Spectral global analytic geometry

A short note

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Abstract

We propose a setting of spectral global analytic geometry, that adapts spectral derived geometry to the analytic setting, and use it to define topological Hoschchild and topological periodic cyclic homology of commutative Banach rings, and a spectral version of global analytic spaces.

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1 Introduction

A naive approach to spectral analytic geometry would be given by the following basic idea: a normed ring spectrum is a ring spectrum A with a norm on the ring $\pi_0(A)$ and a compatible family of norms on the modules $\pi_i(A)$ for $i \neq 0$. It is also natural to ask that all these abelian groups are complete for their given norm.

We would like to work exclusively with normed ring spectra, and to have a natural way to define them from a normed version of homotopy theory, meaning a homotopy theory for normed sets. We use here Durov's homotopy theory of normed sets from [Dur16].

2 Normed stable homotopy theory

2.1 Normed homotopy theory

Let us start with the category NSets of normed sets (given by a pair $(X, |\cdot|_X)$ where X is a set and $|\cdot|_X : X \to \mathbb{R}_+$ is a map) with contracting maps between them, given by maps $f : X \to Y$ such that

$$|f(x)|_Y \leq |x|_X.$$

We embed it, following Durov in [Dur16], in the copresheaf category of functors from $(\mathbb{R}_+, \leq)$ to Sets, using the fully faithful functor

$$i : \text{NSets} \to \text{Sets}^{(\mathbb{R}_+,\leq)}$$

that $(X, |\cdot|_X : X \to \mathbb{R}_+)$ to the functor $i(X, |\cdot|_X) : [r \mapsto \{x \in X, |x|_X \leq r\}]$. Remark that the $\overline{\mathbb{R}}_+$-normed set $(\mathbb{Z}, |\cdot|_\infty^\infty)$ may be seen as a pro-$\mathbb{R}_+$-normed set given by

$$(\mathbb{Z}, |\cdot|_\infty) = \lim_{t\to\infty} (\mathbb{Z}, |\cdot|_\infty^t)$$

so that it corresponds to the copresheaf that sends every number to the pointed set $\{0\}$. We would prefer to see it as an ind-normed set, or a pro-normed set. In this context, it would be necessary to work with the Lipschitz category to make (multiple) addition a morphism. This means that pro-Lipschitz sets may allow to work with $(\mathbb{Z}, |\cdot|_\infty^\infty)$ in a consistent way.

Similarly, one embeds the category sNSets of simplicial normed sets into the category of simplicial copresheaves on $(\mathbb{R}_+, \leq)$. This copresheaf category $s\text{Sets}^{(\mathbb{R}_+,\leq)}$ may be localized with respect to weak equivalences to get the ∞-category of ∞-functors $\mathcal{S}^{(\mathbb{R}_+,\leq)} := \text{Fun}((\mathbb{R}_+, \leq), \mathcal{S})$ where $\mathcal{S}$ is the ∞-category of spaces, obtained by localizing sSets at weak equivalences. This may be presented, as usual, by the projective model structure on simplicial copresheaves. This category is presentable, and has two Day convolution symmetric monoidal structures (see [Gla13]) induced by the addition $+$

[1] and multiplication $\cdot$ monoidal structures on $\mathbb{R}_+$. They correspond to the corresponding additive and multiplicative monoidal structures $\oplus = \oplus_1$ and $\odot$ on normed sets given by

$$|(x,y)|_{X\oplus Y} = |x|_X + |y|_Y \text{ and } |(x,y)|_{X\odot Y} = |x|_X \cdot |y|_Y.$$

The monoidal structure $\odot$ is closed, and we denote by $\underline{\mathrm{Hom}}$ the corresponding inner homomorphisms. If $X = (X, |\cdot|_X)$ and $Y = (Y, |\cdot|_Y)$ are two normed sets, the normed set $\underline{\mathrm{Hom}}(X,Y)$ is the set of Lipschitz maps $f : X \to Y$, i.e., maps such that there exists a constant $C \geq 0$ such that

$$|f(x)|_Y \leq C|x|_X.$$

One may also work directly with the category LipSets of normed sets with Lipschitz maps between them, but it has less good finiteness properties. The basic idea is to embed LipSets in the category ind-$\text{Sets}^{(\mathbb{R}_+,\leq)}$ by sending $(X, |\cdot|_X)$ to the ind-object

$$j(X) := \text{“colim”}_{c>0}[r \mapsto i(X)(cr)].$$

This embedding

$$j : \text{LipSets} \to \text{ind-Sets}^{(\mathbb{R}_+,\leq)}$$

actually gives a way to define the notion of Lipschitz map between arbitrary copresheaves on $(\mathbb{R}_+, \leq)$, i.e., it comes, by construction, by precomposition with a functor

$$j : \text{Sets}^{(\mathbb{R}_+,\leq)} \to \text{ind-Sets}^{(\mathbb{R}_+,\leq)}$$

that is not the usual embedding. We then fix two universes $A \in B$ and consider A-small Lipschitz sets and A-small sets. Then the above embedding gives an embedding of simplicial Lipschitz sets into simplicial objects of the category ind-$\text{Sets}^{(\mathbb{R}_+,\leq)}$. We may (in close analogy with [BBK17]) localize this category along equivalences given by morphisms of simplicial objects $f : X_* \to Y_*$ such that for every object P of the given category, the morphism

$$\mathrm{Hom}(P, f) : \mathrm{Hom}(P, X_*) \to \mathrm{Hom}(P, Y_*)$$

is a weak equivalence of simplicial sets. The obtained localization form presentable ∞-categories in the universe B, and the Day convolution product $\odot$ extend naturally to this category. We will denote this symmetric monoidal ∞-category by $(s\text{ind-Sets}^{(\mathbb{R}_+,\leq)}[W^{-1}], \odot)$.

Both ∞-categories $\mathcal{S}^{(\mathbb{R}_+,\leq)}$ and $s\text{ind-Sets}^{(\mathbb{R}_+,\leq)}[W^{-1}]$ are complete and cocomplete.

[1]This ℓ^1 additive monoidal structure may actually be replaced by the p-addition $a +_p b$ that is used to define ℓ^p spaces for $0 \leq p \leq \infty$, the $p = 0$ case being treated as a limit case in our setting, it may be better to work with $(\overline{\mathbb{R}}_+, \leq)$-presheaves, to deal with $|\cdot|_\infty^\infty$, say, but then the monoidal structure is the ℓ^0 one

2.2 Metric homotopy theory and completeness

The following section may not be useful for the rest of the article, but we may need it to define the notion of completeness of a normed/metrized spectrum if the approach using homotopy groups doesn't work.

Durov also remarked in his work [Dur16] that one may encode an ultrametric space by a normed simplicial set and that a similar construction may be applied to arbitrary metric spaces.

We now proceed to study the homotopy theory of quasi-metric spaces.

Definition 1. A quasi-metric space is a pair (X, d) composed of a set X and a function $d : X \times X \to \mathbb{R}_+$ such that

1. $d(x, x) = 0$ for all $x \in X$,
2. $d(x, y) = d(y, x)$ for all $(x, y) \in X^2$, and
3. $d(x, z) \leq C \max(d(x, y), d(y, z))$ for every $(x, y, z) \in X^3$ and some fixed $C \geq 1$.

A quasi-metric space is a metric space if it fulfils the stronger inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

for all $(x, y, z) \in X^3$. We denote $q\text{MET}$ (resp. MET) the category of quasi-metric (resp. metric) spaces.

In the litterature, quasi-metric (resp. metric) spaces in this sense are sometimes called pseudo-quasi-metric (resp. pseudo-metric) spaces.

One may see the triangle inequality as being related to defining a Lipschitz norm on the groupoid of pairs

$$P(X) := (p_1, p_2 : X \times X \to X : \Delta, m : X^3 \to X^2),$$

where the multiplication m is given by $m(x, y, z) = (x, z)$ and the source and target maps are the two natural projections. More precisely, if we put on X^2 the norm given by $|(x, y)|_1 = d(x, y)$, and on X the norm given by $|x|_0 = d(x, x)$, then the induced norm on

$$X^3 \cong X^2 \underset{p_2, X, p_1}{\times} X^2$$

is given by

$$|(x, y, z)|_2 = \max(d(x, y), d(y, z)).$$

The composition map $m : (x, y, z) \mapsto (x, z)$ is then Lipschitz, because

$$d(x, z) \leq C \max(d(x, y), d(y, z)).$$

More generally, if we put on each level X^{n+1} of the simplicial nerve $N_*(P(X))$ of the groupoid $P(X)$ the norm given by

$$|(x_0, \dots, x_n)| = \max_i d(x_i, x_{i+1}),$$

we get that $N_*(P(X))$ may be seen as a simplicial Lipschitz set.

Similarly, if (X, d) is a metric space, we may associate to it the normed simplicial set whose underlying set is $N_*(P(X))$ and with

$$|(x_0, \dots, x_n)| = \sum_i d(x_i, x_{i+1}).$$

We arrive to the following proposition (that was also essentially proved by Durov).

Proposition 1. *There are fully faithful functors*

$$\text{Met} \to s\text{NSets} \textit{ and } q\text{Met} \to s\text{LipSets}$$

from the category of metric (resp. quasi-metric) spaces to the category of simplicial normed sets (resp. simplicial Lipschitz sets).

As a corollary, we get that there are fully faithful functors

$$s\text{Met} \to ss\text{NSets} \text{ and } sq\text{Met} \to ss\text{LipSets}$$

from the category of simplicial metric (resp. quasi-metric) spaces to the category of bisimplicial normed sets (resp bisimplicial Lipschitz sets).

We will define the homotopy theory of (simplicial) metric spaces to be the ∞-category

$$\text{Met}\mathcal{S} := \text{N}\mathcal{S} := \text{Fun}_\infty((\mathbb{R}_+, \leq), ss\text{Sets}[W^{-1}])$$

where W is given by diagonal weak equivalences of bisimplicial sets and the localization is meant in the ∞-categorical sense. This category is clearly a locally presentable ∞-category.

The homotopy theory of (simplicial) quasi-metric spaces is defined as the ∞-category

$$q\text{Met}\mathcal{S} := \text{Fun}_\infty((\Delta^{op})^2, \text{ind-Sets}^{(\mathbb{R}_+, \leq)})[W^{-1}].$$

It is locally presentable with respect to A-parametrized colimits.

There is a natural "constant" metric and quasi-metric space that gives functors

$$c : \text{Met} \to \text{Met}\mathcal{S} \text{ and } c : q\text{Met} \to q\text{Met}\mathcal{S}.$$

Proposition 2. *The symmetric monoidal structure on $(\mathbb{R}_+, \leq)$ given by multiplication induces a closed symmetric monoidal ∞-category structure on both* Met$\mathcal{S}$ *and* qMet$\mathcal{S}$*, called the multiplicative monoidal structure, and denoted $\otimes^m$.*

Proof. The symmetric monoidal structure is given by Day convolution (see [Gla13]), and extension of a given monoidal structure to the indization. The fact that it is closed is standard. □

Let (X, d) be a quasi-metric space and $x : \mathbb{N}_{>0} \to X$ be a Cauchy sequence. Then x induces (possibly after extracting a subsequence) a morphism of simplicial Lipschitz spaces

$$N(P(x)) : N(P(\mathbb{N}_{>0})) \to N(P(X))$$

where the norm on the left hand side is given by

$$|(i_1, \dots, i_n)| := \frac{1}{\min_k(i_k)} = \max_k(1/i_k),$$

and the norm on the right hand side is the one described above, using the metric. Moreover, the fact that the sequence is Cauchy implies that at every level, the associated map fulfills

$$|(i_1, \dots, i_n)| \to 0 \text{ implies } |N(P)(x)(i_1, \dots, i_n)| \to 0.$$

This condition is automatic, since we have a morphism of Lipschitz normed sets.

The above sequence converges if $N(P(x))$ extends to $N(P(\mathbb{N}_{>0} \cup \{\infty\}))$ with the same norm as above with the convention $1/\infty = 0$. More precisely, it is convergent if it extends to a morphism

$$\overline{N(P(x))} : N(P(\mathbb{N}_{>0} \cup \{\infty\})) \to N(P(X)).$$

Such a Lipschitz morphism fulfils

$$\left|\overline{N(P(x))}(n_0, \dots, n_m)\right| \to 0 \quad \text{whenever} \quad |(n_0, \dots, n_m)| \to 0.$$

We may define a Cauchy sequence in an arbitrary object of $s$ind-$\text{SETS}^{(\mathbb{R}_+, \leq)}[W^{-1}]$ to be a morphism

$$x : N(P(\mathbb{N}_{>0})) \to X$$

and the object is called complete if every such morphism extends to $N(P(\mathbb{N}_{>0} \cup \{\infty\}))$.

2.3 Normed spectra

We will consider the "monoidal stabilization" (see [Rob14], Chapter 4) of the pointed symmetric monoidal ∞-category $(\mathcal{S}_*^{(\mathbb{R}_+, \leq)}, \odot)$ with respect to a circle object S^1. There are various possible choices for the circle object. The usual one is given by $S^1 := B\underline{\mathbb{Z}}$ where $\underline{\mathbb{Z}}$ is the constant presheaf defined by $\underline{\mathbb{Z}}(r) = \mathbb{Z}$ (it is associated to the norm on $\mathbb{Z}$ that sends everything to 0). The corresponding stabilisation is simply the symmetric monoidal ∞-category $(\text{SP}^{(\mathbb{R}_+, \leq)}, \odot)$ of presheaves of spectra. We don't want to use this category because its objects may be informally seen as families of normed spaces $(X_n, |\cdot|_n)$ with contracting maps

$$X_n \wedge^m \mathbb{Z} \to X_{n+1},$$

and such map sends every element to an element of norm 0 by construction. But the Eilenberg-Maclane spectrum of a Banach ring $(R, |\cdot|)$ is not like this.

The monoidal stabilization that we will adopt in this work is by the object $S^1_t := B(\mathbb{Z}, |\cdot|_\infty)$, where the classifying space is defined, as a pointed normed simplicial set, by

$$B(\mathbb{Z}, |\cdot|_\infty)_n := (\mathbb{Z}^n, |\cdot|_{\ell^1,\infty}),$$

where $|(x_1, \ldots, x_n)|_{\ell^1,\infty} := |x_1|_\infty + \cdots + |x_n|_\infty$. We denote this symmetric monoidal ∞-category by $(\mathrm{NS\textsc{p}}, \wedge)$. The associated sphere spectrum $\mathbb{S}_t$ will have

$$\pi_0(\mathbb{S}_t) = (\mathbb{Z}, |\cdot|_\infty),$$

and if $(R, |\cdot|)$ is a Banach ring, we may associate to it a normed Eilenberg-Maclane spectrum $H(R, |\cdot|)$ in $\mathrm{NS\textsc{p}}$.

Definition 2. Let the n-sphere be the object of $\mathrm{NS\textsc{p}}$ defined by $S^n_t := (S^1_t)^{\odot n}$ for $n \in \mathbb{Z}$. The normed n-homotopy group of M is defined by

$$\pi_n(M) := \pi_0(\underline{\mathrm{Hom}}(S^n_t, M)),$$

where $\pi_0 : \mathrm{NS\textsc{p}} \to \mathrm{A\textsc{b}}^{(\mathbb{R}_+,\leq)}$ is obtained by symmetric monoidal localization of the functor

$$\pi_0 : s\mathrm{S\textsc{ets}}_*^{(\mathbb{R}_+,\leq)} \to \mathrm{S\textsc{ets}}_*^{(\mathbb{R}_+,\leq)}.$$

A normed spectrum is called a Banach spectrum if all its normed homotopy group are (represented by) complete abelian groups. We denote by $\mathrm{B\textsc{an}S\textsc{p}}$ the full subcategory of $\mathrm{NS\textsc{p}}$ given by Banach spectra.

Proposition 3. *The natural inclusion of Banach spectra into normed spectra has a natural ∞-adjoint functor called the completion and denoted $M \mapsto \hat{M}$.*

One may extend the monoidal structure $\odot$ on $\mathrm{NS\textsc{p}}$ (by the above adjunction) to a monoidal structure $\hat{\odot}$ on $\mathrm{B\textsc{an}S\textsc{p}}$ called the completed wedge product.

3 Topological Hochschild and periodic cyclic homology of Banach ring spectra

We are now able to define the topological Hochschild cohomology of a Banach commutative ring spectrum.

Definition 3. The topological Hochshild homology of a Banach commutative ring spectrum A is given by the Banach spectrum

$$THH(A) = S^1_t \otimes A$$

where the tensor product is meant as the tensoring of Banach spectra by simplicial normed sets.

Remark that one can't describe S^1_t as one usually describes S^1 to compute Hoschshild homology by $S^1 = \{*\} \coprod_{\{*\}\coprod\{*\}} \{*\}$, so the topological Hoschild homology must be computed by using the definition $S^1_t = B(\mathbb{Z}, |\cdot|_\infty)$, that gives a bigger complex.

Definition 4. The periodic topological cyclic homology of a Banach ring spectrum A is given by the Banach spectrum

$$TP(A) := THH(A)^{tS^1_t}$$

obtained by the Tate construction associated to the natural action of S^1_t on $THH(A)$.

4 Spectral analytic geometry: definitions and basic open questions

We now develop the basics of an analytic geometry based on the category of (connective) Banach ring spectra. First, the analog of the convergent power series algebras are given by the following construction.

Let A be a commutative ring spectrum and X be a finite normed set. We denote $X \otimes^m A$ the associated free A-module and $A\langle X\rangle$ the associated free commutative Banach ring spectrum. The algebra $A\langle X\rangle$ is called the Banach algebra of convergent power series on the normed set X. The ind-Banach algebra $A\langle X\rangle^\dagger$ is defined as the formal colimit of the sequence $A\langle Y\rangle$ where Y has the same underlying set as X and the norm given by $(1 + \epsilon(x))|x|_X$ with varying over maps $\epsilon : X \to \mathbb{R}^*_+$.

A connective ind-Banach ring spectrum over A is called an affinoid algebra if it is a finite colimit of algebras of the form $A\langle X\rangle^\dagger$.

We use Bambozzi and Ben-Bassat's categorical characterization of affinoid embeddings from [BB15] to extend the G-topology to this situation. We may also define the étale topology, if necessary.

Definition 5. The topological Hochschild cohomology of a normed spectral stack X is given by the mapping stack

$$THH(X) := \mathrm{Map}(S^1_t, X)$$

where S^1_t is the constant Banach spectral stack with value the normed spectrum S^1_t. The topological periodic cyclic homology of a spectral stack X is given by the global sections of the Tate construction

$$TP(X) := \Gamma(X, THH^{tS^1_t})$$

for the natural S^1_t action on THH.

Question 1. We would like to show that in the algebraic setting, i.e., if X is a (say proper, and smooth) scheme over $\mathbb{Z}$ (or $\mathbb{Z}[1/N]$), seen as an analytic space over $(\mathbb{Z}, |\cdot|_0)$, then our construction gives back usual topological Hoschshild and periodic cyclic homology. This should be implied by a GAGA-type argument (see Porta and Yue-Yu) but given in the spectral setting.

Question 2. If A is Banach algebra, we would like to give an explicit presentation of $THH(X)$ similar to the usual presentation of algebraic topopological Homology, that uses $S^1 = \{*\} \coprod_{\{*\}\coprod\{*\}} \{*\}$, but of course different. It should be based on an explicit description of a cofibrant resolution of the normed simplicial set $S^1_t = B(\mathbb{Z}, |\cdot|_\infty)$.

Question 3. We would also like to show that if X is an analytic space over $\mathbb{R}$ (or $\mathbb{C}$, or $(\mathbb{Q}, |\cdot|_0)$, or any Banach algebra containing $\mathbb{Q}$), then

$$THH(X) = HH(X)$$

in the sense of completed Hoschild homology, and $TP(X) = HP(X)$ has a direct link with the usual de Rham cohomology of X.

Question 4. We would like to define a global analog of a Hodge structure whose underlying de Rham module would be $TP(X)$. This would entail the definition of a chern class map

$$\mathrm{ch} : K(X) \to TP(X)$$

defined à la Toen-Vezzosi [TV09], but in a normed spectral setting.

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