 2.1 was proved in [3].
Let us underline that the branch p of the complex momentum in Theorem 2.1 need not be the one with respect to which K is canonical.

2.2. **Asymptotics of a meromorphic solution.** Let us turn to the problem discussed in the present paper. Recall that 0 is a simple pole of v. Since $v(z) \to \infty$ as $z \to 0$, reducing somewhat d_x and d_y if necessary, we can and do assume that the set $S \setminus \{0\}$ is regular and that the imaginary part of the complex momentum does not vanish there.

2.2.1. *The solution we study.* Let $S' = S \setminus \mathbb{R}_+$. In S', fix an analytic branch p of the complex momentum satisfying $\operatorname{Im} p(z) < 0$.
Pick a point z_0 in $S' \cap \mathbb{R}_-$. As this point is regular, there exists a solution ψ to equation (1.1) that is analytic in a neighborhood of z_0 independent of h and that admits the asymptotic representation (2.3) in this neighborhood.
Adjusting d_x and d_y if necessary, we can and do assume that there exists $c \in (0, d_x)$ such that ψ is analytic and admits the asymptotic representation (2.3) in the domain $S_c = \{z \in S \,:\, \operatorname{Re} z < -c\}$.
As $\operatorname{Im} p(z) < 0$ in S', the expression $z \mapsto \left| e^{\frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta} \right|$ (compare with the leading term in (2.3)) increases as z in S' moves to the right parallel to $\mathbb{R}$.
If h is sufficiently small, the solution ψ is meromorphic in S; its poles belong to $h\mathbb{N}$ and they are simple.

2.2.2. *The uniform asymptotics of ψ in S.* To describe the asymptotics of ψ, we define an auxiliary function. Clearly, the complex momentum p has a logarithmic branch point at zero. In $\mathbb{C} \setminus \mathbb{R}_+$, we fix the analytic branch of $z \mapsto \ln(-z)$ such that $\ln(-z)|_{z=-1} = 0$. In section 4.2.2 we check

Lemma 2.1. *The function $z \mapsto p(z) - i\ln(-z)$ is analytic in S. The function $z \mapsto z \sin p(z)$ is analytic and does not vanish in S.*

For $z \in S'$, we set

$$G_0(z) = \frac{\sqrt{h/2\pi}}{\sqrt{-z \sin p(z)}}\, e^{\frac{z}{h}\ln\frac{1}{h} + \frac{i}{h}\int_0^z (p(\zeta) - i\ln(-\zeta))\,d\zeta}. \tag{2.4}$$

Here and below, $\sqrt{h/2\pi}$ and $\ln\frac{1}{h}$ are positive; $\sqrt{-z\sin p(z)} = \sqrt{-z}\sqrt{\sin p(z)}$; the branch of $z \mapsto \sqrt{\sin p(z)}$ coincides with the one from (2.3).
In view of Lemma 2.1, G_0 is analytic in S.
Our main result is

Theorem 2.2. *In S, the solution ψ admits the asymptotic representation*

$$\psi(z) = \Gamma\left(1 - \frac{z}{h}\right) G_0(z)\, e^{\frac{i}{h}\int_{z_0}^{0} p\,dz + o(1)}, \qquad \text{as } h \to 0 \tag{2.5}$$

where Γ is the Euler Γ-function and the integration path stays in S.

So, the special function describing the asymptotic behavior of ψ near the poles generated by a simple pole of v is the Γ-function.

2.2.3. *The asymptotics of ψ outside a neighborhood of $\mathbb{R}_+$.* For large values of $|z/h|$, the Γ-function in (2.5) can be replaced with its asymptotics. Let us give more details.
Fix $\epsilon > 0$. We recall that, uniformly in the sector $|\arg\zeta| \le \pi - \epsilon$, one has

$$\Gamma(1+\zeta) = \sqrt{2\pi\zeta}\, e^{\zeta(\ln\zeta - 1) + o(1)}, \quad |\zeta| \to \infty, \tag{2.6}$$

where the functions $\zeta \mapsto \sqrt{\zeta}$ and $\zeta \mapsto \ln\zeta$ are analytic in this sector and satisfy the conditions $\sqrt{1} = 1$ and $\ln 1 = 0$.
Fix δ positive sufficiently small. Using (2.6), one checks that, in S outside the δ-neighborhood of $\mathbb{R}_+$, the representation (2.5) turns into (2.3).
By construction, ψ admits the asymptotic representation (2.3) in S_c. Theorem 2.2

implies that the representation remains valid in S'. This reflects the standard semi-classical heuristics saying that an asymptotic representation of a solution remains valid as long as the leading term is increasing; in the present case, the modulus of the exponential in the leading term in (2.3) increases in S' as long as z moves to the right parallel to $\mathbb{R}$.

2.2.4. *The asymptotics of ψ near $\mathbb{R}_+$ away from* 0. Assume that z is inside the δ-neighborhood of $\mathbb{R}_+$ but outside the δ-neighborhood of 0. In this case, to simplify (2.5), we first use the relation

$$\Gamma(1-\zeta) = \frac{\pi}{\sin(\pi\zeta)}\,\frac{1}{\Gamma(\zeta)} \tag{2.7}$$

and, next, the asymptotic representation (2.6). This yields the asymptotic representation

$$\psi(z) = \frac{e^{\,\frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta + o(1)}}{(1-e^{2\pi i z/h})\,\sqrt{\sin(p(z))}}\,, \qquad h\to 0, \tag{2.8}$$

where p, $z\mapsto \int_0^z p(\zeta)\,d\zeta$ and $\sqrt{\sin(p)}$ are obtained by analytic continuation from $S'\cap\mathbb{C}_+$ into the domain under consideration.

2.3. **A basis of solutions.** The set of solutions to (1.1) is a two-dimensional module over the ring of h-periodic functions (see section 3.1). We now explain how to construct a basis of this module.

2.3.1. As the first solution, we take $f_+(z) = e^{-\frac{i}{h}\int_{z_0}^{0} p(z)\,dz}\,\psi(z)$. In S', it admits the asymptotic representation

$$f_+(z) \sim \frac{1}{\sqrt{\sin p(z)}}\, e^{\,\frac{i}{h}\int_0^z p(\zeta)\,d\zeta}, \qquad h\to 0, \tag{2.9}$$

and has simple poles at the points of $h\mathbb{N}$.
We note that $\left|e^{\frac{i}{h}\int_0^z p(\zeta)\,d\zeta}\right|$ increases when z moves to the right parallel to $\mathbb{R}$.

2.3.2. Fix $z_1\in S\cap\mathbb{R}_+^*$. Possibly reducing S somewhat, similarly to ψ, one constructs a solution ϕ that, in $S\setminus\mathbb{R}_-$, admits the asymptotic representation

$$\phi(z) \sim \frac{1}{\sqrt{\sin(p(z))}}\, e^{-\frac{i}{h}\int_{z_1}^z p(\zeta)\,d\zeta}, \qquad h\to 0, \tag{2.10}$$

and has simple poles at the points of $-h\mathbb{N}$. The branches of the complex momenta appearing in (2.9) and (2.10) coincide in $\mathbb{C}_+$.
Note that $\left|e^{-\frac{i}{h}\int_{z_1}^z p(\zeta)\,d\zeta}\right|$ increases when z moves to the left parallel to $\mathbb{R}$.
The function $z\mapsto 1-e^{2\pi i z/h}$ being h-periodic, we define another solution to equation (1.1) by the formula $f_-(z) := (1-e^{2\pi i z/h})e^{\frac{i}{h}\int_{z_1}^0 p(z)\,dz}\,\phi(z)$. The function f_- is analytic in S; its zeroes are simple and located at the points of $h\mathbb{N}\cup\{0\}$. As we prove in section 6.2, in S', the solution f_- has the asymptotics

$$f_-(z) \sim \frac{1}{\sqrt{\sin(p(z))}}\, e^{-\frac{i}{h}\int_0^z p(\zeta)\,d\zeta + o(1)}, \qquad h\to 0. \tag{2.11}$$

2.3.3. In section 6.2, we shall see that, for sufficiently small h, f_+ and f_- form a basis of the space of solutions to equation (1.1) meromorphic in S (possibly reduced somewhat).

2.4. **The idea of the proof of Theorem 2.2 and the plan of the paper.** To prove Theorem 2.2, we consider the function $z \mapsto f(z) = \psi(z)/\Gamma(1 - z/h)$. It is analytic in S. Using tools of the complex WKB method for difference equations, outside a disk D centered at 0 (and independent of h), we compute the asymptotics of f and obtain $f(z) = e^{\frac{i}{h}\int_{z_0}^{0} p\,dz} G_0(z)(1 + o(1))$. The factor G_0 is analytic and does not vanish in S. Therefore, the function $z \mapsto e^{-\frac{i}{h}\int_{z_0}^{0} p\,dz} f(z)/G_0(z) - 1$ is analytic in D, and, as it is small outside D, the maximum principle implies that it is small also inside D.
The plan of the paper is the following. In section 3 we describe basic facts on equation (1.1) and the main tools of the complex WKB method for difference equations. In section 4, we derive the asymptotics of the solution ψ in S outside a neighborhood of 0. In section 5, we finally prove the asymptotic representation (2.5). In section 6, we briefly discuss the solution ϕ mentioned in section 2.3.

2.5. **Acknowledgments.** This work was supported by the CNRS and the Russian foundation of basic research under the French-Russian grant 17-51-150008.

3. Preliminaries

We first recall basic facts on the space of solutions to equation (1.1); next, we recall basic constructions of the complex WKB method for difference equations and prepare an important tool, Theorem 3.1. We will use it various times to obtain the asymptotics of solutions to (1.1).

3.1. **The space of solutions to equation (1.1).** The observations that we now discuss are well-known in the theory of difference equations and are easily proved. We follow [11].
Fix $(X_1, X_2, Y) \in \mathbb{R}^3$ so that $X_1 + 2h < X_2$. We discuss the set $\mathcal{M}$ of solutions to equation (1.1) on $I := \{z \in \mathbb{C} : X_1 < \operatorname{Re} z < X_2, \operatorname{Im} z = Y\}$.
Let $\psi_\pm \subset \mathcal{M}$. The expression

$$w(\psi_+(z), \psi_-(z)) = \psi_+(z+h)\psi_-(z) - \psi_+(z)\psi_-(z+h), \quad z,\, z+h \in I, \tag{3.1}$$

is called *the Wronskian* of ψ_+ and ψ_-. It is h-periodic in z.
If the Wronskian of ψ_+ and ψ_- does not vanish, they form a basis in $\mathcal{M}$, i.e, $\psi \in \mathcal{M}$ if and only if

$$\psi(z) = a(z)\psi_+(z) + b(z)\psi_-(z), \quad z \in I, \tag{3.2}$$

where a and b are h-periodic complex valued functions. One has

$$a(z) = \frac{w(\psi(z),\, \psi_-(z))}{w(\psi_+(z),\, \psi_-(z))} \quad \text{and} \quad b(z) = \frac{w(\psi_+(z),\, \psi\,(z))}{w(\psi_+(z),\, \psi_-(z))}. \tag{3.3}$$

The set $\mathcal{M}$ is a two-dimensional module over the ring of h-periodic functions.

3.2. **Basic constructions of the complex WKB method.** We begin by defining *canonical curves* and *canonical domains*, the main geometric objects of the method.

3.2.1. *Canonical curves.* For $z \in \mathbb{C}$, we put $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. A connected curve $\gamma \subset \mathbb{C}$ is called *vertical* if it is the graph of a piecewise continuously differentiable function of y.
Define the complex momentum, turning points and regular domains as in section 2.1.
Let γ be a regular vertical curve parameterized by $z = z(y)$, and p be a branch of

the complex momentum that is analytic near γ. We pick $z_0 \in \gamma$. The curve γ is called *canonical* with respect to p if, at the points where $z'(\cdot)$ exists, one has

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z} p(\zeta)\,d\zeta > 0 \quad \text{and} \quad \frac{d}{dy}\mathrm{Im}\int_{z_0}^{z} (p(\zeta)-\pi)\,d\zeta < 0, \tag{3.4}$$

and at the points of jumps of $z'(\cdot)$, these inequalities are satisfied for both the left and right derivatives.

3.2.2. *Canonical domains.* In this paper we discuss only bounded canonical domains.

Let $K \subset \mathbb{C}$ be a bounded simply connected regular domain and let p be a branch of the complex momentum analytic in it. The domain K is said to be *canonical* with respect to p if, on the boundary of K, there are two regular points, say, z_1 and z_2 such that, for any $z \in K$, there exists a curve $\gamma \subset K$ passing through z and connecting z_1 to z_2 that is canonical with respect to p. In this paper, we use the local canonical domains described by

Lemma 3.1. *For any regular point, there exists a canonical domain that contains this point.*

This lemma is an analog of Lemma 5.3 in [9]; mutatis mutandis, their proofs are identical.

3.2.3. *Standard asymptotic behavior.* Let $U \subset \mathbb{C}$ be a regular simply connected domain; pick $z_0 \in U$ and assume $z \mapsto p(z)$ and $z \mapsto \sqrt{\sin p(z)}$ are analytic in U.

We say that a solution ψ to equation (1.1) has the standard (asymptotic) behavior

$$\psi(z) \sim \frac{1}{\sqrt{\sin(p(z))}}\, e^{\frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta} \tag{3.5}$$

in U if, for h sufficiently small, ψ is analytic and admits the asymptotic representation (2.3) in U.

Theorem 2.1 says that, for any given bounded canonical domain K, for any branch of $z \mapsto p(z)$ analytic in K, there exists a solution with the standard asymptotic behavior (3.5) in K. To study its asymptotic behavior outside K, we use the construction described in the next subsection.

3.3. **A continuation principle.** Assume the potential v in equation (1.1) is analytic in a domain in $\mathbb{C}$. Let z_0 be a regular point, V_0 be a regular simply connected domain containing z_0 and p be a branch of the complex momentum analytic in V_0. Finally, let ψ be a solution to (1.1) having the standard asymptotic behavior (3.5) in V_0. One has

Theorem 3.1. *Let $z_1 \in V_0$. Consider the straight line $L = \{z \in \mathbb{C} : \mathrm{Im}\, z = \mathrm{Im}\, z_1\}$. Pick $z_2 \in L$ such that $\mathrm{Re}\, z_2 > \mathrm{Re}\, z_1$. Assume the segment $I = \{z \in L : \mathrm{Re}\, z_1 \leq \mathrm{Re}\, z \leq \mathrm{Re}\, z_2\}$ is regular.*

If $\mathrm{Im}\, p\,(z) < 0$ along I, then there exists $\delta > 0$ such that the δ-neighborhood of I is regular and ψ has the standard behavior (3.5) in this neighborhood.

Theorem 3.1 roughly says that the asymptotic formula (2.3) stays valid along a horizontal line as long as the leading term grows exponentially. It is akin to Lemma 5.1 in [9] that deals with differential equations. The proof of Theorem 3.1 given below follows the plan of the proof of Lemma 5.1 in [9].

Let $\tilde\psi$ be a solution to equation (1.1) with the standard behavior $\tilde\psi(z) \sim \frac{e^{-\frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta}}{\sqrt{\sin(p(z))}}$ in V_0. If $\mathrm{Im}\, p < 0$ in V_0, then the analogue, mutatis mutandis, of Theorem 3.1 on the behavior of $\tilde\psi$ to the left of V_0 holds.

Proof of Theorem 3.1. Clearly, for $\zeta \in I$, there exists an open disk D centered at ζ such that D is regular and $\operatorname{Im} p(z) < 0$ in D. In view of Lemma 3.1, if D is sufficiently small, then there exists two solutions $\psi_\pm$ having the standard behavior $\psi_\pm(z) \sim \frac{e^{\pm \frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta}}{\sqrt{\sin(p(z))}}$ in D (here, we first integrate from z_0 to z_1 in V_0, next from z_1 to ζ along I and finally from ζ to z inside D).
The segment I being compact, we construct finitely many open disks $(D_j)_{0\le j\le J}$, each centered in a point of I, covering I and such that

(1) for $0 \le j \le J$, the disk D_j is regular and one has $\operatorname{Im} p(z) < 0$ in D_j;
(2) $z_1 \in D_0$, $z_2 \in D_J$, and ψ has the standard behavior (3.5) in D_0;
(3) for $0 \le j \le J$, there exists two solutions $\psi_\pm^j$ having the standard behavior $\psi_\pm^j \sim \frac{1}{\sqrt{\sin p(z)}} e^{\pm\frac{i}{h}\int_{z_0}^{z} p d\zeta}$ in the domain D_j.

Denote the rightmost point of the boundary of D_j by w_j. Possibly, excluding some of the disks D_j from the collection $(D_j)_{0\le j\le J}$ and reordering them, we can and do assume that, for $1 \le j \le J$, $D_j \setminus D_{j-1} \neq \emptyset$ and $w_j > w_{j-1}$. Indeed, to choose D_1, consider the point w_0. If w_0 is to the right of z_2, we can keep only D_0 in the collection. Otherwise, in our collection, there is a disk that contains w_0. Denote it by D_1. We then obtain the set of disks by induction.
For $r > 0$ we define

$$S(r) = \{z \in \mathbb{C} : |\operatorname{Im}(z - z_1)| < r\}.$$

For $1 \le j \le J$, let $r_j = \min\{r > 0 : D_j \cap D_{j-1} \subset S(r)\}$. Pick $0 < \delta < \min_{1\le j\le J} r_j$ sufficiently small so that I_δ, the δ-neighborhood of I, be a subset of $\cup_{j=0}^{J} D_j$.
Let us prove that ψ has the standard behavior (3.5) in I_δ.
First we note that, for h sufficiently small, by means of the formula $\psi(z) = -\psi(z - 2h) - v(z)\psi(z-h)$ (i.e., by means of equation (1.1)), ψ can be analytically continued in I_δ. It clearly satisfies equation (1.1) in I_δ.
Let us justify the asymptotic representation (2.3) in I_δ. For $0 \le j \le J$, we define $d_j = D_j \cap I_\delta$. For $j = 1, 2, \dots J$, we consecutively prove that ψ has the standard behavior in d_j to the right of d_{j-1}. Therefore, we let $d^0 = d_{j-1}$, $d = d_j$, $\psi_\pm = \psi_\pm^j$, and then proceed in the following way.
Using the standard asymptotic behavior of ψ_+ and ψ_-, one proves the asymptotic formula

$$w(\psi_+(z), \psi_-(z)) = 2i + o(1), \quad z, z+h \in D_j, \qquad h \to 0. \tag{3.6}$$

As the Wronskians are h-periodic, for sufficiently small h, formula (3.6) is valid uniformly in I_δ. This implies in particular that, for sufficiently small h, in I_δ, the solution ψ is a linear combination of the solutions $\psi_\pm$ with h-periodic coefficients, and one has (3.2) and (3.3).
The leading terms of the asymptotics of ψ and ψ_+ coincide in $d^0 \cap d$. Thus, one has

$$a(z) = 1 + o(1), \quad z, z+h \in d^0 \cap d, \qquad h \to 0. \tag{3.7}$$

Due to the h-periodicity of a, for sufficiently small h, formula (3.7) stays valid in the whole I_δ.
One also has $b(z) = o\left(e^{\frac{2i}{h}\int_{z_0}^{z} p(\zeta)d\zeta}\right)$ for $z \in d^0 \cap d$. For sufficiently small h, the h-periodicity of b yields

$$b(z) = o\left(e^{\frac{2i}{h}\int_{z_0}^{\tilde z} p(\zeta)d\zeta}\right), \quad z \in I_\delta, \tag{3.8}$$

where $\tilde{z} \in d^0 \cap d$ and $\tilde{z} = z \bmod h$.
Estimates (3.7) and (3.8) imply that, in d, one has

$$\psi(z) = a(z)\psi_+(z) + b(z)\psi_-(z) = \frac{e^{\frac{i}{h}\int_{z_0}^{z} p(\zeta)d\zeta}}{\sqrt{\sin p(z)}} \left(1 + o(1) + o\left(e^{-\frac{2i}{h}\int_{\tilde{z}}^{z} p(\zeta)d\zeta}\right)\right).$$

Assume that $z \in d$ is located to the right of d_0. As $\operatorname{Im} p < 0$ in d, one has $\operatorname{Re}\left(i\int_{\tilde{z}}^{z} p(\zeta)d\zeta\right) > 0$. This implies that ψ has the standard behavior (3.5) in d to the right of d^0 and completes the proof of Theorem 3.1. $\square$

4. The asymptotics outside a neighborhood of 0

We consider the solution ψ described in section 2.2.1 and derive its asymptotics outside a neighborhood of 0, the pole of v.

4.1. The asymptotics outside a neighborhood of $\mathbb{R}_+$.

Recall that

- in the rectangle S_c, the solution ψ has the standard behavior (3.5);
- S' is regular;
- in S', the branch p appearing in (3.5) satisfies the inequality $\operatorname{Im} p(z) < 0$.

Theorem 3.1 yields the asymptotics of ψ in S' to the right of S_c, namely

Lemma 4.1. *The solution ψ admits the standard asymptotic behavior (3.5) in the domain S'.*

Let us underline that the obstacle to justify the standard behavior of ψ in the whole domain S is the pole of v at 0.

4.2. Asymptotics in a neighborhood of $\mathbb{R}_+$ outside a neighborhood of 0.

We note that the function $z \mapsto (1 - e^{2\pi i z/h})$ is h-periodic. As ψ satisfies equation (1.1), so does $z \mapsto (1 - e^{2\pi i z/h})\,\psi(z)$. Moreover, as ψ has poles only at the points of $h\mathbb{N}$ and as these poles are simple, the solution $z \mapsto (1 - e^{2\pi i z/h})\,\psi(z)$ is analytic in S.
For $\delta > 0$, let $P(\delta) = \{z \in S \,:\, \operatorname{Re} z > 0,\ |\operatorname{Im} z| < \delta\}$. In this subsection, we prove

Proposition 4.1. *Let $\delta > 0$ be sufficiently small. In $P(\delta)$, the solution $z \mapsto (1 - e^{2\pi i z/h})\,\psi(z)$ has the standard behavior*

$$(1 - e^{2\pi i z/h})\,\psi(z) \sim n_0\, \frac{e^{\frac{i}{h}\int_0^z p_{up}(\zeta)\,d\zeta}}{\sqrt{\sin p_{up}(z)}}\,, \qquad n_0 = e^{\frac{i}{h}\int_{z_0}^{0} p(z)\,dz}; \tag{4.1}$$

here, p_{up} and $\sqrt{\sin p_{up}}$ are respectively obtained from p and $\sqrt{\sin p}$ by analytic continuation from $S' \cap \mathbb{C}_+$ to $P(\delta)$.

To prove Proposition 4.1, it suffices to check that, for any point $z_* \in P(\delta)$, there exists a neighborhood, say, V_* of this point (independent of h) where the solution $z \mapsto (1 - e^{2\pi i z/h})\psi(z)$ has the standard behavior.
As their study is simpler, we begin with the points $z_* \notin \mathbb{R}$.

4.2.1. *Points z_* in $\mathbb{C}_+$.* Let $z_* \in P(\delta) \cap \mathbb{C}_+$. Let $V_* \subset P(\delta) \cap \mathbb{C}_+$ be an open disk (independent of h) centered at z_*. In V_* one has $1 - e^{2\pi i z/h} = 1 + o(1)$ as $h \to 0$. Furthermore, by Lemma 4.1, ψ has the standard behavior (3.5) in V_*. Therefore, in V_*, one computes

$$(1 - e^{2\pi i z/h})\,\psi(z) = \frac{e^{\frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta + o(1)}}{\sqrt{\sin p(z)}} = n_0\, \frac{e^{\frac{i}{h}\int_0^z p_{up}(\zeta)\,d\zeta + o(1)}}{\sqrt{\sin p_{up}(z)}}. \tag{4.2}$$

This implies the standard behavior (4.1) in V_*.

4.2.2. *Points z_* in $\mathbb{C}_-$.* Pick now $z_* \in P(\delta)\cap\mathbb{C}_-$ and let $V_* \subset P(\delta)\cap\mathbb{C}_-$ be an open disk (independent of h) centered at z_*.
We use

Lemma 4.2. *For $z \in P(\delta)\cap\mathbb{C}_-$, one has*

$$p(z) = p_{up}(z) - 2\pi, \qquad \sqrt{\sin p(z)} = -\sqrt{\sin p_{up}(z)}. \tag{4.3}$$

Lemma 4.2 yields

$$\frac{e^{\frac{i}{h}\int_{z_0}^{z} p(\zeta)\,d\zeta}}{\sqrt{\sin p(z)}} = -n_0\,\frac{e^{\frac{i}{h}\int_0^z (p_{up}(z)-2\pi)\,dz}}{\sqrt{\sin p_{up}(z)}}, \quad z\in V_*. \tag{4.4}$$

By Lemma 4.1, ψ has the standard behavior (3.5) in V_*. Therefore, (4.4) implies that, in V_*, one has

$$(1-e^{2\pi iz/h})\psi(z) = n_0\,(1-e^{-2\pi iz/h})\frac{e^{\frac{i}{h}\int_0^z p_{up}(\zeta)\,d\zeta + o(1)}}{\sqrt{\sin p_{up}(z)}} = n_0\,\frac{e^{\frac{i}{h}\int_0^z p_{up}(\zeta)\,d\zeta + o(1)}}{\sqrt{\sin p_{up}(z)}},$$

and $z\mapsto(1-e^{2\pi iz/h})\psi(z)$ has the standard behavior (4.1) in V_*.
To prove Lemma 4.2, we shall use

Lemma 4.3. *In S', one has*

$$p(z) = i\ln(z) + C + g(z) \tag{4.5}$$

where $\ln$ is a branch of the logarithm analytic in $\mathbb{C}\setminus\mathbb{R}_+$, C is a constant, and g is a function analytic in $S\cup 0$ vanishing at 0.

Proof of Lemma 4.3. By definition (see (2.2)), p satisfies $e^{2ip(z)}+v(z)e^{ip(z)}+1=0$. Therefore,

$$e^{ip(z)} = -v(z)/2 + \sqrt{(v(z)/2)^2-1},$$

where the branch of the square root is to be determined. Since $v(z)\to\infty$ as $z\to 0$, we rewrite this formula in the form

$$e^{ip(z)} = -v(z)/2\,(1+\sqrt{1-(2/v(z))^2}). \tag{4.6}$$

As $\operatorname{Im} p(z) < 0$ in S' and as $v(z)\to\infty$ when $z\to 0$, equation (2.2) implies that $e^{ip(z)}\to\infty$ when $z\to 0$. Therefore, in (4.6), the determination of the square root is to be chosen so that $\sqrt{1-(2/v(z))^2} = 1 + o(1/(v(z))^2)$ as $z\to 0$. Then, (4.6) yields the representation

$$e^{ip(z)} = -v(z) + \tilde{g}(z) \tag{4.7}$$

where $\tilde g$ is analytic in a neighborhood of 0 vanishing at 0. By assumption, v has the Laurent expansion $v(z) = v_{-1}/z + v_0 + v_1 z + \dots,\ \ v_{-1}\neq 0$, in a neighborhood of 0. Thus, (4.7) implies representation (4.5) in a neighborhood of 0.
The function $z\mapsto g(z) := p(z) - i\ln(z) - C$ is analytic in S in a neighborhood of $\mathbb{R}_-$. To check that it is analytic in the whole domain S, we consider two analytic continuations of $z\mapsto p(z) - i\ln z - C$ into a neighborhood of $\mathbb{R}_+$ in S, one from $S'\cap\mathbb{C}_+$ and another from $S'\cap\mathbb{C}_-$. As they coincide near 0, they coincide in the whole connected component of 0 in their domain of analyticity. So, g is analytic in S. This completes the proof of Lemma 4.3. □

It now remains to prove Lemma 4.2. Therefore, we first prove Lemma 2.1.

The proof of Lemma 2.1. The statements of Lemma 2.1 on the analyticity of the functions $z\mapsto p(z) - i\ln(-z)$ and $z\mapsto z\sin p(z)$ follow directly from Lemma 4.3. This lemma also implies that the second function does not vanish at 0. Finally, this function does not vanish in $S\setminus\{0\}$ in view of Remark 2.1. The proof of Lemma 2.1 is complete. □

Proof of Lemma 4.2. Let $z \in P(\delta) \cap \mathbb{C}_-$. The first formula in (4.3) follows directly from the representation (4.5). By Lemma 2.1, the function $z \mapsto \sin p(z)$ is analytic and does not vanish in $S \setminus \{0\}$. So, for $z \in P(\delta) \cap \mathbb{C}_-$, $\sqrt{\sin p(z)}$ and $\sqrt{\sin p_{up}(z)}$ either coincide or are of opposite signs. In view of (4.5), we obtain the second relation in (4.3). $\square$

Having proved Lemma 4.2, the analysis for $z \in \mathbb{C}_-$ is complete.

4.2.3. *Real points z_*: construction of two linearly independent solutions.* To treat the case of real points, we define two linearly independent solutions to equation (1.1) that have standard asymptotic behavior to the right of 0 and express ψ in terms of these solutions.

Below, in the proof of Proposition 4.1, we always assume that z_* is a point in $\mathbb{R}_+^* \cap S$. Let $x_\pm \in S$ be two points such that $0 < x_+ < z_* < x_-$.

We recall that the set $S \setminus \{0\}$ is regular (see the beginning of section 2.2). For $\bullet \in \{+, -\}$, the point $x_\bullet$ is regular. By Lemma 3.1 and Theorem 2.1, there exists a regular $\epsilon_\bullet$-neighborhood $V_\bullet$ of $x_\bullet$ in $P(\delta)$ such that there exists a solution $\psi_\bullet$ to (1.1) with the standard asymptotic behavior $\psi_\bullet \sim \frac{1}{\sqrt{\sin p_{up}(z)}} e^{\bullet \frac{i}{h} \int_{x_\bullet}^z p_{up}(\zeta)\, d\zeta}$ in the domain $V_\bullet$.

Let $\epsilon = \min\{\epsilon_+, \epsilon_-\}$ and V be the ϵ-neighborhood of the interval (x_+, x_-).

As the imaginary part of the complex momentum can not vanish in $S \setminus \{0\}$, $\operatorname{Im} p_{up}$ is negative in V. Therefore, by Theorem 3.1, the solution ψ_+ has the standard asymptotic behavior

$$\psi_+(z) \sim \frac{e^{\frac{i}{h} \int_{x_+}^z p_{up}(\zeta)\, d\zeta}}{\sqrt{\sin(p_{up}(z))}} \tag{4.8}$$

in V to the right of V_+.

Similarly, one shows that ψ_- has the standard asymptotic behavior

$$\psi_-(z) \sim \frac{e^{-\frac{i}{h} \int_{x_-}^z p_{up}(\zeta)\, d\zeta}}{\sqrt{\sin(p_{up}(z))}} \tag{4.9}$$

in V to the left of V_-.

Then, (4.8) and (4.9) yield that, as $h \to 0$, one has

$$w(\psi_+(z), \psi_-(z)) = 2i e^{\frac{i}{h} \int_{x_+}^{x_-} p_{up}(z)\, dz + o(1)}, \qquad z \in V. \tag{4.10}$$

At cost of reducing ε, this asymptotic is uniform in V. We note that the error term is analytic in z together with $\psi_\pm$.

In view of (4.10), for sufficiently small h, the solutions $\psi_\pm$ form a basis of the space of solutions to equation (1.1) defined in V; for $z \in V$, we have (3.2) and (3.3). Our next step is to compute the asymptotics of a and b in (3.2).

4.2.4. *The coefficient a.* We prove

Lemma 4.4. *As $h \to 0$,*

$$a(z) = \frac{n_0\, e^{\frac{i}{h} \int_0^{x_+} p_{up}(z)\, dz + o(1)}}{1 - e^{2\pi i z/h}}, \quad z \in V, \tag{4.11}$$

where the error term is analytic in z.

Proof. For $z \in V$, we shall compute the asymptotics of the Wronskian $w_-(z) = w(\psi(z), \psi_-(z))$ appearing in the formula for a in (3.3).

As those of ψ, the poles of w_- in S are contained in $h\mathbb{N}$ and they are simple. We first compute the asymptotics of w_- in V outside the real line. Then, the information on the poles yields a global asymptotic representation for w_- in V and, thus, (4.11).

First, we assume that $z \in V \cap \mathbb{C}_+$. Then, p_{up} and $\sqrt{\sin p_{up}}$ coincide respectively with p and $\sqrt{\sin p}$; using the asymptotics of ψ and ψ_- yields

$$w_-(z) = 2in_0\, e^{\frac{i}{h}\int_0^{x_-} p_{up}(z)\,dz + o(1)}, \quad z \in V \cap \mathbb{C}_+, \quad h \to 0. \tag{4.12}$$

Now, we assume that $z \in V \cap \mathbb{C}_-$. Then, (4.3) implies that

$$\psi(z) = -\frac{n_0\, e^{\frac{i}{h}\int_0^z (p_{up}(\zeta)-2\pi)\,d\zeta + o(1)}}{\sqrt{\sin p_{up}(z)}}, \qquad h \to 0. \tag{4.13}$$

This representation and (4.9) yield

$$w_-(z) = -2in_0\, e^{-2\pi iz/h} e^{\frac{i}{h}\int_0^{x_-} p_{up}(z)\,dz}(1+o(1)), \quad z \in V \cap \mathbb{C}_-, \quad h \to 0. \tag{4.14}$$

Let

$$f\,:\, z \mapsto n_0^{-1}\,(1-e^{2\pi iz/h})\, e^{-\frac{i}{h}\int_0^{x_-} p_{up}(z)\,dz} w_-(z) - 2i.$$

Representations (4.12) and (4.14) imply that

$$f(z) = o(1), \quad z \in V \setminus \mathbb{R}, \quad h \to 0. \tag{4.15}$$

We recall that f is an h-periodic function (as is w_-). Therefore, at the cost of reducing ϵ somewhat, we get the uniform estimate $f(z) = o(1)$ for $|\operatorname{Im} z| = \epsilon$ and h sufficiently small. Moreover, the description of the poles of w_- implies that f is analytic in the strip $\{|\operatorname{Im} z| \le \epsilon\}$.

Now, let us consider f as a function of $\zeta = e^{2\pi iz/h}$. It is analytic in the annulus $\{e^{-2\pi\epsilon/h} \le |\zeta| \le e^{2\pi\epsilon/h}\}$ and, on the boundary of this annulus, it admits the uniform estimate $|f(\zeta)| = o(1)$. By the maximum principle, the estimate $f(\zeta) = o(1)$ holds uniformly in the annulus.

Thus, as a function of z, the function f satisfies the uniform estimate $f(z) = o(1)$ for $|\operatorname{Im} z| \le \epsilon$ and, therefore, in the whole domain V.

This estimate, the representation (4.10) and the definition of a (see (3.3)) imply (4.11). ∎

4.2.5. *The coefficient b.* We estimate b in V for h sufficiently small. To state our result, let γ be the connected component of x_+ in the set of $z \in P(\delta)$ satisfying

$$\operatorname{Im} \int_{x_+}^z p(z)\,dz = 0.$$

As

$$\frac{\partial}{\partial x} \operatorname{Im} \int_{x_+}^z p(\zeta)\,d\zeta = \operatorname{Im} p(z) \ne 0 \quad \text{in } P(\delta),$$

the Implicit Function Theorem guarantees that γ is a smooth vertical curve in a neighborhood of x_+. Reducing ϵ if necessary, we can and do assume that γ intersects both the lines $\{\operatorname{Im} z = \pm\epsilon\}$. We prove

Lemma 4.5. *In V (with ϵ reduced somewhat if necessary), on γ and to the right of γ, one has*

$$b(z) = o(1)\, \frac{n_0\, e^{\frac{i}{h}\int_0^{x_+} p_{up}(z)\,dz - \frac{i}{h}\int_{x_+}^{x_-} p_{up}(z)\,dz}}{1-e^{2\pi iz/h}}, \quad h \to 0, \tag{4.16}$$

where $o(1)$ is analytic in z.

Proof. Let us estimate the Wronskian $w_+(z) = w(\psi_+(z), \psi(z))$, the numerator in (3.3).

First, we assume that $z \in V \cap \mathbb{C}_+$. Then, p_{up} and $\sqrt{\sin p_{up}}$ coincide respectively with p and $\sqrt{\sin p}$. Thus, the leading terms of the asymptotics of ψ and ψ_+ coincide up to a constant factor; this yields

$$w_+ = o(1)\, n_0\, e^{\frac{i}{h}\int_0^{x_+} p_{up}(z)\,dz + \frac{2i}{h}\int_{x_+}^z p_{up}(\zeta)\,d\zeta}, \qquad h \to 0. \tag{4.17}$$

We recall that the Wronskians are h-periodic (see section 3.1). Let us assume additionally that z is either between γ and $\gamma+h$ or on one of these curves. Pick $\tilde z\in\gamma$ such that $\operatorname{Im}\tilde z=\operatorname{Im} z$. In view of the definition of γ, as p_{up} is analytic in $P(\delta)$, one has

$$\left|e^{\frac{2i}{h}\int_{x_+}^{z} p_{up}(z)\,dz}\right| = \left|e^{\frac{2i}{h}\int_{\tilde z}^{z} p_{up}(\zeta)\,d\zeta}\right| \le e^{C},\quad h\to 0;$$

here, C is a positive constant independent of h. This estimate and (4.17) imply that, for $z\in V\cap\mathbb{C}_+$ either between the curves γ and $\gamma+h$ or on one of them, one has

$$w_+ = o(1)\, n_0\, e^{\frac{i}{h}\int_0^{x_+} p_{up}(z)\,dz},\qquad h\to 0. \tag{4.18}$$

Reducing ϵ somewhat if necessary, we can and do assume that (4.18) holds on the line $\operatorname{Im} z=\epsilon$ between γ and $\gamma+h$ or on one of these curves. Then, thanks to the h-periodicity of w_+, it holds for all z on the line $\{\operatorname{Im} z=\epsilon\}$.
Now, we assume that $z\in V\cap\mathbb{C}_-$. Using the asymptotics (4.13) and (4.8), we compute

$$w_+(z) = o(1)\, n_0\, e^{-2\pi i z/h}\, e^{\frac{i}{h}\int_0^{x_+} p_{up}(z)\,dz+\frac{2i}{h}\int_{x_+}^{z} p_{up}(\zeta)\,d\zeta}\qquad h\to 0.$$

Arguing as when proving (4.18), we finally obtain

$$w_+(z) = o(1)\, n_0\, e^{-2\pi i z/h}\, e^{\frac{i}{h}\int_0^{x_+} p_{up}(z)\,dz},\quad \operatorname{Im} z=-\epsilon,\quad h\to 0. \tag{4.19}$$

Let $g\,:\,z\mapsto(1-e^{2\pi i z/h})w_+(z)$. Estimates (4.18) and (4.19) imply that, for $|\operatorname{Im} z|=\epsilon$,

$$n_0^{-1}\, e^{-\frac{i}{h}\int_0^{x_+} p_{up}(z)\,dz} g(z) = o(1),\quad h\to 0. \tag{4.20}$$

As those of the solution ψ do, the poles of w_+ in $P(\delta)$ belong to the set $h\mathbb{N}$ and are simple. So, the function g is analytic in the strip $\{|\operatorname{Im} z|\le\epsilon\}$. Moreover, it is h-periodic as w_+ is. Thus, the maximum principle implies that (4.20) holds in the whole strip $\{|\operatorname{Im} z|\le\epsilon\}$. Estimate (4.20), representation (4.10) and the definition of b (see (3.3)) yield (4.16). This completes the proof of Lemma 4.5. □

4.2.6. *Completing the proof of Proposition 4.1.* Let $V_*\subset V$ be a disk independent of h, centered at z_* and located to the right of γ (i.e., such that, for any $z\in V_*$, there exists $\tilde z\in\gamma$ such that $\operatorname{Re}\tilde z<\operatorname{Re} z$ and $\operatorname{Im}\tilde z=\operatorname{Im} z$).
Below we assume that $z\in V_*$.
Using (4.9) and (4.8), the asymptotic representations for $\psi_\pm$, and (4.11) and (4.16), the representations for a and b, we compute

$$\frac{b(z)\,\psi_-(z)}{a(z)\psi_+(z)} = o(1)e^{-\frac{2i}{h}\int_{x_+}^{z} p_{up}(\zeta)d\zeta},\quad h\to 0.$$

As before, let $\tilde z\in\gamma$ be such that $\operatorname{Im}\tilde z=\operatorname{Im} z$. Then, we have

$$\left|e^{-\frac{2i}{h}\int_{x_+}^{z} p_{up}(\zeta)d\zeta}\right| = \left|e^{-\frac{2i}{h}\int_{\tilde z}^{z} p_{up}(\zeta)d\zeta}\right| \le 1.$$

Here, we used the definition of γ and the fact that $\operatorname{Im} p_{up}<0$ in V. As a result, we have $\frac{b(z)\,\psi_-(z)}{a(z)\psi_+(z)}=o(1)$. Formula (3.2) then yields

$$\psi(z) = a(z)\,\psi_+(z)\left(1+\frac{b(z)\,\psi_-(z)}{a(z)\psi_+(z)}\right) = a(z)\,\psi_+(z)(1+o(1)).$$

where $o(1)$ is analytic in $z\in V_*$. This and the asymptotic representations for ψ_+ and a yields (4.1) in V_*. This completes the proof of Proposition 4.1.

5. Global asymptotics

5.1. **The proof of Theorem 2.2.** We follow the plan outlined in section 2.4. Recall that G_0 is defined in (2.4). We prove

Proposition 5.1. *Let $\delta > 0$ be sufficiently small. In S, outside the δ-neighborhood of 0, one has*

$$\psi(z)/\Gamma(1 - z/h) = n_0\, G_0(z)\,(1 + o(1)), \quad h \to 0. \tag{5.1}$$

Let us check that Theorem 2.2 follows from Proposition 5.1.

We recall that the poles of ψ belong to $h\mathbb{Z}$ and are simple. Furthermore, in view of Lemma 2.1, G_0 is analytic in S. Clearly, G_0 has no zeros in S. These observations imply that the function

$$f\,:\,z \;\longrightarrow\; \frac{\psi(z)}{n_0\,G_0(z)\,\Gamma(1-z/h)} - 1$$

is analytic in S. By Proposition 5.1, it satisfies the estimate $f(z) = o(1)$ in S outside the δ-neighborhood of 0. Therefore, by the maximum principle, it satisfies this estimate in the whole of S. This implies the statement of Theorem 2.2. To complete the proof of this theorem, it now suffices to check Proposition 5.1.

5.2. **The proof of Proposition 5.1.** Fix ε sufficiently small positive. The proof of Proposition 5.1 consists of two parts: first, we prove (5.1) in the sector $S_\pi = \{z \in S\,:\,|z| \geq \delta,\ |\arg z - \pi| \leq \pi - \varepsilon\}$, and, then, in the sector $S_0 = \{z \in S\,:\,|z| \geq \delta,\ |\arg z| \leq \varepsilon\}$.

5.2.1. *The asymptotic in the sector S_π.* If $z \in S_\pi$ and $h \to 0$, we can use formula (2.6) for $\Gamma(1 - z/h)$ and the standard asymptotic representation (2.3) for ψ, see Lemma 4.1. This immediately yields (5.1) in S_π.

5.2.2. *The asymptotic in the sector S_0.* Let ε be so small that S_0 be a subset of $P(\delta)$ (defined just above Proposition 4.1). For $z \in S_0$ and h small, we express $\Gamma(1-z/h)$ in terms of $\Gamma(z/h)$ by formula (2.7), then, we use formula (2.6) for $\Gamma(z/h)$ and the standard asymptotic representation for $(1 - e^{2\pi i/z})\psi(z)$ (see Proposition 4.1). This yields

$$\psi(z)/\Gamma(1 - z/h) = n_0\, \tilde{G}_0(z)\,(1 + o(1)), \quad h \to 0, \tag{5.2}$$

where

$$\tilde{G}_0(z) = \frac{i\,\sqrt{h/2\pi}}{\sqrt{z \sin p_{up}(z)}}\, e^{\,\frac{z}{h}\ln\frac{1}{h} + \frac{i}{h}\int_0^z (p_{up}(\zeta) - i(\ln(\zeta) - i\pi))\,d\zeta}, \tag{5.3}$$

and the functions $z \mapsto \ln z$ and $z \mapsto \sqrt{z}$ are analytic in $\mathbb{C} \setminus \mathbb{R}_-$ and positive respectively if $z > 1$ and $z > 0$. We note that by Lemma 2.1, $\tilde{G}_0$ is analytic in S_0. Define the functions $z \mapsto \sqrt{-z}$ and $z \mapsto \ln(-z)$ as in (2.4), i.e. so that they be analytic in $\mathbb{C} \setminus \mathbb{R}_+$ and positive if $z < 0$ and $z < -1$ respectively. Then, these functions are related to the functions $z \mapsto \ln z$ and $z \mapsto \sqrt{z}$ from (5.3) by the formulas

$$\sqrt{-z} = -i\sqrt{z}, \quad \ln(-z) = \ln z - i\pi, \quad z \in \mathbb{C}_+.$$

Furthermore, in $S_0 \cap \mathbb{C}_+$ the functions p and p_{up} coincide. These two observations imply that one has $\tilde{G}_0 = G_0$ in $S_0 \cap C_+$.

As both G_0 and $\tilde{G}_0$ are analytic in S_0, they coincide in the whole of S_0. This and (5.2) imply the representation (5.1) for $z \in S_0$. This completes the analysis in the sector S_0 and the proof of Proposition 5.1.

6. A basis for the space of solutions

We finally discuss a basis of the space of solutions to equation (1.1) that are meromorphic in S. First, we describe the two solutions forming the basis and, second, we compute their Wronskian.

6.1. **First solution.** As the first solution, we take $f_+(z) = \psi(z)/n_0$. It has the standard behavior (2.9) in S' and simple poles at the points of $h\mathbb{N}$. We recall that the modulus of the exponential factor in (2.9) increases when z moves to the right parallel to $\mathbb{R}$.

6.2. **Second solution.** Let $z_1 > 0$ be a point in S. Mutatis mutandis, in the way we constructed ψ, in S (possibly reduced somewhat), we construct a solution ϕ in $S \setminus \mathbb{R}_-$ that has the standard asymptotic behavior (2.10) and such that the quasi-momenta p (and the functions $\sqrt{\sin p}$) in (2.9) and (2.10) coincide in $\mathbb{C}_+$.
The modulus of the exponential from (2.10) increases when z moves to the left parallel to $\mathbb{R}$.
The solution ϕ has simple poles at the points of $-h\mathbb{N}$ and, in S, it admits the asymptotic representation

$$\phi(z) = n_1\Gamma(1 + z/h)\, G_1(z)(1 + o(1)), \quad n_1 = e^{\frac{i}{h}\int_{z_1}^{0} p(z)\,dz}, \quad h \to 0, \tag{6.1}$$

$$G_1(z) = \frac{\sqrt{h/2\pi}}{\sqrt{z \sin p(z)}}\, e^{-\frac{z}{h}\ln\frac{1}{h} - \frac{i}{h}\int_0^z (p(\zeta) - i\ln(\zeta))\,d\zeta}. \tag{6.2}$$

Here, the functions $z \mapsto \sqrt{z}$ and $z \mapsto \ln z$ are analytic in $\mathbb{C} \setminus \mathbb{R}_-$ and positive, respectively, if $z > 0$ and $z > 1$. The factor G_1 is analytic in S.
We define the second solution to be $f_-(z) = 1/n_1\,(1 - e^{2\pi i z/h})\,\phi(z)$. The solution f_- is analytic in S. It has simple zeros at the points $z \in h\mathbb{N}$ and at 0. By means of (6.1), one can easily check that, in S', it has the standard behavior (2.11).

6.3. **The Wronskian of the basis solutions.** Using the asymptotic representations for $f_\pm$ in S', one easily computes

$$w(f_+(z), f_-(z)) = 2i + o(1), \quad h \to 0. \tag{6.3}$$

As the Wronskian is h-periodic, this representation is valid in the whole domain S. We see that, for sufficiently small h, the leading term of the Wronskian does not vanish, and, thus, $f_\pm$ form a basis in the space of solutions to equation (1.1) in S (possibly reduced somewhat for (6.3) to be uniform).

References

[1] A. Avila and S. Jitomirskaya. The Ten Martini Problem. *Annals Math.,* 170:303-342, 2009.

[2] V. Babich, M. Lyalinov and V. Grikurov. *Diffraction theory: the Sommerfeld-Malyuzhinets technique.* Oxford, Alpha Science, 2008.

[3] V. Buslaev and A. Fedotov. Complex WKB method for Harper equation. *St. Petersburg Math. J.*, 6(3):495-517, 1995.

[4] S. Yu. Dobrokhotov and A. V. Tsvetkova. On Lagrangian manifolds related to asymptotics of Hermite polynomials. *Math. notes*, 104(6):810-822, 2018.

[5] A. Eckstein. Unitary reduction for the two-dimensional Schrödinger operator with strong magnetic field. *Math. Nachr.*, 282(4):504-525, 2009.

[6] A. Fedoryuk. *Asymptotic analysis. Linear ordinary differential equations.* Springer-Verlag, Berlin, Heidelberg, 2009.

[7] A. Fedotov. Monodromization method in the theory of almost-periodic equations. *St. Petersburg Math. J.,* 25:303-325, 2014.

[8] A. Fedotov and F. Klopp. Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case. *Commun. in math. phys.*, 227:1-92, 2002.

[9] A. Fedotov and F. Klopp. On the absolutely continuous spectrum of an one-dimensional quasi-periodic Schrödinger operator in adiabatic limit. *Transactions of AMS*, 357:4481-4516, 2005.

[10] A. Fedotov and F. Klopp. Quasiclassical asymptotics of solutions to difference equations with meromorphic coefficients. In *Proceedings of the conference "Days on Diffraction 2017"*, 110-112, IEEE, St. Petersburg, 2017.

[11] A. Fedotov and F. Klopp. The complex WKB method for difference equations and Airy functions. HAL, https://hal.archives-ouvertes.fr/hal-01892639

[12] A. Fedotov and F. Sandomirskiy. An exact renormalization formula for the Maryland model. *Commun. in math. phys.*, 334(2):1083-1099, 2015.

[13] A. Fedotov and E. Shchetka. A complex WKB method for difference equations in bounded domains. *J. Math. Sciences(New York)*, 224:157-169, 2017.

[14] A. Fedotov and E. Shchetka. Complex WKB method for the difference Schrödinger equation with the potential being a trigonometric polynomial. *St. Petersburg Math. J.*, 29:363-381, 2018.

[15] D. Grempel, S. Fishman, R. Prange. Localization in an incommensurate potential: An exactly solvable model. *Phys. Rev. Letters*, 49:833-836, 1982.

[16] J. S. Geronimo, O. Bruno and W. Van Assche. WKB and turning point theory for second-order difference equations. *Operator theory: advances and app.*, 69:269-301, 1992.

[17] J. P. Guillement, B. Helffer and P. Treton. Walk inside Hofstadter's butterfly. *J. Phys. France,* 50:2019-2058, 1989.

[18] B.Helffer and J.Sjöstrand. Analyse semi-classique pour l'équation de Harper (avec application à l'équationde Schrödinger avec champ magnétique). *Mémoires de la SMF (nouvelle série)*, 34:1-113, 1988.

[19] S. Jitomirskaya, W. Liu. Universal hierarchical structure of quasiperiodic eigenfunctions. *Annals of math.*, 187(3) : 721-776, 2018.

[20] W. Wasow. *Asymptotic expansions for ordinary differential equations.* Dover Publications, New York, 1987.

[21] M. Wilkinson. An exact renormalisation group for Bloch electrons in a magnetic field.*J. Phys. A: Math. Gen.*, 20: 4337-4354, 1987.

[22] R.Wong, Z. Wang. Asymptotic expansions for second-order linear difference equations with a turning point. *Numerische Mathematik*, 94: 147-194, 2003.

(Alexander Fedotov) St. Petersburg State University, 7/9 Universitetskaya nab., St.Petersburg, 199034, Russia

Email address: a.fedotov@spbu.ru

(Frédéric Klopp) Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche , F-75005, Paris, France

Email address: frederic.klopp@imj-prg.fr