ASYMPTOTIC ERGODICITY OF THE EIGENVALUES OF RANDOM OPERATORS IN THE LOCALIZED PHASE

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ABSTRACT. We prove that, for a general class of random operators, the family of the unfolded eigenvalues in the localization region is asymptotically ergodic in the sense of N. Minami (see [31]). N. Minami conjectured this to be the case for discrete Anderson model in the localized regime. We also provide a local analogue of this result. From the asymptotics ergodicity, one can recover the statistics of the level spacings as well as a number of other spectral statistics. Our proofs rely on the analysis developed in [15].

RÉSUMÉ. Nous démontrons que, pour une classe générale d'opérateurs aléatoires, les familles valeurs propres "dépliées" sont asymptotiquement ergodiques au sens de N. Minami (voir [31]). N. Minami à conjecturé que ceci est vrai pour le modèle d'Anderson discret dans le régime localisé. On démontre également un résultat analogue pour les valeurs propres "locales". L'ergodicité asymptotique des valeurs propres permet alors d'en déduire les statistiques des espacements de niveaux ainsi que nombre d'autres statistiques spectrales. Nos preuves reposent sur l'analyse faite dans [15].

0. INTRODUCTION

On $\ell^2(\mathbb{Z}^d)$, consider the Anderson model

$$H_\omega = -\Delta + \lambda V_\omega$$

where

- $-\Delta$ is the free discrete Laplace operator

$$(-\Delta u)_n = \sum_{|m-n|=1} u_m \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d); \tag{0.1}$$

- V_ω is the random potential

$$(V_\omega u)_n = \omega_n u_n \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d). \tag{0.2}$$

  We assume that the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are independent identically distributed and that their common distribution admits a compactly supported bounded density, say g.
- The coupling constant λ is chosen positive.

The Anderson model was introduced in [1] to describe a single electron's motion in a disordered crystal. When one omits particle interactions, it is the paradigmatic model for the behavior of quantum particles in a disordered medium . Since their introduction, random operators have been (and still are) the object of a huge literature both in physics (see e.g. [26, 27]) and mathematics (see e.g. [33, 21]). One of the most studied questions is the occurrence of a localized phase. Physically, this phase corresponds to a region of energies in which the electrons are trapped in the medium i.e. diffusion is suppressed.

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Mathematically, this phase corresponds to a region, say, S, of the spectrum of the Hamiltonian in which the quantum evolution group does not propagate to infinity. This can be expressed by the fact that the operator $X\, e^{-itH_\omega}\mathbf{1}_S(H_\omega)\mathbf{1}_{|x|\leq C}$ stays bounded uniformly in time; here, X is the position operator. Actually the dynamical localization property one can generally show is much stronger (see assumption (Loc) in section 1.1). This dynamical localization property implies that the spectrum of H_ω must consist only of eigenvalues. The associated eigenfunctions are generally exponentially decaying (see Lemma 2.1 in section 2.1.1). This entails that an eigenvalue essentially only depends on the local configuration of the potential i.e. on the local potential in the region where the eigenfunction associated to the eigenvalue does live. So, by virtue of the Heisenberg uncertainty principle, nearby eigenvalues should roughly behave as independent random variables. Thus, properly renormalized, the eigenvalues should look like a Poisson cloud. This has been proved to be true locally near a typical energy (see e.g. [32, 29, 8, 15]) in the sense that the locally renormalized eigenvalue process converges weakly to a Poisson process. In the present paper, we show the asymptotic ergodicity of the renormalized eigenvalues i.e. that the process of the renormalized eigenvalues tested against a uniform random variable converges in law to a Poisson process almost surely (see Theorem 0.1, 1.1, 1.5 and 1.6). This is a global signature of the i.i.d behavior of the eigenvalues in the localized phase. It, in particular, implies the convergence of the empirical distribution of the unfolded level spacings to the exponential function (see Theorem 1.2) or enables one to compute the asymptotics of the empirical distribution of the level spacing (see Theorem 1.4) or that of any marginal of the eigenvalue or renormalized eigenvalues process.

We now recall some well known facts on the Anderson model (see e.g. [21]) and state a version of our main result for this model. We will turn to general operators in the next section.
One has

- for almost every $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$, the spectrum of H_ω is equal to the set $\Sigma := [-2d, 2d]+\operatorname{supp} g$;
- there exists a bounded density of states, say $E \mapsto \nu(E)$, such that, for any continuous function $\varphi:\ \mathbb{R} \to \mathbb{R}$, one has

$$\int_{\mathbb{R}} \varphi(E)\nu(E)dE = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle). \tag{0.3}$$

  Here, and in the sequel, $\mathbb{E}(\cdot)$ denotes the expectation with respect to the random parameters, and $\mathbb{P}(\cdot)$ the probability measure they induce.
  Let N be the integrated density of states of H_ω i.e. N is the distribution function of the measure $\nu(E)dE$. The function ν is only defined E-almost everywhere. In the sequel, when we speak of $\nu(E)$ for some E, we mean that the non decreasing function N is differentiable at E and that $\nu(E)$ is its derivative at E;
- for λ large, ω almost surely, the spectrum of H_ω is pure punctual i.e. made up only of eigenvalues; the associated eigenvalues are exponentially decaying; moreover, one has dynamical localization in the sense described above.

For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L = [-L, L]^d$ be a large box and $|\Lambda| := \#\Lambda = (2L + 1)^d$ be its cardinality. Let $H_\omega(\Lambda)$ be the operator H_ω restricted to Λ with periodic boundary conditions. The notation $|\Lambda| \to +\infty$ is a shorthand for considering $\Lambda = \Lambda_L$ in the limit $L \to +\infty$. Let us denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \cdots \leq E_{|\Lambda|}(\omega, \Lambda)$.

For $t \in [0,1]$, consider the following point process

$$\Xi(\omega,t,\Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{|\Lambda|[N(E_n(\omega,\Lambda))-t]}. \tag{0.4}$$

The values $N(E_1(\omega,\Lambda)) \leq \cdots \leq N(E_n(\omega,\Lambda)) \leq \cdots \leq N(E_{|\Lambda|}(\omega,\Lambda))$ are called the *renormalized or unfolded eigenvalues or levels* (see e.g. [30]).
We prove

Theorem 0.1. *For sufficiently large coupling constant λ, ω-almost surely, when $|\Lambda| \to +\infty$, the probability law of the point process $\Xi(\omega,\cdot,\Lambda)$ under the uniform distribution $\mathbf{1}_{[0,1]}(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.*

This proves a conjecture by N. Minami (see [28, 31]); a weaker version of Theorem 0.1, namely, L^2-convergence in ω when $d = 1$, is proved in [31].
Theorem 0.1, in particular, implies the convergence of the level spacings statistics already obtained for this model under more restrictive assumptions in [15] (see also Theorem 1.4 in the present paper for more details). Indeed, in Theorem 0.1, we do not make any regularity assumption on the density of states $E \mapsto \nu(E)$.
Actually, Theorem 0.1 is a prototype of the general result we state and prove below. Essentially, we prove that the claim in Theorem 0.1 holds in the localization region for any random Hamiltonian satisfying a Wegner and a Minami estimate (see assumptions (W) and (M) in section 1). To do so, we use the analysis made in [15]; in particular, our analysis relies on a slight generalization of one of the approximation theorems proved in [15], namely, Theorem 1.16.
It is also interesting to compare Theorem 0.1 to the local eigenvalue statistics that have been obtained in [29] for the discrete Anderson model (see also [32, 8, 15] for similar results for other models). There, one studies $\Xi(\omega,t,\Lambda)$ for fixed t. It is shown that $\Xi(\omega,t,\Lambda)$ converges weakly to a Poisson process (under the assumption that N is differentiable and has a positive derivative at the energy $E = N^{-1}(t)$). That is, for any $P \in \mathbb{N}^*$, any $(I_p)_{1\leq p\leq P}$ measurable subsets of $\mathbb{R}$ and any $(k_p)_{1\leq p\leq P}$ integers, one has

$$\mathbb{P}\left(\left\{\omega;\ \forall 1 \leq p \leq P,\ \langle \Xi(\omega,t,\Lambda), \mathbf{1}_{I_p}\rangle = k_p\right\}\right) \underset{|\Lambda|\to+\infty}{\to} \prod_{p=1}^{p} e^{-|I_p|}\frac{(|I_p|)^{k_p}}{k_p!} \tag{0.5}$$

Actually, the study done in [15] gives a result stronger than (0.5) under slightly weaker assumptions.
In the present paper, we study the process for a "random" t but obtain an almost sure convergence result. This gives access in particular to the level spacings statistics (see Theorems 1.2 and 1.4) but also all the marginals of the process under very mild assumptions.

1. The results

Consider $H_\omega = H_0 + V_\omega$, a $\mathbb{Z}^d$-ergodic random Schrödinger operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$ (see e.g. [33, 35]). Typically, H_0, the deterministic part of the random Hamiltonian H_ω, is the Laplacian $-\Delta$, possibly perturbed by a periodic potential. Magnetic fields can be considered as well; in particular, the Landau Hamiltonian is also admissible as a background Hamiltonian. We assume that H_0 is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ or $\mathcal{C}_0(\mathbb{Z}^d)$ (the space of sequences with compact support) and that the operator has at most polynomially growing coefficients. For the sake of simplicity, we assume that V_ω is almost surely bounded; hence, almost surely, H_ω has the same domain as H_0.

1.1. **The setting and the assumptions.** For Λ, a cube in either $\mathbb{R}^d$ or $\mathbb{Z}^d$, we let $H_\omega(\Lambda)$ be the self-adjoint operator H_ω restricted to Λ with periodic boundary conditions. As in [15], our analysis stays valid for Dirichlet boundary conditions.
In the sequel, we shall denote by $\mathbf{1}_J(H)$ the spectral projector of the operator H on the energy interval J. $\mathbb{E}(\cdot)$ denotes the expectation with respect to ω.

1.1.1. *Independence at a distance.* Our first assumption will be an independence assumption for local Hamiltonians that are far away from each other, that is,

(IAD): There exists $R_0 > 0$ such that for any two cubes Λ and Λ' such that $\mathrm{dist}(\Lambda, \Lambda') > R_0$, the random Hamiltonians $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are stochastically independent.

Remark 1.1. This assumption may be relaxed to assume that the correlation between the random Hamiltonians $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ decays sufficiently fast as $\mathrm{dist}(\Lambda, \Lambda') \to +\infty$. We refer to [15] for more details.

1.1.2. *Eigenvalue estimates.* Let Σ be the almost sure spectrum of H_ω; its existence is guaranteed by the ergodicity assumption of H_ω. Pick I a relatively compact open subset of Σ. Assume the following holds:

(W): a Wegner estimate holds in I, i.e. there exists $C > 0$ such that, for $J \subset I$, and Λ, a cube in $\mathbb{R}^d$ or $\mathbb{Z}^d$, one has

$$\mathbb{E}\left[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))\right] \leq C|J|\,|\Lambda|. \tag{1.1}$$

(M): a Minami estimate holds in I, i.e. there exists $C > 0$ and $\rho > 0$ such that, for $J \subset I$, and Λ, a cube in $\mathbb{R}^d$ or $\mathbb{Z}^d$, one has

$$\mathbb{E}\left[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \cdot [\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda))) - 1]\right] \leq C(|J|\,|\Lambda|)^{1+\rho}. \tag{1.2}$$

Remark 1.2. The Wegner estimate (W) has been proved for many random Schrödinger models e.g. for both the discrete and the continuous Anderson models under rather general conditions on the single site potential and on the randomness (see e.g. [19, 21, 22, 36]) but also for other models (see e.g. [17, 24]). The left hand side in (1.1) can be lower bounded by the probability to have at least one eigenvalue in J.
Weaker forms of assumption (W) i.e. when the right hand side is replaced with $C|J|^\alpha\,|\Lambda|^\beta$ for some $\alpha \in (0, 1]$ and $\beta \geq 1$, are known to hold also for some non monotonous models (see e.g. [23, 20, 16]). This is sufficient for our proofs to work if one additionally knows that the integrated density of states is a Hölder continuous function.

On the Minami estimate (M), much less is known. For the discrete Anderson model, it holds in arbitrary dimension with $I = \Sigma$ (see [29, 18, 4, 7]). For the continuous Anderson model in any dimension, in [8], it is shown to hold at the bottom of the spectrum under more restrictive conditions on the single site potential than needed to prove the Wegner estimate (W). These proofs yield an optimal exponent $\rho = 1$. In dimension 1, regardless of the random model under consideration, in [25], it is shown that the Minami estimate holds at energies in the localization region (see assumption (Loc) below) provided a Wegner estimate is known. In this case, the exponent ρ can be taken arbitrarily close to 1.
Finally, let us note that the left hand side in (1.2) can be lower bounded by the probability to have at least two eigenvalues in J. So, (M) can be interpreted as a measure of the independence of nearby eigenvalues.

The integrated density of states of H_ω (see the introduction) can also be defined as the limit

$$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ less than E}\}}{|\Lambda|} \tag{1.3}$$

By (W), $N(E)$ is the distribution function of a measure that is absolutely continuous with respect to to the Lebesgue measure on $\mathbb{R}$. Let ν be the density of state of H_ω i.e. the distributional derivative of N. In the sequel, for a set I, $|N(I)|$ denotes the Lebesgue measure of $N(I)$ i.e. $|N(I)| = \displaystyle\int_I \nu(E)dE$.

1.1.3. *The localization region.* Let us now describe what we call the localized regime in the introduction. For $L \geq 1$, Λ_L denotes the cube $[-L/2, L/2]^d$ in either $\mathbb{R}^d$ or $\mathbb{Z}^d$. In the sequel, we write Λ for Λ_L i.e. $\Lambda = \Lambda_L$ and when we write $|\Lambda| \to +\infty$, we mean $L \to +\infty$. For $\mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$ and a vector φ in $\mathcal{H}$, we define

$$\|\varphi\|_x = \begin{cases} \|\mathbf{1}_{\Lambda(x)}\varphi\|_2 \text{ where } \Lambda(x) = \{y;\ |y-x| \leq 1/2\} & \text{if } \mathcal{H} = L^2(\mathbb{R}^d), \\ |\varphi(x)| & \text{if } \mathcal{H} = \ell^2(\mathbb{Z}^d). \end{cases} \tag{1.4}$$

Let I be a compact interval. We assume that I lies in the region of complete localization (see e.g. [12, 13]) for which we use the following finite volume version:

(Loc): for all $\xi \in (0,1)$, one has

$$\sup_{L>0} \sup_{\substack{\text{supp} f\subset I \\ |f|\leq 1}} \mathbb{E}\left(\sum_{\gamma\in\mathbb{Z}^d} e^{|\gamma|^\xi} \|\mathbf{1}_{\Lambda(0)} f(H_\omega(\Lambda_L))\mathbf{1}_{\Lambda(\gamma)}\|_2\right) < +\infty. \tag{1.5}$$

Remark 1.3. Such regions of localization have been shown to exist and described for many random models (see e.g. [13, 3, 2, 35, 23, 20, 16, 17, 24]); a fairly recent review can be found in [21]; other informational texts include [33, 12].
Once a Wegner estimate is known (though it is not an absolute requirement see e.g. [6, 11, 10, 14]), the typical regions where localization holds are vicinities of the edges of the spectrum. One may have localization over larger regions (or the whole) of the spectrum if the disorder is large like in Theorem 0.1.
The assumption (Loc) may be relaxed; we refer to Remark 1.3 of [15] for more details.

For $L \in \mathbb{N}$, recall that $\Lambda = \Lambda_L$ and that $H_\omega(\Lambda)$ is the operator H_ω restricted to Λ with periodic boundary conditions. The notation $|\Lambda| \to +\infty$ is a shorthand for considering $\Lambda = \Lambda_L$ in the limit $L \to +\infty$.
Finally, let $E_1(\omega,\Lambda) \leq E_2(\omega,\Lambda) \leq \cdots \leq E_N(\omega,\Lambda) \leq \cdots$ denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity. Actually, the Minami estimate (M) implies that ω almost surely the eigenvalues are simple.

1.2. **The results.** For the finite volume approximations, we state our results in two cases. In the first case described in section 1.2.1, we consider a macroscopic energy interval i.e. the energy interval in which we study the eigenvalues is a fixed compact interval where all the above assumptions hold. In the second case described in section 1.2.2, the energy interval shrinks to a point but not too fast so as to contain enough eigenvalues that is asymptotically infinitely many eigenvalues.
We also consider another point of view on the random Hamiltonian. Namely, under assumption (Loc), in I, one typically proves that the spectrum of H_ω is made only of eigenvalues and that to these eigenvalues, one associates exponentially decaying eigenfunctions

(exponential or Anderson localization) (see e.g. [33, 12, 13, 21]). One can then enumerate these eigenvalues in an energy interval by considering only those with localization center (i.e. with most of their mass) in some cube Λ and study the thus obtained eigenvalue process. This is done in section 1.2.3.

1.2.1. *Macroscopic energy intervals.* For $J = [a, b]$ a compact interval such that $N(b) - N(a) = |N(J)| > 0$ and a fixed configuration ω, consider the point process

$$\Xi_J(\omega, t, \Lambda) = \sum_{E_n(\omega,\Lambda)\in J} \delta_{|N(J)||\Lambda|[N_J(E_n(\omega,\Lambda))-t]} \tag{1.6}$$

under the uniform distribution in $[0, 1]$ in t; here we have set

$$N_J(\cdot) := \frac{N(\cdot) - N(a)}{N(b) - N(a)} = \frac{N(\cdot) - N(a)}{|N(J)|}. \tag{1.7}$$

Our main result is

Theorem 1.1. *Assume (IAD), (W), (M) and (Loc) hold. Assume that $J \subset I$, the localization region, is such that $|N(J)| > 0$.*
Then, ω-almost surely, as $|\Lambda| \to +\infty$, the probability law of the point process $\Xi_J(\omega, \cdot, \Lambda)$ under the uniform distribution $\mathbf{1}_{[0,1]}(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.

First, let us note that Theorem 0.1 is an immediate consequence of Theorem 1.1 as it is well known that, for the discrete Anderson model at large disorder, the whole spectrum is localized in the sense of (Loc) (see e.g. [21]).
A number of spectral statistics for the unfolded eigenvalues are immediate consequences of Theorem 1.1 and the results of [30]. For example, by [30, Proposition 4.4], Theorem 1.1 implies the convergence of the empirical distribution of unfolded level spacings to e^{-x} (see [30, 31, 15]), namely,

Theorem 1.2. *Assume (IAD), (W), (M) and (Loc) hold. Assume that $J \subset I$, the localization region, is such that $|N(J)| > 0$. Let $N(J, \omega, \Lambda)$ be the random number of eigenvalues of $H_\omega(\Lambda)$ in J. Define the renormalized eigenvalue (or level) spacings in the following way*

$$\forall n, \qquad \delta N_n(\omega, \Lambda) = |\Lambda|(N(E_{n+1}(\omega, \Lambda)) - N(E_n(\omega, \Lambda))) \geq 0.$$

Define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DRLS(x; J, \omega, \Lambda) = \frac{\#\{E_n(\omega, \Lambda) \in J \text{ s.t. } \delta N_n(\omega, \Lambda) \geq x\}}{N(J, \omega, \Lambda)}.$$

Then, ω-almost surely, as $|\Lambda| \to +\infty$, $DRLS(x; J, \omega, \Lambda)$ converges uniformly to $e^{-x}\mathbf{1}_{x\geq 0}$.

We refer to [30] for more results on the statistics of asymptotically ergodic sequences.
As in [15], one can also study the statistics of the levels themselves i.e. before unfolding. Using classical results on transformations of point processes (see [5, 34]) and the fact that N is Lipschitz continuous and increasing, one obtains

Theorem 1.3. *Assume (IAD), (W), (M) and (Loc) hold. Assume that $J = [a, b] \subset I$ is a compact interval in the localization region satisfying $|N(J)| > 0$.*
Define

- *the probability density $\nu_J(\cdot) := \dfrac{1}{|N(J)|}\nu(\cdot)\mathbf{1}_J(\cdot)$ where $n = \dfrac{dN}{dE}$ is the density of states of H_ω;*

- *the point process* $\tilde{\Xi}_J(\omega,t,\Lambda)=\sum_{E_n(\omega,\Lambda)\in J}\delta_{\nu(t)|\Lambda|[E_n(\omega,\Lambda)-t]}$.

Then, ω-almost surely, the probability law of the point process $\tilde{\Xi}_J(\omega,\cdot,\Lambda)$ under the distribution $\nu_J(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.

We note that, in Theorem 1.3, we do not make any regularity assumption on N except for the Wegner estimate. This enables us to remove the regularity condition imposed on the density of states ν in the proof of the almost sure convergence of the level spacings statistics given in [15]. Thus, we prove

Theorem 1.4. *Assume (IAD), (W), (M) and (Loc) hold. Pick $J\subset I$ a compact interval in the localization region such that $|N(J)|>0$. Let $N(J,\omega,\Lambda)$ be the random number of eigenvalues of $H_\omega(\Lambda)$ in J. Define the eigenvalue (or level) spacings as*

$$\forall n,\quad \delta E_n(\omega,\Lambda)=|\Lambda|(E_{n+1}(\omega,\Lambda)-E_n(\omega,\Lambda))\geq 0$$

and the empirical distribution of these spacings to be the random numbers, for $x\geq 0$

$$DLS(x;J,\omega,\Lambda)=\frac{\#\{E_n(\omega,\Lambda)\in J \text{ s.t. } \delta E_n(\omega,\Lambda)\geq x\}}{N(J,\omega,\Lambda)}.$$

Then, ω-almost surely, as $|\Lambda|\to+\infty$, $DLS(x;J,\omega,\Lambda)$ converges uniformly to the distribution $x\mapsto g_{\nu,J}(x)$ where $g_{\nu,J}(x)=\dfrac{1}{|N(J)|}\displaystyle\int_J e^{-|N(J)|\cdot x\cdot\nu(\lambda)}\nu(\lambda)d\lambda$.

1.2.2. *Microscopic energy intervals.* One can also prove a version of Theorem 1.1 that is local in energy. One proves

Theorem 1.5. *Assume (IAD), (W), (M) and (Loc) hold in I. Pick $E_0\in I$.*
Fix $(I_\Lambda)_\Lambda$ a decreasing sequence of intervals such that $\sup_{I_\Lambda}|x|\underset{|\Lambda|\to+\infty}{\to}0$.

Let us assume that

$$\text{(1.8)}\qquad \text{if } \ell'=o(L) \text{ then } \frac{|N(E_0+I_{\Lambda_{L+\ell'}})|}{|N(E_0+I_{\Lambda_L})|}\underset{L\to+\infty}{\to}1.$$

Then, there exists $\tau=\tau(\rho)$ such that, if, for Λ large, one has

$$\text{(1.9)}\qquad |N(E_0+I_\Lambda)|\cdot|I_\Lambda|^{-1-\tilde\rho}\geq 1 \quad\text{and}\quad |\Lambda|^\delta\cdot|N(E_0+I_\Lambda)|\underset{|\Lambda|\to+\infty}{\to}+\infty$$

for some $\delta\in(0,1)$ and $\tilde\rho>0$ satisfying

$$\text{(1.10)}\qquad \frac{\delta\,\tilde\rho}{1+\tilde\rho}<\tau$$

then, ω-almost surely, the probability law of the point process $\Xi_{E_0+I_\Lambda}(\omega,\cdot,\Lambda)$ under the uniform distribution $\mathbf{1}_{[0,1]}(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.

The exponent $\tau=\tau(g)$ can be computed explicitly (see (3.64)). The first condition in (1.9) requires that N is not too flat at E_0. How flat it may be depends on the exponent $\tilde\rho$, thus, in part on the value of τ if δ is not less that τ. Indeed, if $\delta<\tau$, then (1.10) is satisfied for any $\tilde\rho>0$ and actually, we can take $\tilde\rho=+\infty$ i.e. drop the first condition in (1.9); note that this is what happens in the case of macroscopic intervals. If $\delta\geq\tau$, a condition on the flatness of N kicks in.
Condition (1.8) is necessary as we don't impose anything else on how the density of states

of the intervals $E_0 + I_\Lambda$ be have; they could oscillate which could presumably ruin convergence.
As a consequence of Theorem 1.5, using the results of [30], one shows that one has convergence of the unfolded local level spacings distribution at any point of the almost sure spectrum if one looks at "large" enough neighborhoods of the point; here, "large" does not mean that the neighborhood needs to be large: it merely needs not to shrink too fast to 0 (see (1.9)). In particular, the conclusions of Theorem 1.2 hold true for the eigenvalues in $E_0 + I_\Lambda$ satisfying the assumptions of Theorem 1.5.

1.2.3. *Results for the random Hamiltonian on the whole space.* In our previous results, we considered the eigenvalues of the random Hamiltonian restricted to a box. As in [15], one can also consider the operator H_ω on the whole space. Therefore, we recall

Proposition 1.1 ([15]). *Assume (IAD), (W) and (Loc). Fix $q > 2d$. Then, for any $\xi \in (0,1)$, ω-almost surely, there exists $C_\omega > 1$ such that $\mathbb{E}(C_\omega) < \infty$, such that*

(1) *with probability 1, if $E \in I \cap \sigma(H_\omega)$ and φ is a normalized eigenfunction associated to E then, for any $x(E,\omega) \in \mathbb{R}^d$ or $\mathbb{Z}^d$ that is a maximum of $x \mapsto \|\varphi\|_x$, one has, for $x \in \mathbb{R}^d$,*
$$\|\varphi\|_x \leq C_\omega (1 + |x(E,\omega)|^2)^{q/2} e^{-|x - x(E,\omega)|^\xi}$$
where $\|\cdot\|_x$ is defined in (1.4).
We define $x(E,\omega)$ to be **a center of localization** *for E or φ.*

(2) *Pick $J \subset I$ such that $|N(J)| > 0$. Let $N^f(J,\Lambda,\omega)$ denotes the number of eigenvalues of H_ω having a center of localization in Λ. Then, there exists $\beta > 0$ such that, for Λ sufficiently large, one has*
$$\left| \frac{N^f(J,\Lambda,\omega)}{|N(J)|\,|\Lambda|} - 1 \right| \leq \frac{1}{\log^\beta |\Lambda|}.$$

In view of Proposition 1.1, ω-almost surely, for L sufficiently large, there are only finitely many eigenvalues of H_ω in J having a localization center in Λ_L. Thus, we can enumerate these eigenvalues as $E_1^f(\omega,\Lambda) \leq E_2^f(\omega,\Lambda) \leq \cdots \leq E_N^f(\omega,\Lambda)$ where we repeat them according to multiplicity. As in the finite volume case, ω almost surely, these eigenvalues are simple.
For $t \in [0,1]$, define the point process $\Xi_J^f(\omega,t,\Lambda)$ by (1.6) and (1.7) for those eigenvalues. As a corollary of Theorem 1.1, we obtain

Theorem 1.6. *Assume (IAD), (W), (M) and (Loc) hold. Assume that $J \subset I$, the localization region, that $|N(J)| > 0$.*
Then, ω-almost surely, the probability law of the point process $\Xi_J^f(\omega,\cdot,\Lambda)$ under the uniform distribution $\mathbf{1}_{[0,1]}(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.

Theorem 1.6 also admits an corresponding analogue that is local in energy i.e. a counterpart of Theorem 1.5.

1.3. **Outline of the paper.** Let us briefly outline the remaining parts of the paper. In section 2, we recall some results from [15] that we build our analysis upon. The strategy of the proof will be roughly to study the eigenvalues of the random operator where the integrated density of states, $N(\cdot)$, takes value close to t. Most of those eigenvalues, as in shown in [15], can be approximated by i.i.d. random variables the distribution law of which is roughly uniform on $[0,1]$ when properly renormalized. We then show that this

approximation is accurate enough to obtain the almost sure convergence announced in Theorem 1.1.
Theorem 1.5 is proved in the same way and we only make a few remarks on this proof in section 3.7. Theorem 1.6 is deduced from Theorem 1.1 approximating the eigenvalues of H_ω by those of $H_\omega(\Lambda)$ for sufficiently large Λ; this is done in section 3.8.
Section 4 is devoted to the proof of Theorems 1.3 and 1.4 . It relies on point process techniques, in particular, on transformations of point processes (see e.g. [5, 34]).

2. The spectrum of a random operator in the localized regime

Let us now recall some results describing the spectrum of a random operator in the localized regime that we will use in our proofs. They are mostly taken from [15].

2.1. **I.I.D approximations to the eigenvalues.** The first ingredient of our proof is a description of most of the eigenvalues of $H_\omega(\Lambda)$ in some small interval, say, I_Λ: it holds with a probability that tends to 1 faster than any negative power of $|\Lambda|$. The description is given in terms of i.i.d. random variables that we construct explicitly: they are the unique eigenvalue inside I_Λ of the restrictions of $H_\omega(\Lambda)$ to disjoint cubes that are much smaller than Λ. The distribution of these random variables is computed in Lemma 2.2 in section 2.2.

2.1.1. *Localization estimates and localization centers.* We first recall a result of [15] defining and describing localization centers, namely,

Lemma 2.1 ([15]). *Under assumptions (W) and (Loc), for any $p>0$ and $\xi\in(0,1)$, there exists $q>0$ such that, for $L\geq 1$ large enough, with probability larger than $1-L^{-p}$, if*

(1) *$\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to $E_{n,\omega}\in I$,*
(2) *$x_n(\omega)\in\Lambda_L$ is a maximum of $x\mapsto\|\varphi_{n,\omega}\|_x$ in Λ_L,*

then, for $x\in\Lambda_L$, one has

$$\|\varphi_{n,\omega}\|_x\leq L^q e^{-|x-x_n(\omega)|^\xi}$$

where $\|\cdot\|_x$ is defined in (1.4).
Moreover, define $C(\varphi)=\{x\in\Lambda;\ \|\varphi\|_x=\max_{\gamma\in\Lambda}\|\varphi\|_\gamma\}$ to be the set of localization centers for φ. Then, the diameter of $C(\varphi_j(\omega,\Lambda))$ is less than $C_q(\log|\Lambda|)^{1/\xi}$.

For each eigenfunction φ, we define its localization center in a unique way by ordering the set $C(\varphi)$ lexicographically and taking its supremum.

2.1.2. *An approximation theorem for eigenvalues.* Pick $\xi\in(0,1)$, $R>1$ large and $\rho'\in(0,\rho)$ where ρ is defined in (M). For a cube Λ, consider an interval $I_\Lambda=[a_\Lambda,b_\Lambda]\subset I$. Set $\ell'_\Lambda=(R\log|\Lambda|)^{\frac{1}{\xi}}$. We say that the sequence $(I_\Lambda)_\Lambda$ is (ξ,R,ρ'')-admissible if, for any Λ, one has

$$|\Lambda|\,|N(I_\Lambda)|\geq 1,\quad |N(I_\Lambda)|\,|I_\Lambda|^{-1-\rho''}\geq 1,\quad |N(I_\Lambda)|^{\frac{1}{1+\rho''}}(\ell'_\Lambda)^d\leq 1. \tag{2.1}$$

The reduction theorem we will use is a modified version of [15, Theorem 1.15], namely,

Theorem 2.1. *Assume (IAD), (W), (M) and (Loc) hold. Let $\Lambda=\Lambda_L$ be the cube of center 0 and side length L.*
Pick $\rho'\geq\rho$ and $\rho''\in\left(0,\frac{\rho}{1+d(\rho'+1)}\right)$ where ρ is defined in (M). Pick a sequence of intervals that is (ξ,R,ρ'')-admissible, say, $(I_\Lambda)_\Lambda$ such that $\ell'_\Lambda\ll\tilde\ell_\Lambda\ll L$ and $|N(I_\Lambda)|^{\frac{1}{1+\rho''}}\tilde\ell_\Lambda^d\to 0$ as $|\Lambda|\to\infty$.

For any $p > 0$, for L sufficiently large (depending only on (ξ, R, ρ'', p) but not on the admissible sequence of intervals), there exists

- *a decomposition of Λ_L into disjoint cubes of the form $\Lambda_{\ell_\Lambda}(\gamma_j) := \gamma_j + [0, \ell_\Lambda]^d$, where $\ell_\Lambda = \tilde{\ell}_\Lambda(1 + \mathcal{O}(\tilde{\ell}_\Lambda/|\Lambda_L|)) = \tilde{\ell}_\Lambda(1 + o(1))$ such that*
  - $\cup_j \Lambda_{\ell_\Lambda}(\gamma_j) \subset \Lambda_L$,
  - $\text{dist}\,(\Lambda_{\ell_\Lambda}(\gamma_j), \Lambda_{\ell_\Lambda}(\gamma_k)) \geq \ell'_\Lambda$ *if* $j \neq k$,
  - $\text{dist}\,(\Lambda_{\ell_\Lambda}(\gamma_j), \partial\Lambda_L) \geq \ell'_\Lambda$,
  - $|\Lambda_L \setminus \cup_j \Lambda_{\ell_\Lambda}(\gamma_j)| \lesssim |\Lambda_L|\ell'_\Lambda/\ell_\Lambda$,
- *a set of configurations $\mathcal{Z}_\Lambda$ such that*
  - *$\mathcal{Z}_\Lambda$ is large, namely,*

$$\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - |\Lambda|^{-p} - \exp\left(-c|N(I_\Lambda)|^{(1+\rho)/(1+\rho'')}|\Lambda|\ell_\Lambda^{d\rho'}\right) - \exp\left(-c|\Lambda||N(I_\Lambda)|^{1/(1+\rho'')}\ell'_\Lambda\ell_\Lambda^{-1}\right) \tag{2.2}$$

so that

- *for $\omega \in \mathcal{Z}_\Lambda$, there exists at least $\dfrac{|\Lambda|}{\ell_\Lambda^d}\left(1 + O\left(|N(I_\Lambda)|^{1/(1+\rho'')}\ell_\Lambda^d\right)\right)$ disjoint boxes $(\Lambda_{\ell_\Lambda}(\gamma_j))_j$ satisfying the properties:*
  1. *the Hamiltonian $H_\omega(\Lambda_{\ell_\Lambda}(\gamma_j))$ has at most one eigenvalue in I_Λ, say, $E_n(\omega, \Lambda_{\ell_\Lambda}(\gamma_j))$;*
  2. *$\Lambda_{\ell_\Lambda}(\gamma_j)$ contains at most one center of localization, say $x_{k_j}(\omega, L)$, of an eigenvalue of $H_\omega(\Lambda)$ in I_Λ, say $E_{k_j}(\omega, \Lambda)$;*
  3. *$\Lambda_{\ell_\Lambda}(\gamma_j)$ contains a center $x_{k_j}(\omega, \Lambda)$ if and only if $\sigma(H_\omega(\Lambda_{\ell_\Lambda}(\gamma_j))) \cap I_\Lambda \neq \emptyset$; in which case, one has*

$$|E_{k_j}(\omega, \Lambda) - E_n(\omega, \Lambda_{\ell_\Lambda}(\gamma_j))| \leq |\Lambda|^{-R} \text{ and } \text{dist}(x_{k_j}(\omega, L), \Lambda_L \setminus \Lambda_{\ell_\Lambda}(\gamma_j)) \geq \ell'_\Lambda \tag{2.3}$$

where we recall that $\ell'_\Lambda = (R \log |\Lambda|)^{\frac{1}{\xi}}$;

- *the number of eigenvalues of $H_\omega(\Lambda)$ that are not described above is bounded by*

$$C|N(I_\Lambda)||\Lambda|\left(|N(I_\Lambda)|^{\frac{\rho-\rho''}{1+\rho''}}\ell_\Lambda^{d(1+\rho')} + |N(I_\Lambda)|^{-\frac{\rho''}{1+\rho''}}(\ell'_\Lambda)^{d+1}\ell_\Lambda^{-1}\right); \tag{2.4}$$

this number is $o(|N(I_\Lambda)||\Lambda|)$ provided

$$|N(I_\Lambda)|^{-\frac{\rho''}{1+\rho''}}(\ell'_\Lambda)^{d+1} \ll \ell_\Lambda \ll |N(I_\Lambda)|^{-\frac{\rho-\rho''}{d(1+\rho')(1+\rho'')}}. \tag{2.5}$$

We first note that the assumptions on $(I_\Lambda)_\Lambda$ in Theorem 2.1 imply that $|I_\Lambda| \to 0$ and $|N(I_\Lambda)|$ must go to 0 faster than logarithmically in $|\Lambda|$ (see the right hand side of (2.5)). We note that the statement of Theorem 2.1 is essentially void except if the probability lower bounded in (2.2) does not go to 0. This will depend on the choice one makes for the length scales ℓ_Λ and on the size of $N(I_\Lambda)$ (resp. $|I_\Lambda|$ the two being linked by (2.1)). How to make this choice depends on the problem one wants to analyze. An example of such a choice is given in [15].

Let us now briefly explain how the length-scale $\ell = \ell_\Lambda$ will be chosen in the present analysis (see section 3.2). We will use Theorem 2.1 on intervals I_Λ such that $|N(I_\Lambda)| \asymp |\Lambda|^{-\alpha}$ (for some $\alpha \in (0, 1)$ to be chosen) and set $\ell_\Lambda \asymp |N(I_\Lambda)|^{-\mu}$ for some $\mu \in (0, 1)$. Thus, $\log \ell'_\Lambda \asymp \log(\log |\Lambda|) \ll \log |\Lambda| \asymp \log \ell_\Lambda$ and (2.5) is satisfied if the exponent μ is chosen so that

$$\frac{\rho''}{1+\rho''} < \mu < \frac{\rho - \rho''}{d(1+\rho')(1+\rho'')}. \tag{2.6}$$

This is possible as $\rho' \geq \rho$ and $\rho'' \in \left(0, \frac{\rho}{1+d(\rho'+1)}\right)$.
When comparing Theorem 2.1 with [15, Theorem 1.15], we see that we have introduced a new parameter $\rho' > 0$. The main benefit is that this will enable us to take ρ'' and μ small (at the expense of taking ρ' large) and, thus, take α close to 1.
Note that the left hand side inequality in (2.5) implies that $|N(I_\Lambda)|^{\frac{1}{1+\rho''}}\tilde{\ell}^d_\Lambda \to 0$ as $\frac{\rho-\rho''}{1+\rho} < 1$.
With these choices, there exists $\chi \in (0,1)$ such that the bound (2.4) then becomes

$$C|N(I_\Lambda)||\Lambda|^{1-\chi} \tag{2.7}$$

Notice that (2.6), $\rho' \geq \rho$ and $\rho'' > 0$ imply that $\mu \in (0, 1/d)$.
We still have one parameter to choose, namely, α. We will choose it in such way that the lower bound in (2.2) tends quickly to 1. Pick $\alpha \in (0,1)$ so that

$$\max\left(\frac{1+\rho}{1+\rho''} - d\mu\rho', \mu + \frac{1}{1+\rho''}\right) < \frac{1}{\alpha}. \tag{2.8}$$

This is possible as $\rho' \geq \rho$, $\rho'' \in \left(0, \frac{\rho}{1+d(\rho'+1)}\right)$ and (2.6) imply

$$1 < \mu + \frac{1}{1+\rho''} \quad \text{and} \quad 1 < \frac{1+\rho}{1+\rho''} - d\mu\rho'.$$

Thus, as $\ell_\Lambda \asymp |N(I_\Lambda)|^{-\mu}$ and $|N(I_\Lambda)| \asymp |\Lambda|^{-\alpha}$, one computes that

$$|N(I_\Lambda)|^{(1+\rho)/(1+\rho'')}|\Lambda|\ell_\Lambda^{d\rho} \gtrsim |N(I_\Lambda)|^{(1+\rho)/(1+\rho'')-d\rho'\mu}|\Lambda| \geq |\Lambda|^{1-\alpha[(1+\rho)/(1+\rho'')-d\rho'\mu]},$$

and

$$|\Lambda||N(I_\Lambda)|^{1/(1+\rho'')}\ell'_\Lambda\ell_\Lambda^{-1} \gtrsim |N(I_\Lambda)|^{1/(1+\rho'')+\mu}|\Lambda|(\log|\Lambda|)^{1/\xi} \geq |\Lambda|^{1-\alpha[1/(1+\rho'')+\mu]}(\log|\Lambda|)^{1/\xi}.$$

Hence, as, by (2.8), $1-\alpha\left[\frac{1+\rho}{1+\rho''} - d\rho'\mu\right] > 0$ and $1-\alpha\left[\frac{1}{1+\rho''}+\mu\right] > 0$, the lower bound in estimate (2.2) becomes

$$\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - |\Lambda|^{-p}. \tag{2.9}$$

In the proofs of Theorem 1.1 and 1.5, we will use Theorem 2.1 on intervals I_Λ of weight $N(I_\Lambda)$ and with length-scales ℓ_Λ chosen as just explained (see section 3.2.1). We will need some additional restrictions on the exponents ρ', μ and α that will be introduced in section 3.2.1.

We will not give a self-contained proof of Theorem 2.1. In the appendix, section 5, we only indicate the (very few) modifications that are to be made to the proof of [15, Theorem 1.15] to obtain Theorem 2.1.

2.2. **Distribution of the unfolded eigenvalues.** The second ingredient of our proof is the distribution of the unfolded eigenvalues for the operator H_ω restricted to the small cubes $(\Lambda_\ell(\gamma))_\gamma$ constructed in Theorem 2.1.
Pick $1 \ll \ell' \ll \ell$. Consider a cube $\Lambda = \Lambda_\ell$ centered at 0 of side length ℓ. Pick an interval $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$ (i.e. I_Λ is contained in the localization region) for ℓ sufficiently large.
Consider the following random variables:

- $X = X(\Lambda, I_\Lambda) = X(\Lambda, I_\Lambda, \ell')$ is the Bernoulli random variable

$$X = \mathbf{1}_{H_\omega(\Lambda) \text{ has exactly one eigenvalue in } I_\Lambda \text{ with localization center in } \Lambda_{\ell-\ell'}}$$

- $\tilde{E} = \tilde{E}(\Lambda, I_\Lambda)$ is this eigenvalue conditioned on $X = 1$.

Let ϑ be the distribution function of $\tilde{E}$. We know

Lemma 2.2 ([15]). *Assume (W), (M) and (Loc) hold.*
For $\kappa\in(0,1)$, one has

$$|\mathbb{P}(X=1)-|N(I_\Lambda)||\Lambda||\lesssim(|\Lambda||I_\Lambda|)^{1+\rho}+|N(I_\Lambda)||\Lambda|\ell'\ell^{-1}+|\Lambda|e^{-(\ell')^\kappa} \tag{2.10}$$

where $N(E)$ denotes the integrated density of states of H_ω.
One has

$$|(\vartheta(x)-\vartheta(y))\,P(X=1)|\lesssim|x-y||I_\Lambda||\Lambda|.$$

Moreover, setting $N(x,y,\Lambda):=[N(a_\Lambda+x|I_\Lambda|)-N(a_\Lambda+y|I_\Lambda|)]|\Lambda|$, one has

$$\begin{aligned}|(\vartheta(x)-\vartheta(y))\,P(X=1)-N(x,y,\Lambda)|&\\ \lesssim(|\Lambda||I_\Lambda|)^{1+\rho}+|N(x,y,\Lambda)|\ell'\ell^{-1}+|\Lambda|e^{-(\ell')^\kappa}.&\end{aligned} \tag{2.11}$$

Estimates (2.10) and (2.11) are of interest mainly if their right hand side, which are to be understood as error terms, are smaller than the main terms. In (2.10), the main restriction comes from the requirement that $N(I_\Lambda)|\Lambda|\gg(|\Lambda||I_\Lambda|)^{1+\rho}$ which is essentially a requirement that $N(I_\Lambda)$ should not be too small with respect to $|I_\Lambda|$. Lemma 2.2 will be used in conjunction with Theorems 2.1. The cube Λ in Lemma 2.2 will be the cube Λ_ℓ in Theorem 2.1. Therefore, the requirements induced by the other two terms are less restrictive. The second term is an error term if $\ell'\ll\ell$ which is guaranteed by assumption; this induces no new requirement. To guarantee that the third term in the right hand side of (2.10) be small compared to $|N(I_\Lambda)||\Lambda|$, one requires that $|N(I_\Lambda)||\Lambda|\gg\ell^d e^{-(\ell')^\kappa}$. This links the size of the cube $\Lambda=\Lambda_\ell$ where we apply Lemma 2.2 to the size of $|N(I_\Lambda)|$. The right choice for ℓ (that is conditioned by Theorem 2.1) is $\ell\asymp|N(I_\Lambda)|^{-\mu}$. In our application, we will pick $\ell'\asymp(\log\ell)^{1/\xi}$ for some $\xi\in(0,1)$ coming from the localization estimate (Loc); so taking $\kappa>\xi$ ensures that the third term in the right hand side of (2.10) is small compared to $|N(I_\Lambda)||\Lambda|$. For further details, we refer to the comments following the statement of Theorem 2.1 and section 3.2.1 for details.
In (2.11), the main restriction comes from the requirement that $N(x,y,\Lambda)\gg(|\Lambda||I_\Lambda|)^{1+\rho}$. This is essentially a requirement on the size of $|x-y|$: it should not be too small. On the other hand, we expect the spacing between the eigenvalues of $H_\omega(\Lambda_L)$ to be of size $|\Lambda_L|^{-1}$. Note that, here, we keep the notations of Theorem 2.1. Recall that the cube Λ in Lemma 2.2 will be the cube Λ_ℓ in Theorem 2.1, hence, a cube much smaller than Λ_L. So to distinguish between the eigenvalues, one needs to be able to know ϑ up to a resolution $|x-y||I_\Lambda|\sim|\Lambda_L|^{-1}$. This will force us to use Lemma 2.2 on intervals I_Λ such that $|N(I_\Lambda)|\asymp|\Lambda|^{-\alpha}$ for some $\alpha\in(0,1)$ close to 1 (see the discussion following Theorem 2.1 and section 3.2.1). Moreover, the approximation of $\vartheta(x)-\vartheta(y)$ by $N(x,y,\Lambda)/P(X=1)$ will be good if $|x-y|\gg(|\Lambda_L||I_\Lambda|)^{-1}\asymp|\Lambda_L|^{-\chi}$ for some $\chi>0$.

2.3. A large deviation principle for the eigenvalue counting function.

Define the random numbers

$$N(I_\Lambda,\Lambda,\omega):=\#\{n;\ E_n(\omega,\Lambda)\in I_\Lambda\}. \tag{2.12}$$

Write $I_\Lambda=[a_\Lambda,b_\Lambda]$ and recall that $|N(I_\Lambda)|=N(b_\Lambda)-N(a_\Lambda)$ where N is the integrated density of states. Using Theorem 2.1 and standard large deviation estimates for i.i.d. random variables, one shows that $N(I_\Lambda,\Lambda,\omega)$ satisfies a large deviation principle, namely,

Theorem 2.2. *Assume (IAD), (W), (M) and (Loc) hold. Recall that ρ is defined in Assumption (M).*
For any $\rho''\in(0,\rho/(1+(1+\rho)d))$, $\delta\in(0,1)$ and $\delta'\in(0,1-\delta)$, there exists $\delta''>0$ such that, if $(I_\Lambda)_\Lambda$ is a sequence of compact intervals in the localization region I satisfying

(1) $|N(I_\Lambda)||\Lambda|^\delta \to 0$ *as* $|\Lambda| \to +\infty$
(2) $|N(I_\Lambda)|\,|\Lambda|^{1-\delta'} \to +\infty$ *as* $|\Lambda| \to +\infty$
(3) $|N(I_\Lambda)|\,|I_\Lambda|^{-1-\rho''} \to +\infty$ *as* $|\Lambda| \to +\infty$,

then, for any $p > 0$, *for* $|\Lambda|$ *sufficiently large (depending on* ρ'', δ *and* δ' *but not on the specific sequence* $(I_\Lambda)_\Lambda$*), one has*

$$\mathbb{P}\left(|N(I_\Lambda, \Lambda, \omega) - |N(I_\Lambda)||\Lambda|| \geq |N(I_\Lambda)||\Lambda|^{1-\delta''}\right) \leq |\Lambda|^{-p}. \tag{2.13}$$

This result is essentially Theorem 1.8 in [15]; the only change is a change of scale for $|N(I_\Lambda)|$ in terms of $|\Lambda|$ (see point (1)). Up to this minor difference, the proofs of the two results are the same.
Assume that, for J, an interval in the region of localization I, one has the lower bound $|N(x) - N(y)| \gtrsim |x - y|^{1+\rho''}$ for $(x, y) \in I^2$ and some $\rho'' \in (0, \rho/(1 + (1 + \rho)d))$. Then, as $K \mapsto |N(K)|$ is a measure, thus, additive, for $K \subset J$ the region of localization, one may split K into intervals $(K_k)_k$ such that $|N(K_k)| \asymp |\Lambda|^{-\delta}$, and sum the estimates given by Theorem 2.2 on each K_k to obtain that

$$\mathbb{P}\left(|N(K, \Lambda, \omega) - |N(K)||\Lambda|| \geq |N(K)||\Lambda|^{1-\delta'}\right) \lesssim |\Lambda|^{-p}.$$

Though we will not need it, this gives an interesting large deviation estimate for intervals of macroscopic size.

3. The proofs of Theorems 1.1, 1.6 and 1.5

We first prove Theorem 1.1. Theorem 1.6 is then a immediate consequence of Theorem 1.1 and the fact that most of the eigenvalues of $H_\omega(\Lambda)$ and those of H_ω having center of localization in Λ differ at most by L^{-p} for any p and L sufficiently large (see section 3.8). Theorem 1.5 is proved in the same way as Theorem 1.1 in section 3.8; thus, we skip most of the details of this proof.
We shall use the following standard notations: $a \lesssim b$ means there exists $c < \infty$ so that $a \leq cb$; $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. We write $a \asymp b$ when $a \lesssim b$ and $b \lesssim a$.
From now on, to simplify notations, we write N instead of N_J so that the density of states increases from 0 to 1 on J. We also write Ξ instead of Ξ_J
For $\varphi :\ \mathbb{R} \to \mathbb{R}$ continuous and compactly supported, set

$$\mathcal{L}_{\omega,\Lambda}(\varphi) := \mathcal{L}_{\omega,J,\Lambda} := \int_0^1 e^{-\langle \Xi(\omega,t,\Lambda),\varphi\rangle} dt \tag{3.1}$$

and

$$\langle \Xi(\omega, t, \Lambda), \varphi\rangle := \sum_{E_n(\omega,\Lambda)\in J} \varphi(|\Lambda|[N(E_n(\omega, \Lambda)) - t]) \tag{3.2}$$

To prove Theorems 1.1 and 1.5, it suffices (see [31]) to prove

Theorem 3.1. *For* $\varphi :\ \mathbb{R} \to \mathbb{R}^+$ *continuously differentiable and compactly supported, ω-almost surely,*

$$\mathcal{L}_{\omega,\Lambda}(\varphi) \underset{|\Lambda|\to+\infty}{\to} \exp\left(-\int_{-\infty}^{+\infty} \left(1 - e^{-\varphi(x)}\right) dx\right). \tag{3.3}$$

Then, a standard dense subclass argument shows that the limit (3.3) holds for compactly supported, continuous, non negative functions. This completes the proof of Theorem 1.1.

3.1. **The proof of Theorem 3.1.** The integrated density of states N is non decreasing. By assumption (W), it is Lipschitz continuous. One can partition $[0,1] = \bigcup_{m\in\mathcal{M}} I_m$ where $\mathcal{N}$ is at most countable and $(I_m)_{m\in\mathcal{M}}$ are intervals such that either

- I_m is open and N is strictly increasing on the open interval $N^{-1}(I_m)$; we then say that $m \in \mathcal{M}^+$;
- I_m reduces to a single point and N is constant on the closed interval $N^{-1}(I_m)$; we then say that $m \in \mathcal{M}^0$.

We prove

Lemma 3.1. *For the limit (3.3) to hold ω-almost surely, it suffices that, for any $m \in \mathcal{M}^+$, for $\varphi : \mathbb{R} \to \mathbb{R}^+$ continuously differentiable and compactly supported, ω-almost surely, one has*

$$\left|\mathcal{L}_{\omega,I_m,\Lambda}(\varphi) - \exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right)\right| \underset{|\Lambda|\to+\infty}{\to} 0. \tag{3.4}$$

Proof. As for $m \in \mathcal{M}^0$, I_m is a single point, one computes

$$\mathcal{L}_{\omega,\Lambda}(\varphi) = \sum_{m\in\mathcal{M}^+}\int_{I_m} e^{-\langle\Xi(\omega,t,\Lambda),\varphi\rangle}dt. \tag{3.5}$$

Assume $J = [a,b]$. Fix $t \in I_m = (N(a_m), N(b_m))$ for some $m \in \mathcal{M}^+$. For $m \in \mathcal{M}^0$, N is constant equal to, say, N_m on I_m. Assume that φ has its support in $(-R,R)$. Then, for $|\Lambda|$ large (depending only on R), one computes

$$\begin{aligned}
\langle\Xi(\omega,t,\Lambda),\varphi\rangle &= \sum_{m\in\mathcal{M}^0}\#\{E_n(\omega,\Lambda)\in I_m\}\varphi(|\Lambda|[N_m - t)]) \\
&\quad + \sum_{m\in\mathcal{M}^+}\sum_{E_n(\omega,\Lambda)\in I_m}\varphi(|\Lambda|[N(E_n(\omega,\Lambda)) - t)]) \\
&= \sum_{E_n(\omega,\Lambda)\in I_m}\varphi(|\Lambda|[N(E_n(\omega,\Lambda)) - t)]) \\
&= \sum_{E_n(\omega,\Lambda)\in I_m}\varphi(|N(I_m)||\Lambda|[N_{I_m}(E_n(\omega,\Lambda)) - (t-N(a_m))/|N(I_m)|)]) \\
&= \langle\Xi_{I_m}(\omega,(t-N(a_m))/|N(I_m)|,\Lambda),\varphi\rangle
\end{aligned}$$

On the other hand

$$\int_{N(a_m)}^{N(b_m)} e^{-\langle\Xi_{I_m}(\omega,(t-N(a_m))/|N(I_m)|,\Lambda),\varphi\rangle}dt = |N(I_m)|\int_0^1 e^{-\langle\Xi_{I_m}(\omega,t,\Lambda),\varphi\rangle}dt.$$

Recall that, as the measure defined by N is absolutely continuous with respect to the Lebesgue measure, we have

$$\sum_{m\in\mathcal{M}^+}|N(I_m)| = |N(J)| = 1.$$

Thus, by Lebesgue's dominated convergence theorem, as $\mathcal{N}^+$ is at most countable, we get that, if the necessary condition given in Lemma 3.1 is satisfied, then ω-almost surely, we get

$$\lim_{|\Lambda|\to+\infty}\mathcal{L}_{\omega,\Lambda}(\varphi) = \sum_{m\in\mathcal{M}^+}|N(I_m)|\lim_{|\Lambda|\to+\infty}\mathcal{L}_{\omega,I_m,\Lambda}(\varphi).$$

Thus, we have proved Lemma 3.1. □

From now on, we assume that N is a strictly increasing one-to-one mapping from J to $[0,1]$ and prove Theorem 3.1 under this additional assumption.

Therefore, we first bring ourselves back to proving a similar result for "local" eigenvalues i.e. eigenvalues of restrictions of $H_\omega(\Lambda)$ to cubes much smaller than Λ that lie inside small intervals i.e. much smaller than J. The "local" eigenvalues are those described by points (1), (2), (3) of Theorem 2.1. Using Lemma 2.2 then essentially brings ourselves back to the case of i.i.d. random variables uniformly distributed on $[0,1]$.

Theorem 2.1 does not give control on all the eigenvalues. To control the integral (3.1), this is not necessary: a good control of most of the eigenvalues is sufficient as Lemma 3.8 below shows. Theorem 2.2, which is a corollary of Theorem 2.1 and Lemma 2.2, is used to obtain good bounds on the number of controlled eigenvalues in the sense of Lemma 3.8.

3.2. **Reduction to the study of local eigenvalues.** Assume we are in the setting of Theorem 1.1 and that N is as above i.e. N is a strictly increasing Lipschitz continuous function from J to $[0,1]$. Recall that ν is its derivative, the density of states.

3.2.1. *Choosing the right scales.* To obtain our results, we will use Theorem 2.1 and Lemma 2.2. Therefore, we split the interval I into small intervals and choose the length scale $\ell = \ell_\Lambda$ so that we can apply both Theorem 2.1 and Lemma 2.2 to these intervals. We now explain how this choice is done.

Recall that ρ is defined in (M) and pick (ρ', ρ'') such that

$$\rho' \geq \rho + 1 + \max\left(\rho, \frac{1}{d}\right) \quad \text{and} \quad \frac{\rho}{1+2d(\rho'-\rho)} < \rho'' < \frac{\rho}{1+d(\rho'+1)}. \tag{3.6}$$

The computations done right after Theorem 2.1 (see also [15, section 4.3.1]) show that, for $\alpha \in (0,1)$ satisfying (2.8) and $\mu \in (0, 1/d)$ satisfying (2.6) for I_Λ (in the localization region) and $\ell = \ell_\Lambda$ such that

$$|N(I_\Lambda)| \asymp |\Lambda|^{-\alpha} \quad \text{and} \quad \ell_\Lambda \asymp |N(I_\Lambda)|^{-\mu} \tag{3.7}$$

if, in addition, I_Λ satisfies

$$|N(I_\Lambda)| \geq |I_\Lambda|^{1+\rho''}, \tag{3.8}$$

we can apply Theorem 2.1 and Lemma 2.2 to I_Λ.

In addition to (2.6) and (2.8), we now require μ and α to satisfy

$$\frac{\rho-\rho''}{d(1+\rho'')(2\rho'-\rho)} < \mu \quad \text{and} \quad 1 + \frac{1}{2}\left(\frac{\rho-\rho''}{1+\rho''} - d\mu\rho\right) > \frac{1}{\alpha}. \tag{3.9}$$

This is possible as (3.6) implies that

$$\frac{\rho-\rho''}{1+\rho''} - d\mu\rho' < \frac{1}{2}\left(\frac{\rho-\rho''}{1+\rho''} - d\mu\rho\right) \text{ and } \frac{\rho-\rho''}{d(1+\rho'')(2\rho'-\rho)} < \frac{\rho-\rho''}{d(1+\rho')(1+\rho'')}. \tag{3.10}$$

We now define two more exponents that will be useful in the sequel:

- define β by

$$\beta := 1 - \frac{1}{\alpha} + \frac{1+\rho}{1+\rho''} - d\mu\rho - \frac{1}{\alpha} > 2 - \frac{2}{\alpha} + \frac{\rho-\rho''}{1+\rho''} - d\mu\rho > 0 \tag{3.11}$$

  using the second inequality in (3.9);

- pick κ satisfying

$$\max\left(1, \frac{1}{d\alpha\beta}\right) < \kappa < \frac{1+\rho''}{d\alpha\rho''} \tag{3.12}$$

which is possible by (3.6) as $\alpha \in (0,1)$.

3.2.2. *Reduction to small energy intervals.* Partition $J = [a,b]$ into disjoint intervals $(J_{j,\Lambda})_{1\leq j\leq j_\Lambda}$ of weight $|N(J_{j,\Lambda})| \sim |\Lambda|^{-\alpha}$ so that $j_\Lambda \asymp |\Lambda|^\alpha$.
Define the sets

$$B = \left\{1 \leq j \leq j_\Lambda;\ |N(J_{j,\Lambda})| \leq |J_{j,\Lambda}|^{1+\rho''}\right\} \quad \text{and} \quad G = \{1, \cdots, j_\Lambda\} \setminus B. \tag{3.13}$$

The set B is the set of "bad" indices j for which the interval $J_{j,\Lambda}$ does not satisfy the assumptions of Theorem 2.1, more precisely, does not satisfy the second condition in (2.1) (that is (3.8)) for the exponent ρ''.
For $j \in B$, one has

$$|J_{j,\Lambda}| \geq |N(J_{j,\Lambda})|^{1/(1+\rho'')} = |\Lambda|^{-\alpha/(1+\rho'')}.$$

Thus, one gets

$$\#B \lesssim |\Lambda|^{\alpha/(1+\rho'')} \tag{3.14}$$

Fix $\alpha' \in (\alpha, \min[1, \alpha(1+2\rho'')/(1+\rho'')])$. For $j \in G$, write $J_{j,\Lambda} = [a_\Lambda, b_\Lambda)$ and define

$$K_{j,\Lambda} := [a'_\Lambda, b'_\Lambda] \subset J_{j,\Lambda} \text{ where } \begin{cases} a'_\Lambda = \inf\left\{a \geq a_\Lambda; N(a) - N(a_\Lambda) \geq |\Lambda|^{-\alpha'}\right\}, \\ b'_\Lambda = \sup\left\{b \leq b_\Lambda; N(b_\Lambda) - N(b) \geq |\Lambda|^{-\alpha'}\right\}. \end{cases} \tag{3.15}$$

that is, $K_{j,\Lambda}$ is the interval $J_{j,\Lambda}$ where small neighborhoods of the endpoints have been remove.
Thus, our construction yields that

1. the total density of states of the set we have remove is bounded by

$$\sum_{j\in B} |N(J_{j,\Lambda})| + \sum_{j\in G} |N(J_{j,\Lambda} \setminus K_{j,\Lambda})| \lesssim |\Lambda|^{-\alpha+\alpha/(1+\rho'')} + |\Lambda|^{-\alpha'+\alpha} \lesssim |\Lambda|^{-(\alpha'-\alpha)}; \tag{3.16}$$

2. for $j \in G$, $t \in N(K_{j,\Lambda})$ and $E \in J_{j',\Lambda}$ for $j' \neq j$, one has

$$|\Lambda||N(E) - t| \gtrsim |\Lambda|^{1-\alpha'}.$$

Note that one has

$$1 = |N(J)| = \sum_{j\in G} |N(J_{j,\Lambda})| + \sum_{j\in B} |N(J_{j,\Lambda})| = \sum_{j\in G} |N(K_{j,\Lambda})| + O\left(|\Lambda|^{-(\alpha'-\alpha)}\right). \tag{3.17}$$

Recall (3.2). Thus, for Λ sufficiently large, by point (1) above, as φ is non negative, one has

$$\begin{aligned}\int_0^1 e^{-\langle \Xi(\omega,t,\Lambda),\varphi\rangle} dt &= \sum_{j\in G} \int_{N(K_{j,\Lambda})} e^{-\langle \Xi(\omega,t,\Lambda),\varphi\rangle} dt + O\left(|\Lambda|^{-(\alpha'-\alpha)}\right) \\ &= \sum_{j\in G} \int_{N(K_{j,\Lambda})} e^{-\langle \Xi_j(\omega,t,\Lambda),\varphi\rangle} dt + O\left(|\Lambda|^{-(\alpha'-\alpha)}\right)\end{aligned}$$

where, as φ is compactly supported, by point (2) above, for $|\Lambda|$ large, one has

$$\langle \Xi_j(\omega,t,\Lambda),\varphi\rangle = \sum_{E_n(\omega,\Lambda)\in J_{j,\Lambda}} \varphi(|\Lambda|[N(E_n(\omega,\Lambda)) - t]).$$

Point (1) and (3.17) then yield

$$\begin{aligned}\int_0^1 e^{-\langle \Xi(\omega,t,\Lambda),\varphi\rangle}dt &= \sum_{j\in G}\int_{N(J_{j,\Lambda})} e^{-\langle \Xi_j(\omega,t,\Lambda),\varphi\rangle}dt + O\left(|\Lambda|^{-(\alpha'-\alpha)}\right)\\ &= \sum_{j\in G}|N(J_{j,\Lambda})|\int_0^1 e^{-\langle \Xi_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle}dt + O\left(|\Lambda|^{-(\alpha'-\alpha)}\right)\end{aligned} \tag{3.18}$$

where $\Xi_{J_{j,\Lambda}}(\omega,t,\Lambda)$ is defined by (1.6) for $J = J_{j,\Lambda}$.

3.2.3. *Asymptotic ergodicity uniformly for the small intervals.* Following the proof of Lemma 3.1, the above computation shows that the limit (3.4) will hold ω-almost surely if we prove that, ω almost surely, one has

$$\sup_{j\in G}\left|\int_0^1 e^{-\langle \Xi_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle}dt - \exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right)\right| \underset{|\Lambda|\to+\infty}{\to} 0. \tag{3.19}$$

To prove (3.19), we first prove a weaker result, namely, almost sure convergence along a subsequence.

Lemma 3.2. *Pick $(\alpha_L)_{L\geq 1}$ any sequence valued in $[1/2,2]$ such that $\alpha_L \to 1$ when $L \to +\infty$.*
For $\kappa > 1$ satisfying (3.12) and for $\varphi : \mathbb{R}\to\mathbb{R}^+$ continuously differentiable and compactly supported, ω-almost surely, one has

$$\sup_{j\in G}\left|\int_0^1 e^{-\langle \Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L^\kappa}),\varphi_{\alpha_L}\rangle}dt - \exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right)\right| \underset{L\to+\infty}{\to} 0 \tag{3.20}$$

where, for $\alpha > 0$, we have set, $\varphi_\alpha(\cdot) = \varphi(\alpha\,\cdot)$.

Indeed, Lemma 3.2, (3.18) and (3.17) clearly imply the claimed almost sure convergence on a subsequence; more precisely, it implies that, for $(\alpha_L)_{L\geq 1}$ a sequence such that $\alpha_L \to 1$ when $L \to +\infty$, ω-almost surely,

$$\left|\mathcal{L}_{\omega,\Lambda_{L^\kappa}}(\varphi_{\alpha_L}) - \exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right)\right| \underset{L\to+\infty}{\to} 0. \tag{3.21}$$

which is the claimed almost sure convergence on a subsequence for the choice of sequence $\alpha_L = 1$.
To obtain the almost sure convergence on the whole sequence, we prove

Lemma 3.3. *Fix κ satisfying (3.12). Then, for some $\kappa' > 0$, for $\varphi : \mathbb{R}\to\mathbb{R}^+$ continuously differentiable and compactly supported, ω-almost surely, for L sufficiently large, one has*

$$\sup_{L^\kappa\leq L'\leq (L+1)^\kappa}\left|\mathcal{L}_{\omega,\Lambda_{L'}}(\varphi) - \mathcal{L}_{\omega,\Lambda_{L^\kappa}}(\varphi_{\alpha_{L'}})\right| \lesssim L^{-\kappa'} \tag{3.22}$$

where $\alpha_{L'} = |\Lambda_{L'}|/|\Lambda_{L^\kappa}|$.

As $\alpha_L \to 1$ when $L \to +\infty$, equation (3.19) and, thus, Theorem 3.1, are immediate consequences of (3.21) and (3.22).

3.3. **The proof of Lemma 3.2.** The proof of Lemma 3.2 will consist in reducing the computation of the limit (3.20) to the case of i.i.d. random variables that have a distribution close to the uniform one. The number of these random variables will be random as well but large; it is essentially controlled by Theorem 2.2.
The reduction is done in three steps. First, using Theorem 2.1, we introduce a family of i.i.d. random variables, the distribution of which is controlled by Lemma 2.2. Second, in Lemma 3.4, we show that the Laplace transform of the process defined by these random variables is close to the Laplace transform of the process we want to compute; therefore, we use th description given by Theorem 2.1. Finally, in Lemma 3.5, we show that Laplace transform of the process defined by the new random variables converges to that of the Poisson process; therefore, we use the distribution computed using Lemma 2.2.

Pick R large in Theorem 2.1. The construction done in section 3.2.2 with the choice of scales ℓ_Λ and exponents μ, α, ρ' and ρ''explained in section 3.2.1 implies that, for $j\in G$ (see (3.13)), one can apply

- Theorem 2.1 to the energy interval $I_\Lambda := J_{j,\Lambda}$ for $H_\omega(\Lambda_L)$, the small cubes being of side length $\ell=\ell_\Lambda$;
- Lemma 2.2 to the energy interval $I_\Lambda := J_{j,\Lambda}$ and any of the cubes $\Lambda_\ell(\gamma)$ of the decomposition obtained in Theorem 2.1.

For $j\in G$ and $(\Lambda_\ell(\gamma_k))_k$, the cubes constructed in Theorem 2.1 (we write $\ell=\ell_\Lambda$), define the random variables:

- $X_{j,k}=X(\Lambda_\ell(\gamma_k),J_{j,\Lambda})$ is the Bernoulli random variable
  $$X_{j,k}=\mathbf{1}_{H_\omega(\Lambda_\ell(\gamma_k))\text{ has exactly one eigenvalue in }J_{j,\Lambda}\text{ with localization center in }\Lambda_{\ell-\ell'}}$$
  where $\ell=\ell_\Lambda$ and $\ell'=\ell'_\Lambda$ are chosen as in Theorem 2.1 and section 3.2.1; thus, $\ell_\Lambda\asymp|\Lambda|^{\alpha\mu}$ and $\ell'_\Lambda\asymp(\log|\Lambda|)^\xi$ (for some $\xi\in(0,1)$);
- $\tilde E_{j,k}=\tilde E(\Lambda_\ell(\gamma_k),J_{j,\Lambda})$ is this eigenvalue conditioned on the event $\{X_{j,k}=1\}$;

and the point measure

$$\Xi^{app}_{J_{j,\Lambda}}(\omega,t,\Lambda):=\sum_{k;\ X_{j,k}=1}\delta_{|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(\tilde E_{j,k})-t]}. \tag{3.23}$$

We prove

Lemma 3.4. *There exists $\chi>0$ such that, for any $p>0$ and $R>0$, there exists a set of configurations, say, $\mathcal{Z}_\Lambda$ such that $\mathbb{P}(\mathcal{Z}_\Lambda)\geq 1-|\Lambda|^{-p}$ and, for Λ sufficiently large, one has*

$$\sup_{\varphi\in\mathcal{C}^+_{1,R}}\sup_{\substack{j\in G\\ \omega\in\mathcal{Z}_\Lambda}}\left|\int_0^1 e^{-\langle\Xi_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle}dt-\int_0^1 e^{-\langle\Xi^{app}_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle}dt\right|\lesssim|\Lambda|^{-\chi}. \tag{3.24}$$

where we have defined

$$\mathcal{C}^+_{1,R}=\left\{\varphi:\ \mathbb{R}\to\mathbb{R}^+;\ \begin{array}{c}\varphi\text{ is continuously differentiable s.t.}\\ \operatorname{supp}\varphi\subset(-R,R)\text{ and }\|\varphi\|_{\mathcal{C}^1}\leq R\end{array}\right\}. \tag{3.25}$$

and

Lemma 3.5. *For $\kappa>1$ satisfying (3.12), for $\varphi\in\mathcal{C}^+_{1,R}$ and for any $(\alpha_L)_{L\geq1}$ a sequence valued in $[1/2,2]$, one has*

$$\sum_{j\in G}\sum_{L\geq1}\mathbb{E}\left(\left[\int_0^1 e^{-\langle\Xi^{app}_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L^\kappa}),\varphi_{\alpha_L}\rangle}dt-\exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi_{\alpha_L}(x)}\right)dx\right)\right]^2\right)<+\infty.$$

Let us now complete the proof of Lemma 3.2 using Lemmas 3.4 and 3.5.
Fix κ, φ and $(\alpha_L)_L$ as in Lemma 3.2. Picking $p>1$, as all the integrands are bounded by 1 and as $\mathbb{P}(\mathcal{Z}_\Lambda)\geq 1-L^{-p}$, (3.24) and the Borel-Cantelli Lemma imply that

$$\mathbb{E}\left(\limsup_{L\geq 1}\sup_{j\in G}\left|\int_0^1 e^{-\langle \Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L^\kappa}),\varphi_{\alpha_L}\rangle}dt-\int_0^1 e^{-\langle \Xi^{app}_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L^\kappa}),\varphi_{\alpha_L}\rangle}dt\right|\right)=0.$$

Moreover, as $\alpha_L\to 1$ when $L\to+\infty$, Lemma 3.5 and the Dominated Convergence Theorem clearly imply that, for $\varphi\in\mathcal{C}^+_{1,R}$, one has

$$\mathbb{E}\left(\limsup_{L\geq 1}\sup_{j\in G}\left|\int_0^1 e^{-\langle \Xi^{app}_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,j,\Lambda_{L^\kappa}),\varphi_{\alpha_L}\rangle}dt-\exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi_{\alpha_L}(x)}\right)dx\right)\right|\right)=0,$$

and

$$\lim_{L\to+\infty}\exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi_{\alpha_L}(x)}\right)dx\right)=\exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right).$$

These three estimates clearly imply (3.20) and complete the proof of Lemma 3.2.

3.4. **The proof of Lemma 3.4.** For $j\in G$, we let $\mathcal{Z}^j_\Lambda$ be the set of configurations ω defined by Theorem 2.1 for the energy interval $I_\Lambda=J_{j,\Lambda}$. Then, for any p, if Λ is sufficiently large (independently of $j\in G$), (2.9) gives a lower bound on $\mathbb{P}(\mathcal{Z}^j_\Lambda)$; this uniformity is warranted by Theorem 2.1: the size of the cube Λ necessary for the result to hold does not depend on the admissible sequence; it depends only on the parameters of admissibility that are the same for all the intervals $(J_{j,\Lambda})_{j\in G}$..
Let $\mathcal{N}^b_{\omega,j,\Lambda}$ be the set of indices n of the eigenvalues $(E_n(\omega,\Lambda))_n$ of $H_\omega(\Lambda)$ in $J_{j,\Lambda}$ that are not described by (1)-(3) of Theorem 2.1. Let $\mathcal{N}^g_{\omega,j,\Lambda}$ be the complementary set. Both sets are random. By (2.5) and our choice of length-scales (see the comments following Theorem 2.1), the number of eigenvalues not described by (1), (2) and (3) of Theorem 2.1, say, $N^b_{\omega,j,\Lambda}:=\#\mathcal{N}^b_{\omega,j,\Lambda}$ is bounded by, for some $\chi>0$,

$$N^b_{\omega,j,\Lambda}\leq |N(J_{j,\Lambda})||\Lambda|^{1-\chi} \tag{3.26}$$

whereas, by (2.13) in Theorem 2.2, the total number of eigenvalue of $H_\omega(\Lambda)$ in $J_{j,\Lambda}$, say, $N(J_{j,\Lambda},\Lambda,\omega)$ satisfies, for some $\delta>0$, for any $p>0$ and $|\Lambda|$ sufficiently large (independent of $j\in G$),

$$\mathbb{P}\left(\left|\frac{N(J_{j,\Lambda},\Lambda,\omega)}{|N(J_{j,\Lambda})||\Lambda|}-1\right|\geq|\Lambda|^{-\delta}\right)\leq|\Lambda|^{-p}. \tag{3.27}$$

Let now $\mathcal{Z}^j_\Lambda$ be the set of configurations ω where one has both the conclusions of Theorem 2.1 and the bound

$$\left|\frac{N(J_{j,\Lambda},\Lambda,\omega)}{|N(J_{j,\Lambda})||\Lambda|}-1\right|\leq|\Lambda|^{-\delta}. \tag{3.28}$$

By (2.9) and (3.27), this new set still satisfies (2.9).
Define the point measure:

$$\Xi^g_{J_{j,\Lambda}}(\omega,t,\Lambda):=\sum_{n\in\mathcal{N}^g_{\omega,j,\Lambda}}\delta_{|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(E_n(\omega,\Lambda))-t]}$$

and recall that, $(\Lambda_\ell(\gamma_k))_k$ are the cubes constructed in Theorem 2.1 (we write $\ell=\ell_\Lambda$) and we have defined the random variables:

- $X_{j,k} = X(\Lambda_\ell(\gamma_k), J_{j,\Lambda})$ is the Bernoulli random variable
  $$X_{j,k} = \mathbf{1}_{H_\omega(\Lambda_\ell(\gamma_k))\text{ has exactly one eigenvalue in } J_{j,\Lambda} \text{ with localization center in } \Lambda_{\ell-\ell'}}$$
  where $\ell = \ell_\Lambda$ and $\ell' = \ell'_\Lambda$ are chosen as described above;
- $\tilde E_{j,k} = \tilde E(\Lambda_\ell(\gamma_k), J_{j,\Lambda})$ is this eigenvalue conditioned on the event $\{X_{j,k} = 1\}$;

and the point measure $\Xi^{app}_{J_{j,\Lambda}}(\omega, t, \Lambda)$ by (3.23).
Let us now give an estimate of the number

$$N^{app}_{\omega,j,\Lambda} := \{k;\ X_{j,k} = 1\}. \tag{3.29}$$

It is provided by

Lemma 3.6. *For any $p > 0$, for $|\Lambda|$ sufficiently large (independent of $j \in G$), one has*

$$\mathbb{P}\left(\left|N^{app}_{\omega,j,\Lambda} - |N(J_{j,\Lambda})||\Lambda|\right| \geq [|N(J_{j,\Lambda})||\Lambda|]^{2/3}\right) \leq e^{-\left[|N(J_{j,\Lambda})||\Lambda|\right]^{1/3}/3} \leq |\Lambda|^{-p}.$$

Proof. Lemma 3.6 follows by a standard large deviation argument for the i.i.d. Bernoulli random variables $(X_{j,k})_k$ as, by Lemma 2.2 and our choice of $J_{j,\Lambda}$ and (ℓ', ℓ) (for $\mu \in (\xi, 1)$ in Lemma 2.2, ξ being the exponent fixing $\ell' = \ell'_\Lambda$ in Theorem 2.1), their common distribution satisfies

$$P(X_{j,k} = 1) = |N(J_{j,\Lambda})||\Lambda_\ell|(1 + o(1)).$$

The proof of Lemma 3.6 is complete. □

Thus, one may restrict once more the set of configurations ω to those such that, for some $\delta > 0$,

$$\left|\frac{N^{app}_{\omega,j,\Lambda}}{|N(J_{j,\Lambda})||\Lambda|} - 1\right| \leq |\Lambda|^{-\delta}. \tag{3.30}$$

and call this set again $\mathcal{Z}^j_\Lambda$. By Lemma 3.6 and (2.9), the probability of this set also satisfies (2.9) for any $p > 0$ provided $|\Lambda|$ is sufficiently large (independent of $j \in G$).

We now define the set $\mathcal{Z}_\Lambda$ of Lemma 3.4 as $\mathcal{Z}_\Lambda = \bigcap_{j\in G} \mathcal{Z}^j_\Lambda$. As for $|\Lambda|$ sufficiently large, all the sets $\{\mathcal{Z}^j_\Lambda\}_{j\in G}$ satisfy (2.9) and as $\#G \lesssim |\Lambda|$, we obtain that $\mathcal{Z}_\Lambda$ satisfies (2.9). Moreover, for $\omega \in \mathcal{Z}_\Lambda$, for all $j \in G$, one has the conclusions on Theorem 2.1 in $J_{j,\Lambda}$ as well as (3.28) and (3.30).
We now prove

Lemma 3.7. *For some $\chi > 0$, for Λ sufficiently large, one has,*

$$\sup_{\varphi\in\mathcal{C}^+_{1,R}} \sup_{\substack{j\in G\\ \omega\in\mathcal{Z}^j_\Lambda}} \left|\int_0^1 e^{-\langle \Xi_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle} dt - \int_0^1 e^{-\langle \Xi^g_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle} dt\right| \lesssim |\Lambda|^{-\chi}, \tag{3.31}$$

and

$$\sup_{\varphi\in\mathcal{C}^+_{1,R}} \sup_{\substack{j\in G\\ \omega\in\mathcal{Z}^j_\Lambda}} \left|\int_0^1 e^{-\langle \Xi^g_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle} dt - \int_0^1 e^{-\langle \Xi^{app}_{J_{j,\Lambda}}(\omega,t,\Lambda),\varphi\rangle} dt\right| \lesssim |\Lambda|^{-\chi}. \tag{3.32}$$

Clearly, by summing (3.31) and (3.32), we obtain (3.24). Thus, we will have completed the proof of Lemma 3.4 when we will have completed the proof of Lemma 3.7. Before proving Lemma 3.7, we state and prove a simple but useful result, namely,

Lemma 3.8. *Pick a sequence of scale $(L_p)_{p\geq 1}$ such that $L_p \to +\infty$. For $p \geq 1$, consider two finite sequences $(x_n^p)_{1\leq n\leq N_p}$ and $(y_m^p)_{1\leq m\leq M_p}$ such that there exists $1 \leq K_p \leq \min(N_p, M_p)$ and sets $X_p \subset \{1,\cdots,N_p\}$ and $Y_p \subset \{1,\cdots,M_p\}$ s.t.*

1. *$\#X_p = \#Y_p = K_p$ and $[(N_p - K_p) + (M_p - K_p)]/L_p =: a_p \to 0$,*
2. *there exists a one-to-one map, say $\Psi_p: X_p \mapsto Y_p$ such that, for $n \in X_p$, one has $|x_n^p - y_{\Psi_p(n)}^p| \leq \varepsilon_p/L_p$, $\varepsilon_p \in [0,1]$*

Fix $\eta \in (0,1)$. Set $\Xi_p^x(t) = \sum_{n=1}^{N_p} \delta_{L_p[x_n^p - t]}$ and $\Xi_p^y(t) = \sum_{m=1}^{M_p} \delta_{L_p[y_m^p - t]}$. Then, for p such that $a_p^{\eta-1} > R$, one has

$$\sup_{\varphi\in\mathcal{C}_{1,R}^+}\left|\int_0^1 e^{-\langle\Xi_p^x(t),\varphi\rangle}dt - \int_0^1 e^{-\langle\Xi_p^y(t),\varphi\rangle}dt\right| \leq 4a_p^\eta + e^{R\,\varepsilon_p\,K_p} - 1. \tag{3.33}$$

Proof of Lemma 3.8. Let $\tilde{X}_p = \{1,\cdots,N_p\}\setminus X_p$ and $\tilde{Y}_p = \{1,\cdots,M_p\}\setminus Y_p$. For $(n,m) \in \tilde{X}_p \times \tilde{Y}_p$, define

$$I_n^x = \begin{cases} x_n^p + a_p^\eta[N_p - K_p]^{-1}[-1,1] \text{ if } \tilde{X}_p \neq \emptyset \text{ i.e. } N_p - K_p \geq 1,\\ \emptyset \text{ if not;}\end{cases}$$

$$I_m^y = \begin{cases} y_m^p + a_p^\eta[M_p - K_p]^{-1}[-1,1] \text{ if } \tilde{Y}_p \neq \emptyset \text{ i.e. } M_p - K_p \geq 1,\\ \emptyset \text{ if not.}\end{cases}$$

Then, by point (1) of our assumptions on the sequences $(x_n^p)_n$ and $(y_m^p)_m$, one has

$$\begin{aligned} 0 &\leq \int_0^1 e^{-\langle\Xi_p^x(t),\varphi\rangle}dt - \int_{[0,1]\setminus[(\cup_{n\in\tilde{X}_p}I_n^x)\cup(\cup_{m\in\tilde{Y}_p}I_m^y]} e^{-\langle\Xi_p^x(t),\varphi\rangle}dt \\ &\leq (N_p - K_p)a_p^\eta[N_p - K_p]^{-1} + (M_p - K_p)a_p^\eta[M_p - K_p]^{-1} = 2a_p^\eta \end{aligned} \tag{3.34}$$

and, similarly

$$0 \leq \int_0^1 e^{-\langle\Xi_p^y(t),\varphi\rangle}dt - \int_{[0,1]\setminus[(\cup_{n\in\tilde{X}_p}I_n^x)\cup(\cup_{m\in\tilde{Y}_p}I_m^y]} e^{-\langle\Xi_p^y(t),\varphi\rangle}dt \leq 2a_p^\eta \tag{3.35}$$

On the other hand, for $t \in [0,1]\setminus[(\cup_{n\in\tilde{X}_p}I_n^x)\cup(\cup_{m\in\tilde{Y}_p}I_m^y]$ and p such that $a_p^{\eta-1} > R$, one has

$$L_p\,\mathrm{dist}(t, \tilde{X}_p \cup \tilde{Y}_p) \geq a_p^\eta\,L_p\,\sup\left([N_p - K_p]^{-1}, [N_p - K_p]^{-1}\right) \geq a_p^{\eta-1} > R.$$

Thus, for p such that $a_p^{\eta-1} > R$, for $t \in [0,1]\setminus[(\cup_{n\in\tilde{X}_p}I_n^x)\cup(\cup_{m\in\tilde{Y}_p}I_m^y]$ and $\varphi \in \mathcal{C}_{1,R}^+$ (see (3.25)), one has

$$\langle\Xi_p^x(t),\varphi\rangle = \sum_{n\in X_p}\varphi(L_p[x_n^p - t]) \quad \text{and} \quad \langle\Xi_p^y,\varphi\rangle = \sum_{m\in Y_p}\varphi(L_p[y_m^p - t]).$$

Now, by point (2) of our assumptions on the sequences $(x_n^p)_n$ and $(y_m^p)_m$, one has

$$\sup_{\varphi\in\mathcal{C}_{1,R}^+}\ \sup_{\substack{t\in[0,1]\\ t\notin(\cup_{n\in\tilde{X}_p}I_n^x)\\ t\notin(\cup_{m\in\tilde{Y}_p}I_m^y)}} \left|\langle\Xi_p^x(t),\varphi\rangle - \langle\Xi_p^y(t),\varphi\rangle\right| \leq \varepsilon_p\,K_p\cdot\sup_{\varphi\in\mathcal{C}_{1,R}^+}\|\varphi'\|_\infty \leq R\,\varepsilon_p\,K_p. \tag{3.36}$$

Hence, as φ is non negative, we obtain, for p such that $a_p^{\eta-1} > R$,

$$\sup_{\varphi\in\mathcal{C}^+_{1,R}}\left|\int_{[0,1]\setminus[(\cup_{n\in\tilde X_p} I_n^x)\cup(\cup_{n\in\tilde Y_p} I_n^y)]}\left(e^{-\langle\Xi_p^x(t),\varphi\rangle}-e^{-\langle\Xi_p^y(t),\varphi\rangle}\right)dt\right|\le e^{R\,\varepsilon_p\,K_p}-1 \tag{3.37}$$

Combining (3.34), (3.35) and (3.37) completes the proof of Lemma 3.8. □

Remark 3.1. Lemma 3.8, and, in particular, the error term coming from (3.36), can be improved if one assumes that the points in the sequences are not too densely packed. This is the case in the applications we have in mind. Though we do not use it here, it may be useful to treat the case of long range correlated random potentials where the error estimates of the local approximations of eigenvalues given by Theorem 2.1 can not be that precise anymore.

The proof of Lemma 3.7. As underlined above, the statements of Lemma 3.7 are corollaries of Lemma 3.8.
To obtain (3.31), for $p = |\Lambda|$, it suffices to take

- $x_n^p = E_n(\omega,\Lambda)$ for $n\in\mathcal{N}^g_{\omega,j,\Lambda}\cup\mathcal{N}^b_{\omega,j,\Lambda}$,
- $y_n^p = E_n(\omega,\Lambda)$ for $n\in\mathcal{N}^g_{\omega,j,\Lambda}$.

Assumption (2) in Lemma 3.8 is clearly fulfilled as $(y_n^p)_n$ is a subsequence of $(x_n^p)_n$. Assumption (1) is an immediate consequence (3.26) and (3.28). Moreover, by (3.28), in the notations of Lemma 3.8, using (3.7), we get that $a_p \asymp p^{-\chi}$ for some $\chi > 0$ independent of $j\in G$. Thus, we can apply Lemma 3.8 uniformly in $j\in G$ and obtain (3.31).
Let us now prove (3.32). Notice that, by Theorem 2.1, one has $N^{app}_{\omega,j,\Lambda}\ge N^g_{\omega,j,\Lambda}$. Moreover, to each $n\in\mathcal{N}^g_{\omega,j,\Lambda}$, one can associate a unique $k(n)\in[\![1,N^{app}_{\omega,j,\Lambda}]\!]$ such that $X_{j,k(n)}=1$ and the first part of (2.3) hold.
To prove (3.32), for $p=|\Lambda|$, it suffices to set

- $x_n^p = \tilde E_{j,k(n)}$ for $k(n)$ such that $X_{j,k(n)}=1$,
- $y_n^p = E_n(\omega,\Lambda)$ for $n\in\mathcal{N}^g_{\omega,j,\Lambda}$.

So we may take $K_p = N^g_{\omega,j,\Lambda}$. By the first part of (2.3), we know that assumption (2) of Lemma 3.8 is satisfied with $\varepsilon_p = |\Lambda|^{-2}$. Thus, $\varepsilon_p\cdot K_p\lesssim|\Lambda|^{-1}$.
That assumption (1) is satisfied follows immediately from (3.26) and (3.30). The uniformity in $j\in G$ is obtained as in the proof of (3.31) except that one uses (3.30) instead of (3.28).
This completes the proof of Lemma 3.7 and, thus, of Lemma 3.4. □

3.5. **The proof of Lemma 3.5.** Let us recall a few facts that will be of use in this proof. Write $\Lambda_\ell=\Lambda_\ell(0)$ and define the random variables X and $\tilde E$ as in the beginning of section 2.2 for $I_\Lambda = J_{j,\Lambda}$ and the cube Λ_ℓ. Recall that the cube $\Lambda=\Lambda_L$ is much larger than Λ_ℓ. Now, pick $N^{app}_{\omega,j,\Lambda}$ independent copies of $\tilde E$, say $(\tilde E_k)_{1\le k\le N^{app}_{\omega,j,\Lambda}}$ (see the beginning of section 3.3). Then, the random process $\Xi^{app}_{J_{j,\Lambda}}$ is the process

$$\Xi^{app}_{J_{j,\Lambda}}(\omega,t,\Lambda):=\sum_{1\le k\le N^{app}_{\omega,j,\Lambda}}\delta_{|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(\tilde E_k)-t]}.$$

By Lemma 3.8 and (3.30), it thus suffices to study the point process

$$\Xi(\omega,t,j,\Lambda):=\sum_{1\le k\le|\Lambda||N(J_{j,\Lambda})|}\delta_{|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(\tilde E_k)-t]}. \tag{3.38}$$

Recall that $N_{J_{j,\Lambda}}$ is defined by (1.7) for $J = J_{j,\Lambda}$. Pick $\varphi \in \mathcal{C}^+_{1,R}$ (see (3.25)). As the random variables $(\tilde{E}_k)_{1\leq k\leq |N(J_{j,\Lambda})||\Lambda|}$ are i.i.d., one computes

$$\mathbb{E}\left(\int_0^1 e^{-\langle \Xi(\omega,t,j,\Lambda),\varphi\rangle}dt\right) = \int_0^1 \Phi(t,\Lambda,J_{j,\Lambda},\varphi)dt \tag{3.39}$$

and

$$\mathbb{E}\left(\left[\int_0^1 e^{-\langle \Xi(\omega,t,j,\Lambda),\varphi\rangle}dt\right]^2\right) = \int_0^1\int_0^1 \Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)dtdt' \tag{3.40}$$

where

$$\Phi(t,\Lambda,J_{j,\Lambda},\varphi) = \left[1-\mathbb{E}\left(1-e^{-\varphi(|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(\tilde{E})-t])}\right)\right]^{|N(J_{j,\Lambda})||\Lambda|} \tag{3.41}$$

and

$$\begin{aligned}&\Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)\\ &= \left[1-\mathbb{E}\left(1-e^{-\varphi(|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(\tilde{E})-t])-\varphi(|N(J_{j,\Lambda})||\Lambda|[N_{J_{j,\Lambda}}(\tilde{E})-t'])}\right)\right]^{|N(J_{j,\Lambda})||\Lambda|}.\end{aligned} \tag{3.42}$$

If $E\mapsto N_{J_{j,\Lambda}}(E)$ were the distribution function of the random variable $\tilde{E}$, the random variables $N_{J_{j,\Lambda}}(\tilde{E})$ would be distributed uniformly on $[0,1]$ and the desired result would be standard and follow e.g. from the computations done in the appendix of [31]. The distribution function of $\tilde{E}$ is described by Lemma 2.2. As we only consider $j\in G$, we know that $|N(J_{j,\Lambda})|\geq |J_{j,\Lambda}|^{1+\rho''}$ for some ρ'' satisfying (3.6). Recall that $\xi\in(0,1)$ is the exponent defining ℓ'_Λ from Theorem 2.1. Choosing $\kappa\in(\xi,1)$ in Lemma 2.2, for $x\in J_{j,\Lambda}$ (take $y=0$), using (2.10) and (3.9), the estimation (2.11) becomes,, for some $\beta'>0$, for $|\Lambda|=|\Lambda_L|$ sufficiently large,

$$\begin{aligned}\left|(1+\kappa_\Lambda)\cdot|N(J_{j,\Lambda})||\Lambda|\,\vartheta(x)-|N(J_{j,\Lambda})||\Lambda|\,N_{J_{j,\Lambda}}(x)\right| &\lesssim |\Lambda||N(J_{j,\Lambda})|^{\frac{1+\rho}{1+\rho''}}|\Lambda_\ell|^\rho\\ &\lesssim |N(J_{j,\Lambda})|^{\frac{1+\rho}{1+\rho''}-d\mu\rho-\alpha^{-1}}\end{aligned} \tag{3.43}$$

where, by (2.10) and the same computation as in (3.43), one has

$$\kappa_\Lambda := \frac{\mathbb{P}(X(\Lambda_{\ell_\Lambda},J_{j,\Lambda},\ell'_\Lambda)=1)}{|N(J_{j,\Lambda})||\Lambda_\ell|}-1 \text{ and } |\kappa_\Lambda|\lesssim |N(J_{j,\Lambda})|^{(\rho-\rho'')/(1+\rho'')-d\mu\rho}. \tag{3.44}$$

Using (3.43), as $\varphi\in\mathcal{C}^+_{1,R}$, from (3.41) we derive

$$\begin{aligned}&\left|\frac{\log\Phi(t,\Lambda,J_{j,\Lambda},\varphi)}{|N(J_{j,\Lambda})||\Lambda|}-\log\left[1-\mathbb{E}\left(1-e^{-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\cdot\vartheta(\tilde{E})-t])}\right)\right]\right|\\ &\qquad\lesssim |N(J_{j,\Lambda})|^{\frac{1+\rho}{1+\rho''}-d\mu\rho-\alpha^{-1}}.\end{aligned} \tag{3.45}$$

The random variable $\vartheta(\tilde{E})$ is uniformly distributed on $[0,1]$; thus, we compute

$$\begin{aligned}\mathbb{E}\left(1-e^{-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\cdot\vartheta(\tilde{E})-t])}\right) &= \int_0^1\left(1-e^{-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\,u-t])}\right)du\\ &= \frac{1}{(1+\kappa_\Lambda)|N(J_{j,\Lambda})||\Lambda|}\int_{-|N(J_{j,\Lambda})||\Lambda|t}^{|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)-t]}\left(1-e^{-\varphi(u)}\right)du\\ &= \frac{1}{(1+\kappa_\Lambda)|N(J_{j,\Lambda})||\Lambda|}\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(u)}\right)du\end{aligned} \tag{3.46}$$

if we assume that t satisfies

$$\frac{R}{|N(J_{j,\Lambda})||\Lambda|}\leq t\leq 1+\kappa_\Lambda-\frac{R}{|N(J_{j,\Lambda})||\Lambda|}. \tag{3.47}$$

The last equality in (3.46) holds as φ has its support in $[-R,R]$.
Recall that, by (3.7), one has

$$|N(J_{j,\Lambda})||\Lambda|\asymp|N(J_{j,\Lambda})|^{1-\frac{1}{\alpha}},\quad |N(J_{j,\Lambda})||\Lambda||N(J_{j,\Lambda})|^{\frac{1+\rho}{1+\rho''}-d\mu\rho-\alpha^{-1}}\asymp|N(J_{j,\Lambda})|^{\beta} \tag{3.48}$$

where β is defined by (3.50).
Moreover, by (3.44), one has

$$|\kappa_\Lambda|\,|N(J_{j,\Lambda})|\,|\Lambda|\lesssim|N(J_{j,\Lambda})|^{\beta} \tag{3.49}$$

as, by (3.10), one has

$$1-\frac{1}{\alpha}+\frac{\rho-\rho''}{1+\rho''}-d\mu\rho=\beta+\frac{1}{\alpha}-1>\beta>0. \tag{3.50}$$

Thus, recalling that $|N(J_{j,\Lambda})|\to 0$ as $|\Lambda|\to+\infty$, (3.42), (3.43) and (3.47) yield, for $|\Lambda|$ sufficiently large,

$$\log\Phi(t,\Lambda,J_{j,\Lambda},\varphi)=\frac{1}{(1+\kappa_\Lambda)}\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(u)}\right)du+O\left(|N(J_{j,\Lambda})|^{\beta}\right) \tag{3.51}$$

if we assume that t satisfies

$$\frac{R}{|N(J_{j,\Lambda})||\Lambda|}\leq t\leq 1-\frac{R+1}{|N(J_{j,\Lambda})||\Lambda|}. \tag{3.52}$$

Note that, in (3.51), $O\left(|N(J_{j,\Lambda})|^{\beta}\right)$ is independent of j, thus, so is the smallest size of Λ one should choose for (3.51) to hold under assumption (3.52).
Let us now estimate $\log\Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)$. We proceed as above. Using (3.43) and $\varphi\in\mathcal{C}^+_{1,R}$, from (3.42), we derive

$$\begin{aligned}\frac{\log\Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)}{|N(J_{j,\Lambda})||\Lambda|}&\\ =\log\Big[1-\mathbb{E}\Big(1-e^{-\varphi(|N(J_{j,\Lambda})||[(1+\kappa_\Lambda)\cdot\vartheta(\tilde E)-t])-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\vartheta(\tilde E)-t'])}\Big)&\\ +O(|N(J_{j,\Lambda})|^{\frac{1+\rho}{1+\rho''}-d\mu\rho-\alpha^{-1}})\Big].&\end{aligned} \tag{3.53}$$

Moreover, for t and t' satisfying (3.47) such that additionally

$$\frac{2R}{|N(J_{j,\Lambda})||\Lambda|}\leq|t-t'|, \tag{3.54}$$

as above, one computes

$$\begin{aligned}&\mathbb{E}\left(1-e^{-\varphi(|N(J_{j,\Lambda})||[(1+\kappa_\Lambda)\cdot\vartheta(\tilde E)-t])-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\vartheta(\tilde E)-t'])}\right)\\ &=\int_0^1\left(1-e^{-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\,u-t])-\varphi(|N(J_{j,\Lambda})||\Lambda|[(1+\kappa_\Lambda)\,u-t'])}\right)du\\ &=\frac{2}{(1+\kappa_\Lambda)|N(J_{j,\Lambda})||\Lambda|}\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(u)}\right)du.\end{aligned} \tag{3.55}$$

As above, we obtain that, for $|\Lambda|$ sufficiently large,

$$\log\Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)=2\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(u)}\right)du+O\left(|N(J_{j,\Lambda})|^{\beta}\right) \tag{3.56}$$

if we assume that t and t' both satisfy (3.52) and (3.54). Again, in (3.56), $O\left(|N(J_{j,\Lambda})|^{\beta}\right)$ is independent of j, thus, so is the smallest size of Λ one should choose for (3.56) to hold under assumptions (3.52) and (3.54).
Finally notice that $\Phi(t,\Lambda,J_{j,\Lambda},\varphi)$ and $\Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)$ are both bounded by 1 and that the measure of the sets of $t\in[0,1]$ satisfying (3.52) and the measure of the sets of $(t,t')\in[0,1]^2$ satisfying (3.52) for t and t' and (3.54) are both larger than $1-O\left(|N(J_{j,\Lambda})|^{\alpha^{-1}(1-\alpha)}\right)$.
Thus, thus, taking (3.7) into account, we have proved

Lemma 3.9. *Fix $R>0$. Fix ρ' and ρ'' satisfying (3.6). Fix $\alpha\in(0,1)$ and $\mu\in(0,1/d)$ satisfying (2.6), (2.8) and (3.9).*
For $|\Lambda|$ sufficiently large (depending only on R, ρ', ρ'', α and μ), one has

$$\sup_{\varphi\in\mathcal{C}^+_{1,R}}\sup_{j\in G}\left|\int_0^1\Phi(t,\Lambda,J_{j,\Lambda},\varphi)dt-\exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right)\right|\lesssim|\Lambda|^{-\beta\alpha} \tag{3.57}$$

and

$$\sup_{\varphi\in\mathcal{C}^+_{1,R}}\sup_{j\in G}\left|\int_0^1\int_0^1\Phi(t,t',\Lambda,J_{j,\Lambda},\varphi)dtdt'-\exp\left(-2\int_{-\infty}^{+\infty}\left(1-e^{-\varphi(x)}\right)dx\right)\right| \lesssim|\Lambda|^{-\beta\alpha} \tag{3.58}$$

where β is defined in (3.50).

Let us use Lemma 3.9 to complete the proof of Lemma 3.5. For $L\geq1$, let $\Lambda=\Lambda_L$. Fix $(\alpha_L)_{L\geq1}$ a sequence valued in $[1/2,2]$. Then, for $\varphi\in\mathcal{C}^+_{1,R}$, the sequence $(\varphi_{\alpha_L})_{L\geq1}$ is bounded in $\mathcal{C}^+_{1,2R}$. Thus, by Lemma 3.9, for $\kappa>1$ such that $\kappa\alpha\beta d>1$ and $(\alpha_L)_{L\geq1}$, any sequence valued in $[1/2,2]$, we have that

$$\sum_{j\in G}\sum_{L\geq1}\mathbb{E}\left(\left[\int_0^1 e^{-\langle\Xi(\omega,t,j,\Lambda_{L^\kappa}),\varphi_{\alpha_L}\rangle}dt-\exp\left(-\int_{-\infty}^{+\infty}\left(1-e^{-\varphi_{\alpha_L}(x)}\right)dx\right)\right]^2\right)<+\infty.$$

Thus, we have proved Lemma 3.5. □
The additional restriction we impose on κ in (3.12), namely the upper bound, is not used in Lemma 3.5. It will be of use in Lemma 3.3.

3.6. **The proof of Lemma 3.3.** Fix κ satisfying (3.12). Clearly, by (3.17) and (3.18), to prove Lemma 3.3, it suffices to show that, for some $\kappa'>0$, ω-almost surely, one has

$$\sup_{\substack{j\in G\\ L^\kappa\leq L'\leq(L+1)^\kappa}}\left|\int_0^1 e^{-\langle\Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L'}),\varphi\rangle}dt-\int_0^1 e^{-\langle\Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L^\kappa}),\varphi_{\alpha_{L'}}\rangle}dt\right|\lesssim L^{-\kappa'} \tag{3.59}$$

where $\alpha_{L'}=|\Lambda_{L'}|/|\Lambda_{L^\kappa}|$. Notice here that we chose the same partition of J into $(J_{j,\Lambda_{L^\kappa}})_j$ for all $L^\kappa\leq L'\leq(L+1)^\kappa$ which is possible as $|\Lambda_{L'}|=|\Lambda_{L^\kappa}|(1+o(1))$.
The strategy of the proof of (3.59) goes as follows. In Lemma 3.10 below, we prove that, with a good probability, for all $j\in G$, most eigenvalues of $H_\omega(\Lambda_{L'})$ and of $H_\omega(\Lambda_{L^\kappa})$ in $J_{j,\Lambda_{L^\kappa}}$ have center of localization in $\Lambda_{(L-1)^\kappa}$; this will be obtained as a consequence of the description given by Theorem 2.1. Thus, by Lemma 2.1, the Minami and Wegner estimates (M) and (W), with a good probability, these eigenvalues of $H_\omega(\Lambda_{L'})$ and of $H_\omega(\Lambda_{L^\kappa})$ are close to one another. We then use Lemma 3.8 to compare the point measures $\Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L'})$ and $\Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega,t,\Lambda_{L^\kappa})$ and, thus, derive (3.59).
We prove

Lemma 3.10. *Pick $\kappa > 1$ satisfying (3.12) and $p > 0$ arbitrary. There exists $\kappa' > 0$ such that, with probability at least $1 - L^{-p}$, for L sufficiently large, one has*

(1) *if $L^\kappa \leq L' \leq (L+1)^\kappa$ and $j \in G$, to each eigenvalue of $H_\omega(\Lambda_{L'})$ in $J_{j,\Lambda_{L^\kappa}}$ with localization center in $\Lambda_{(L-1)^\kappa}$, say, E, one can associate a unique eigenvalue of $H_\omega(\Lambda_{L^\kappa})$ in $J_{j,\Lambda_{L^\kappa}}$, say, E', such that $|E - E'| \leq L^{-3d\kappa}$;*

(2) *if $L^\kappa \leq L' \leq (L+1)^\kappa$ and $j \in G$, to each eigenvalue of $H_\omega(\Lambda_{L^\kappa})$ in $J_{j,\Lambda_{L^\kappa}}$ with localization center in $\Lambda_{(L-1)^\kappa}$, say, E, one can associate a unique eigenvalue of $H_\omega(\Lambda_{L'})$ in $J_{j,\Lambda_{L^\kappa}}$, say, E', such that $|E - E'| \leq L^{-3d\kappa}$.*

(3)

$$\sup_{\substack{L^\kappa \leq L' \leq (L+1)^\kappa \\ j \in G}} \left[\left| \frac{N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \Lambda_{(L-1)^\kappa}, \omega)}{N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L^\kappa}, \Lambda_{(L-1)^\kappa}, \omega)} - 1 \right| + \left| \frac{N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \Lambda_{(L-1)^\kappa}, \omega)}{N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \omega)} - 1 \right| \right] \lesssim L^{-\kappa'};$$

We postpone the proof of Lemma 3.10 and use it to apply Lemma 3.8 to $\Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega, t, \Lambda_{L'})$ and $\Xi_{J_{j,\Lambda_{L^\kappa}}}(\omega, t, \Lambda_{L^\kappa})$. By Lemma 3.10, with probability at least $1 - L^{-p}$, the assumptions of Lemma 3.8 will be satisfied if, using the notations of Lemma 3.8, we take

- X_p to be the eigenvalues of $H_\omega(\Lambda_{L'})$ in $J_{j,\Lambda_{L^\kappa}}$ with localization center in $\Lambda_{(L-1)^\kappa}$,
- Y_p to be the eigenvalues of $H_\omega(\Lambda_{L^\kappa})$ in $J_{j,\Lambda_{L^\kappa}}$ with localization center in $\Lambda_{(L-1)^\kappa}$.

Indeed, Lemma 3.10 then provides the estimates

$$0 \leq a_p \lesssim L^{-\kappa'}, \quad 0 \leq K_p \leq C L^{d\kappa+1} \quad \text{and} \quad 0 \leq \varepsilon_p \leq L^{-3d\kappa}.$$

Hence, with probability at least $1 - L^{-p}$, (3.22) is an immediate consequence of Lemma 3.8 (where one of the functions φ has been replaced with $\varphi_{\alpha_{L'}}$). Taking $p > 1$ and applying the Borel-Cantelli lemma, this completes the proof of Lemma 3.3. □

Proof of Lemma 3.10. To prove Lemma 3.10, our main ingredients will be the Minami estimate (M), the Wegner estimate (W), Lemma 2.1 and Theorem 2.1. Pick L' such that $L^\kappa \leq L' \leq (L+1)^\kappa$. Slicing the interval J into intervals of size $L^{-p\rho^{-1} - \kappa d(1+\rho^{-1})}$, for each slice, the Minami estimate tells us that the probability to find two eigenvalues in this slice is bounded by $CL^{-(p+\kappa d)(1+\rho^{-1})}$. As the number of slices is bounded by $CL^{p\rho^{-1} + \kappa d(1+\rho^{-1})}$, we know that, there exists $C > 0$ such that, with probability at least $1 - L^{-p}$,

(P1): no two eigenvalues of $H_\omega(\Lambda_{L'})$ in J are at a distance from each other smaller than $C^{-1} L^{-p\rho^{-1} - \kappa d(1+\rho^{-1})}$.

Fix $\xi \in (0, 1)$ arbitrary. By Lemma 2.1 and the Wegner estimate, we know that, for L sufficiently large, with probability at least $1 - L^{-p}$,

(P2): for any $j \in G$, if E is any eigenvalue of $H_\omega(\Lambda_{L'})$ (resp. $H_\omega(\Lambda_{L^\kappa})$) in $J_{j,\Lambda_{L^\kappa}}$ associated to a localization center in $\Lambda_{(L-1)^\kappa}$, then there exists E' an eigenvalue of $H_\omega(\Lambda_{L^\kappa})$ (resp. $H_\omega(\Lambda_{L'})$) in $J_{j,\Lambda_{L^\kappa}}$ such that $|E - E'| \leq e^{-L^{(\kappa-1)\xi/3}}$.

Indeed, if E is an eigenvalue of $H_\omega(\Lambda_{L'})$ associated to the normalized eigenfunction φ and the localization center x_E, letting ψ_L be a (smooth) cut-off supported in the ball $B(x_E, L^{(\kappa-1)/2})$, we have

$$\left| \|\psi_L \varphi\|_{L^2(\Lambda_{L^\kappa})} - 1 \right| + \|(H_\omega(\Lambda_{L^\kappa}) - E)(\psi_L \varphi)\|_{L^2(\Lambda_{L^\kappa})} \leq e^{-L^{(\kappa-1)\xi/3}} \|\psi_L \varphi\|_{L^2(\Lambda_{L^\kappa})}.$$

Thus, $H_\omega(\Lambda_{L^\kappa})$ has an eigenvalue, say, E' at distance at most $e^{-L^{(\kappa-1)\xi/3}}$ from E. Moreover, by the Wegner estimate (W), the probability that any eigenvalue of $H_\omega(\Lambda_{L^\kappa})$ falls into $\left(J_{j,\Lambda_{L^\kappa}} + e^{-L^{(\kappa-1)\xi/3}}[-1, 1]\right) \setminus J_{j,\Lambda_{L^\kappa}}$ is bounded by $CL^{d\kappa} e^{-L^{(\kappa-1)\xi/3}}$ as, by (3.7), one has

$|J_{j,\Lambda_{L^\kappa}}| \geq N(J_{j,\Lambda_{L^\kappa}}) \asymp L^{-d\alpha}$. Thus, inverting the roles of $H_\omega(\Lambda_{L^\kappa})$ and $H_\omega(\Lambda_{L'})$, we get (P2).

Combining (P1) and (P2), as for L large one has $e^{-L^{(\kappa-1)\xi/3}} \ll L^{-p\rho^{-1}-\kappa d(1+\rho^{-1})}$, we see that, with probability at least $1-3L^{-p}$, for any $j \in G$, there exists Ψ_j a bijection between the eigenvalues of $H_\omega(\Lambda_{L'})$ in $J_{j,\Lambda_{L^\kappa}}$ associated to a loc. center in $\Lambda_{(L-1)^\kappa}$ and the eigenvalues of $H_\omega(\Lambda_{L^\kappa})$ in $J_{j,\Lambda_{L^\kappa}}$ associated to a loc. center in $\Lambda_{(L-1)^\kappa}$ satisfying $|\Psi(E)-E| \leq e^{-L^{(\kappa-1)\xi/3}}$. Thus, we get that, with probability at least $1-3L^{-p}$, one has (1) and (2) of Lemma 3.10 as well as

$$N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \Lambda_{(L-1)^\kappa}, \omega) = N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L^\kappa}, \Lambda_{(L-1)^\kappa}, \omega). \tag{3.60}$$

thus, the first part of point (3) in Lemma 3.10.

To complete the proof of Lemma 3.10, we use Theorem 2.1 with $R > 0$ satisfying $R > d^{-1}(p\rho^{-1} + \kappa d(1+\rho^{-1}))$. For all $j \in G$, we apply Theorem 2.1 to $H_\omega(L)$ and $J_{j,\Lambda_{L^\kappa}}$ where $H_\omega(L)$ is either $H_\omega(\Lambda_{L'})$ or $H_\omega(\Lambda_{L^\kappa})$. We then know that, with probability at least $1-L^{-p}$, the eigenvalues of $H_\omega(L)$ corresponding to localization centers in $\Lambda_{(L-1)^\kappa}$ are described by (1), (2), (3) except for at most $N(J_{j,\Lambda_{L^\kappa}})L^{d\kappa(1-\chi)}$ of them. The number of cubes $(\Lambda_\ell(\gamma)))_\gamma$ constructed in Theorem 2.1 that intersect $\Lambda_{L'} \setminus \Lambda_{(L-1)^\kappa}$ or $\Lambda_{L^\kappa} \setminus \Lambda_{(L-1)^\kappa}$ is of order $|\Lambda_{L'}|^{(d-1/\kappa)/d}\ell^{-d}$ (where $\ell = \ell_{L'}$), that is using (3.7), of order $|\Lambda_{L'}|^{(d-1/\kappa)/d-d\alpha\mu}$. The condition (3.12) guarantees that $1 - \frac{1}{\kappa d} - d\alpha\mu > 0$.

Moreover, for each such cube $\Lambda_\ell(\gamma)$, the operator $H_\omega(\Lambda_\ell(\gamma))$ puts at most one eigenvalue in $J_{j,\Lambda_{L^\kappa}}$. The Wegner estimate implies that the probability that $H_\omega(\Lambda_\ell(\gamma))$ puts at least one eigenvalue in $J_{j,\Lambda_{L^\kappa}}$ is bounded by $C|J_{j,\Lambda_{L^\kappa}}|\ell^d$.

Hence, by a standard large deviation principle for independent random variables (see e.g. [9]), with a probability at least $1-L^{-p}$, the number of eigenvalues described by by (1), (2), (3) of Theorem 2.1 with localization center in $\Lambda_{L'} \setminus \Lambda_{(L-1)^\kappa}$ is bounded by $C|J_{j,\Lambda_{L^\kappa}}|\,|\Lambda_{L'}|^{(d-1/\kappa)/d}$. Using (3.7) and the definition of G (see (3.13)), for $j \in G$, we have

$$\begin{aligned}|J_{j,\Lambda_{L^\kappa}}|\,|\Lambda_{L'}|^{(d-1/\kappa)/d} &\lesssim |N(J_{j,\Lambda_{L^\kappa}})|^{1/(1+\rho'')}|\Lambda_{L'}|^{(d-1/\kappa)/d} \\ &\asymp |\Lambda_{L'}||N(J_{j,\Lambda_{L^\kappa}})||\Lambda_{L'}|^{-1/(\kappa d)+\alpha\rho''/(1+\rho'')} \lesssim |\Lambda_{L'}||N(J_{j,\Lambda_{L^\kappa}})||\Lambda_{L'}|^{-\chi}\end{aligned}$$

as, using (3.12), (2.6), $\alpha < 1$ and $1+\rho < \rho'$, one computes $\frac{1}{d\kappa} - \frac{\alpha\rho''}{1+\rho''} > 0$.

Thus, we obtain that there exists $\chi > 0$ such that, with probability at least $1-L^{-p}$,

(P3): except for at most $C|\Lambda_{L^\kappa}||N(J_{j,\Lambda_{L^\kappa}})||\Lambda_{L'}|^{-\chi}$ of them, the eigenvalues of $H_\omega(\Lambda_{L'})$ and those of $H_\omega(\Lambda_{L^\kappa})$ are associated to a center of localization in $\Lambda_{(L-1)^\kappa}$.

Now, with a probability at least $1-3L^{-p}$, using (P1), (P2) and (P3), we see that

$$\left|N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \Lambda_{(L-1)^\kappa}, \omega) - N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \omega)\right| \lesssim |\Lambda_{L^\kappa}|\,|N(J_{j,\Lambda_{L^\kappa}})|\,|\Lambda_{L'}|^{-\chi}$$

and

$$\left|N(J_{j,\Lambda_{L^\kappa}}, \Lambda_{L'}, \Lambda_{(L-1)^\kappa}, \omega) - |\Lambda_{L^\kappa}|\,|N(J_{j,\Lambda_{L^\kappa}})|\right| \lesssim |\Lambda_{L^\kappa}|\,|N(J_{j,\Lambda_{L^\kappa}})|\,|\Lambda_{L'}|^{-\chi}.$$

This completes the proof of point (3) of Lemma 3.10, thus, the proof of Lemma 3.10. □

3.7. **The proof of Theorem 1.5.** This proof follows the same analysis as the proof of Theorem 1.1. The only difference comes in the second step of the reduction when one splits the interval $E_0 + I_\Lambda$ into smaller intervals (see section 3.2.2). As $|I_\Lambda| \to 0$ and, thus, $|N(E_0+I_\Lambda)| \to 0$, one has to modify this part of the reduction and, as we will see now, a new condition comes up because of the possible difference of asymptotics for $|N(E_0+I_\Lambda)|$

and $|I_\Lambda|$.
Let us first follow the construction done in section 3.2.2 under an assumption more restrictive than the second assumption in (1.9), namely, that $|N(E_0+I_\Lambda)| \asymp |\Lambda|^{-\delta}$. Pick $\alpha>\delta$ such that the conditions on the exponents ρ', ρ'', μ, α, β and κ in section 3.2.1 be satisfied; this can be done by picking ρ' large. Split the interval $N(E_0+I_\Lambda)$ into subintervals of size $|\Lambda|^{-\alpha}$ (see section 3.2.2). Then, the estimate of the size of B defined by (3.13) for E_0+I_Λ becomes

$$\#B \leq |\Lambda|^{\alpha(1+\rho'')^{-1}-\delta(1+\tilde\rho)^{-1}} \tag{3.61}$$

where $\tilde\rho$ is defined in Theorem 1.5.
On the other hand, for the density of states measure of the sets of energies that we don't control (i.e. those corresponding to the indices in B) to be much smaller than $|N(E_0+I_\Lambda)|$, we need to require that

$$\#B\cdot|\Lambda|^{-\alpha} \ll |\Lambda|^{-\delta} \tag{3.62}$$

that is, using (3.61), it is sufficient that the exponents satisfy the inequality

$$\frac{\tilde\rho\delta}{1+\tilde\rho} < \frac{\alpha\rho''}{1+\rho''}. \tag{3.63}$$

Define $\mathcal{E}$ to be the set of (ρ',ρ'',μ) satisfying (3.6), (2.6) and (3.9) and

$$\tau := \sup_{(\rho',\rho'',\mu)\in\mathcal{E}} \frac{\rho''}{1+\rho-d\mu\rho'(1+\rho'')} > 0 \tag{3.64}$$

As α can be chosen arbitrary in the interval defined by (2.8) and (3.9), we will be able to find (α,ρ'') satisfying (3.63) if $\dfrac{\tilde\rho\delta}{1+\tilde\rho}<\tau$.
Once this condition, hence, condition (3.62), is fulfilled, the analysis is the same as in the macroscopic case. In particular, the analogues of Lemmas 3.2 and 3.3 hold.
Let us return to the general case when we only know that for some $\delta\in(0,1)$ satisfying (1.10), one has $|N(E_0+I_\Lambda)||\Lambda|^\delta \to +\infty$. Then, we define δ_0 as

$$\delta_0 = \inf\{\delta'>0;\ |N(E_0+I_\Lambda)||\Lambda|^{\delta'}\to+\infty\}.$$

Thus, by assumption (1.9), one has $\delta_0\in[0,\delta]$ where δ is defined in Theorem 1.5. Pick any $\delta_1>\delta_0$ such that $\dfrac{\tilde\rho\delta_1}{1+\tilde\rho}<\tau$. We can then analyze the process associated to the energies in E_0+I_Λ by splitting this interval into intervals of size (computed with respect to the density of states) of order $|\Lambda|^{-\delta_1}$, with at most $O(|\Lambda|^{\delta_1-\delta_0})$ such intervals. The exponent $\delta_1-\delta_0$ can be made arbitrarily small, we can glue the results in the same way as in the macroscopic case.

Let us complete this section with a remark on how condition (1.8) is used. It is needed to obtain the results corresponding to Lemmas 3.3 and 3.10. In Lemmas 3.3 and 3.10, the number of eigenvalues we take into account is asymptotic to $|N(E_0+I_{\Lambda_{L'}})||\Lambda_{L'}|$ and we want these number to be close to each other for all the cube of side-length L' in $[L^\kappa,(L+1)^\kappa]$. Therefore, we need that $|N(E_0+I_{\Lambda_{L'}})|\sim|N(E_0+I_{\Lambda_{L^\kappa}})|$ which is (1.8). This will now imply that the error estimate in the analogues of Lemmas 3.3 and 3.10, instead of being of size an inverse power of L, will simply be $o(1)$ (coming from condition (1.8)). But this does not modify the final result.

3.8. **The proof of Theorem 1.6.** Theorem 1.6 follows from Theorem 1.1, Lemma 3.8 and the fact that most eigenvalues of H_ω in J with localization center in Λ are very well approximated by an eigenvalue of $H_\omega(\Lambda)$ in J, and vice versa.
Write $J = [a, b]$. Using the techniques of the proof of Lemma 3.10, one proves the following result for the eigenvalues of H_ω in J having localization center in Λ

Lemma 3.11. *Fix $\chi \in (0, 1)$. There exists $\chi' > 0$ such that, ω-almost surely, for L sufficiently large, one has*

(1)
$$\left|\frac{N^f(J,\Lambda,\omega)}{N(J,\Lambda,\omega)} - 1\right| \le |\Lambda|^{-\chi'};$$

(2) *to each eigenvalue of $H_\omega(\Lambda_L)$ in $J_L := [a+L^{-3d/2}, b-L^{-3d/2}]$ with localization center in Λ_{L-L^χ}, say, E, one can associate an eigenvalue of H_ω in J with localization center in Λ_L, say, E', such that $|E - E'| \le L^{-2d}$;*

(3) *to each eigenvalue of H_ω in J_L with localization center in Λ_{L-L^χ}, say, E, one can associate an eigenvalue of $H_\omega(\Lambda_L)$ in J, say, E', that satisfies $|E - E'| \le L^{-2d}$.*

One then uses this to combine Theorem 1.1 and Lemma 3.8 to obtain Theorem 1.6.

4. The proof of Theorems 1.3 and 1.4

These proofs are simple and rely on general theorems on transformations of point processes (see e.g. [5, Chap. 5.5] and [34, Chap. 3.5]).

4.1. **The proof of Theorem 1.3.** As in the proof of Theorem 1.1, it suffices to consider the case when J is an interval in the essential support of ν, that is, N is strictly increasing on J. In particular, one has $\nu(t) > 0$ for almost every $t \in J$.
If t is a random variable distributed according to the law $\nu_J(t)dt$, then $\tilde{t} := N_J(t)$ is uniformly distributed on $[0, 1]$. Thus, the process $\Xi_J(\omega, \tilde{t}, \Lambda)$ under the uniform law in $\tilde{t}$ has the same law as the process $\Xi_J(\omega, N_J(t), \Lambda)$ under the law $\nu_J(t)dt$.
Rewrite the point measures $\Xi_J(\omega, N_J(t), \Lambda)$ and $\tilde{\Xi}_J(\omega, t, \Lambda)$ as

$$\Xi_J(\omega, N_J(t), \Lambda) = \sum_{E_n(\omega,\Lambda)\in J} \delta_{x_n(\omega,t)} \quad \text{and} \quad \tilde{\Xi}_J(\omega, t, \Lambda) = \sum_{E_n(\omega,\Lambda)\in J} \delta_{\tilde{x}_n(\omega,t)}$$

where

$$x_n(\omega, t) := |N(J)||\Lambda|[N_J(E_n(\omega, \Lambda)) - N_J(t)] = |\Lambda|[N(E_n(\omega, \Lambda)) - N(t)]$$

and

$$\tilde{x}_n(\omega, t) := \nu(t)|\Lambda|[E_n(\omega, \Lambda) - t].$$

Thus, one has

$$x_n(\omega, t) = \varpi_\Lambda(\tilde{x}_n(\omega, t); t) \quad \text{and} \quad \tilde{x}_n(\omega, t) = \chi_\Lambda(x_n(\omega, t); t) \tag{4.1}$$

where

$$\varpi_\Lambda(x; t) = |\Lambda| \left[N\left(t + \frac{x}{\nu(t)|\Lambda|}\right) - N(t)\right]$$

and

$$\chi_\Lambda(x; t) = \nu(t)|\Lambda| \left[N^{-1}\left(N(t) + \frac{x}{|\Lambda|}\right) - t\right]$$

where N^{-1} is the inverse of the strictly increasing Lipschitz continuous function N. Note that, if $N(J,\Lambda,\omega)$ denotes the number of eigenvalues of $H_\omega(\Lambda)$ in J, one has

$$t = \frac{1}{N(J,\Lambda,\omega)} \cdot N^{-1}\left(\sum_{E_n(\omega,\Lambda)\in J} N(E_n(\omega)) - \frac{x_n}{|\Lambda|}\right). \tag{4.2}$$

Following the notations of [34], let $\mathcal{M}_p(\mathbb{R})$ denote the space of point measures on the real line endowed with its standard metric structure. Actually, by Minami's estimate (M), we could restrict ourselves to working with simple point measures.
The point processes $\Xi_J(\omega, N_J(t), \Lambda)$ and $\tilde\Xi_J(\omega, t, \Lambda)$ under the law $\nu_J(t)dt$ are the random processes (i.e. the Borelian random variables) obtained as push-forwards of the probability measure $\nu_J(t)dt$ through the maps $t \in \mathbb{R} \mapsto \Xi_J(\omega, N_J(t), \Lambda) \in \mathcal{M}_p(\mathbb{R})$ and $t \in \mathbb{R} \mapsto \tilde\Xi_J(\omega, t, \Lambda) \in \mathcal{M}_p(\mathbb{R})$. We denote them respectively by $\Xi_J(\omega,\Lambda)$ and $\tilde\Xi_J(\omega,\Lambda)$.
One can extend the mapping $x \in \mathbb{R} \mapsto \chi_\Lambda(x,t) \in \mathbb{R}$ to a map, say, $\chi_{\omega,\Lambda}$ on point measures in $\mathcal{M}_p(\mathbb{R})$ on the real line by mapping the supports pointwise onto one another and computing t using (4.2) i.e.

$$\chi_{\omega,\Lambda}\left(\sum_n a_n\delta_{x_n}\right) = \sum_n a_n \delta_{\chi_{\omega,\Lambda}(x_n; t(\sum_n a_n\delta_{x_n}))}$$

where $t\left(\sum_n a_n\delta_{x_n}\right)$ is defined as

$$t\left(\sum_n a_n\delta_{x_n}\right) = \frac{1}{N(J,\Lambda,\omega)} \sum_{E_n(\omega,\Lambda)\in J} N^{-1}\left(N(E_n(\omega)) - \frac{x_n}{|\Lambda|}\right).$$

For fixed Λ and ω, the map $\chi_{\omega,\Lambda}: \mathcal{M}_p(\mathbb{R}) \to \mathcal{M}_p(\mathbb{R})$ is measurable as the map $t \mapsto \chi_\Lambda(x,t)$ is. Moreover, by the computations made above (see (4.1) and (4.2)), one has

$$\chi_{\omega,\Lambda}(\Xi_J(\omega,\Lambda)) = \tilde\Xi_J(\omega,\Lambda). \tag{4.3}$$

For any $x \in R$, t almost surely, one has $\chi_\Lambda(x;t) \to x$ as $|\Lambda| \to +\infty$. Hence, as $|\Lambda| \to +\infty$, $\chi_{\omega,\Lambda}$ tends to the identity except on at most a set of measure 0 in $\mathcal{M}_p(\mathbb{R})$. On the other hand, Theorem 1.3 tells us that, ω almost surely, $\Xi_J(\omega,\Lambda)$ converges in law to the Poisson process of intensity 1 on the real line. Thus, we can apply [5, Theorem 5.5] to obtain that, ω-almost surely, $\tilde\Xi_J(\omega,\Lambda)$, that is, $\tilde\Xi_J(\omega,t,\Lambda)$ under the measure $\nu_J(t)dt$, converges in law to the Poisson process of intensity 1 on the real line. This completes the proof of Theorem 1.3. □

4.2. **The proof of Theorem 1.4.** To complete this proof, recalling the notations of Theorem 1.4, we notice that, for $x > 0$,

$$\begin{aligned}\{E_n \in J;\ |\Lambda|(E_{n+1}(\omega,\Lambda) - E_n(\omega,\Lambda)) \geq x\} \\ = \{E_n \in J;\ \nu(t)|\Lambda|(E_{n+1}(\omega,\Lambda) - E_n(\omega,\Lambda)) \geq \nu(t)\,x\}.\end{aligned}$$

Thus, integration with respect to $\nu_J(t)dt$ over J, Theorem 1.3 and the same computations as those made to obtain Proposition 4.4 in [30] lead to, ω-almost surely

$$\begin{aligned}DLS(x; J,\omega,\Lambda) &= \int_J \frac{\#\{E_n \in J;\ \nu(t)|\Lambda|(E_{n+1}(\omega,\Lambda) - E_n(\omega,\Lambda)) \geq \nu(t)\,x\}}{N(J,\omega,\Lambda)}\nu_J(t)dt \\ &\underset{|\Lambda|\to+\infty}{\to} \int_J e^{-|N(J)|\,x\,\nu(t)}\nu_J(t)dt.\end{aligned}$$

This completes the proof of Theorem 1.4. □

5. Appendix

We now indicate how one should modify the proof of [15, Theorem 1.15] to obtain Theorem 2.1.

One just needs to modify the way one estimates the set $\mathcal{S}_{\ell_\Lambda,L}$ that is the set of disjoint boxes of the decomposition $\Lambda_\ell(\gamma_j)\subset\Lambda_L$ containing at least 2 centers of localization of $H_\omega(\Lambda_L)$. Here, $\Lambda=\Lambda_L$. It follows from [15, Lemma 3.1] (taking into account $\ell'_\Lambda \ll \ell_\Lambda$) that, using independence and Stirling's formula,

$$\begin{aligned}\mathbb{P}(\sharp(\mathcal{S}_{\ell_\Lambda,L}\geq k) &\lesssim \binom{|\Lambda_L|/n}{k}(|I_\Lambda|\ell_\Lambda^d)^{(1+\rho)k}\\ &\lesssim \left(\mathrm{e}\frac{|\Lambda_L|}{k\ell_\Lambda^d}\right)^k(|I_\Lambda|\ell_\Lambda^d)^{(1+\rho)k} = \left(\frac{\mathrm{e}|\Lambda_L|}{k}N(I_\Lambda)^{\frac{1+\rho}{1+\rho''}}\ell_\Lambda^{d\rho}\right)^k \lesssim 2^{-k},\end{aligned}$$

if we choose

$$k\geq K := \left[2\mathrm{e}N(I_\Lambda)|\Lambda_L|\left(N(I_\Lambda)^{\frac{\rho-\rho''}{1+\rho''}}\ell_\Lambda^{d\rho'}\right)\right]+1. \tag{5.1}$$

Note that,

$$K\asymp\frac{|\Lambda_L|}{\ell_\Lambda^d}\left(N(I_\Lambda)^{\frac{1}{1+\rho''}}\ell_\Lambda^{d\frac{1+\rho'}{1+\rho}}\right)^{1+\rho} = o\left(\frac{|\Lambda_L|}{\ell_\Lambda^d}\right) \tag{5.2}$$

if the right hand side inequality in (2.5) holds. As a consequence, as $\rho'>\rho$, we get that

$$\mathbb{P}(\#(\mathcal{S}_{\ell,L})\geq K)\lesssim 2^{-K}.$$

So that, with probability larger than $1-2^{-K}$, we can assume that the boxes $\Lambda_\ell(\gamma_j)$, except at most K of them, contain at most one center of localization.

We now control the number of centers of localization that may be contained in these K exceptional boxes. In a box of size ℓ, the deterministic a priori bound on the number of eigenvalues guarantees that this number is bounded by ℓ_Λ^d (up to a constant). Using this crude estimate the number of eigenvalues we miss with these K boxes is bounded by

$$K\ell_\Lambda^d\lesssim N(I_\Lambda)|\Lambda_L|\left(N(I_\Lambda)^{\frac{\rho-\rho''}{1+\rho''}}\ell_\Lambda^{d(1+\rho')}\right) = o(N(I_\Lambda)|\Lambda_L|),$$

provided the right hand side inequality in (2.5) holds.

The remaining part of the proof of Theorem 2.1 is identical to that of [15, Theorem 1.15]. □

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