Berkovich analytic geometry and dynamical systems

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Abstract

We combine the tools of Berkovich's analytic geometry, of geometric Galois actions and of class field theory to define topological dynamical systems associated to varieties over finite fields and to schemes over $\mathbb{Z}$. The associated dynamical zeta functions, that count attracting periodic orbits for the flow, correspond to the associated (non-completed) arithmetic zeta functions in the variable $s \in \mathbb{C}$, so that our construction hints to a possible relation between Berkovich's analytic spaces and Deninger's conjectural cohomological formalism for the study of arithmetic zeta functions. [1]

1 Introduction

In [Den98] and [Den94], Deninger developped a cohomological formalism to describe the local archimedean and non-archimedean zeta functions of arithmetic schemes in a unified framework, using regularized determinants of some operators on (typically infinite dimensional) complex cohomology theories. He also proposed a conjectural cohomological formalism for the study of global arithmetic zeta functions, that would give a global analog of the now well developped Weil cohomology theories. He also shows some analogies between the global cohomology one is looking for in his program and the leafwise cohomology of a foliated space with a transversal flow.

In [Ber90] and [LP22], Berkovich, Lemanissier and Poineau developped a theory of global analytic spaces that allows the study of archimedean and non-archimedean components of arithmetic schemes in a unified framework.

In the local case of a scheme X over a finite field $\mathbb{F}_p$, a very naïve description of a dynamical system with properties similar to the one Deninger is seeking for is given by the suspension of the action of the Frobenius map F on the set $|X_{\bar{\mathbb{F}}_p}|$ of closed points of X on a chosen algebraic closure of $\mathbb{F}_p$. One may embed this set theoretic suspension into the suspension

$$X^{an}_{\bar{\mathbb{F}}_p} \times_{F^{\mathbb{Z}}} \mathbb{R}^*_+$$

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of the corresponding Berkovich analytic space. Our first result is to show that the twisted suspension flow

$$\sigma_t(x, s) = (x^t, ts)$$

has exactly the right attracting closed orbits, corresponding to the closed points of X. Of course, a deeper geometric study of these topological dynamical systems would involve the use of the analytic étale topology on $X^{an}_{\bar{\mathbb{F}}_p}$, leading to a space very similar to (the suspension of) the local Weil-étale topos (see Morin's article [Mor10] for a comparison between the Weil-étale and the Deninger approach to zeta functions).

In the global situation, we adapt our local construction by replacing $\bar{\mathbb{F}}_p$ by $\bar{\mathbb{Z}}$ and the Frobenius map by (a version of) the global Weil group $\tilde{W}_{\mathbb{Q}}$ together with a map $|\cdot|_f : \tilde{W}_{\mathbb{Q}} \to \mathbb{Q}^*_+$, defining our global space to be

$$X^{an}_{\bar{\mathbb{Z}}} \times_{\tilde{W}_{\mathbb{Q}}} \mathbb{R}^*_+.$$

We also show that, in this global situation, the twisted suspension flow has exactly the right attracting closed orbits, corresponding to closed points of X.

Those global topological dynamical systems only take care of the non-archimedean components of arithmetic zeta functions, and do not encode any Hodge theoretical information at the archimedean place, so that there is no hope to use them directly for the cohomological study of global zeta functions. However, we find it quite stricking that a simple combination of geometric Galois actions, class field theory and Berkovich's global analytic geometry already gives nice topological dynamical systems that are directly related to global arithmetic zeta functions.

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2 Attracting fixed points for the power flow

We refer to Berkovich's book [Ber90] and to Lemanissier and Poineau's book [LP22] for the theory of global analytic spaces.

If X is a scheme over $\mathbb{Z}$, we denote X^{an} the full associated global analytic space, that is an analytic space over $\mathcal{M}(\mathbb{Z})$, the set of multiplicative seminorms on $\mathbb{Z}$ fulfilling a generalized triangular inequality

$$|x + y| \leq C \max(|x|, |y|)$$

for some $C \geq 1$.

We follow Poineau in [Poi10] to define the power flow on points of X^{an}.

Definition 1. The power flow on points of X^{an} sends a point $x \in X^{an}$, corresponding to a morphism

$$\mathcal{M}(\mathcal{K}(x), |\cdot(x)|) \to X^{an},$$

and a real number $t \in \mathbb{R}_+^*$, to the point x^t corresponding to the morphism

$$\mathcal{M}(\mathcal{K}(x), |\cdot(x)|^t) \to X^{an}.$$

Remark that the only fixed points in X^{an} by the flow of powers are given by trivial norms $|\cdot|_{\mathfrak{p},0}$ on residue fields of arbitary point $\mathfrak{p}$ of the scheme X. We now want to have a criterion, in term of the flow of powers, for such a trivial norm to correspond to a closed point $\mathfrak{p} \in |X|$.

We recall for the reader's convenience the following definition.

Definition 2. Let M be a topological space and $\sigma : \mathbb{R}\times M \to M$ be a topological flow. A closed orbit γ (in particular, a fixed point) for the flow σ on M is called *attracting* if there exists a neighborhood U of γ such that any point x in U tends to γ, i.e., for every $y \in U$, and every neighborhood V of γ, there exists T_V such that $\sigma_t(x) \in V$ for all $t > T_V$.

Proposition 1. *Let X be a scheme of finite type over $\mathbb{Z}$ or over an algebraic closure of a finite field. The attracting fixed points of the power flow on X^{an} are given by the closed points of X.*

Proof. Let $\mathfrak{p} \in X$ be a point and $\mathrm{Spec}(A)$ be a noetherian affine neighborhood of $\mathfrak{p}$. Let $(f_1, \ldots, f_m) = \mathfrak{p}$ be generators of the ideal $\mathfrak{p}$. Then if $0 < \epsilon < 1$, the open subset

$$U_\epsilon := \{x \in \mathrm{Spec}(A)^{an},\ \forall i,\ |f_i(x)| < \epsilon\}$$

is an open neighborhood of the trivial norm $|\cdot|_{0,\mathfrak{p}}$ that tends by the flow to

$$\mathrm{Spec}(A/\mathfrak{p})^{an} = \{x \in \mathrm{Spec}(A)^{an},\ \forall i,\ |f_i(x)| = 0\}.$$

If $\mathfrak{p}$ is maximal, $A/\mathfrak{p}$ is a finite field or the algebraic closure of a finite field, and the left hand side is reduced to the trivial norm $|\cdot|_{0,\mathfrak{p}}$. This shows that $\mathfrak{p}$ is an attracting point for the power flow on X^{an}. If $\mathfrak{p}$ is not maximal, let U be an open neighborhood of $|\cdot|_{0,\mathfrak{p}}$. Then U induces an open neighborhood V of $|\cdot|_0$ in $\mathrm{Spec}(A/\mathfrak{p})^{an}$. We are thus reduced to suppose that we are working with the generic point of $X = \mathrm{Spec}(A/\mathfrak{p})$, i.e., with the generic point of the spectrum of an integral ring that is not a field. The corresponding trivial norm is the only norm such that $|f| = 1$ for every non-zero f. Since $A/\mathfrak{p}$ is a non-trivial integral ring, it has a maximal ideal, so that X^{an} has a point different from $|\cdot|_0$. Let x be an arbitrary point in X^{an} different from $|\cdot|_0$. Then there exists a non-zero f such that $|f(x)| \neq 1$. If $|f(x)| < 1$, then $|f(x)|^t$ tends to zero so that x^t cannot tend to $|\cdot|_0$. If $|f(x)| > 1$, then $|f(x)|^t$ tends to infinity, so that x^t cannot tend to $|\cdot|_0$ either. This shows that $|\cdot|_{0,\mathfrak{p}}$ is not an attracting point for the flow of powers. □

Corollary 1. *Let $\bar{\mathbb{Z}} \subset \mathbb{C}$ be the integral closure of $\mathbb{Z}$ and X be a scheme of finite type over $\mathbb{Z}$. The attracting fixed points of the power flow on $X_{\bar{\mathbb{Z}}}^{an}$ are given by the closed points of $X_{\bar{\mathbb{Z}}}$.*

Proof. Remark that $\bar{\mathbb{Z}} = \operatorname{colim}_{K/\mathbb{Q}} \mathcal{O}_K$ where the colimit is taken over all finite Galois extensions of $\mathbb{Q}$ in $\mathbb{C}$. Let $p_K : X^{an}_{\bar{\mathbb{Z}}} \to X^{an}_{\mathcal{O}_K}$ be the natural projection. It is compatible to the flow. Moreover, it is given by the quotient of $X^{an}_{\bar{\mathbb{Z}}}$ by the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ so that it is also open. Indeed, If U is an open in $X^{an}_{\bar{\mathbb{Z}}}$, then $p_K^{-1}(p_K(U)) = \mathrm{Gal}(\bar{\mathbb{Q}}/K) \cdot U$ is open. The Galois action commutes to the flow. Suppose that $x \in X^{an}_{\bar{\mathbb{Z}}}$ is an attracting fixed point for the flow. There exists a neighborhood U of x such that for all $y \in U$, we have that y^t tends to x as t tends to infinity. Then for each σ in $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$, $\sigma(y)^t = \sigma(y^t)$ tends to $\sigma(x)$ as t tends to infinity. This shows that $p_K(U)$ is a neighborhood of $p_K(x)$ in $X^{an}_{\mathcal{O}_K}$ such that all $z \in p_K(U)$ converge to $p_K(x)$. So attracting fixed points for the flow on $X^{an}_{\bar{\mathbb{Z}}}$ correspond exactly to compatible systems of attracting fixed points for the flow on the $X^{an}_{\mathcal{O}_K}$, that correspond, by Proposition 1 to compatible systems of closed points of the $X_{\mathcal{O}_K}$, that correspond, classically, to closed points of $X_{\bar{\mathbb{Z}}}$. This shows the corollary. □

3 Varieties over finite fields and dynamical systems

Let X be a scheme of finite type over $\mathbb{Z}$. The arithmetic zeta function of X is given by the Euler product

$$\zeta_X(s) := \prod_{x \in |X|} \frac{1}{1 - N(x)^{-s}}$$

where the product ranges over closed points $x \in |X|$ and $N(x) = \mathrm{card}(k(x))$. If the scheme X is defined over a finite field $\mathbb{F}_p$, then one may write

$$\zeta_X(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} (p^{-s})^m \right),$$

where $N_m := X(\mathbb{F}_{p^m})$ for some choice of $\mathbb{F}_{p^m}/\mathbb{F}_p$. One usually translates this counting of points with values in finite fields into the counting of fixed points by powers F^m of the Frobenius map F on the closed points $|X_{\bar{\mathbb{F}}_p}|$ for some choice of an algebraic closure of $\mathbb{F}_p$.

Deninger explained in [Den94] that it is possible to compute directly $\zeta_X(s)$ in this finite field case by the use of Grothendieck's trace formula and of regularized determinants of an operator on a given (typically infinite dimensional) $\mathbb{C}$-linear cohomology associated to étale cohomology with its action of Frobenius.

The first aim of this note is to give a natural topological dynamical expression for the function ζ_X in the finite field case. The basic idea is to embed the set theoretical suspension of the Frobenius dynamical system $F \circlearrowright |X_{\bar{\mathbb{F}}_p}|$ into the suspension of the corresponding analytic space by the natural map

$$\tilde{X}_{\mathbb{F}_p} := X^{an}_{\bar{\mathbb{F}}_p} \times_{F^{\mathbb{Z}}} \mathbb{R}^*_+ \hookleftarrow |X_{\bar{\mathbb{F}}_p}| \times_{F^{\mathbb{Z}}} \mathbb{R}^*_+.$$

We then replace the standard suspension flow $(x, s) \mapsto (x, ts)$ by the "twisted suspension flow"

$$\sigma_t(x, s) = (x^t, ts)$$

where $x \mapsto x^t$ is the usual $\mathbb{R}_+^*$-power flow on multiplicative seminorms and $t \mapsto ts$ is the standard action of $\mathbb{R}_+^*$ on itself.

The following results give an adjustment to our setting of a classical result about suspension flows. To distinguish the closed orbits in $\tilde{X}_{\mathbb{F}_p}$ that correspond to closed points, we use the notion of attracting closed orbits from Definition 2.

Proposition 2. *There is a natural bijection $x \mapsto \gamma_x$ between points of the scheme X and primitive closed orbits γ_x of the flow σ on $\tilde{X}_{\mathbb{F}_p}$. By this bijection, closed points correspond to primitive attracting closed orbits of the flow, and the minimal length of such an orbit γ_x is* $\log(\mathrm{card}(k(x)))$.

Proof. Remark first that, by Proposition 1, we know that the flow of powers $x \mapsto x^t$ on $X^{an}_{\bar{\mathbb{F}}_p}$ has as only fixed points the trivial norms $|\cdot|_{0,\mathfrak{p}}$ corresponding to all points $\mathfrak{p}$ of the scheme $X_{\bar{\mathbb{F}}_p}$ and as attracting fixed points those corresponding to closed points. The points of the scheme X corresponds to finite Frobenius orbits on the set of points of the scheme $X_{\bar{\mathbb{F}}_p}$. This implies that closed orbits for the twisted suspension flow on $\tilde{X}_{\mathbb{F}_p}$ correspond to points of the scheme X, and that the attracting ones correspond to closed points of X. The statement on the length is the classical result on suspension flows, adapted to our setting. □

Corollary 2. *Let X be of finite type over $\mathbb{F}_p$. The zeta function may be described as a dynamical zeta function of the given flow σ on $\tilde{X}$*

$$\zeta_X(s) \equiv \zeta_{(\tilde{X},\sigma)}(s) := \prod_{\gamma \in acOrb} \frac{1}{1 - e^{-\ell(\gamma)s}}$$

where pacOrb denotes the primitive attracting closed *orbits for the flow and $\ell(\gamma)$ is the minimal length of the given closed orbit.*

4 Schemes over $\mathbb{Z}$ and non-archimedean dynamical systems

Having defined a nice dynamical system associated to a variety over a finite field, we proceed by trying to adapt this construction to the global situation of a flat scheme X over $\mathbb{Z}$. We get our inspiration from the various works of Deninger on this subject (see [Den98] for an introduction). Remark however that our dynamical system doesn't take care of the archimedean local factors and of any Hodge theoretical information, so that there is no hope to use it for the cohomological study of the corresponding global zeta functions.

As explained by Deninger in his works, the dynamical system he is looking for in this global situation may look like

$$M \times_{\mathbb{Q}_+^*} \mathbb{R}_+^*$$

for M some space equipped with an action of $\mathbb{Q}_+^*$.

We will replace the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$ used in the previous construction by the integral closure $\bar{\mathbb{Z}} \subset \mathbb{C}$ of $\mathbb{Z}$ in $\mathbb{C}$. Then $F^{\mathbb{Z}}$ will be replaced by a version of the Weil group that maps to $\mathbb{Q}_+^*$, and that we will now define.

4.1 The action of Weil groups on closed points

Consider the sub-extensions $\mathbb{Z}^{ab} \subset \mathbb{C}$ and $\bar{\mathbb{Z}} \subset \mathbb{C}$. Recall that for each prime number p and each prime $\mathfrak{p}/p$ in $\bar{\mathbb{Z}}$, there is a corresponding decomposition subgroup

$$D(\mathfrak{p}) = \{\sigma,\ \sigma(\mathfrak{p}) = \mathfrak{p}\} \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

and that the Galois group acts transitively on all $\mathfrak{p}/p$, so that the set $\{\mathfrak{p}/p\}$ may be identified with the quotient $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})/D(\mathfrak{p}_0)$ once chosen one $\mathfrak{p}_0/p$.

Proposition 3. *Let X be a finite type scheme over $\mathbb{Z}$, p be a prime number and $\{\mathfrak{p}/p\}$ denote the primes of $\bar{\mathbb{Z}}$ over p. Then if $\mathfrak{p}_0/p$ is a prime, there is a $D(\mathfrak{p}_0)$-equivariant commutative diagram*

$$\begin{array}{ccc} |X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}| & \longrightarrow & |X_{\bar{\mathbb{Z}}/p}| = \coprod_{\mathfrak{p}/p} |X_{\bar{\mathbb{Z}}/\mathfrak{p}}| \\ & \searrow \quad \swarrow & \\ & |X_{\mathbb{F}_p}| & \end{array}$$

and the set of closed points $|X_{\mathbb{F}_p}|$ is both the quotient of $|X_{\bar{\mathbb{Z}}/p}|$ by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and of $X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}$ by $D(\mathfrak{p}_0)$.

Proof. This follows from the definition of the decomposition group and from $\{\mathfrak{p}/p\} \cong \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})/D(\mathfrak{p}_0)$. □

We need to replace the Galois and decomposition group in the above considerations by the corresponding Weil groups, because they are easier to relate to the group $\mathbb{Q}_+^*$ (at finite places).

We refer to [Tat79] for an introduction to the theory of Weil groups. The Weil group W_E of a local or global field E together with a fixed algebraic closure $\bar{E}$ is (in particular) a topological group together with:

1. a morphism $\varphi : W_E \to \mathrm{Gal}(\bar{E}/E)$ with dense image and open kernel,
2. an isomorphism, called the reciprocity map of class field theory

$$r_E : C_E \xrightarrow{\sim} W_E^{ab},$$

where $C_E = E^\times$ if E is local and $C_E = \mathbb{A}_E^*/E^*$ if E is global.

Just recall that the Weil group is formally obtained as the projective limit along Galois subextensions $F \subset \bar{E}$ of the finite level Weil groups $W_{F/E}$, that are given by extensions

$$1 \to C_F^* \to W_{F/E} \to \mathrm{Gal}(F/E) \to 1$$

corresponding to a compatible system of fundamental classes of class field theory in the groups $H^*(\mathrm{Gal}(F/E), C_F)$.

The choice of a prime $\mathfrak{p}$ of $\bar{\mathbb{Z}}$ over a prime p defines an algebraic closure of the local field $\mathbb{Q}_p$ denoted $\bar{\mathbb{Q}}_{\mathfrak{p}}$, and it induces a natural continuous homomorphism

$$\iota_{\mathfrak{p}} : W_{\mathfrak{p},\mathbb{Q}_p} \to W_{\mathbb{Q}}$$

between the corresponding Weil groups whose image in the Galois group is dense in the decomposition group $D(\mathfrak{p})$.

The inverse of the reciprocity map allows us to define, a map

$$|\cdot| \circ r_{\mathbb{Q}}^{-1} : W_{\mathbb{Q}}^{ab} \to \mathbb{R}_+^*.$$

We would like to replace the group $\mathbb{R}_+^*$ by $\mathbb{Q}_+^*$, so that we need to use the finite idèles norm. However, it is only defined on $\mathbb{A}^*$. This motivates the following definition.

Definition 3. The adelic covering of the global Weil group is defined as the fiber product

$$\widetilde{W}_{\mathbb{Q}} := \mathbb{A}^* \times_{W_{\mathbb{Q}}^{ab}} W_{\mathbb{Q}}$$

along the global reciprocity map $\mathbb{A}^* \to \mathbb{A}^*/\mathbb{Q}^* \xrightarrow{r_{\mathbb{Q}}} W_{\mathbb{Q}}$. If $\mathfrak{p}/p$ is a prime of $\bar{\mathbb{Z}}$, we denote

$$\tilde{\iota}_{\mathfrak{p}} : \tilde{W}_{\mathfrak{p},\mathbb{Q}_p} \to \widetilde{W}_{\mathbb{Q}}$$

the inverse image of the corresponding local Weil group in $\widetilde{W}_{\mathbb{Q}}$.

The adelic covering of the global Weil group has a natural morphism

$$\tilde{\varphi} : \widetilde{W}_{\mathbb{Q}} \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

with dense image and a natural morphism (induced by the finite idèles norm)

$$|\cdot|_f : \widetilde{W}_{\mathbb{Q}} \to \mathbb{A}^* \to \mathbb{Q}_+^*.$$

We now adapt Proposition 3 to the setting of Weil group actions.

Proposition 4. *Let X be a finite type scheme over $\mathbb{Z}$, p be a prime number and $\{\mathfrak{p}/p\}$ denote the primes of $\bar{\mathbb{Z}}$ over p. Then if $\mathfrak{p}_0/p$ is a prime, there is a $\tilde{W}_{\mathfrak{p}_0,\mathbb{Q}_p}$-equivariant commutative diagram*

$$\begin{array}{ccc} |X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}| & \longrightarrow & |X_{\bar{\mathbb{Z}}/p}| = \coprod_{\mathfrak{p}/p} |X_{\bar{\mathbb{Z}}/\mathfrak{p}}| \\ & \searrow \quad \swarrow & \\ & |X_{\mathbb{F}_p}| & \end{array}$$

and the set of closed points $|X_{\mathbb{F}_p}|$ is both the quotient of $|X_{\bar{\mathbb{Z}}/p}|$ by $\widetilde{W}_{\mathbb{Q}}$ and of $X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}$ by $\widetilde{W}_{\mathfrak{p}_0,\mathbb{Q}_p}$.

Proof. This follows from the density of the images of the Weil groups in the corresponding Galois groups and from Proposition 3. □

4.2 The global dynamical systems

We now define the global dynamical system $\tilde{X}$ associated to X to be the quotient

$$\tilde{X} := X_{\bar{\mathbb{Z}}}^{an} \underset{\tilde{\varphi}, \tilde{W}_{\mathbb{Q}}, |\cdot|_f}{\times} \mathbb{R}_+^*.$$

The flow is defined by the same formula as before, i.e., by

$$\sigma_t(x, s) = (x^t, ts).$$

Proposition 5. *The attracting closed orbits of the dynamical system $\tilde{X}$ are in bijective correspondence with closed points of X, and the length of the attracting closed orbit γ_x associated to a point $x \in |X|$ is* $\log(\text{card}(k(x)))$*.*

Proof. By Proposition 1, the only attracting points for the flow of powers on $X_{\bar{\mathbb{Z}}}^{an}$ are given by its closed points. Because of that, and since there is a Galois-equivariant partition

$$|X_{\bar{\mathbb{Z}}}| = \coprod_p |X_{\bar{\mathbb{Z}}/p}|,$$

we may reduce our study of attracting orbits to the component

$$X_{\bar{\mathbb{Z}}/p}^{an} \underset{\tilde{\varphi}, \tilde{W}_{\mathbb{Q}}, |\cdot|_f}{\times} \mathbb{R}_+^*$$

of the given dynamical system. Using the natural morphisms $\bar{\mathbb{Z}}/p \to \bar{\mathbb{Z}}/\mathfrak{p}$ for $\mathfrak{p}/p$, we define a morphism

$$\coprod_{\mathfrak{p}/p} \text{Spec}(\bar{\mathbb{Z}}/\mathfrak{p}) \to \text{Spec}(\bar{\mathbb{Z}}/p)$$

that induces a Galois-equivariant morphism

$$\coprod_{\mathfrak{p}/p} X_{\bar{\mathbb{Z}}/\mathfrak{p}}^{an} \to X_{\bar{\mathbb{Z}}/p}^{an}$$

that is a bijection on closed points. Since the periodic orbits for our flow are concentrated on closed points, we are reduced to saying that the choice of $\mathfrak{p}_0/p$ induces a map

$$X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}^{an} \times_{F^{\mathbb{Z}}} \mathbb{R}_+^* \cong X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}^{an} \times_{\tilde{W}_{\mathfrak{p}_0, \mathbb{Q}_p}} \mathbb{R}_+^* \longrightarrow X_{\bar{\mathbb{Z}}/p}^{an} \underset{\tilde{\varphi}, \tilde{W}_{\mathbb{Q}}, |\cdot|_f}{\times} \mathbb{R}_+^*$$

that is compatible to the flow and gives an identification between the attracting periodic orbits of it on both sides. This last fact follows from the corresponding set theoretic identification

$$|X_{\bar{\mathbb{Z}}/\mathfrak{p}_0}| \times_{F^{\mathbb{Z}}} \mathbb{R}_+^* \longrightarrow |X_{\bar{\mathbb{Z}}/p}| \underset{\tilde{\varphi}, \tilde{W}_{\mathbb{Q}}, |\cdot|_f}{\times} \mathbb{R}_+^*.$$

□

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