Appendix A

Norms of smoothing functions

Our aim here is to give bounds on the norms of some smoothing functions. They are all based on the Gaussian $e^{-t^2/2}$ in one way or the other.

A.1 THE FUNCTIONS η AND η_1

We will work with functions $\eta, \eta_1 : [0, \infty) \to \mathbb{R}$ defined by

$$\eta_1(x) = \begin{cases} \sqrt{\frac{2}{\pi}} x^2 e^{-x^2/2} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases} \tag{A.1}$$

and

$$\begin{aligned} \eta(x) &= (2 \cdot 1_{[1/2,1]}) *_M \sqrt{\frac{2}{\pi}} x^2 e^{-x^2/2} = \int_x^{2x} 2\sqrt{\frac{2}{\pi}} w^2 e^{-w^2/2} \frac{dw}{w} \\ &= \sqrt{\frac{8}{\pi}} \cdot (e^{-x^2/2} - e^{-2x^2}) \end{aligned}$$

for $x \geq 0$; we let $\eta(x) = 0$ for $x < 0$.

Since, as is well-known, $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, we know that

$$|\eta|_1 = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left(e^{-x^2/2} - e^{-2x^2}\right) dx = \sqrt{\frac{2}{\pi}} (\sqrt{2\pi} - \sqrt{\pi/2}) = 1.$$

Of course, the factor $\sqrt{8/\pi}$ in the definition of η is there so as to make $|\eta|_1$ equal 1. Taking derivatives, we see that $\eta(x)$ has its only local maximum on $[0, \infty)$ at $x = 2\sqrt{(\log 2)/3}$, and that that $\lim_{x\to\infty} \eta(x) = \eta(0) = 0$. Hence

$$\begin{aligned} |\eta'|_1 &= 2\eta\left(2\sqrt{\frac{\log 2}{3}}\right) = 4\sqrt{\frac{2}{\pi}}\left(e^{-4\frac{\log 2}{2\cdot 3}} - e^{-4\frac{2\log 2}{3}}\right) \\ &= 4\sqrt{\frac{2}{\pi}}\left(\frac{1}{2^{2/3}} - \frac{1}{2^{8/3}}\right) = \frac{3}{2^{1/6}\sqrt{\pi}}. \end{aligned}$$

By the same token,

$$|\eta|_\infty = \frac{3}{2^{7/6}\sqrt{\pi}}.$$

The Fourier transform is a little harder to bound.

Lemma A.1. *Let*

$$\eta(x) = \begin{cases} \sqrt{8/\pi} \cdot (e^{-x^2/2} - e^{-2x^2}) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{A.2}$$

Then

$$|\widehat{\eta''}|_\infty = 2.73443691486 + O^*(3 \cdot 10^{-11}).$$

Proof. Let

$$f_a(x) = \begin{cases} e^{-ax^2} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then, for $a > 0$, $\widehat{f_a}(t)$ equals

$$\int_0^\infty e^{-ax^2} e^{-2\pi i x t} dx = e^{-\frac{\pi^2}{a}t^2} \int_0^\infty e^{-a(x+i\pi t/a)^2} dx = e^{-\frac{\pi^2}{a}t^2} \int_{\frac{i\pi t}{a}}^{\frac{i\pi t}{a}+\infty} e^{-az^2} dz.$$

We shift the contour of integration, and obtain

$$\begin{aligned} \widehat{f_a}(t) &= e^{-\frac{\pi^2}{a}t^2} \left(- \int_0^{\frac{i\pi t}{a}} e^{-az^2} dz + \int_0^\infty e^{-az^2} dz \right) \\ &= e^{-\frac{\pi^2}{a}t^2} \left(-\frac{1}{\sqrt{a}} \int_0^{\frac{i\pi t}{\sqrt{a}}} e^{-z^2} dz + \frac{\sqrt{\pi/a}}{2} \right) \\ &= \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-\frac{\pi^2}{a}t^2} \left(1 - i \operatorname{erfi}\left(\frac{\pi t}{\sqrt{a}} \right) \right), \end{aligned}$$

where erfi is the imaginary error function (4.4) . This formula is of course standard; see [AS64, 7.4.6–7.4.7].

Now, recalling the standard rule $\widehat{g'}(t) = (2\pi i t)\widehat{g}(t)$ (§in 2.4.1; valid when g and g' are both in L^1), we see that

$$\begin{aligned} &\widehat{\eta''}(t) = (2\pi i t)^2 \widehat{\eta}(t) = -2^{7/2}\pi^{3/2}t^2(\widehat{f_{1/2}}(t) - \widehat{f_2}(t)) \\ &= 4\pi^2 t^2 e^{-\pi^2 t^2/2} \left(\left(1 - 2e^{-\frac{3}{2}\pi^2 t^2} \right) - i \left(\operatorname{erfi}\left(\frac{\pi t}{\sqrt{2}} \right) - 2e^{-\frac{3}{2}\pi^2 t^2} \operatorname{erfi}(\sqrt{2}\pi t) \right) \right). \end{aligned} \tag{A.3}$$

Before we use the expression (A.3), let us give a somewhat crude bound, useful for t large. The function η'' has a jump (from 0 to $3\sqrt{8/\pi}$) at the origin, but $\eta^{(3)}$ is integrable and defined outside the origin. Hence

$$|\widehat{\eta''}(t)| \leq \frac{|\widehat{\eta^{(3)}}(t)|}{2\pi|t|} \leq \frac{|\eta^{(3)}|_\infty}{2\pi|t|} = \frac{1}{2\pi|t|} \left(3\sqrt{\frac{8}{\pi}} + \lim_{x_0 \to 0^+} \int_{x_0}^\infty |\eta^{(3)}(x)| dx \right).$$

Since we are just deriving a crude bound for now, we can use the inequality $|\eta^{(3)}(x)| \leq \sqrt{8/\pi}(|f_{1/2}^{(3)}(x)| + |f_2^{(3)}(x)|)$:

$$\lim_{x_0 \to 0^+} \int_{x_0}^\infty |\eta^{(3)}(x)| dx = \sqrt{\frac{8}{\pi}} \left(\lim_{x_0 \to 0^+} \int_{x_0}^\infty |f_{1/2}^{(3)}(x)| dx + \lim_{x_0 \to 0^+} \int_{x_0}^\infty |f_2^{(3)}(x)| dx \right)$$

We can easily see that $f_a^{(3)}(x) = (-8a^3x^3 + 12a^2x)e^{-ax^2}$ is positive for $0 < x < \sqrt{3/2a}$ and negative for $x > \sqrt{3/2a}$, and that $f_a''(0) = -2a$ and $\lim_{x\to\infty} f_a''(x) = 0$. Hence

$$\lim_{x_0\to 0^+} \int_{x_0}^{\infty} |f_a^{(3)}(x)|dx = 2a + 2|f_a''(\sqrt{3/2a})| = 2a + 8ae^{-3/2},$$

and so

$$\lim_{x_0\to 0^+} \int_{x_0}^{\infty} |\eta^{(3)}(x)|dx = \sqrt{\frac{8}{\pi}} \left(2(1/2+2) + 8(1/2+2)e^{-3/2}\right) = \frac{5+20e^{-3/2}}{\sqrt{\pi/8}}.$$

We conclude that

$$|\widehat{\eta''}(t)| \leq \frac{4+10e^{-3/2}}{(\pi/2)^{3/2}|t|}. \tag{A.4}$$

We will use this bound for $t > 6/5$, say.

Now we apply the bisection method as in §4.1.1, with 5 initial iterations followed by 35 more iterations, to obtain that the maximum of $|\widehat{\eta''}(t)|$ for $t \in [0, 1.2]$ lies in the interval

$$[2.734436914842, 2.734436914882] \tag{A.5}$$

Since $2.73443\ldots$ is greater than $(4+10e^{-3/2})/((6/5)(\pi/2)^{3/2}) = 2.63765\ldots$, and $|\widehat{\eta''}(t)| = |\widehat{\eta''}(-t)|$, we conclude that the maximum of $|\widehat{\eta''}(t)|$ for all $t \in \mathbb{R}$ lies in (A.5). □

We will now bound $|\eta''|_1$. Note that it is substantially greater, i.e., worse, than the bound on $|\widehat{\eta''}|_\infty$ given by Lemma A.1. Thus we may stand to gain something by using Lemma 3.4 rather than Lemma 3.3.

Lemma A.2. *Let η be as in (A.2). Then*

$$|\eta''|_1 = 3.884903382586 + O(2 \cdot 10^{-12}).$$

The procedure of proof will be a little simpler than in later lemmas of this kind, such as Lemma A.4.

Proof. Clearly $\lim_{t\to\infty} \eta'(t) = \eta'(0) = 0$. Since

$$\frac{\eta''(x)}{\sqrt{8/\pi}} = (x^2-1)e^{-x^2/2} - (16x^2-4)e^{-2x^2},$$

$\eta'(x)$ can have a local extremum only when $e^{3x/2} = 16 - 12/(1-x)$. Since $\exp(3x/2)$ is increasing and $16 - 12/(1-x)$ decreases monotonically from 4 to $-\infty$ as x goes from 0 to 1 and decreases monotonically from ∞ to 16 as x goes from 1 to ∞, we see that $e^{3x/2} = 16 - 12/(1-x)$ has exactly two roots, one in $(0, 1)$ and one in $(1, 3)$, say.

The bisection method shows that $\eta'(x)$ does have local extrema in these intervals, and that $\eta'(x)$ takes the following values at them:

$$\begin{aligned} y_1 &= 1.27071184712 + O^*\left(4 \cdot 10^{-13}\right), \\ y_2 &= 0.6717398441732 + O^*\left(2 \cdot 10^{-13}\right). \end{aligned} \tag{A.6}$$

Hence

$$\begin{aligned} |\eta''|_1 &= 2(1.27071184712 + 0.6717398441732 + O^*(6 \cdot 10^{-13})) \\ &= 3.884903382586 + O(2 \cdot 10^{-12}). \end{aligned}$$

□

* * *

Let

$$\eta_*(x) = (\log x)\eta(x) = \begin{cases} \sqrt{8/\pi} \cdot (\log x)(e^{-x^2/2} - e^{-2x^2}) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{A.7}$$

We need to know a couple of norms involving η_*. Thanks are due to N. Elkies, K. Conrad and R. Israel for help with several integrals.

Lemma A.3. *Let $\eta_*(x)$ be as in (A.7). Then*

$$|\eta_*|_1 = 0.415495256376802 + O^*(3 \cdot 10^{-15}).$$

Proof. First of all,

$$\begin{aligned} \int_0^\infty x^a e^{-x^2} dx &= \int_0^\infty u^{a/2} e^{-u} \frac{du}{2\sqrt{u}} \\ &= \frac{1}{2} \int_0^\infty u^{\frac{a+1}{2}-1} e^{-u} du = \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right). \end{aligned}$$

Taking the derivative with respect to a at $a = 0$, we see that

$$\int_0^\infty (\log x) e^{-x^2} dx = \frac{1}{4}\Gamma'(1/2) = \frac{-\sqrt{\pi}(\gamma + \log 4)}{4}, \tag{A.8}$$

where we obtain the value of $\Gamma'(1/2)$ from (3.38) and (3.49). Hence

$$\begin{aligned} &\sqrt{\frac{8}{\pi}} \int_0^\infty (\log x) \left(e^{-x^2/2} - e^{-2x^2}\right) dx \\ &= \sqrt{\frac{8}{\pi}} \left(\sqrt{2} \int_0^\infty (\log \sqrt{2}u) e^{-u^2} du - \frac{1}{\sqrt{2}} \int_0^\infty \log \frac{u}{\sqrt{2}} \cdot e^{-u^2} du\right) \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty (\log u) e^{-u^2} du + \frac{2 \cdot \frac{3}{2} \log 2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du \\ &= -\frac{\gamma + \log 4}{2} + \frac{3 \log 2}{2} = \frac{\log 2 - \gamma}{2}, \end{aligned}$$

where we use (A.8) in the last step.

Now

$$|\eta_*|_1 = -2 \cdot \sqrt{\frac{8}{\pi}} \int_0^1 (\log x) \left(e^{-x^2/2} - e^{-2x^2}\right) dx + \sqrt{\frac{8}{\pi}} \int_0^\infty (\log x) \left(e^{-x^2/2} - e^{-2x^2}\right) dx.$$

For $r > -1$,

$$\begin{aligned}\int_0^1 (\log x) x^r dx &= \int_0^1 (\log x) x^{r+1} d\log x = \int_{-\infty}^0 u e^{(r+1)u} du \\ &= \left(\left(\frac{u}{r+1} - \frac{1}{(r+1)^2}\right) e^{(r+1)u}\right) |_{-\infty}^0 = -\frac{1}{(r+1)^2}.\end{aligned} \tag{A.9}$$

Expanding exp into a Taylor series, we see that

$$\begin{aligned}\int_0^1 (\log x)\left(e^{-x^2/2} - e^{-2x^2}\right) dx &= \int_0^1 (\log x) \left(\sum_{k=0}^\infty \frac{(-x^2/2)^k - (-2x^2)^k}{k!}\right) dx \\ &= -\sum_{k=0}^\infty (-1)^k \cdot \frac{2^k - 2^{-k}}{k!} \int_0^1 (\log x) x^{2k} dx \\ &= \sum_{k=0}^\infty (-1)^k \frac{2^k}{k!(2k+1)^2} - \sum_{k=0}^\infty (-1)^k \frac{2^{-k}}{k!(2k+1)^2} \\ &= \sum_{k=0}^K (-1)^k \frac{2^k}{k!(2k+1)^2} - \sum_{k=0}^{K-1} (-1)^k \frac{2^{-k}}{k!(2k+1)^2} + O^*\left(\frac{2^K + 2^{-K}}{K!(2K+1)^2}\right)\end{aligned}$$

for any even $K \geq 0$, since these are alternating sums. Setting $K = 20$, we obtain

$$\begin{aligned}|\eta_*|_1 &= -2 \cdot \sqrt{\frac{8}{\pi}} \left(-0.112024193759256 + O^*(6 \cdot 10^{-16})\right) + \frac{\log 2 - \gamma}{2} \\ &= 0.415495256376802 + O^*(3 \cdot 10^{-15}).\end{aligned}$$

□

It ought to be possible to prove results such as Lemma A.3 by pressing a button: symbolic integration gives an expression involving a generalized hypergeometric function. Generalized hypergeometric functions are now at least partly implemented in ARB [Joh19]. Of course, one can also prove Lemma A.3 by rigorous numerical integration (§4.1.3), though that feels a little brutal for such a simple integrand.

The following kind of procedure also ought to be completely automated.

Lemma A.4. *Let η_* be as in (A.7). Then*

$$|\eta'_*|_1 = 1.02010539081 + O^*(10^{-11}).$$

Proof. It is clear that $\lim_{t\to 0^+} \eta_*(x) = \lim_{t\to\infty} \eta_*(x) = 0$. It will thus be enough to identify and estimate the local maxima and minima of η_* on $(0,\infty)$. We apply the bisection method using interval arithmetic as explained at the end of §4.1.1, and obtain η_* has two local extrema within $[1/3, 3]$, and that the values of η_* at these extrema are

$$\begin{aligned} y_1 &= -0.305340693793 + O^*\left(2\cdot 10^{-12}\right), \\ y_2 &= 0.204712001611 + O^*\left(2\cdot 10^{-12}\right). \end{aligned} \tag{A.10}$$

Now, for $x > 0$,

$$\frac{\eta_*'(x)}{\sqrt{8/\pi}} = \frac{e^{-x^2/2} - e^{-2x^2}}{x} + (\log x)\left(-xe^{-x^2/2} + 4xe^{-2x^2}\right). \tag{A.11}$$

For $x \leq 1/e$ (say), the first two terms add up to an alternating sum

$$\frac{3}{2}x - \frac{15}{8}x^3 + \ldots \leq \frac{3}{2}x.$$

In the same way and for the same range of x,

$$-\exp(-x^2/2) + 4e^{-2x^2} \geq 3 - (15/2)x^2.$$

Hence, for $x \leq 1/e$,

$$\begin{aligned} \frac{\eta_*'(x)}{\sqrt{8/\pi}} &\leq -|\log x|\left(3x - \frac{15}{2}x^3\right) + \frac{3}{2}x \\ &\leq -|\log x|\left(\frac{3x}{2} - \frac{15}{2}x^3\right) < 0. \end{aligned}$$

For $x \geq e$, it is the third term in (A.11) that dominates:

$$\frac{\eta_*'(x)}{\sqrt{8/\pi}} \leq -(\log x)xe^{-x^2/2}\left(1 - \frac{1}{x^2(\log x)} - \frac{4}{e^{3x^2/2}}\right) < 0.$$

Hence $\eta_*(x)$ has no local extrema in $(0, 1/e)$ or (e, ∞).

We conclude that

$$\begin{aligned} |\eta_*'|_1 &= 2(0.305340693793 - 0.20471200611) + O^*(8\cdot 10^{-12}) \\ &= 1.020105390808 + O^*(8\cdot 10^{-12}) = 1.02010539081 + O^*(10^{-11}). \end{aligned}$$

□

We would also like to have a bound for $|\widehat{\eta_*''}|_\infty$. If we are to proceed as in the proof of Lemma A.1, we need to have an expression for $\widehat{\eta_*}(t)$. Since $\log(x)$ is the derivative of x^ν with respect to ν at $\nu = 0$,

$$\widehat{\eta_*}(t) = \frac{d}{d\nu}\int_0^\infty x^\nu e^{-ax^2} e(-tx)dx, \tag{A.12}$$

and we do have an expression for the integral in the right side of (A.12) in terms of $\Gamma(\nu/2)$, $\Gamma((\nu+1)/2$ and two values of a hypergeometric function ${}_1F_1$ [GR94, 3.952, 8–9]. The function ${}_1F_1$ is now implemented in ARB. (See also [Pea09], [POP17].) The derivative of ${}_1F_1$ with respect to the first variable is given by a generalized hypergeometric function. We could leave it to ARB, or implement it ourselves in the range we need by a Taylor series.

Let us not take that route here. It will turn out that we do not actually need an exact value for $|\eta_*|_1$. We will actually be happy with the coarse bound $|\widehat{\eta_*''}|_\infty \leq |\eta_*''|_1$ and the following estimate, which we will obtain by the same procedure as in Lemma A.4.

Lemma A.5. *Let η_* be as in (A.7). Then*

$$|\eta_*''|_1 = 3.908021634825 + O^*\left(10^{-11}\right).$$

Proof. Clearly, $\lim_{t\to 0^+} \eta_*'(x) = \lim_{t\to\infty} \eta_*'(x) = 0$. Let us find the local maxima and minima of η_*' on $(0,\infty)$. We apply the bisection method using interval arithmetic as explained at the end of §4.1.1, and obtain that η_*' has three local extrema with $[0.01, 3]$, and that the values of η_*' at these extrema are

$$\begin{aligned} y_1 &= -0.94877018055 + O^*\left(4\cdot 10^{-12}\right),\\ y_2 &= 0.815167328066 + O^*\left(8\cdot 10^{-13}\right),\\ y_3 &= -0.1900733087965 + O^*\left(2\cdot 10^{-13}\right). \end{aligned} \tag{A.13}$$

It is easy to see that $\eta_*''(x) \neq 0$ for $x \in (0, 0.01)$ and for $x \in (3,\infty]$, as then one of the terms of

$$\begin{aligned} \frac{\eta_*''(\sqrt{x})}{\sqrt{8/\pi}} &= (\log x)\left((x^2-1)e^{-x^2/2} - (16x^2-4)e^{-2x^2}\right)\\ &\quad + \frac{2}{x}\left(-xe^{-x^2/2} + 4xe^{-2x^2}\right) - \frac{1}{x^2}\left(e^{-x^2/2} - e^{-2x^2}\right). \end{aligned} \tag{A.14}$$

dominates all the others. (For $x \in (0, 0.01)$, it is the term $4(\log x)e^{-2x^2}$; for $x \in (3,\infty)$ it is the term $(x^2-1)(\log x)e^{-x^2/2}$.)

Hence, (A.13) is the full list of extrema of η_*' in $(0,\infty)$. We conclude that

$$|\eta_*''|_1 = -2y_1 + 2y_2 - 2y_3 = 3.908021634825 + O^*\left(10^{-11}\right).$$

□

We also need some bounds involving the function η_1. First of all,

$$|\eta_1|_1 = \sqrt{\frac{2}{\pi}}\int_0^\infty x^2 e^{-x^2/2}dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2/2}dx = 1.$$

Taking derivatives, we see that $x^2e^{-x^2/2}$ has one critical point, at $x = \sqrt{2}$; the value of $x^2e^{-x^2/2}$ at $x = \sqrt{2}$ equals $2/e$. Hence

$$|\eta_1'|_1 = 2|\eta_1(\sqrt{2})|_1 = 2\sqrt{\frac{2}{\pi}}\frac{2}{e} = \frac{\sqrt{32}}{e\sqrt{\pi}}.$$

We gather our results in one place: for η_1 as in (A.1), η as in (A.2) and η_* as in (A.7),

$$\begin{aligned}
&|\eta|_1 = 1, && |\eta'|_1 = \frac{3}{2^{1/6}\sqrt{\pi}} = 1.5079073303\dots,\\
&|\eta''|_1 = 3.88490338258\dots, && |\widehat{\eta''}|_\infty = 2.7344369148\dots,\\
&|\eta_1|_1 = 1, && |\eta_1'|_1 = \frac{\sqrt{32}}{e\sqrt{\pi}} = 1.1741013053\dots,\\
&|\eta_*|_1 = 0.4154952563768\dots, && |\eta_*'|_1 = 1.0201053908\dots,\\
&|\widehat{\eta_*''}|_\infty \le |\eta_*''|_1 = 3.9080216348\dots, && |\eta|_\infty = \frac{3}{2^{7/6}\sqrt{\pi}} = 0.7539536651\dots
\end{aligned} \tag{A.15}$$

We still need a few more bounds.

Lemma A.6. *Let $\eta_1 : \mathbb{R} \to [0,\infty)$ be as in (A.1). Let $\eta_{1,W}(x) = (\log Wx)\eta_1(x)$. Then, for $W \ge 1$,*

$$\begin{aligned}
|\eta_{1,W}|_1 &= \log W + \left(1 - \frac{\gamma + \log 2}{2}\right) + O^*\left(\frac{\sqrt{8/\pi}}{9W^3}\right)\\
|\eta_{1,W}'|_1 &\le \sqrt{\frac{2}{\pi}}\left(\frac{4}{e}\log W + \frac{1}{eW^2}\right) + 0.608238.
\end{aligned} \tag{A.16}$$

In particular, for $W \ge 136$,

$$\frac{|\eta_{1,W}'|_1}{|\eta_{1,W}|_1} \le \frac{3}{4}e^{0.50136} - \frac{3}{100}. \tag{A.17}$$

The form in which we have put the bound (A.17) may seem peculiar, but it will show itself to be convenient.

Proof. Clearly

$$\begin{aligned}
|\eta_{1,W}|_1 &= \int_0^\infty |(\log Wx)\eta_1(x)|dx\\
&= -\int_0^{1/W} (\log Wx)\eta_1(x)dx + \int_{1/W}^\infty (\log Wx)\eta_1(x)dx\\
&= -2\int_0^{1/W} (\log Wx)\eta_1(x)dx + \int_0^\infty (\log Wx)\eta_1(x)dx.
\end{aligned}$$

We can simply bound

$$-\int_0^{1/W} (\log Wx)\eta_1(x)dx \leq \sqrt{\frac{2}{\pi}} \int_0^{1/W} (-\log W - \log x)x^2 dx$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{9W^3}(3\log W + 1) - \frac{\log W}{3W^3} \right) = \frac{\sqrt{2/\pi}}{9W^3}.$$

Of course, $\int_0^\infty (\log W)\eta_1(x)dx = \log W$. By integration by parts and (A.8),

$$\begin{aligned}
&\int_0^\infty (\log x)x^2 e^{-x^2/2}dx = \sqrt{2}\int_0^\infty (\log \sqrt{2}u)\cdot 2u^2 e^{-u^2}du \\
&= \frac{\log 2}{\sqrt{2}} \int_0^\infty u \cdot 2ue^{-u^2}du + \sqrt{2}\int_0^\infty (u\log u)\cdot 2ue^{-u^2}du \\
&= \frac{\log 2}{\sqrt{2}} \int_0^\infty e^{-u^2}du + \sqrt{2}\int_0^\infty (1+\log u)e^{-u^2}du \\
&= \frac{2+\log 2}{\sqrt{2}}\cdot\frac{\sqrt{\pi}}{2} + \sqrt{2}\cdot\frac{-\sqrt{\pi}(\gamma+\log 4)}{4} = \sqrt{\frac{\pi}{2}}\cdot\left(1 - \frac{\gamma+\log 2}{2}\right),
\end{aligned} \tag{A.18}$$

where γ is Euler's constant. Thus, the bound on $|\eta_{1,W}|$ in (A.16) holds.

Since

$$((\log Wx)x^2 e^{-x^2/2})' = (\log Wx)x^2 \cdot (-x)e^{-x^2/2} + (x + 2(\log Wx)x)e^{-x^2/2}$$
$$= ((\log Wx)(2-x^2)+1)\cdot xe^{-x^2/2},$$

the function $\eta_{1,W}$ has its critical points at the roots of

$$(\log Wx)(2-x^2) + 1 = 0. \tag{A.19}$$

Now,

$$((\log Wx)(2-x^2))' = \frac{2-x^2}{x} - 2x(\log Wx) > 0$$

for $x \leq 1/W$. Since the left side of (A.19) equals 1 for $x = 1/W$ and tends to $-\infty$ as $x \to 0^+$, we see that (A.19) has exactly one root x_0 in $[0, 1/W]$, and that $\eta_{1,W}$ is decreasing on $[0, x_0]$. Since $\log Wx > 0$ for $x > 1/W$, we also see that (A.19) has no roots on $[1/W, \sqrt{2}]$. It is also to see that $(\log Wx)(2-x^2)$ decreases from 0 to $-\infty$ as x ranges from $\sqrt{2}$ to ∞. Thus, (A.19) has exactly one root x_1 greater than $\sqrt{2}$; the function $\eta_{1,W}$ is increasing on $[x_0, x_1]$ and decreasing on $[x_1, \infty)$. Since $x_0 < 1/W < x_1$, $\eta_{1,W}(x_0) < 0 < \eta_{1,W}(x_1)$. Hence

$$|\eta'_{1,W}|_1 = -2\eta_{1,W}(x_0) + 2\eta_{1,W}(x_1) = -2\eta_{1,W}(x_0) + \max_{x\geq 0} 2\eta_{1,W}(x).$$

Since $-\eta_{1,W}(x_0) = -\sqrt{2/\pi}(\log Wx)x^2 e^{-x^2/2} \leq -\sqrt{2/\pi}(\log Wx)x^2$ and since $((\log Wx)x^2)' = -x(1 + 2\log Wx)$, which is 0 for $x = 1/\sqrt{e}W$, we see that

$$-2\eta_{1,W}(x_0) \leq 2\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\cdot\frac{1}{eW^2} = \frac{\sqrt{2/\pi}}{eW^2}.$$

Recall that $\eta_{1,W}(x) = \sqrt{2/\pi}((\log W)x^2e^{-x^2/2} + (\log x)x^2e^{-x^2/2})$ and that the maximum of $x \mapsto \sqrt{2/\pi}\cdot x^2e^{-x^2/2}$ equals $\sqrt{2/\pi}\cdot 2/e$. We bound the maximum of $x \mapsto (\log x)x^2e^{-x^2/2}$ on $[\sqrt{2},\infty)$ by the bisection method (applied to the interval $(1.41, 5)$, with 30 iterations). We obtain that the bound on $|\eta'_{1,W}|_1$ in (A.16) holds.

Lastly, let us bound $|\eta'_{1,W}|_1/|\eta_{1,W}|_1$, using the bounds we have just proved. Since $0.608238/((4/e)\sqrt{2/\pi}) = 0.51804\ldots > 0.36481\ldots = 1-(\gamma+\log 2)/2$, the function

$$W \mapsto \frac{\sqrt{\frac{2}{\pi}}\left(\frac{4}{e}\log W + \frac{1}{eW^2}\right) + 0.608238}{\log W + \left(1 - \frac{\gamma+\log 2}{2}\right) - \frac{\sqrt{8/\pi}}{9W^3}}$$

is decreasing for $W \geq 1$. Thus, its value for $W \geq 136$ is at most its value at 5, viz., $1.208193\ldots$. Note, finally, that $(3/4)\cdot e^{0.50136} - 3/100 = 1.208223\ldots > 1.208193\ldots$. □

A.2 THE FUNCTIONS $\eta_\circ$, η_+, h AND h_H

A.2.1 Definitions and basic properties

Define

$$h : x \mapsto \begin{cases} x^2(2-x)^3e^{x-1/2} & \text{if } 0 < x \leq 2,\\ 0 & \text{otherwise}\end{cases} \tag{A.20}$$

We will work with an approximation η_+ to the function $\eta_\circ : (0,\infty)\to\mathbb{R}$ given by

$$\eta_\circ(x) = h(x)\eta_\diamond(x) = \begin{cases} x^3(2-x)^3e^{-(x-1)^2/2} & \text{for } 0 < x \leq 2,\\ 0 & \text{otherwise,}\end{cases} \tag{A.21}$$

where $\eta_\diamond : (0,\infty)\to\mathbb{R}$ is defined by

$$\eta_\diamond(x) = xe^{-x^2/2}. \tag{A.22}$$

The approximation η_+ is defined by

$$\eta_+(x) = h_H(x)xe^{-x^2/2}, \tag{A.23}$$

where

$$h_H(x) = \frac{1}{2\pi i}\int_{-iH}^{iH}(Mh)(s)x^{-s}ds \tag{A.24}$$

and $H > 0$ will be set later.

It is easy to see that $h_H(x)$ is continuous, and in fact in $C^\infty((0,\infty))$, since it is defined in (A.24) as an integral on a compact segment, with an integrand depending smoothly on x. Thus, η_+ is C^∞.

It is also clear from (A.24) that $h_H(x)$ is bounded, and so $\eta_+(x)$ is of fast decay as $x \to \infty$ and decays at least as fast as x for $x \to 0^+$. In the same way, (A.24) implies that $h'_H(x)$ is bounded (as are all higher derivatives), and so $\eta'_+(x)$ is of fast decay as $x \to \infty$ as well as being bounded.

Notice, however, that h_H is not in L^1 with respect to dx/x (or dx), because the integral in (A.24) has sharp cutoffs at iH and $-iH$: integration by parts shows that the dominant term of $h_H(x)$ as $x \to \infty$ is $c(x)/\log x$, where $c(x) = \Re(Mh(iH) \cdot x^{-iH})/\pi$ oscillates between $-|Mh(iH)|/\pi$ and $|Mh(iH)|/\pi$. Thus, we will abstain from writing Mh_H, say, even though is fair enough to think of Mh_H as the truncation of Mh at iH and $-iH$. (We could justifying writing Mh_H by developing an L^2 theory for the Mellin transform, in analogy to the L^2 theory of the Fourier transform, but we will not need to.)

Lemma A.7. *Let h and h_H be as in (A.20) and (A.24). Then*

$$h_H(x) = h *_M \frac{\sin(H \log x)}{\pi \log x}. \tag{A.25}$$

Proof. Clearly, for $I = [-H, H]$,

$$h_H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (Mh)(it) 1_I(t) e^{-it \log x} dt,$$

which equals the value of the Fourier transform of $Mh(it) \cdot 1_I(t)$ at $\log x/2\pi$. Since $1_{[-H,H]}$ is in L^2 and both $Mh(it)$ and its Fourier transform $x \mapsto 2\pi h(e^{2\pi x})$ are in L^1, we may apply (2.12), and obtain that

$$\begin{aligned} h_H(x) &= \frac{1}{2\pi} \left(2\pi h(e^{2\pi x}) * \widehat{1}_I\right) \left(\frac{\log x}{2\pi}\right) \\ &= \int_{-\infty}^{\infty} h(e^{2\pi(\frac{\log x}{2\pi} - u)}) \widehat{1}_I(u) du = \frac{1}{2\pi} \int_0^{\infty} h\left(\frac{x}{w}\right) \widehat{1}_I \left(\frac{\log w}{2\pi}\right) \frac{dw}{w} \end{aligned}$$

almost everywhere. Now, $\widehat{1}_I(t) = \sin(2\pi Ht)/\pi t$, and so (A.25) holds almost everywhere.

We know that $h_H(x)$ is continuous, and the right side of (A.25) is continuous as well (since $h(e^{2\pi x})$ is a function of fast decay, and $\widehat{1}_H$ is uniformly continuous). Therefore, equation (A.25) actually holds everywhere. □

Figures A.1–A.3 show h_H and η_+ for different values of H. The plot for $H = 100$ is indistinguishable from that of $\eta_\circ$. Figures A.3 and A.4 shows the range $x \geq 2$, where $h(t)$ and $\eta_\circ$ are identically zero, for higher H; notice the scale.

Lemma A.8. *Let η_+ be as (A.23). Then $M\eta_+$ is holomorphic for $\Re s > -1$.*

Proof. Let h, h_H and $\eta_\diamond$ be as in (A.20), (A.24) and (A.22). Since $t \to Mh(it)$ is in L^1, so is its truncation to $[-H, H]$, and hence h_H is in L^∞. Therefore, $\eta_+(x)x^{\sigma-1} = h_H(x) \cdot \eta_\diamond(x)x^{\sigma-1}$ is in L^1 for any σ for which $\eta_\diamond(x)x^{\sigma-1}$ is in L^1; that is, the strip

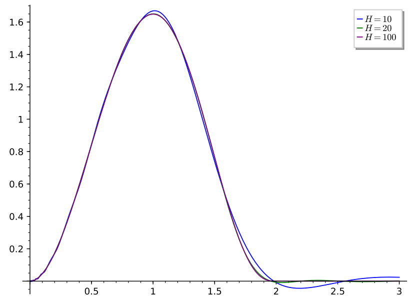

Figure A.1: $h_H(x)$ on $[0, 3]$

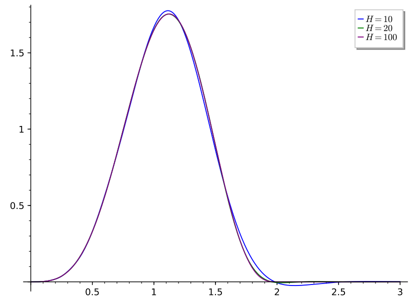

Figure A.2: $\eta_+(x)$ on $[0, 3]$

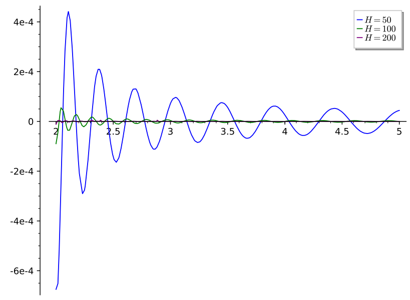

Figure A.3: $h_H(x)$ on $[2, 5]$

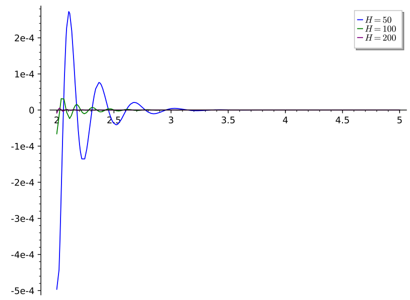

Figure A.4: $\eta_+(x)$ on $[2, 5]$

of holomorphy of $M\eta_+$ contains that of the Mellin transform $M\eta_\diamondsuit$ of $\eta_\diamondsuit(x)$. It is easy to see that the strip of holomorphy of $M\eta_\diamondsuit$ is $\{s : \Re s > -1\}$. □

Lemma A.9. *For $\delta \in \mathbb{R}$, let $\eta_{\diamondsuit,\delta}(x) = \eta_\diamondsuit(x)e(\delta x)$ and $\eta_{+,\delta}(x) = \eta_+ e(\delta x)$, where $\eta_\diamond$ and η_+ are as in (A.22) and (A.23). Then*

$$M\eta_{+,\delta}(s) = \frac{1}{2\pi i}\int_{-iH}^{iH} Mh(z)M\eta_{\diamondsuit,\delta}(s-z)dz \tag{A.26}$$

for $\Re s > -1$.

Proof. By (2.32), $\eta_{+,\delta}$ equals the inverse Mellin transform of

$$\frac{1}{2\pi i}\int_{-iH}^{iH} Mh(z)M\eta_{\diamondsuit,\delta}(s-z)dz \tag{A.27}$$

for $\Re s > -1$. The function in (A.27) is in L^1 on vertical lines $\sigma + i\mathbb{R}$, $\sigma > -1$. Since $\eta_{+,\delta}(x)x^{\sigma-1}$ is in L^1 for $\sigma > -1$, it follows from Fourier inversion (applied to the function defined by (A.27)), together with a change of variables, that $M\eta_+$ equals the function in (A.27) for $\Re s > -1$. □

Part of our work from now on will consist in expressing norms of h_H and η_+ in terms of norms of h, $\eta_\circ$ and Mh.

A.2.2 The Mellin transform Mh

Consider the Mellin transform Mh of the function h. By symbolic integration,

$$Mh(s) = e^{-\pi i s - \frac{1}{2}}(8\gamma(s+2,-2) + 12\gamma(s+3,-2) + 6\gamma(s+4,-2) + \gamma(s+5,-2)), \tag{A.28}$$

where $\gamma(s,x)$ is the lower incomplete Gamma function, as in §4.2.2. Unfortunately, (A.28) leads to catastrophic cancellation. In ARB, or double-precision in general, the error term is already large for $t = 8$, and the result becomes useless for $t \geq 12$ or so. Thus, we are better off deriving our own series development for $Mh(s)$, either using (4.12) or, as we shall do, proceeding as in the derivation of (4.12).

Lemma A.10. *Let $h : (0,\infty) \to \mathbb{R}$ be defined as in (A.20). Then, for any $s \in \mathbb{C}$ other than $0, -1, -2, \ldots$ and any $l \geq \max(8, -\Re s + 3)$,*

$$\begin{aligned} Mh(s) = e^{3/2} 2^s \sum_{k=3}^{l-1} (-1)^{k+1} 2^k \frac{k(k-1)(k-2)(k^2-3k+4)}{s(s+1)\cdots(s+k)} \\ + O^*\left(\frac{l(l-1)(l-2)(l^2-3l+4)}{|s||s+1|\cdots|s+l|} \cdot \frac{e^{3/2} 2^{\Re s + l}}{1-\rho_l(s)}\right), \end{aligned} \tag{A.29}$$

where $r_l(s) = 3.96/|s+l+1|$. Moreover, for any s such that $|s+4| \geq 100$,

$$|Mh(s)| \leq \frac{1025 \cdot 2^{\Re s}}{|s||s+1|\cdots|s+3|}. \tag{A.30}$$

Proof. Write $P(t) = t^2(2-t)^3$. Since $P(2) = P'(2) = P''(2)$, integration by parts yields

$$\begin{aligned} e^{-3/2} Mh(s) &= \int_0^\infty P(t) e^{t-2} t^{s-1} dt = -\frac{1}{s}\int_0^\infty (P(t)e^{t-2})' t^s dt \\ &= -\frac{1}{s(s+1)(s+2)} \int_0^\infty (P(t)e^{t-2})^{(3)} t^{s+2} dt \\ &= \sum_{k=3}^\infty (-1)^k \frac{(P(t)e^{t-2})^{(k)}(2) \cdot 2^{s+k}}{s(s+1)\cdots(s+k)}. \end{aligned}$$

Because P is of degree 5,

$$\begin{aligned} (P(t)e^{t-2})^{(k)}(2) = \sum_{j=0}^5 \binom{k}{j} P^{(j)}(2) e^{2-2} = -24\binom{k}{3} - 96\binom{k}{4} - 120\binom{k}{5} \\ = -k(k-1)(k-2)(k^2 - 3k + 4), \end{aligned}$$

and so

$$Mh(s) = e^{3/2} 2^s \sum_{k=3}^{\infty} (-1)^{k+1} 2^k \frac{k(k-1)(k-2)(k^2-3\,k+4)}{s(s+1)\cdots(s+k)}. \tag{A.31}$$

It is clear that, for $k > 3$, the ratio of the $(k+1)$th to the kth term in (A.31) is at most $\rho_k(s) = 2(k+1)(k^2-k+2)/((k-2)(k^2-3\,k+4)|s+k+1|)$, which is < 1 for $k \geq 8$, $|s+k+1| \geq 4$, or for $k = 3$ and $|s+k+1| > 16$. Hence, equation (A.29) holds with $\rho_l(s)$ instead of $r_l(s)$ for any $l \geq \max(8, -\Re s + 3)$, $s \neq 0, -1, -2, \ldots$. It is easy to verify that $r_l(s) \geq \rho_l(s)$ for all $l \geq 8$, and so (A.29) holds as it stands. It also holds, once again with $\rho_l(s)$ instead of $r_l(s)$, for $l = 3$ and s such that $|s+4| \geq 16$. Hence

$$|Mh(s)| \leq \frac{24 \cdot 8e^{3/2}}{1-\rho_3(s)} \cdot \frac{2^{\Re s}}{|s||s+1|\cdots|s+3|} \leq \frac{1025 \cdot 2^{\Re s}}{|s||s+1|\cdots|s+3|} \tag{A.32}$$

for s such that $|s+4| \geq 100$. □

The following lemma is somewhat crude. It will be used only to bound error terms in numerical integration.

Lemma A.11. *Let $h : (0, \infty) \to \mathbb{R}$ be defined as in (A.20). Then, for any $s \in \mathbb{C}$ with $\Re s \geq -1/2$, $|s+4| \geq 5$,*

$$\left|\frac{d^2}{ds^2} Mh(s)\right| \leq \frac{1.22 \cdot 10^7 \cdot 2^{\Re s}}{|s|^3|s+1||s+2||s+3|}. \tag{A.33}$$

Moreover, for any $s \in i\mathbb{R}$,

$$\left|\frac{d^2}{ds^2} Mh(s)\right| \leq \int_0^2 x(2-x)^3 e^{x-1/2} (\log x)^2 x^s dx \leq 1.0431. \tag{A.34}$$

Proof. Differentiating equation (A.29) and taking $l \to \infty$, we see that

$$\begin{aligned}\left|\frac{d^2}{ds^2} Mh(s)\right| &\leq e^{3/2} 2^{\Re s} \sum_{k=3}^{\infty} 2^k \frac{(k+1)^2 k(k-1)(k-2)(k^2-3k+4)}{|s|^3|s+1|\cdots|s+k|} \\ &\leq \frac{e^{3/2} 2^{3+\Re s} \cdot \frac{105}{8}}{|s|^3|s+1||s+2||s+3|} \sum_{j=0}^{\infty} \frac{2^j}{|s+4|^j} (j+1)(j+2)\ldots(j+6).\end{aligned}$$

Since, for $r \geq 0$,

$$\sum_{j=0}^{\infty} (j+1)\ldots(j+r) x^j = \left(\frac{1}{1-x}\right)^{(r)} = \frac{r!}{(1-x)^{r+1}},$$

we conclude that (A.33) holds. More precisely,

$$\left|\frac{d^2}{ds^2} Mh(s)\right| \leq \frac{c \cdot 2^{\Re s}}{|s|^3|s+1||s+2||s+3|},$$

where $c = \frac{105}{8} e^{3/2} 2^3 6!/(1-2/|s+4|)^7 = 1.2103326 \cdot 10^7$.

To prove (A.34), simply proceed from the definition of Mh, and then use (rigorous) numerical integration. □

We can use (A.29) to compute $Mh(it)$ (though using (A.28) is preferable for $|t|$ small) and (A.30) to bound $Mh(\sigma + it)$ for σ fixed and $|t|$ large. We use midpoint integration, as in (4.2), with the bounds from Lemma A.11 as an input. We obtain via ARB, using (A.28) and Lemma A.11, that

$$\int_0^1 |Mh(it)| dt \leq 1.9054814,$$

and, via D. Platt's `int_double` package, together with (A.29) and Lemma A.11,

$$\int_1^{5000} |Mh(it)| dt \leq 4.09387319.$$

Hence, by (A.30),

$$|Mh(it)|_1 \leq 2(1.9054814 + 4.09387319) + O^*\left(\int_{5000}^{\infty} \frac{2050}{t^4} dt\right) \leq 11.99871. \tag{A.35}$$

A.3 NORMS OF $\eta_\circ$ AND η_+

A.3.1 The difference $\eta_+ - \eta_\circ$ in L^2 norm.

We wish to estimate the distance in L^2 norm between $\eta_\circ$ and its approximation η_+.

Lemma A.12. *Let $\eta_\circ, \eta_+ : (0,\infty) \to \mathbb{R}$ be as in (A.21) and (A.23), with $H > 0$. Then*

$$\int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t} = \frac{1}{\pi} \int_H^\infty |Mh(it)|^2 dt. \tag{A.36}$$

Proof. The inverse Mellin transform is an isometry for the same reason that the Mellin transform is: both are Fourier transforms under a change of variables. Recall that h_H was defined in (A.24) as the inverse Mellin transform of Mh on the imaginary axis truncated by $|\Im s| \leq H$. Hence $h(t) - h_H(t)$ is the inverse Mellin transform of Mh on the imaginary axis truncated by $|\Im s| > H$. Then we get (A.36) by isometry. □

Lemma A.13. *Let $\eta_\circ, \eta_+ : (0,\infty) \to \mathbb{R}$ be as in (A.21) and (A.23), with $H \geq 100$. Then*

$$|\eta_+ - \eta_\circ|_2 \leq \frac{140}{H^{7/2}}, \quad |(\eta_+ - \eta_\circ)(x) \log x| \leq \frac{61.5}{H^{7/2}}.$$

Proof. By (A.21), (A.23) and Lemma A.12,

$$\begin{aligned} |\eta_+ - \eta_\circ|_2^2 &= \int_0^\infty \left| h_H(t) t e^{-t^2/2} - h(t) t e^{-t^2/2} \right|^2 dt \\ &\leq \left(\max_{t \geq 0} e^{-t^2} t^3 \right) \cdot \int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t} \\ &= \left(\max_{t \geq 0} e^{-t^2} t^3 \right) \cdot \frac{1}{\pi} \int_H^\infty |Mh(it)|^2 dt. \end{aligned} \tag{A.37}$$

The maximum $\max_{t \geq 0} t^3 e^{-t^2}$ is $(3/2)^{3/2} e^{-3/2}$.

By (A.30), under the assumption that $H \geq 100$,

$$\int_H^\infty |Mh(it)|^2 dt \leq \int_H^\infty \frac{1025^2}{t^8} dt \leq \frac{1025^2}{7H^7}. \tag{A.38}$$

We conclude that

$$|\eta_+ - \eta_\circ|_2 \leq \frac{1025}{\sqrt{7\pi}} \left(\frac{3}{2e} \right)^{3/4} \cdot \frac{1}{H^{7/2}} \leq \frac{140}{H^{7/2}}. \tag{A.39}$$

We could do better by computing the difference between h_+ and $h_\circ$ directly for given H, using (A.25), but we will not bother to.

We must now bound

$$\left| \int_0^\infty (\eta_+(t) - \eta_\circ(t))^2 (\log t)^2 \, dt \right|.$$

This quantity is at most

$$\left(\max_{t \geq 0} e^{-t^2} t^3 (\log t)^2 \right) \cdot \int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t}.$$

By the bisection method with 23 iterations (see §4.1),

$$\max_{t \geq 0} e^{-t^2} t^3 (\log t)^2 = \max_{t \in [10^{-6}, 10]} e^{-t^2} t^3 (-\log t) = 0.07892\ldots.$$

Hence, by (A.36) and (A.38), again under the assumption that $H \geq 100$,

$$\int_0^\infty (\eta_+(t) - \eta_\circ(t))^2 (\log t)^2 dt \leq 0.078925 \cdot \frac{1025^2}{7\pi H^7} \leq \left(\frac{61.5}{H^{7/2}} \right)^2. \tag{A.40}$$

□

A.3.2 Norms involving $\eta_\circ$ and η_+

Let us first prove a general result.

Lemma A.14. *Let $\eta_+ : (0,\infty) \to \mathbb{R}$ be as in (A.23), with $H > 0$ arbitrary. Then, for any $\sigma \geq -3/2$, $\eta_+(t)t^\sigma$ is in L^2, and, for any $\sigma > -2$, $\eta_+(t)t^\sigma$ is in L^1.*

Proof. Just as in the proof of Lemma A.13, by (A.21), (A.23) and Lemma A.12,

$$\begin{aligned} |(\eta_+(t) - \eta_\circ(t))t^\sigma|_2^2 &= \int_0^\infty \left| h_H(t)te^{-t^2/2} - h(t)te^{-t^2/2}\right|^2 t^{2\sigma} dt \\ &\leq \left(\max_{t\geq 0} e^{-t^2} t^{2\sigma+3}\right) \cdot \int_0^\infty |h_H(t) - h(t)|^2 \frac{dt}{t} \qquad \text{(A.41)} \\ &= \left(\max_{t\geq 0} e^{-t^2} t^{2\sigma+3}\right) \cdot \frac{1}{\pi} \int_H^\infty |Mh(it)|^2 dt. \end{aligned}$$

Here

$$\frac{1}{\pi}\int_H^\infty |Mh(it)|^2 dt \leq \frac{1}{2\pi}\int_{-\infty}^\infty |Mh(it)|^2 dt = \int_0^\infty |h(t)|^2 \frac{dt}{t} < \infty,$$

and, if $\sigma > -3/2$, $\max_{t\geq 0} e^{-t^2} t^{2\sigma+3}$ is also finite. Then $\eta_+(t)t^\sigma$ is in L^2.

By Cauchy-Schwarz, for $\sigma > -2$,

$$\int_0^1 |\eta_+(t)t^\sigma| dt + \int_1^\infty |\eta_+(t)t^\sigma| dt$$

is at most

$$\sqrt{\int_0^1 |\eta_+(t)t^{-3/2}|^2 dt}\sqrt{\int_0^1 t^{2\sigma+3} dt} + \sqrt{\int_1^\infty |\eta_+(t)t^{\sigma+1}|^2 dt}\sqrt{\int_0^1 t^{-2} dt}.$$

Since $|\eta_+(t)t^{-3/2}|_2, |\eta_+(t)t^{\sigma+1}|_2 < \infty$, it follows that $|\eta_+(t)t^\sigma|_1 < \infty$. □

Let us now bound some L^1- and L^2-norms involving η_+. First, by rigorous numerical integration via ARB,

$$|\eta_\circ|_2 = 0.800128, \qquad |\eta_\circ(t)\log t|_2 = 0.213868. \qquad \text{(A.42)}$$

(Integrating symbolically is also an option in the first case.) Hence, by Lemma A.13,

$$|\eta_+|_2 \leq |\eta_\circ|_2 + |\eta_+ - \eta_\circ|_2 \leq 0.800129 + \frac{140}{H^{7/2}} \qquad \text{(A.43)}$$

and

$$|\eta_+(t)\log t|_2 \leq |\eta_\circ(t)\log t|_2 + |(\eta_+ - \eta_\circ)(t)\log t|_2 \leq 0.213869 + \frac{61.5}{H^{7/2}}. \qquad \text{(A.44)}$$

In general, for any $f : (0,\infty) \to \mathbb{C}$ such that $\eta_\circ(t)f(t) \in L^1$,

$$\begin{aligned} |\eta_+(t)f(t)|_1 &= |\eta_\circ(t)f(t)|_1 + O^*\left(|(\eta_+(t) - \eta_\circ(t))f(t)|_1\right) \\ &= |\eta_\circ(t)f(t)|_1 + O^*\left(|(h_H(t) - h(t))te^{-t^2/2}f(t)|_1\right) \end{aligned}$$

and, by Cauchy-Schwarz, (A.36) and (A.38),

$$\begin{aligned}|(h_H(t)-h(t))te^{-t^2/2}f(t)|_1 &\leq \left|\frac{h_H(t)-h(t)}{\sqrt{t}}\right|_2 \cdot |t^{3/2}e^{-t^2/2}f(t)|_2\\ &\leq \frac{1025}{\sqrt{7\pi}H^{7/2}}|t^{3/2}e^{-t^2/2}f(t)|_2.\end{aligned}$$

For instance, by numerical integration via ARB,

$$|\eta_\circ(t)t^{-1/2}|_1 = 0.909875\dots, \quad |\eta_\circ(t)\log t|_1 = 0.245205\dots.$$

and, by symbolic integration,

$$|t^{3/2}e^{-t^2/2}t^{-1/2}|_2 = \frac{\pi^{1/4}}{2}, \quad |t^{3/2}e^{-t^2/2}\log t|_2 = \sqrt{\frac{\pi^2}{48}+\frac{\gamma^2}{8}-\frac{\gamma}{4}}.$$

Thus

$$|\eta_+(t)t^{-1/2}|_1 \leq 0.909876 + \frac{146}{H^{7/2}}, \quad |\eta_+(t)\log t|_1 \leq 0.245206 + \frac{71}{H^{7/2}}. \tag{A.45}$$

A.3.3 Norms involving η_+'

Lemma A.15. *Let $\eta_+ : (0,\infty) \to \mathbb{R}$ be defined as in (A.23), with $H \geq 100$. Then*

$$|(\eta_+ - \eta_\circ)'|_2 = O^*\left(\frac{109}{H^2}\right),$$

and so

$$|\eta_+'|_2 = |\eta_\circ'|_2 + O^*\left(\frac{109}{H^2}\right) = 1.65454\ldots + O^*\left(\frac{109}{H^2}\right).$$

As will become clear, we could provide a better error term, one inversely proportional to $H^{5/2}$, but we will not need to, as we will use Lemma A.15 only to bound rather minor error terms.

Proof. We wish to estimate $|\eta_+'|_2$. Clearly

$$|\eta_+'|_2 = |\eta_\circ'|_2 + O^*(|(\eta_+ - \eta_\circ)'|_2).$$

By symbolic integration, $|\eta_\circ'|_2 = 1.65454\ldots$.

Since η_+' and $\eta_\circ'$ are bounded and both $\eta_+(x)$ and $\eta_\circ$ decay at least as fast as x for $x \to 0^+$, we may apply the transformation rule $M(f')(s) = -(s-1)\cdot Mf(s-1)$

from (2.33) to $f(x) = (\eta_+ - \eta_\circ)(x)$ for any $\sigma > -1$. Since $\eta_+ - \eta_\circ$ is in L^2, we can apply the Mellin transform as an isometry for $\sigma = 1/2$, and obtain

$$\begin{aligned} |(\eta_+ - \eta_\circ)'|_2^2 &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |M((\eta_+ - \eta_\circ)')(s)|^2 \, ds \\ &= \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |s \cdot M(\eta_+ - \eta_\circ)(s)|^2 \, ds. \end{aligned} \tag{A.46}$$

Recall that $\eta_+(t) = h_H(t)\eta_\diamond(t)$, where $\eta_\diamond(t) = te^{-t^2/2}$. Since $\eta_\diamond(t)$ is the inverse Mellin transform of $M\eta_\diamond$ on any line $\Re s = \sigma$ with $\sigma > -1$, we see from (2.32) that

$$\begin{aligned} M(\eta_+ - \eta_\circ)(s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} M(h - h_H)(ir) M\eta_\diamond(s - ir) dr \\ &= \frac{1}{2\pi} \int_{|r|>H} Mh(ir) M\eta_\diamond(s - ir) dr \end{aligned} \tag{A.47}$$

for $\Re s > -1$.

Recall from (A.30) that $|Mh(ir)| \leq 1025/r^4$. By a substitution $t = x^2/2$,

$$M\eta_\diamond(s) = \int_0^\infty e^{-x^2/2} x^s dx = \int_0^\infty e^{-t} (2t)^{\frac{s-1}{2}} dt = 2^{\frac{s-1}{2}} \Gamma\left(\frac{s+1}{2}\right).$$

We could now use the decay properties of Γ to obtain a bound on $M(\eta_+ - \eta_\circ)(s)$. In the interests of a quick and clean solution, let us proceed instead as follows. In general, for $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$,

$$\begin{aligned} |f * g|_2^2 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y) g(x - y) dy \right|^2 dx \\ &= \int_{-\infty}^{\infty} |f(y)| dy \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)| |g(x - y)|^2 dy dx = |f|_1^2 |g|_2^2 \end{aligned} \tag{A.48}$$

by Cauchy-Schwarz. (This is a special case of Young's inequality.) By the easy inequality $|a + b|^2 \leq 2|a|^2 + 2|b|^2$,

$$\begin{aligned} |(f * g)(x)x|_2^2 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} |f(y)| |g(x - y)| \cdot (y + (x - y)) dy \right|^2 dx \\ &= 2 \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} |f(y)y| |g(x - y)| dy \right|^2 dx \\ &+ 2 \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} |f(y)| |g(x - y)| |x - y| dy \right|^2 dx, \end{aligned} \tag{A.49}$$

and so, by (A.48)

$$|(f * g)(x)x|_2^2 \leq 2(|f(x)x|_1^2 |g|_2^2 + |f|^2 |g(x)x|_2^2). \tag{A.50}$$

Applying (A.48) and (A.50) to (A.46), we see that

$$|(\eta_+ - \eta_\circ)'|_2^2 \leq \frac{1}{(2\pi)^3}\left(\frac{1}{4}|f|_1^2|g|_2^2 + 2(|f(x)x|_1^2|g|_2^2 + |f|^2|g(x)x|_2^2)\right),$$

where $f(x) = Mh(ix)$ for $|x| > H$, $f(x) = 0$ for $|x| \leq H$, and $g(x) = M\eta_\diamondsuit(-1/2 + ix)$.

By Plancherel,

$$|g|_2^2 = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |M\eta_\diamondsuit(s)|^2 ds = 2\pi|\eta_\diamondsuit(x)/x|_2^2 = 2\pi|e^{-x^2/2}|_2^2 = \pi^{3/2}$$

and, since $sM\eta_\diamondsuit(s) = -(M(x\eta_\diamondsuit'(x)))(s)$,

$$|g(x)x|_2^2 + \frac{1}{4}|g(x)|_2^2 = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |(M(x\eta_\diamondsuit'(x)))(s)|^2 ds = 2\pi|x\eta_\diamondsuit'(x)|_2^2 = \frac{7}{8}\pi^{3/2}.$$

Since $|f(x)| \leq 1025/r^4$, we see that $|f|_1 \leq 1025/3H^3$ and $|f(x)x|_1 \leq 1025/2H^2$. Hence

$$|(\eta_+ - \eta_\circ)'|_2^2 \leq \frac{1}{4\pi^3}\left(\frac{1025^2}{4H^4}\pi^{3/2} + \frac{1025^2}{9H^6}\cdot\frac{7\pi^{\frac{3}{2}}}{8}\right) \leq \frac{11792.43}{H^4} + \frac{4586}{H^6} \leq \frac{11793}{H^4}$$

under the assumption $H \geq 100$, and so $|(\eta_+ - \eta_\circ)'|_2 \leq 109/H^2$. □

Lemma A.16. *Let $\eta_+ : (0,\infty) \to \mathbb{R}$ be defined as in (A.23), with $H \geq 100$. Then*

$$|(\eta_+ - \eta_\circ)'(x)x|_2 = O^*\left(\frac{77}{H^2}\right),$$

Proof. Proceeding just as in the proof of Lemma A.15, we obtain that

$$|(\eta_+ - \eta_\circ)'(x)x|_2^2 \leq \frac{1}{(2\pi)^3}\left(\frac{1}{4}|f|_1^2|g|_2^2 + 2(|f(x)x|_1^2|g|_2^2 + |f|^2|g(x)x|_2^2)\right),$$

where $f(x) = Mh(ix)$ for $|x| > H$, $f(x) = 0$ for $|x| \leq H$, and $g(x) = M\eta_\diamondsuit(1/2 + ix)$.

By Plancherel,

$$|g|_2^2 = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |M\eta_\diamondsuit(s)|^2 ds = 2\pi|\eta_\diamondsuit(x)|_2^2 = 2\pi|xe^{-x^2/2}|_2^2 = \frac{\pi^{3/2}}{2}$$

and, since $sM\eta_\diamondsuit(s) = -(M(x\eta_\diamondsuit'(x)))(s)$,

$$|g(x)x|_2^2 + \frac{1}{4}|g(x)|_2^2 = \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} |(M(x\eta_\diamondsuit'(x)))(s)|^2 ds = 2\pi|x^2\eta_\diamondsuit'(x)|_2^2 = \frac{33}{16}\pi^{\frac{3}{2}}.$$

Since $|f(x)| \leq 1025/r^4$, we get that

$$|(\eta_+ - \eta_\circ)'(x)x|_2^2 \leq \frac{1}{4\pi^3}\left(\frac{1025^2}{4H^4}\frac{\pi^{3/2}}{2} + \frac{1025^2}{9H^6}\cdot\frac{33\pi^{\frac{3}{2}}}{16}\right) \leq \frac{5896.3}{H^4} + \frac{10810}{H^6} \leq \frac{5898}{H^4}$$

under the assumption $H \geq 100$, and so $|(\eta_+ - \eta_\circ)'|_2 \leq 77/H^2$. □

Lemma A.17. *Let $\eta_+ : (0,\infty) \to \mathbb{R}$ be defined as in (A.23), with $H > 0$. Then, for every $\sigma > -1/2$, $\eta_+'(x)x^\sigma$ is in L^2, and, for every $\sigma > -1$, $\eta_+'(x)x^\sigma$ is in L^1.*

Proof. Assume $\sigma > -1/2$. Yet again, we proceed as in the proof of Lemma A.15, and get that

$$|(\eta_+ - \eta_\circ)'(x)x^\sigma|_2^2 \leq \frac{1}{(2\pi)^3}\left(\frac{1}{4}|f|_1^2|g|_2^2 + 2(|f(x)x|_1^2|g|_2^2 + |f|_1^2|g(x)x|_2^2)\right),$$

where $f(x) = Mh(ix)$ for $|x| > H$, $f(x) = 0$ for $|x| \leq H$, and $g(x) = M\eta_\diamondsuit(-1/2 + \sigma + ix)$. Much as usual, f and $f(x)x$ are in L^1 by (A.30), and g and $g(x)x$ are in L^2 because $\eta_\diamondsuit(x)x^{\sigma-1}$ and $\eta_\diamondsuit'(x)x^\sigma$ are in L^2. Hence $|(\eta_+ - \eta_\circ)'(x)x^\sigma|_2 < \infty$, and so $\eta_+'(x)x^\sigma$ is in L^2.

We deduce that $\eta_+'(x)x^\sigma$ is in L^1 using Cauchy-Schwarz in the same way as in the proof of Lemma A.14. □

Lemma A.18. *Let $\eta_+ : (0,\infty) \to \mathbb{R}$ be defined as in (A.23), with $H \geq 100$. Then*

$$|\eta_+' \log t|_1 = 0.99637\ldots + O^*\left(\frac{336}{H^2}\right).$$

Proof. Since $\eta_\circ$ is increasing for $t \in [0,1]$ and decreasing for $t \in [1,2]$,

$$|\eta_\circ' \log t|_1 = \int_0^1 \eta_\circ'(t)(-\log t)dt + \int_1^2 (-\eta_\circ'(t))\log t dt = \int_0^2 \frac{\eta_\circ(t)}{t}dt = 0.99637\ldots.$$

By Cauchy-Schwarz, for any $\rho > 0$,

$$\begin{aligned}|(\eta_+ - \eta_\circ)'(t)\log t|_1 &\leq |(\eta_+ - \eta_\circ)'(t)(\rho t + 1)|_2 \cdot \left|\frac{\log t}{\rho t + 1}\right|_2 \\ &= (|(\eta_+ - \eta_\circ)'|_2 + |(\eta_+ - \eta_\circ)'(t)t|_2)\cdot\sqrt{\frac{\pi^2}{3\rho} + \frac{(\log\rho)^2}{\rho}}.\end{aligned}$$

We apply Lemmas A.15 and A.16, set $\rho = 7/6$, and obtain that

$$|(\eta_+ - \eta_\circ)'(t)\log t|_1 \leq \frac{314}{H^2}.$$

□

A.3.4 The L^∞-norm of η_+

Lemma A.19. *Let $\eta_+ : (0,\infty) \to \mathbb{R}$ be defined as in (A.23), with $H \geq 100$. Then*

$$|\eta_+|_\infty = 1 + O^*\left(\frac{66}{H^2}\right).$$

Proof. Recall that $\eta_+(x) = h_H(x)\eta_\diamondsuit(x)$, where $\eta_\diamondsuit(x) = xe^{-x^2/2}$. Clearly

$$|\eta_+|_\infty = |\eta_\circ|_\infty + O^*\left(|\eta_+ - \eta_\circ|_\infty\right) = |\eta_\circ|_\infty + O^*\left(|h(x) - h_H(x)|_\infty |\eta_\diamondsuit(x)|_\infty\right). \tag{A.51}$$

Taking derivatives, we easily see that

$$|\eta_\circ|_\infty = \eta_\circ(1) = 1, \qquad |\eta_\diamondsuit(x)|_\infty = 1/\sqrt{e}$$

It remains to bound $|h(x) - h_H(x)|_\infty$. By definition (A.24), for any $x > 0$,

$$h(x) - h_H(x) = \frac{1}{2\pi}\int_{|t|\geq H} (Mh)(it)x^{-it}dt.$$

Hence, by (A.30),

$$|h(x) - h_H(x)| \leq \frac{1}{2\pi}\int_{|t|\geq H} |(Mh)(it)|dt \leq \frac{1}{\pi}\int_H^\infty \frac{1025}{t^4}dt \quad = \frac{1}{3\pi}\frac{1025}{H^3} \leq \frac{108.8}{H^3},$$

and so

$$|\eta_+ - \eta_\circ|_\infty \leq \frac{1}{\sqrt{e}}|h - h_H|_\infty \leq \frac{66}{H^3}.$$

□