RESONANCES FOR "LARGE" ERGODIC SYSTEMS IN ONE DIMENSION: A REVIEW

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Abstract. The present note reviews recent results on resonances for one-dimensional quantum ergodic systems constrained to a large box (see [14]). We restrict ourselves to one dimensional models in the discrete case. We consider two types of ergodic potentials on the half-axis, periodic potentials and random potentials. For both models, we describe the behavior of the resonances near the real axis for a large typical sample of the potential. In both cases, the linear density of their real parts is given by the density of states of the full ergodic system. While in the periodic case, the resonances distribute on a nice analytic curve (once their imaginary parts are suitably renormalized), in the random case, the resonances (again after suitable renormalization of both the real and imaginary parts) form a two dimensional Poisson cloud.

0. Introduction

On $\ell^2(\mathbb{N})$, consider V a bounded potential and the operator $H = -\Delta + V$ satisfying the Dirichlet boundary condition at 0.
The potentials V we will consider are of two types:

- V periodic;
- $V = V_\omega$ random e.g. a collection of i.i.d. random variables.

The spectral theory of such models has been studied extensively (see e.g. [11]) and it is well known that, when considered on $\ell^2(\mathbb{Z})$, the spectrum of H is purely absolutely continuous when V is periodic ([27]) while it is pure point when $V = V_\omega$ is the Anderson potential ([2, 22]). On $\ell^2(\mathbb{N})$, the picture is the same except for possible discrete eigenvalues outside the essential spectrum which coincides and is of the same nature as the essential spectrum of the operator on $\ell^2(\mathbb{Z})$.
Let $L > 0$. The object of our study is the following operator on $\ell^2(\mathbb{N})$

$$H_L = -\Delta + V\mathbf{1}_{[\![0,L]\!]} \tag{0.1}$$

when L becomes large. Here, $-\Delta$ is the free Laplace operator defined by $-(\Delta u)(n) = u(n+1) + u(n-1)$ for $n \geq 0$ where $u = (u(n))_{n\geq 0} \in \ell^2(\mathbb{N})$ and $u(-1) = 0$ (Dirichlet boundary condition at 0), and $[\![0, L]\!] = \{0, 1, \cdots, L\} \subset \mathbb{N}$.
Clearly, the essential spectrum of H_L is that of the discrete Laplace operator, that is, $[-2, 2]$, and it is absolutely continuous. Moreover, outside this absolutely continuous spectrum, H_L has only discrete eigenvalues associated to exponentially decaying eigenfunctions.
We are interested in the resonances of the operator H_L. These can be defined as the poles of the meromorphic continuation of the resolvent of H_L through the continuous spectrum of H_L (see e.g. [28]). One proves that

2010 *Mathematics Subject Classification.* 47B80, 47H40, 60H25, 82B44, 35B34.
Key words and phrases. Resonances; random operators; periodic operators.
The author is supported by the grant ANR-08-BLAN-0261-01.

Theorem 1. *The operator valued holomorphic function* $z \in \mathbb{C}^+ \mapsto (z - H_L)^{-1}$ *admits a meromorphic continuation from* $\mathbb{C}^+$ *to* $\mathbb{C}\backslash((-\infty, -2] \cup [2, +\infty))$ *(see Fig. 1) with values in the operators from* $\ell^2_{comp}(\mathbb{N})$ *to* $\ell^2_{loc}(\mathbb{N})$.
Moreover, the number of poles of this meromorphic continuation in the lower half-plane is at most equal to L.

As said, we define the resonances as the poles of this meromorphic continuation. The resonance widths, the imaginary part of the resonances, play an important role in the large time behavior of e^{-itH_L}, especially the smallest width that gives the leading order contribution (see [28, 29, 19]).

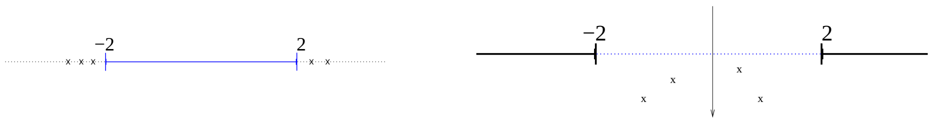

FIGURE 1. The spectrum of H_L and the analytic continuation of $(z - H_L)^{-1}$

As $L \to +\infty$, H_L converges to H in the strong resolvent sense. Thus, it is natural to expect that the differences in the spectral nature between the cases V periodic and V random should reflect into differences in the behavior of the resonances. As we shall see, this is the case.
Our goal is to describe the resonances or, rather, their statistical properties and relate them (the distribution of the resonances, the distribution of their widths) to the spectral characteristics of $H = -\Delta + V$. In the periodic case, we expect that the Bloch-Floquet data for the operator $-\Delta + V$ on $\ell(\mathbb{Z})$ will be of importance; in the random case, this role should be taken over by the distribution of the eigenvalues of $-\Delta + V_\omega$.

The scattering theory or the closely related study of resonances for the operator (0.1) or for similar one-dimensional models has already been discussed in various works both in the mathematical and physical literature [6, 5, 16, 25, 3, 17, 1, 15, 26, 18]. The proofs of the result we present below will be released elsewhere ([14]). Though we will restrict ourselves to the discrete model, the continuous model can be dealt with in a very similar way.

Let us now describe our results. We start with the periodic case and turn to the random case in the next section.

1. The periodic case

We assume that, for some $p > 0$, one has

$$V_{n+p} = V_n \quad \text{for all} \quad n \geq 0. \tag{1.1}$$

Let Σ' be the spectrum of H acting on $\ell^2(\mathbb{N})$ and Σ_0 be the spectrum of $-\Delta + V$ acting on $\ell^2(\mathbb{Z})$. One then has the following description for the spectra:

- on $\ell^2(\mathbb{Z})$, $\Sigma_0 = \bigcup_{j=1}^{p} [a_j^-, a_j^+]$ for some $a_j^- < a_j^+$ $(p \geq 1)$ and the spectrum is purely absolutely continuous (see e.g. [27]); the spectral resolution can be obtained via a Bloch-Floquet decomposition;
- on $\ell^2(\mathbb{N})$, one has (see e.g. [23])

– $\Sigma' = \Sigma_0 \cup \{v_j; 1 \leq j \leq n\}$ and Σ_0 is the a.c. spectrum of H;
– the $(v_j)_{0\leq j\leq n}$ are isolated simple eigenvalues associated to exponentially decaying eigenfunctions.

When L gets large, it is natural to expect that the interesting phenomena are going to happen near energies in Σ'. In $\Sigma' \cap [(-\infty, -2) \cup (2, +\infty)]$, one can check that H_L has only discrete eigenvalues. We will now describe what happens for the resonances near $[-2, 2]$. Therefore, we introduce an auxiliary operator.

1.1. **An auxiliary operator.** On $\ell^2(\mathbb{Z}_-)$ (where $\mathbb{Z}_- = \{n \leq 0\}$), consider the operator $H_k^- = -\Delta + \tau_k V$ with Dirichlet boundary condition at 0; $\tau_k V$ is the potential V shifted k times to the left i.e. $\tau_k V(\cdot) = V(\cdot + k)$. Let $\Sigma_k^- = \sigma(H_k^-)$.
As is the case for H, the essentially spectrum of H_k^- is purely absolutely continuous and one has $\sigma_{\text{ess}}(H_k^-) = \Sigma_0$ (see e.g. [24, Chapter 7]). H_k^- may also have discrete eigenvalues in $\mathbb{R} \setminus \Sigma_0$. Let dN_k^- be the spectral measure associated to H_k^- and the vector δ_0 i.e.

$$\text{for } \operatorname{Im} E \neq 0, \quad \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E} := \langle \delta_0, (H_k^- - E)^{-1} \Delta_0 \rangle.$$

Then, dN_k^- is a positive measure that is absolutely continuous on Σ_0. Moreover, its density is real analytic on $\overset{\circ}{\Sigma}_0$.
Let I be a compact interval in $(-2, 2) \cap \overset{\circ}{\Sigma}_0$. For $E \in \overset{\circ}{\Sigma}_0$, define

$$S_k^-(E) = \text{p.v.}\left(\int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E}\right) := \lim_{\varepsilon \downarrow 0}\left(\int_{-\infty}^{E_0-\varepsilon} \frac{dN_k^-(\lambda)}{\lambda - E} - \int_{E_0+\varepsilon}^{+\infty} \frac{dN_k^-(\lambda)}{\lambda - E}\right) \tag{1.2}$$

The existence and regularity of the Cauchy principal value S_k^- on $\overset{\circ}{\Sigma}_0$ is guaranteed by the regularity of dN_k^- in $\overset{\circ}{\Sigma}_0$ (see e.g. [10]).
In the lower half-plane $\{\operatorname{Im} E < 0\}$, define the function

$$\Xi_k(E) := \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E} + e^{-i \arccos(E/2)}. \tag{1.3}$$

Here, the function $z \mapsto \arccos z$ is the analytic continuation to the lower half-plane of the determination taking values in $[-\pi, 0]$ over the interval $[-1, 1]$.
The function Ξ_k vanishes at infinity and is analytic in $\{\operatorname{Im} E < 0\}$ and in a neighborhood of $(-2, 2) \cap \overset{\circ}{\Sigma}_0$. It vanishes identically if and only if V vanishes identically (see [14]). So, if $V \not\equiv 0$, Ξ_k has only isolated zeros of finite multiplicity in $\{\operatorname{Im} E < 0\}$ and on $(-2, 2) \cap \overset{\circ}{\Sigma}_0$.

1.2. **Resonance free regions.** We start with a description of the resonance free region.

Theorem 2. *Let I be a compact interval in $(-2, 2)$. Then,*

- *if $I \subset \mathbb{R} \setminus \Sigma'$, then, there exists $C > 0$ such that, for L sufficiently large, there are no resonances in $\{Re\, z \in I,\ Im\, z \geq -1/C\}$;*
- *if $I \subset \Sigma_0$, then, there exists $C > 0$ such that, for L sufficiently large, there are no resonances in $\{Re\, z \in I,\ Im\, z \geq -1/(CL)\}$;*

- *if* $\{v_j\} = \overset{\circ}{I} \cap \Sigma' = I \cap \Sigma'$ *and* $I \cap \Sigma_0 = \emptyset = I \cap \Sigma_k^-$ *, then, for* L *sufficiently large s.t.* $L = k \mod (p)$ *, there exists a unique resonance in* $\{Re\, z \in I,\ Im\, z \geq -1/C\}$*; moreover, this resonance, say* z_j*, satisfies, for some* ρ_j *independent of* L*,*

$$Im\, z_j \asymp -e^{-\rho_j L} \quad \text{and} \quad |z_j - v_j| \asymp e^{-\rho_j L}. \tag{1.4}$$

So, in the complex strip below the spectrum of the Laplace operator i.e the interval $(-2, 2)$, except at the discrete spectrum of H, there exists a resonance free region of width at least of order L^{-1}. Each discrete eigenvalue of H generates a resonance that is exponentially close to the real axis.

1.3. **Description of the resonances closest to Σ_0.** Let I be a compact interval in $(-2, 2) \cap \overset{\circ}{\Sigma}_0$ such that I does not contain any zero of Ξ_k (see (1.3)).
Let $(\lambda_j)_j = (\lambda_j^L)_j$ be the Dirichlet eigenvalues of $(-\Delta + V)_{|[\![0,L]\!]}$ in increasing order (see [27]). We then prove the

Theorem 3. *There exists* $C_0 > 0$ *such that, for* $C > C_0$*, there exists* $L_0 > 0$ *such that for* $L > L_0$*, for* $\lambda_j \in I$ *such that* $\lambda_{j+1} \in I$*, there exists a unique resonance in* $[\lambda_j, \lambda_{j+1}] + i[-CL^{-1}, 0]$*, say* z_j*. It satisfies*

$$z_j = \lambda_j + \frac{f_k(\lambda_j)}{L} \cot^{-1}\left(\left[e^{-i \arccos(\lambda_j/2)} + S_k^-(\lambda_j)\right] g_k(\lambda_j)\right) + o\left(\frac{1}{L}\right)$$

where $k \equiv L \mod p$ *(*p *is the period of* V*) and* $(f_k)_{0 \leq k \leq p-1}$ *and* $(g_k)_{0 \leq k \leq p-1}$ *are real analytic functions defined by the Floquet theory of* H *on* $\mathbb{Z}$*.*

The functions $(f_k)_k$ and $(g_k)_k$ can be computed explicitly in terms of the Floquet reduction (see [14]).
One can also analyze wat happens near the real zeros of Ξ (see [14]): one obtains that the resonances leave the real axis to go up to distances of size $\dfrac{\log L}{L}$.

From this quite explicit description of the resonances, one shows that, in $I + i[-C/L, 0]$, for L sufficiently large,

- the resonances when rescaled to have imaginary parts of order 1 accumulate on a real analytic curve;
- the local (linear) density of resonances is given by the density of states of H.

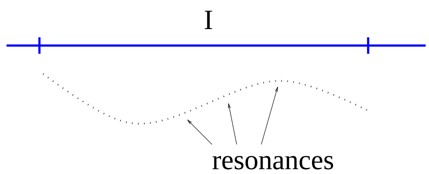

FIGURE 2. The rescaled resonances for a periodic potential

More precisely, one proves

Corollary 1. *Fix* I *and* k *as above. Then, there exist* $V \supset I$*, a neighborhood of* I *and* h_k *real analytic on* V *such that, for* C *sufficiently large and* L *sufficiently large s.t.* $L \equiv k \mod (p)$*,*

- *if* $z \in I + i[-CL^{-1}, 0]$ *is a resonance of* H_L*, then*

$$L \cdot Im\, z = h_k(Re\, z) + o(1); \tag{1.5}$$

- *for $J\subset I$, any interval one has*

$$\frac{\#\{z\in J+i[-CL^{-1},0], z \text{ resonance of } H_L\}}{L+1}=\int_J dN(E)+o(1) \tag{1.6}$$

where N is the integrated density of states (see e.g. [22]*) defined by*

$$N(E)=\lim_{L\to+\infty}\frac{\#\{\text{eigenvalues of } (-\Delta+V)_{|[\![0,L]\!]} \text{ in } (-\infty,E]\}}{L+1}. \tag{1.7}$$

Fig. 2 pictures the resonances after rescaling their width by L: these are nicely spaced points interpolating a smooth curve.

1.4. **Description of the low lying resonances.** One can also study what happens below the lines $\text{Im} z=-C/L$. As above, let I be a compact interval in $(-2,2)\cap\overset{\circ}{\Sigma}_0$ such that I does not contain any zero of Ξ_k (see (1.3)).
One proves

Theorem 4. *There exists $c>0$ s.t., for L sufficiently large s.t. $L=k \mod (p)$,*

- *the resonances of H in $I+i[-c,0]$ are those described in Theorem 3;*
- *the resonances of H in $I+i(-\infty,-c]$ satisfy $|\Xi_k(z)|\leq e^{-cL}$. In particular, there are finitely many of them and they are all exponentially close to some zero of Ξ_k.*

So, by Theorem 3, except for a finite number of resonances that converge to the zeros of Ξ_k, all the resonances of H_L converge to the spectrum of H.

2. The random case

Let now $V=V_\omega$ where $V_\omega(n)=\omega_n$ and $(\omega_n)_{n\geq 0}$ are bounded independent and identically distributed random variables. Assume that the common law of the random variables admits a bounded density, say, g.
Set $H_\omega=-\Delta+V_\omega$ on $\ell^2(\mathbb{N})$. Let $\sigma(H_\omega)$ be the spectrum of H_ω and Σ be the almost sure spectrum of $-\Delta+V_\omega$ acting on $\ell^2(\mathbb{Z})$ (see [11]); one knows that

$$\Sigma=[-2d,2d]+\text{supp}\, g$$

One has the following description for the spectra:

- on $\ell^2(\mathbb{Z})$. ω-almost surely, $\sigma(-\Delta+V_\omega)=\Sigma$; the spectrum is purely punctual; it consists of simple eigenvalues associated to exponentially decaying eigenfunctions (Anderson localization, see e.g. [22, 11]); one can prove that the whole spectrum is dynamically localized;
- on $\ell^2(\mathbb{N})$, one has (see e.g. [22, 2])
  - ω-almost surely, $\sigma(H_\omega)=\Sigma\cup K_\omega$;
  - Σ is the essential spectrum of H_ω; it consists of simple eigenvalues associated to exponentially decaying eigenfunctions;
  - the set K_ω is the discrete spectrum of H_ω which may be empty and depends on ω.

2.1. **The integrated density of states and the Lyapunov exponent.** It is defined by (1.7) where $-\Delta+V$ is replaced by $-\Delta+V_\omega$; the limit then exists ω-a.s. and is ω-a.s. independent of ω. N is the distribution function of a probability measure supported on Σ. As the common law of the random variables $(\omega_n)_{n\geq 0}$ admits a bounded density, the integrated density of states $N(E)$ is known to be Lipschitz continuous ([22, 11]). Let $n(E)=\dfrac{dN}{dE}(E)$ be its derivative; it exists for almost every E.
One also defines the Lyapunov exponent, say $\rho(E)$ as follows

$$\rho(E)=\lim_{L\to+\infty}\frac{1}{L+1}\log\left\|\prod_{n=L}^{0}\begin{pmatrix}E-V_\omega(n) & -1\\ 1 & 0\end{pmatrix}\right\|. \tag{2.1}$$

For any E, ω-almost surely, this limit is known to exist and to be independent of ω (see e.g. [22, 2]). Moreover, it is positive and continuous for all E and the Thouless formula states that it is the harmonic conjugate of $n(E)$ (see e.g. [4]).

2.2. **Resonance free regions.** Define $H_{\omega,L}$ by (0.1) for $V=V_\omega$. We again start with a description of a resonance free region near the spectrum of $-\Delta$. As in the periodic case, the size of this region will depend on whether an energy belongs to the essential spectrum of H_ω or not. We prove

Theorem 5. *Let I be a compact interval in $(-2,2)$. Then, one has*

- *there exists $C>0$ such that, ω-a.s., if $I\subset\mathbb{R}\setminus\sigma(H_\omega)$, then, for L sufficiently large, there are no resonances of $H_{\omega,L}$ in $\{Re\, z\in I,\ Im\, z\geq -1/C\}$;*
- *there exists $C>0$ such that, ω-a.s., if $\{v_j\}=\{v_j(\omega)\}=\overset{\circ}{I}\cap K_\omega=I\cap K_\omega$ and $I\cap\Sigma=\emptyset$, then, for L sufficiently large, there exists a unique resonance in $\{Re\, z\in I,\ Im\, z\geq -1/C\}$; moreover, this resonance, say z_j, satisfies (1.4) for some $\rho_j=\rho_j(\omega)$ independent of L.*
- *if $I\subset\overset{\circ}{\Sigma}$, then, there exists $C>0$ such that, ω-a.s., for L sufficiently large, there are no resonances of $H_{\omega,L}$ in $\{Re\, z\in I,\ Im\, z\geq -e^{-2\rho L(1+o(1))})\}$ where ρ is the maximum of the Lyapunov exponent $\rho(E)$ on I.*

When comparing this result with Theorem 2, it is striking that the width of the resonance free region below Σ is much smaller in the random case than in the periodic case. This a consequence of the localized nature of the spectrum i.e. of the exponential decay of the eigenfunction.

2.3. **Description of the resonances close to Σ.** We will now see that below the resonance free strip exhibited in Theorem 5 one does find resonances, actually, many of them. We prove

Theorem 6. *Let I be a compact interval in $(-2,2)\cap\overset{\circ}{\Sigma}$. Then, ω-a.s.,*

- *for any $\kappa\in(0,1)$, one has*
  $$\frac{1}{L}\#\left\{z \text{ resonance of } H_{\omega,L} \text{ s.t. } Re\, z\in I,\ Im\, z\geq -e^{-L^\kappa}\right\}\to\int_I dN(E);$$
- *fix $E\in I$ such that $n(E)>0$; then, for $\delta>0$, there exits $\varepsilon>0$ such that*
  $$\liminf_{L\to+\infty}\frac{1}{L}\#\left\{\text{resonances } z \text{ s.t. } \begin{array}{c} Re\, z\in[E-\varepsilon,E+\varepsilon],\\ Im\, z\geq -e^{-2(\rho(E)-\delta)L}\end{array}\right\}>0.$$

The first striking fact is that the resonances are much closer to the real axis than in the periodic case; the lifetime of these resonances is much larger. The resonant states are quite stable with lifetimes that are exponentially large in the width of the random perturbation.

The structure of the set of resonances is also very different from the one observed in the periodic case (see Fig. 2) as we will see now. Let I be a compact interval in $(-2,2)\cap\overset{\circ}{\Sigma}$ and $\kappa\in(0,1)$. Fix $E_0\in I$ such that $n(E_0)>0$.

Let $(z_j^L(\omega))_j$ be the resonances of $H_{\omega,L}$ in $K_L:=[E_0-\varepsilon,E_0+\varepsilon]+i\left[-e^{-L^\kappa},0\right]$. We first rescale the resonances: define

$$\begin{aligned} x_j &= x_j^L(\omega)=n(E_0)\,L\,(\mathrm{Re}\,z_j^L(\omega)-E_0)\\ y_j &= y_j^L(\omega)=-\frac{1}{2\rho(E_0)\,L}\log|\mathrm{Im}\,z_j^L(\omega)|. \end{aligned} \tag{2.2}$$

Let us note that the scaling of the real and of the imaginary of the resonances are very different. According to the conclusions of Theorem 6, this scaling essentially sets the mean spacing between the real parts of the resonances to 1 and the imaginary parts to be of order 1.

Consider now the two-dimensional point process $\xi_L(E_0,\omega)$ defined

$$\xi_L(E_0,\omega)=\sum_{z_j^L\in K_L}\delta_{(x_j,y_j)}. \tag{2.3}$$

We prove

Theorem 7. *The point process ξ_L converges weakly to a Poisson process in $\mathbb{R}\times[0,1]$ with intensity 1. That is, for any $p\geq0$, if $(I_n)_{1\leq n\leq p}$ resp. $(C_n)_{1\leq n\leq p}$, are disjoint intervals of the real line $\mathbb{R}$ resp. of $[0,1]$, then*

$$\lim_{L\to+\infty}\mathbb{P}\left(\left\{\omega;\ \begin{matrix} \#\left\{j;\ \begin{matrix}x_j(\omega,\Lambda)\in I_1\\ y_j(\omega,\Lambda)\in C_1\end{matrix}\right\}=k_1\\ \vdots\qquad\qquad\vdots\\ \#\left\{j;\ \begin{matrix}x_j(\omega,\Lambda)\in I_p\\ y_j(\omega,\Lambda)\in C_p\end{matrix}\right\}=k_p\end{matrix}\right\}\right)=\prod_{n=1}^{p}e^{-\mu_n}\frac{(\mu_n)^{k_n}}{k_n!},$$

where $\mu_n:=|I_n||C_n|$ for $1\leq n\leq p$.

Hence, after rescaling the picture of the resonances (see Fig 3) is that of points chosen randomly independently of each other in $\mathbb{R}\times[0,1]$. This is the analogue of the celebrated result on the Poisson structure of the eigenvalues for a random system (see e.g. [21, 20, 7])

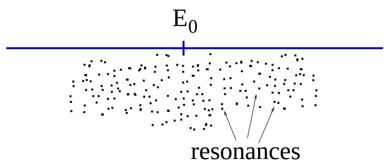

FIGURE 3. The rescaled resonances for a random potential

In [12], we proved decorrelation estimates that can be used in the present setting to prove

Theorem 8. *Fix $E_0\neq E_0'$ such that $n(E_0)>0$ and $n(E_0')>0$. Then, the limits of the processes $\xi_L(E_0,\omega)$ and $\xi_L(E_0',\omega)$ are stochastically independent.*

Due to the rescaling, the above results give a
picture of the resonances in a zone of the type

$$E_0 + L^{-1}\left[-\varepsilon^{-1}, \varepsilon^{-1}\right] - i\left[e^{-2(1-\varepsilon)\rho(E_0)L}, e^{-2\varepsilon\rho(E_0)L}\right]$$

For $\varepsilon > 0$ small fixed, when L gets large, this rectangle is of a very small width and located very close to the real axis.
One can get a number of other statistics using the techniques developed for the study of the spectral statistics of the eigenvalues of a random system in the localized phase (see [14, 9, 7, 8, 13]).

2.4. **The description of the low lying resonances.** It is natural to question what happens deeper in the complex plane. To answer this question, fix two increasing sequences of scales $\ell = (\ell_L)_L$ and $\ell' = (\ell'_L)_L$ such that

$$\ell'_L \leq \ell_L, \quad \frac{\ell'_L}{\log L} \underset{L\to+\infty}{\to} +\infty \quad \text{and} \quad \frac{\ell_L}{L} \underset{L\to+\infty}{\to} 0. \tag{2.4}$$

Fix $\kappa \in (0,1)$. Fix $x_0 \in [0,1]$ and $E_0 \in I$ so that $n(E_0) > 0$. Let $(z_j^L(\omega))_j$ be the resonances of $H_{\omega,L}$ in $\tilde K_L := [E_0-\varepsilon, E_0+\varepsilon] + i\left[-e^{-\ell'_L}, 0\right)$. Note that $\tilde K_L$ is much wider than K_L defined above. We first rescale the resonances using the scale ℓ: define

$$x_j = x_j^{\ell_L}(\omega) = (\operatorname{Re} z_j^L(\omega) - E_0)\ell_L \quad \text{and} \quad y_j = y_j^{\ell_L}(\omega) = \frac{1}{2\ell_L}\log|\operatorname{Im} z_j^L(\omega)|. \tag{2.5}$$

Consider now the two-dimensional point process

$$\xi_{L,\ell,\ell'}(x_0, E_0, \omega) = \sum_{z_j^L \in K_L} \delta_{(x_j, y_j)}. \tag{2.6}$$

We prove

Theorem 9. *For $x_0 \in [0,1)$ and $E_0 \in I$ so that $\nu(E_0) > 0$, the point process $\xi_{L,\ell,\ell'}(x_0, E_0, \omega)$ converges weakly to a Poisson process in $\mathbb{R}\times\mathbb{R}^+$ with density $\frac{n(E_0)}{\rho(E_0)}dxdy$.*

!!Note that, for any $\alpha > 1$, due to the scaling (2.5), all the resonances that live above the line $\operatorname{Im} z = -e^{-\ell_\Lambda^\alpha}$ are pushed off to infinity and those below the line $\operatorname{Im} z = -e^{-\ell_\Lambda^{1/\alpha}}$ pushed off to the real line. So in Theorem 9, we described the resonances near the line $\operatorname{Im} z = -e^{-\ell_\Lambda}$. By (2.4), these resonances lie deeper into the lower half-plane than those studied in Theorem 7.
For the processes $(\xi_{L,\ell,\ell'}(x_0, E_0, \omega))_{x_0,E_0}$, one gets an asymptotic independence result analogous to Theorem 8, namely,

Theorem 10. *Fix E_0 and E'_0 such that $n(E_0) > 0$ and $n(E'_0) > 0$ and x_0 and x'_0 such that $(x_0, E_0) \neq (x'_0, E'_0)$.*
Then, the limits of the processes $\xi_{L,\ell,\ell'}(x_0, E_0, \omega)$ and $\xi_{L,\ell,\ell'}(x'_0, E'_0, \omega)$ are stochastically independent.

One can also study the resonances that are even further away from the real axis in a way similar to Theorem 4 in the periodic case. Define

$$\tilde V_{\omega,L}(n) = \begin{cases} \omega_{L-n} \text{ for } 0 \leq n \leq L, \\ 0 \text{ for } L+1 \leq n. \end{cases}$$

Consider the operator $\tilde{H}_{\omega,L} = -\Delta + \tilde{V}_{\omega,L}$ on $\ell^2(\mathbb{N})$ with Dirichlet boundary condition at 0. As the $(V_\omega(n))_n \geq 0$ are i.i.d. random variables. $\tilde{H}_{\omega,L}$ clearly has the same distribution as $H_{\omega,L}$. Define the measure dN_ω by its Borel transform

$$\text{for } \operatorname{Im} E \neq 0, \quad \int_{\mathbb{R}} \frac{dN_\omega(\lambda)}{\lambda - E} = \langle \delta_0, (H_\omega - E)^{-1} \delta_0 \rangle.$$

We prove

Theorem 11. *Fix $\kappa \in (0,1)$ and $I \subset \overset{\circ}{\Sigma}$. With probability 1, for L sufficiently large, in $I + i\left(-\infty, e^{-L^\kappa}\right]$, a resonance of $\tilde{H}_{\omega,L}$, say E, satisfies*

$$\int_{\mathbb{R}} \frac{dN_\omega(\lambda)}{\lambda - E} + e^{-i \arccos(E/2)} = O\left(e^{-L^\kappa}\right).$$

This should be compared with the results of [16, 17].

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