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MÉMOIRE D'HABILITATION À DIRIGER DES RECHERCHES

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Approches géométriques de la résolution des singularités et des partitions des nombres entiers

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Articles presented in this memoir

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- **Resolving singularities of reducible curves with one toric morphism** (with Ana Bélen de Felipe and Pedro González Pérez), Preprint.

- **Jet schemes and generating sequences of divisorial valuations in dimension two**, Michigan Math. J., Volume 66, Issue 1 (2017), 155-174.

- **Jet schemes and minimal embedded desingularization of plane branches** (with Monique Lejeune-Jalabert and Ana Reguera), Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., special issue dedicated to Professor H. Hironaka.

In section 3

- **Embedding dimension of the arc space at a stable point and Mather log discrepancy** (with Ana Reguera), Publ. Res. Inst. Math. Sci. 54 (2018), no. 1, 105-139.

- **The motivic zeta function of a space monomial curve with a plane semigroup,** (with Wim Veys and Lena Vos), 32 pages, to appear in Advances in Geometry.

- **Note on the monodromy conjecture for a space monomial curve with a plane semigroup ,** (with Jorge Martin-Morales, Wim Veys and Lena Vos), to appear in C. R. Math. Acad. Sci. Paris, 11 pages.

- **Newton non-degenerate mu-constant deformations admit simultaneous embedded resolutions** (with Maximiliano Leyton -Alvarez and Mark Spivakovsky), Submitted.

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- **Jet schemes of quasi-ordinary surface singularities** (with Helena Cobo), to appear in Nagoya Journal of Mathematics, 88 pages.

- **Jet schemes of normal toric surfaces**, Bull. Soc. Math. France 145 (2017), no. 2, 237-266.

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- **The embedded Nash problem of birational models of rational triple point singularities,** (with B. Karadeniz, C. Plénat, M. Tosun), 35 pages, to appear in Journal of Singularities.

- **Jet schemes and minimal embedded toric resolution of rational double point singularities** (with Camille Plénat), Comm. Algebra 46 (2018), no. 3, 1314-1332.

- **Jet schemes of rational double point surface singularities** Valuation Theory in Interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Sept. 2014, pp: 373-388.

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- **Partitions identities and application to infinite dimensional Groebner basis and viceversa** (with Pooneh Afsharijoo), Arc Schemes and Singularities, World Scientific Publishing, pp. 145-161 (2020). .

- **Arc spaces and Rogers-Ramanujan Identities** (with Clemens Bruschek and Jan Schepers) The Ramanujan Journal, January 2013, Volume 30, Issue 1, pp 9-38.

- **Arc spaces and Rogers-Ramanujan Identities** (with Clemens Bruschek and Jan Schepers), Discrete Mathematics and Theoretical Computer Science Proceedings,FPSAC (2011), 211-220.

Other publications

- **Jet schemes of complex plane branches and equisingularity** Annales de l'Institut Fourier, Tome 61, numéro 6 (2011), p. 2313-2336.

- **Jet schemes of toric surfaces** C. R. Math. Acad. Sci. Paris 349 (2011), no. 9-10, 563-566.

- **Computing Hironaka's invariants: ridge and directrix** (with Jérémy Berthomieu and Pascal Hivert), Contemporary Mathematics, vol. 521, Amer. Math. Soc., Providence, RI, 2010, pp. 9-20.

- **Algebraic Geometry and Number Theory**, Lecture notes of a summer school in Istanbul (**Edited** with Celal Cem Sarioglu, Christophe Soulé, Ayberk Zeytin), Progress in Mathematics, 321. Birkhauser/Springer, Cham, 2017.

CHAPTER II

HDR

1. A preamble and a geometric approach to resolution of singularities

The main objects studied in this memoir are the singularities of an algebraic variety. Roughly speaking, these are the points where the tangent space is not well defined or simply these are the special points, those that one remarks at first when "looking" at the variety. Despite of the fact that the study of a singular "point" is often a local study, singular points affect the global shape and the global "character" of a variety; this can be remarked for instance when studying "some" zeta functions which counts "points" on a curve having an isolated singular point; the shape of such a zeta function, which is a global invariant, is very much influenced by the properties of the singular point. Singularities arise for example when considering hyperplane sections of non-necessarily singular varieties [**152**]; when considering quotients of non-necessarily singular varieties by group actions [**146**]; when considering natural moduli spaces; when compactifying some non necessarily singular varieties or moduli spaces [**84**]...

A guiding problem in singularity theory and in algebraic geometry is the problem of existence and of understanding how to determine a resolution of singularities:

A (abstract) resolution of singularities of an algebraic variety X is a modification (a proper birational morphism: an isomorphism on a open subvariety of Y) $\mu : Y \longrightarrow X$ such that Y is non-singular.

Another more involved version of resolution of singularities is the embedded resolution of a singular variety $X \subset Z$:

An embedded resolution of singularities of an algebraic variety $X \subset Z$ is a a proper birational morphism $\mu : Y \longrightarrow Z$ such that Y is non singular and the strict transform of X by μ is non-singular and transversal to the exceptional locus of μ (the locus where μ is not an isomorphism).

Resolution of singularities has applications that range from Algebraic Geometry to Analysis, Dynamical systems, Differential Geometry, Number theory... In Algebraic Geometry or real and complex analytic geometry, it is used to transform some problems concerning singular spaces to problems concerning non singular spaces; it allows to define invariants of singularities which help in problems of classification of singularities; it also serves as a change of variables when computing integrals. An embedded resolution gives an abstract resolution by looking at its restriction to the strict transform; it contains and gives (much) more information than the information encoded in an abstract resolution. A celebrated theorem proved by Hironaka gives the existence of embedded resolution of singularities of varieties defined over a field of characteristic zero [68]. In positive characteristics, the existence of embedded resolution of singularities is proved only for varieties in dimension 2; in dimension 3, there is a proof of the existence of abstract resolution of singularities in [32]. This is (with local uniformization, which is a "super" local version of resolution of singularities) a very active research subject, see e.g. [7,20,21,32,37,38,65,81,125,143,150].

The traditional approach to resolve singularities is to iterate blowing ups at smooth centers in order to make an invariant drop. This invariant should take values in a discrete ordered set with a smallest element (which detects smoothness). It should not only detect smoothness, but also should be easy to compute so that its behavior can be followed when iterating the blowing ups. The big advantage of this approach is that it has worked in characteristic zero and that it gives an algorithm. But the construction of such a resolution is rarely linked to the deep geometry of the singularities: such a resolution is obtained as a composition of maybe one million blowups which are not related in general to the deep geometry of the singularities of the starting variety.

The unifying theme whose shadow is present almost all over this memoir (except in the chapter on integer partitions which is more algebraic even though it is at least in the beginning motivated by singularities) is a geometric approach to resolution of singularities; an approach which is based on a dialog between the following two themes:

- I A reverse Nash problem.
- II Teissier's conjecture on embedded resolution of singularities with one toric morphism.

Reverse Nash problem

Before reversing it, let us say two words about the (direct) Nash problem [73,120,131]. Given a singular variety X, if X has an abstract resolution of singularities (for example when X is defined over a field of characteristic 0), it has infinitely many other abstract resolutions of singularities; Nash searched in the arc space for the intrinsic information which is common to all these resolutions. Since we are interested in embedded resolution of singularities, we need to also consider the jet schemes which are finite dimensional approximations of the arc space. The arc space and jet schemes of X are respectively the space of germs of formal curves drawn on X and (up to a locally trivial fibration) the spaces of germs of curves drawn in an ambient space containing X and which have a contact (indexed by natural numbers) large enough with X; If $X \subset \mathbf{A}^n$ is an affine variety, one can think of an arc as a $n-$tuple of series

$x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbf{K}[[t]]$ which satisfies the defining equation of X, and of an $m-$jet ($m \in \mathbf{N}$) as $n-$tuple of polynomials $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbf{K}[[t]]/(t^{m+1})$ which satisfies the defining equations of X modulo (t^{m+1}). The center of an arc or of a jet is the point $x(0)$ obtained when setting $t = 0$. We denote by X_m (respectively X_∞) the $m-$th jet scheme (respectively the arc space) of X; for $A \subset X$, we denote by X_m^A, X_∞^A respectively the space of $m-$jets and of arcs whose center belongs to A. Nash defined a correspondence between the irreducible components of the arc space centered at the singular locus of X (that we denote by X_∞^{Sing}) and the essential valuations of X; these latter correspond to the divisorial valuations whose center on every resolutions of singularities is an irreducible component of the exceptional locus (if the reader is not familiar with divisorial valuations, he may think of the irreducible components of the exceptional locus). It is important to keep in mind that the (except for $0-$dimensional schemes) arc space is not noetherian; hence the finiteness of the number of irreducible components of X_∞^{Sing} is not guaranteed (when we have a resolution of singularities, the finiteness was proved by Nash). The Nash correspondence associated to a variety X is not bijective in general [**44**, **75**, **77**] (it is bijective for surface singularities [**41**] and for toric varieties [**75**]); even when it is bijective, it is not possible, in general, to find a resolution of singularities where the only appearing divisorial valuations are the essential ones [**26**]. Still, every divisorial valuation associated with a prime divisor appearing on some resolution of singularities of X corresponds to an irreducible family of arcs traced on X.

What we call the reverse Nash problem is the following question:

Can we construct (or describe) a (abstract or embedded) resolution of singularities of X from its arc space and jet schemes ?

Teissier's conjecture on embedded resolution of singularities with one toric morphism

As we mentioned above, the traditional way to resolve singularities is to blowup a "permissible" center in order to make an adapted invariant drop and hence to define an algorithm which stops after finitely many steps. Such an algorithm exists in characteristic 0, thanks to the existence of a hypersurface of maximal contact (which allows an induction on the dimension of the variety) which does not exist when working in positive characteristics. Teissier asked the following question:

Given a singular variety $X \subset \mathbf{A}^n$, does there exist an embedding $X \subset \mathbf{A}^n \hookrightarrow \mathbf{A}^N, N \geq n$, and a toric structure on $\mathbf{A}^N$ such that $X \subset \mathbf{A}^N$ has an embedded resolution by one toric morphism ?

We will call such an embedding torific. When an embedded resolution of singularities exists, a torific embedding exists [**154**]. If the reader is not familiar with the theory of toric varieties, he can think of a toric morphism as a morphism which is locally defined my monomials: a monomial morphism. In general, it is an open conjecture that the answer is yes. If true, this conjecture would imply the existence of resolution of singularities. Teissier made deep advances in the super local version of this conjecture: the embedded local uniformization problem, which is also an important problem motivating a part of the work in this memoir; we will get to this part later.

Let us explain in more details what we called a geometric approach to resolution of singularities: we would like to use the reverse Nash problem to construct a torific embedding; the word geometry is used since this approach is based on the geometry of the arc space and jet schemes (and sometimes of the space of valuations). Let us consider $X \subset \mathbf{A}^n$; we are interested in finding a torific embedding of X. We divide the problematic into two questions or again two problematics [106]:

(1) Given a divisorial valuation v centered at $0 \in \mathbf{A}^n$, determine whether there exist an embedding $e : \mathbf{A}^n \hookrightarrow \mathbf{A}^N$, (where N depends on v) and a toric proper birational morphism $\mu : X_\Sigma \longrightarrow \mathbf{A}^N$ such that:

$$\begin{array}{ccc} \widetilde{\mathbf{A}^n} & \longrightarrow & X_\Sigma \\ \downarrow & & \downarrow \mu \\ \mathbf{A}^n & \overset{e}{\hookrightarrow} & \mathbf{A}^N \end{array}$$

• X_Σ is a smooth toric variety (i.e., Σ is a fan which is obtained by a regular subdivision of the positive quadrant $\mathbb{R}^N_+$, this quadrant is the cone defining $\mathbf{A}^N$ as a toric variety),
• the strict transform $\widetilde{\mathbf{A}^n}$ of $\mathbf{A}^n$ by μ is smooth,
• there exists a toric divisor $E' \subset X_\Sigma$ which intersects $\widetilde{\mathbf{A}^n}$ transversally along a divisor E,
• the valuation defined by the divisor E is v.

Note that a toric divisor E' centered at the origin 0 of $\mathbf{A}^N = \mathrm{Spec}\mathbf{K}[x_1,\ldots,x_N]$ corresponds to a divisorial valuation v' which is monomial, i.e., there exists a vector $\alpha \in \mathbf{N}^N$ such that $v' = v_\alpha$ where

$$v_\alpha : \mathbf{K}[x_1,\ldots,x_N] \longrightarrow \mathbf{N}$$

is defined by: for $h \in \mathbf{K}[x_1,\ldots,x_N]$,

$$h = \sum_{m=(m_1,\ldots,m_N)} a_m x_1^{m_1} \cdots x_N^{m_N}, \qquad v_\alpha(h) = \min_{\{m | a_m \neq 0\}} < \alpha, m >; \tag{1.1}$$

where $< \alpha, m >$ is the usual scalar product on $\mathbf{R}^N$.
Then one can formulate the conditions above by saying that there exists an embedding $\mathbf{A}^n \hookrightarrow \mathbf{A}^N$ such that v is the trace of a monomial valuation defined on $\mathbf{A}^N$.

(2) Determine a finite number of significant divisorial valuations $v_1, \ldots, v_r$ on $\mathbf{A}^n$ from the geometry of the jet schemes and the arc space of X (this step is to compare with the Nash problem that we mentioned above: very roughly speaking, as the Nash problem search for divisorial valuations that will "appear" on every resolution of singularities, here we are searching for divisorial valuations whose torifications in the sense of problematic (1) is essential to obtain a global torification), then embed as above $\mathbf{A}^n$ in a larger affine space $\mathbf{A}^N$ in such a way that all the valuations $v_1, \ldots, v_r$ can be seen as the traces of monomial valuations on $\mathbf{A}^N$.

If $v_1, \ldots, v_r$, are well chosen, this should guarantee that the embedding $X \subset \mathbf{A}^N$ is torific. Let us discuss this last sentence which probably for now looks a bit prophetic. Let $v = v_\alpha$ be the monomial valuation defined on $\mathbf{A}^n = \mathrm{Spec}\mathbf{K}[x_1, \ldots, x_n]$ by a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{N}, i = 1, \ldots, n$. Let $I \subset \mathbf{K}[x_1, \ldots, x_n]$ be an ideal such that the origin O belongs to the variety $V(I) \subset \mathbf{A}^n = \mathrm{Spec}\mathbf{K}[x_1, \ldots, x_n]$ defined by it. We will say that I or $V(I)$ is non-degenerate with respect to ν at O if the singular locus of the variety defined by the initial ideal $in_\nu(I)$ of I does not intersect the torus $(\mathbf{K}^*)^n$. Note that in this context, the initial ideal of I relative to ν is defined by

$$in_\nu(I) = \{in_\nu(f), f \in I\},$$

where for $f = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{K}[x_1, \ldots, x_n]$,

$$in_\nu(f) = \sum_{a_{i_1,\ldots,i_n} \neq 0, i_1\alpha_1 + \cdots + i_n\alpha_n = \nu(f)} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

It follows from [16],[153] (see also [160] for the hypersurface case) that if for every $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbf{N}, i = 1, \ldots, n$, I is Newton non-degenerate with respect to ν_α at O, then we can construct a proper toric birational morphism $Z \longrightarrow \mathbf{A}^n$ that resolves the singularities of $V(I)$ in a neighborhood of O. Notice that I can be degenerate with respect to a valuation defined by a vector α if there exists an irreducible family of jets (having a large contact with $V(I)$) or arcs on $V(I)$ such that for a generic $\gamma = (\gamma_1(t), \ldots, \gamma_n(t))$ in this family, its **order vector** $(ord_t\gamma_1(t), \ldots, ord_t\gamma_n(t)) = \alpha$: indeed, by a Newton-Puiseux type theorem (or the fundamental theorem of tropical geometry [99]), if this is not satisfied, i.e. if there is no arc $in_{\nu_\alpha}(f)$ will contain monomials, hence by definition I will be non-degenerate with respect to ν_α. This suggests that arcs detect Newton degeneration, and wherever there is a Newton degeneration, there is a degenerate arc passing there in the following sense: An arc defines a germ of a curve; we call an arc degenerate whenever the associated curve germ cannot be resolved with one toric morphism (this can also be detected from the properties of the arc, for instance using the notion of Nash multiplicity [88]). There are degenerate arcs that can be traced on a smooth variety: think of a (relatively) nasty plane curve like the germ of curve which is defined by $(\{(y^2-x^3)^2-4x^5y-x^7=0\},0)\subset(\mathbf{A}^2,0)=\{z=y\}\subset(\mathbf{A}^3,0)$; it is associated with the arc (t^4,t^6+t^7,t^6+t^7) which is traced on $\mathbf{A}^2$. The arc is degenerate but $\mathbf{A}^2\subset\mathbf{A}^3$ is Newton non-degenerate. The moral of this part of the story is first that Newton degeneration is detected by degenerate arcs, and second that not all degenerate arcs cause Newton degeneration. Moreover, the notion of degeneration along an arc can be quantified by an invariant that one can call depth and which in the case of a plane curve is the number of Puiseux pairs minus one. Problematic (1) above takes care of this notion of depth and it allows by embedding in higher dimension the elimination of degeneration along a family of arcs (or jets) that defines a divisorial valuation. Problematic (2) concerns the determination of those families of arcs that cause Newton degeneration.

We will now give a presentation of results concerning these two problematics, with some digressions in order to give applications, links between the two problematics and expand a bit some problems that appear inside these problematics and which are interesting for their own sakes.

Let us begin by discussing one aspect of problematic (1). While this problematic was exposed as a geometric problem, it is related to an "algebraic" problem which makes sense for any valuations: determining a generating sequence of a valuation. Let us for a moment stick to the case of a divisorial valuation centered at the origin $X=\mathbf{A}^d=\mathrm{Spec}R$, where $R=\mathbf{K}[x_1,\dots,x_n]$ is a polynomial ring over an algebraically closed field $\mathbf{K}$. A valuation v is then given by a mapping $v:R\longrightarrow\mathbf{N}$ which is the order of vanishing along a divisor $E\subset Z$ which satisfies $\mu(E)$ is the origin of $\mathbf{A}^n$, μ being a birational map $\mu:Z\longrightarrow\mathbf{A}^n$. Let us explain what is a generating sequence of v.

For $\alpha\in\mathbf{N}$, let

$$\mathcal{P}_\alpha=\{h\in R\mid v(h)\geq\alpha\}.$$

We define the $\mathbf{K}$-graded algebra

$$gr_vR=\bigoplus_{\alpha\in\mathbf{N}}\frac{\mathcal{P}_\alpha}{\mathcal{P}_{\alpha+1}}.$$

We call in_v the natural application

$$in_v:R\longrightarrow gr_vR,\ h\mapsto h \bmod \mathcal{P}_{v(h)+1}.$$

DEFINITION 1. [147] A generating sequence of v is a set of elements of R such that their image by in_v generates gr_vR as a $\mathbf{K}$-algebra.

This notion (for any valuation) is central in an earlier version of Spivakovsky's approach [147] to local uniformization and in the present approach of Teissier to the same problem [150], with the difference that Teissier restricts his analysis to minimal generating sequences for rational valuations. In general, it is very difficult to determine a generating sequence of a given valuation, apart in dimensions 1 and 2; an abstract approach follows from the valuative Cohen theorem [150]. A remarkable advance in this direction was done for (rational) valuations in [39, 40], as discussed below. We will show below, at least on an example, the relation between this notion and problematic (1) . We will discuss first our new approach from [106] for the study of generating sequences of divisorial valuations defined as above. For that, we will use the representation of a divisorial valuation as the order of vanishing along a family of arcs.

2. Generating sequences of divisorial valuations from jet schemes and torification of curves

Let $X = \mathbf{A}^n = \operatorname{Spec} R$, as above. We have a natural truncation morphism $X_\infty \longrightarrow X$, that we denote by Ψ_0; for a n-tuple of series, this simply gives the n-tuple of constant terms of these series. For $p \in \mathbf{N}$ and $Y = V(I) \subset X$ a subscheme defined by an ideal $I \subset R$, we consider the subset (contact locus) of arcs in X_∞ that have an order of contact p with Y, this is

$$Cont^p(Y) = \{\gamma \in X_\infty \mid ord_t\gamma^*(I) = p\},$$

where $\gamma^* : R \longrightarrow \mathbf{K}[[t]]$ is the $\mathbf{K}-$algebra homomorphism associated with γ and

$$ord_t\gamma^*(I) = \min_{h\in I}\{ord_t\gamma^*(h)\}.$$

With an irreducible component $\mathbb{W}$ of $Cont^p(Y)$, which is included in the fibre $\Psi_0^{-1}(0)$ above the origin, we associate a valuation $\nu_{\mathbb{W}} : R \longrightarrow \mathbf{N}$ as follows:

$$\nu_{\mathbb{W}}(h) = \min_{\gamma\in\mathbb{W}}\{ord_t\gamma^*(h)\},$$

for $h \in R$. It follows from [51] (see also [47], [137], prop. 3.7 (vii)), that $\nu_{\mathbb{W}}$ is a divisorial valuation centered at the origin $0 \in X$, and that all divisorial valuations centered at $0 \in X$, can be obtained in this way for varying ideals I. We are interested in determining a generating sequence of a valuation of the form $\nu_{\mathbb{W}}$ with an irreducible component $\mathbb{W}$ of $Cont^p(Y)$. Recall from [23] the functorial definition of the arc space X_∞ : for any algebraic variety X, the arc space X_∞ represents the functor that to a $\mathbf{K}-$algebra A associates the set of $A-$valued arcs

$$X(A[[t]]) := \operatorname{Hom}_{\mathbf{K}}(\operatorname{Spec}(A[[t]]), X).$$

Hence, for a $\mathbf{K}-$algebra A we have a bijection

$$\operatorname{Hom}_{\mathbf{K}}(\operatorname{Spec}(A), X_\infty) \simeq \operatorname{Hom}_{\mathbf{K}}(\operatorname{Spec}(A[[t]]), X).$$

In particular, in our case $X = \mathbf{A}^n = \operatorname{Spec}R$, we have $X_\infty = \operatorname{Spec}(R_\infty)$, and to the identity in $\operatorname{Hom}_{\mathbf{K}}(\operatorname{Spec}(R_\infty), X_\infty)$ corresponds the universal family $\Lambda : R \longrightarrow R_\infty[[t]]$.

Let us consider the case $n = 2, R = \mathbf{K}[x_1, x_2]$. We have

$$R_\infty = \mathbf{K}[x_i^{(j)}; i = 1, 2; j \geq 0],$$

and Λ is given by
$$\Lambda(x_i) = x_i^{(0)} + x_i^{(1)}t + x_i^{(2)}t^2 + \cdots, \quad i = 1,2.$$
The procedure that we give can be thought as an elimination algorithm with respect to Λ in the sense that from the equations (that we can see in R_∞) of the irreducible component of $Cont^p(Y)$ defining our valuation we will obtain elements in R that constitute the generating sequence. Let us show this on an example: Assume that the characteristic is not equal to 2. Let us consider the divisorial valuation associated with one irreducible component of $Cont^{27}(Y)$, where Y is the curve defined by the equation $(x_1^2 - x_0^3)^2 - x_0^5 x_1 = 0$. The contact locus $Cont^{27}(Y)$ has two irreducible component, the interesting one (the other one gives a monomial valuation), that we call $\mathbb{W}$ is defined in $\mathbf{A}^2_\infty$ by the ideal
$$\left(x_0^{(0)},\ldots,x_0^{(3)},x_1^{(0)},\ldots,x_1^{(5)},{x_1^{(6)}}^2 - {x_0^{(4)}}^3,(2x_1^{(6)}x_1^{(7)} - 3{x_0^{(4)}}^2 x_0^{(5)})^2 - {x_0^{(4)}}^5 x_1^{(6)}\right)$$
and two inequalities, the most important one of them is $x_0^{(4)} \neq 0$. Noticing that the first equation which is not that of a coordinate hyperplane being not linear, this gives us the first three elements of a generating sequence
$$x_0, x_1, x_2 = x_1^2 - x_0^3.$$
The last element was obtained by what we called an elimination process which corresponds here to dropping the indices in the parentheses from ${x_1^{(6)}}^2 - {x_0^{(4)}}^3$. Note that modulo $x_0^{(0)} = \cdots = x_0^{(3)} = x_1^{(0)} = \ldots = x_1^{(5)} = 0$, $\Lambda(x_2) = ({x_1^{(6)}}^2 - {x_0^{(4)}}^3)t^12 + t^13\phi$, with $\phi \in \mathrm{R}_\infty[[t]]$. The remaining equation, modulo the other equations, can then be rewritten
$$(2x_1^{(6)}x_1^{(7)} - 3{x_0^{(4)}}^2 x_0^{(5)})^2 - {x_0^{(4)}}^5 x_1^{(6)} = {x_2^{(13)}}^2 - {x_0^{(4)}}^5 x_1^{(6)}.$$
Again, the elimination process with respect to Λ corresponds to dropping the indices in the parentheses. The 4th and last element of the generating sequence of $\nu_\mathbb{W}$ which is then:
$$x_3 = x_2^2 - x_0^5 x_1.$$
The valuation $\nu_\mathbb{W}$ is completely determined by its generating sequence x_0, x_1, x_2, x_3 and the values $\nu_\mathbb{W}(x_0) = 4, \nu_\mathbb{W}(x_1) = 6, \nu_\mathbb{W}(x_2) = 13, \nu_\mathbb{W}(x_3) = 27$. By construction, for $i = 2,3$ we have polynomials f_i such that
$$x_i = f_i(x_0, \ldots, x_{i-1}).$$

The functions f_i's provide an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^4$, which is the geometric counterpart of the following morphism
$$\mathbf{K}[x_0,x_1,x_2,x_3] \longrightarrow \frac{\mathbf{K}[x_0,x_1,x_2,x_3]}{(x_2 - f_2(x_0,x_1), x_3 - f_3(x_0,x_1,x_2))} \simeq \mathbf{K}[x_0,x_1].$$

This embedding solves problematic (1) for the valuation $\nu_\mathbb{W}$ and realizes this latter as the trace of the monomial valuation centered at $(\mathbf{A}^4, O)$ and associated with the vector $\alpha = (4,6,13,27)$. Here we only gave the feeling of this, but the reason why the second and the third points of problematic (1) are satisfied follows from the fact that if $\nu = \nu_\alpha$ then the initial ideal of $(x_2 - f_2(x_0,x_1), x_3 - f_3(x_0,x_1,x_2))$ with respect to ν is given by

$$(x_1^2 - x_0^3, x_2^2 - x_0^5 x_1),$$

which is a toric (prime) ideal and its singular locus is a point. More generally we have

THEOREM II.A. [106]*(Mourtada) For* $n = 2$, *there is a constructive solution of problematic (1).*

It is important to mention here that determining a generating sequence is not necessary to solve problematic (1) for a given valuation. In [109], we generalize this theorem in some way which we try to explain; this work ([109]) was on standby for 3 years now because we were unable to find the obstruction to constructing a generating sequence as above: the procedure, in any dimension, gives finitely many elements while in dimension larger of equal to three the graded algebra of a divisorial valuation need not to be finitely generated. The advance comes from [39, 112] and made us understand that our procedure helps to find a generating sequence up to some blow ups.

We can give now an example of our geometric approach to the resolution of singularities. Let $Y \subset \mathbf{A}^2$ be again the curve defined by $(x_1^2 - x_0^3)^2 - x_0^5 x_1 = 0$. The interesting divisorial valuation is the one associated with the irreducible component of Y_{25} (or equivalently of $Cont^{26}(Y)$) which is defined by the ideal

$$\left(x_0^{(0)}, \ldots, x_0^{(3)}, x_1^{(0)}, \ldots, x_1^{(5)}, {x_1^{(6)}}^2 - {x_0^{(4)}}^3\right).$$

We do not explain here in detail why we choose this divisor but we can say that this is the most natural choice which arises from the geometry of the jet schemes, which will be discussed below. But we can say that the space of arcs (on Y) centered at the singular point of Y has one irreducible component whose geometry is reflected by the geometry of this irreducible component of Y_{25}. Applying the procedure that we explained above, we find an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^3$, which is the geometric counterpart of the following morphism

$$\mathbf{K}[x_0,x_1,x_2] \longrightarrow \frac{\mathbf{K}[x_0,x_1,x_2]}{(x_2 - ({x_1}^2 - {x_0}^3))} \simeq \mathbf{K}[x_0,x_1].$$

Our curve Y seen in $\mathbf{A}^3$ is then defined by the ideal

$$I = (x_2 - ({x_1}^2 - {x_0}^3, x_2^2 - x_0^5 x_1).$$

Its (local) tropical variety (with respect to the embedding in $\mathbf{A}^3$) is the half line along the vector $(4,6,13)$ (see [135] for the notion of local tropical variety). The initial ideal of I with respect to the monomial valuation associated with the vector $(4,6,13)$ is given by the ideal

$$J = (x_1^2 - x_0^3, x_2^2 - x_0^5 x_1).$$

The singular locus of the variety defined by this latter ideal (which actually defines a monomial curve) is just a point so that this ideal is non-degenerate and can be resolved with one toric morphism. Hence, this embedding is torific; more generally, this gives another proof of torification for analytically irreducible plane curves [59]. Now applying our geometric approach to resolution of singularities to a reducible plane curve we were able to prove in [42] the following:

THEOREM II.B. *(de Felipe, González-Pérez, Mourtada) For a reducible plane curve singularity, the geometric approach to resolution of singularities yields a torific embedding.*

We can actually construct a torification for curves of any embedding dimension. What makes things more complicated in higher dimensions, is that the initial ideal which is the counter part of the initial ideal that we called J above, is not toric, but it still has the structure of a $T-$ variety (i.e.) a variety which is equiped with an action of a torus of smaller dimension. Some work in this direction is in the ongoing project [25].

3. Torific embeddings in action

3.1. On some local rings of the arc space. In the preceding section, we used the representation of a divisorial valuation v as the order of vanishing along a family of arcs and the equations defining this family in the arc space to obtain a generating sequence of v and hence a representation of the graded algebra associated with v. Here we go in the other direction: from the "graded algebra" we give a presentation of the completion $\widehat{O_{X_\infty,P_v}}$ of the localization of the arc space at the point P_v associated with the family of arcs. One motivation of this study is to understand the dimension of this ring which is noetherian [138]. This dimension is related to the Nash problem (in the sense that if it is equal to one, this ensures that the divisorial valuation is in the image of the Nash map [137, 138]. Another issue is that when working on such questions, one has a foot in the Noetherian world and the other in the non-Noetherian world; this latter has been very little explored.

We have put "graded algebra" between quotation marks because this is not in general the same graded algebra that we have described above. To give the presentation cited above, we use first Noether's normalization lemma in order to see v as the (finite) extension of a divisorial valuation v_n defined on $\mathbf{K}[x_1,\ldots,x_n]$, n being the dimension of X. The restriction v_2 of v_n to $\mathbf{K}[x_1,x_2]$ is a divisorial valuation and using the same method as in the previous section, we can determine its graded algebra (or generating sequence). Then, we look at the restriction v_3 of v_n to $\mathbf{K}(x_1,x_2)[x_3]$; this is also the extension of v_2. Our study of the graded algebra of v_3 used resolution of singularities; this is why our results were only proved in characteristic 0, but these results can be extended to the positive characteristic case using the theory of key polynomials [48, 100, 157]; this is an ongoing project. The important thing that we wanted to point out, is that the key polynomials of v_3 extending v_2 are rational functions in $\mathbf{K}(x_1,x_2)[x_3]$ and not polynomials in $\mathbf{K}[x_1,x_2,x_3]$; this meets the discussion that we had in section 2. By induction, we can determine a "graded algebra", or key polynomials associated with v. Still, from these key polynomials, we are able to give a presentation of $\widehat{O_{X_\infty,P_v}}$; this presentation is not sufficient to understand the dimension even if we are able to bound it thanks to a tricky computation of its embedding dimension: this latter can be expressed in term of the Mather discrepancy (which is an invariant defined using any resolution of singularities of X which factors through its Nash blowing up) $\hat{k}_E$ of the divisor E as follows:

THEOREM II.C. *(Mourtada, Reguera) The embedding dimension of $\widehat{O_{X_\infty,P_E}}$ is given by*

$$\text{embdim}\ \widehat{O_{X_\infty,P_E}} = \hat{k}_E + 1.$$

The proof is done first in the case of a "monomial" valuation for which the presentation of $\widehat{O_{X_\infty,P_v}}$ is simpler (actually the study of the graded algebra allows to reduce the embedding dimension computation to the monomial case). These are the results in [112]. Recently a new proof of this theorem has been found in [45].

3.2. Motivic zeta functions and the monodromy conjecture in family. This section is an example that we have in mind for applications of the torific embedding or the resolutions of singularities that it induces. It is also an application of the jet components graph introduced in section 4.

The history of the *motivic Igusa zeta function* goes back to the seventies when Igusa studied the *p-adic Igusa zeta function*, which is related to the classical problem in number theory of computing the number of solutions of congruences. More precisely, the original Igusa zeta function counts, for a non-constant polynomial $f \in \mathbf{Z}[x_1,\dots,x_n]$ and a prime number p, the $\mathbf{Z}/(p^{m+1}\mathbf{Z})$-points of $X = \{f = 0\}$, when m varies in $\mathbf{N}$. It was introduced by Weil [165], and its basic properties, such as rationality, were first investigated by Igusa [69, 70]. In analogy with the p-adic zeta function, Denef and Loeser [50] introduced the 'more general' motivic Igusa zeta function in which $f \in \mathbf{C}[x_1,\dots,x_n]$ is a complex polynomial, and the $\mathbf{Z}/(p^{m+1}\mathbf{Z})$-points of $X = \{f = 0\}$ are replaced by its $\mathbf{C}[t]/(t^{m+1})$-points, i.e the $\mathbf{C}$-points of X_m. It is more general in the sense that the p-adic Igusa zeta function can be obtained from the motivic one.

The motivic Igusa zeta function $Z_X^{mot}(T)$ associated with X (or with f) can be written as

$$Z_X^{mot}(T) = 1 - \frac{1-T}{T} J_X(T),$$

where $J_X(T)$ is the Poincaré series

$$J_X(T) := \sum_{m\geq 0} [X_m](\mathbb{L}^{-n}T)^{m+1} \in \mathcal{M}_{\mathbf{C}}[[T]].$$

Here, $\mathcal{M}_{\mathbf{C}}$ is a localization of the *Grothendieck ring* of complex varieties, and $[X_m]$ and $\mathbb{L}$ are the classes of X_m and of the affine line $\mathbb{C}$ in this Grothendieck ring, respectively. Clearly, this expression also makes sense when X is any subscheme of $\mathbf{C}^n$ given by some ideal I in

$\mathbb{C}[x_1, \ldots, x_n]$, instead of just a hypersurface. Furthermore, the motivic zeta function turns out to be a rational function in T, and it is natural to study its poles.

The motivic Igusa zeta function for one polynomial $f \in \mathbf{C}[x_1, \ldots, x_n]$ can also be expressed in terms of an *embedded resolution of singularities* of f; the analogous expression for an ideal is in terms of a *principalization* of the ideal. This formula in the hypersurface case can be found in [50], and its generalization to ideals is mentioned in [162]. It is the most classical way to compute the motivic zeta function and allows to determine a complete list of candidates poles of this zeta function. However, it is in general very difficult to calculate a principalization and to verify whether the candidate-poles are actual poles; usually, 'most' of the candidates are in fact not actual poles. In this article, in order to determine the motivic zeta function and its poles, we will compute the Poincaré series $J_X(T)$ from the structure of the jet schemes, making use of the jet components graph.

The poles of the motivic Igusa zeta function associated with a complex polynomial $f \in \mathbf{C}[x_1, \ldots, x_n]$ are the subject of an intriguing open problem, the *monodromy conjecture*, which relates number theoretical invariants and topological invariants of f. Roughly speaking, it predicts a relation between the poles of the motivic zeta function and the action of the monodromy of f, seen as a function $f : \mathbf{C}^n \to \mathbf{C}$, on the cohomology of its Milnor fiber at some point $x \in X \subset \mathbf{C}^n$. For an ideal I, one can state the *generalized monodromy conjecture* [156] in which Verdier monodromy [161] replaces the classical monodromy.

I have two questions in the spirit of the torific embedding :

- To compare the monodromy conjectures after changing embeddings: now that we have a monodromy conjecture in higher codimensions, the first question is whether the veracity of the monodromy conjecture depends on a smooth ambient space; this question has not been addressed yet even if we expect yes as an answer. If so, one can choose a torific embedding (which exists in characteristic 0 but one needs to control it without using a resolution of singularities, otherwise it does not give any new information), hence a toric resolution, and as it is mentioned below, this gives "formulas" for motivic Igusa zeta functions and that of the monodromy zeta function. Some work in this direction for the computation of the motivic Igusa zeta function has been done in [124]). Still these formulas are not enough to solve the monodromy conjecture in general since it is difficult to determine the actual poles of these Zeta functions from these formulas. In the work described below, we compensate this by a deep understanding of the jet schemes via the jet components graph.

- How does the veracity of the monodromy conjecture vary in an equisingular family?

With Wim Veys and Lena Vos we applied this approach to the family of plane branches that degenerate to a monomial curve; we were mainly interested in the monodromy conjecture of such a monomial curve which in general is not a hypersurface, the plane curve case being known since Loeser [98] in the nineties. More precisely, let $\mathcal{C} := \{f = 0\} \subset (\mathbf{C}^2, 0)$ be a germ of a complex plane curve defined by an irreducible series $f \in \mathbf{C}[[x_0, x_1]]$ with $f(0) = 0$, and let

$$\nu_{\mathcal{C}} : R := \frac{\mathbf{C}[[x_0, x_1]]}{(f)} \longrightarrow \mathbf{N}$$

be the associated valuation, where for any $h \in \mathbf{C}[[x_0, x_1]] \setminus (f), \nu_{\mathcal{C}}(h) = (f, h)_0$ is the local intersection multiplicity of the curve $\mathcal{C}$ and the curve $\{h = 0\}$. The semigroup $\Gamma(\mathcal{C}) := \{\nu_{\mathcal{C}}(h) \mid h \in R \setminus \{0\}\} \subset \mathbf{N}$ is finitely generated, and we can identify a unique minimal system of generators $(\bar{\beta}_0, \ldots, \bar{\beta}_g)$ of $\Gamma(\mathcal{C})$. Let $(Y, 0) \subset (\mathbf{C}^{g+1}, 0)$ be the image of the monomial map $M : (\mathbf{C}, 0) \to (\mathbf{C}^{g+1}, 0)$ given by $M(t) = (t^{\bar{\beta}_0}, t^{\bar{\beta}_1}, \ldots, t^{\bar{\beta}_g})$. It is an irreducible curve with the 'plane' semigroup $\Gamma(\mathcal{C})$ as its semigroup and it is the special fiber of a flat family $\eta : (\mathcal{X}, 0) \subset (\mathbf{C}^{g+1} \times \mathbf{C}, 0) \to (\mathbf{C}, 0)$ whose generic fiber is isomorphic to $\mathcal{C}$. We call Y the *monomial curve associated with* $\mathcal{C}$, and the explicit equations defining Y in $\mathbf{C}^{g+1}$ are of the form

$$\begin{cases} x_1^{n_1} & - & x_0^{n_0} & = 0 \\ x_2^{n_2} & - & x_0^{b_{20}} x_1^{b_{21}} & = 0 \\ & \vdots & & \\ x_g^{n_g} & - & x_0^{b_{g0}} x_1^{b_{g1}} \cdots x_{g-1}^{b_{g(g-1)}} & = 0, \end{cases}$$

where $n_i > 1$ and $b_{ij} \geq 0$ are integers that are defined in terms of $(\bar{\beta}_0, \ldots, \bar{\beta}_g)$.

We studied the jet schemes Y_m of Y for every $m \in \mathbf{N}$. From the study of the jet component graph we were able to compute a closed formula for the motivic zeta function in [116]:

THEOREM II.D. *(Mourtada, Veys, Vos)*

$$\begin{aligned} Z_Y^{mot}(T) = & \frac{1 - (\mathbb{L} - 1)\mathbb{L}^{-(g+1)} - \mathbb{L}^{-(g+1)}T}{1 - \mathbb{L}^{-g}T} + \frac{P_0(T)}{1 - \mathbb{L}^{-\nu_1}T^{N_1}} \\ & + \sum_{i=1}^{g-1} \frac{P_i(T)}{(1 - \mathbb{L}^{-\nu_i}T^{N_i})(1 - \mathbb{L}^{-\nu_{i+1}}T^{N_{i+1}})} - \frac{(\mathbb{L} - 1)\mathbb{L}^{-(\nu_g + g + 1)}(1 - T)T^{N_g}}{(1 - \mathbb{L}^{-g}T)(1 - \mathbb{L}^{-\nu_g}T^{N_g})}, \end{aligned}$$

where $P_i(T)$ for $i = 0, \ldots, g-1$ are concrete polynomials with coefficients in the ring $\mathbf{Z}[\mathbb{L}, \mathbb{L}^{-1}]$, and (N_i, ν_i) for $i = 1, \ldots, g$ are couples of known positive integers with

$$\frac{\nu_i}{N_i} = \frac{1}{n_i \bar{\beta}_i} \left(\sum_{l=0}^{i} \bar{\beta}_l - \sum_{l=1}^{i-1} n_l \bar{\beta}_l \right) + (i - 1) + \sum_{l=i+1}^{g} \frac{1}{n_l}.$$

Furthermore, we obtain only $g + 1$ candidate poles:

$$\mathbb{L}^g, \qquad \mathbb{L}^{\frac{\nu_i}{N_i}}, \quad i = 1, \ldots, g.$$

Using residues and the related *topological Igusa zeta function*, we prove that, contrary to formulas that one could obtain using a principalization, all these candidate poles are actual poles. We also get the *log canonical threshold* of $Y \subset \mathbf{C}^{g+1}$ given by $\frac{\nu_1}{N_1} = \sum_{l=0}^{g} \frac{1}{n_l}$. Note that the number of poles of the motivic zeta function of Y is equal to the number of poles of the motivic zeta function of the plane branch $\mathcal{C}$. This implies that the motivic zeta function associated with the special fiber of the family $\eta : (\chi, 0) \to (\mathbf{C}, 0)$ has the same number of poles as the motivic zeta function associated with the generic fiber. This is remarkable as the induced family on the level of jet schemes is not flat. More precisely, let $S := (\mathbf{C}, 0)$ and consider, for every $m \in \mathbf{N}$, the *relative m-th jet scheme* $((\mathcal{X}, 0)/S)_m$ of $\eta : (\mathcal{X}, 0) \to S$ with the natural morphism $\eta_m : ((\mathcal{X}, 0)/S)_m \to S$, whose fibers are isomorphic to the mth jet schemes of the fibers of η. Then, although the family η is equisingular (in particular, flat), we show that the family η_m is not flat for m large enough. We would like to point out that, in the hypersurface case, an equisingular family of hypersurfaces does induce a flat family on the jet schemes (with their reduced structures) [**92**, Theorem 3.4].

These results led to a separate work by Martin-Morales, Veys and Vos containing a proof of the monodromy conjecture for the monomial curve singularity [**102**].

3.3. On $\mu-$constant deformations. In this section, we are somehow in an earlier stage comparing to other sections, since we prove a theorem where a ("simultaneous") torification exists in its simplest form. This section, which is substantial itself, is a preparation of an approach to a far reaching application: The Ramanujam-Lê conjecture [**94**]. This latter states that in a family of singularities where the fibers have a constant Milnor number (μ), all the fibers have same topological type; the case which is still open of this conjecture is the case of families of surfaces; this is explained a bit more below.

Before stating and discussing the main problem we will give some brief preliminaries and introduce the notation that will be used in the main theorem.

Preliminaries on μ-constant deformations

Let

$$\mathcal{O}^x_{n+1} := \mathbb{C}\{x_1, \ldots, x_{n+1}\}, \quad n \geq 0,$$

be the $\mathbf{C}$-algebra of analytic function germs at the origin o of $\mathbf{C}^{n+1}$ and $\mathbf{C}^{n+1}_o$ the complex-analytic germ of $\mathbf{C}^{n+1}$. By abuse of notation we denote by o the origin of $\mathbf{C}^{n+1}_o$. Let V be a hypersurface of $\mathbf{C}^{n+1}_o$, $n \geq 1$, given by an equation $f(x) = 0$, where f is irreducible in $\mathcal{O}^x_{n+1}$. Assume that V has an isolated singularity at o. One of the important topological invariants of the singularity $o \in V$ is the Milnor number $\mu(f)$, defined by

$$\mu(f) := \dim_{\mathbf{C}} \mathcal{O}^x_{n+1}/J(f),$$

where $J(f) := (\partial_1 f, \ldots, \partial_{n+1} f) \subset \mathcal{O}^x_{n+1}$ is the Jacobian ideal of f. In this article we will consider deformations of f that preserve the Milnor number. Let F be a deformation of f:

$$F(x,s) := f(x) + \sum_{i=1}^{l} h_i(s) g_i(x)$$

where $h_i \in \mathcal{O}^s_m := \mathbb{C}\{s_1,...,s_m\}$, $m \geq 1$, and $g_i \in \mathcal{O}^x_{n+1}$ satisfy

$$h_i(o) = g_i(o) = 0.$$

Take a sufficiently small open set $\Omega \subset \mathbf{C}^m$ containing o, and representatives of the analytic function germs $h_1,...,h_l$ in Ω. By a standard abuse of notation we will denote these representatives by the same letters $h_1,...,h_l$. We use the notation $F_{s'}(x) := F(x,s')$ when $s' \in \Omega$ is fixed. We will say that the deformation F is μ-constant if the open set Ω can be chosen so that $\mu(F_{s'}) = \mu(f)$ for all $s' \in \Omega$.

Let us write $g \in \mathbf{C}\{x_1,...,x_{n+1}\}$ as

$$g(x) = \sum_{\alpha \in Z} a_\alpha x^\alpha, \quad Z := \mathbf{Z}^{n+1}_{\geq 0} \setminus \{o\},$$

in the multi-index notation. The *Newton polyhedron* $\Gamma_+(g)$ is the convex hull of the set $\bigcup_{\alpha \in \mathrm{Supp}(g)} (\alpha + \mathbf{R}^n_{\geq 0})$, where $\mathrm{Supp}(g)$ (short for "the support of g") is defined by $\mathrm{Supp}(g) := \{\alpha \mid a_\alpha \neq 0\}$. The *Newton boundary* of $\Gamma_+(g)$, denoted by $\Gamma(g)$, is the union of the compact faces of $\Gamma_+(g)$. We will say that $g(x) = \sum_{\alpha \in Z} a_\alpha x^\alpha$, $Z := \mathbf{Z}^{n+1}_{\geq 0} \setminus \{o\}$, is *non-degenerate with respect to its Newton boundary (or Newton non-degenerate)* if for every compact face γ of the Newton polyhedron $\Gamma_+(g)$ the polynomial $g_\gamma = \sum_{\alpha \in \gamma} a_\alpha x^\alpha$ does not have singularities in $(\mathbf{C}^\star)^{n+1}$.

We say that a deformation F of f is *non-degenerate* if the neighborhood Ω of o in $\mathbf{C}^m$ can be chosen so that for all $s' \in \Omega$ the germ $F_{s'}$ is non-degenerate with respect to its Newton boundary $\Gamma(F_{s'})$.

Preliminaries on Simultaneous Embedded Resolutions

Let us keep the notation from the previous paragraph. We denote $S := \mathbf{C}^m_o$, and W the deformation of V given by F. Then we have the following commutative diagram:

$$\begin{array}{ccccc} V & \hookrightarrow & W & \hookrightarrow & \mathbf{C}^{n+1}_o \times S \\ \downarrow & \square & \downarrow \rho & \swarrow & \\ o & \hookrightarrow & S & & \end{array}$$

where the morphism ρ is flat. We use the notation $W_{s'} := \rho^{-1}(s')$, $s' \in S$.

In what follows we will define what we mean by *Simultaneous Embedded Resolution* of W.

We consider a proper bimeromorphic morphism $\varphi : \widetilde{\mathbf{C}_o^{n+1} \times S} \to \mathbf{C}_o^{n+1} \times S$ such that $\widetilde{\mathbf{C}_o^{n+1} \times S}$ is formally smooth over S, and we denote by $\widetilde{W}^s$ and $\widetilde{W}^t$ the strict and the total transform of W in $\widetilde{\mathbf{C}_o^{n+1} \times S}$, respectively.

DEFINITION 2. The morphism $\widetilde{W}^s \to W$ is a *very weak simultaneous resolution* if $\widetilde{W}^s_{s'} \to W_{s'}$ is a resolution of singularities for each $s' \in S$.

DEFINITION 3. We say that $\widetilde{W}^t$ is a *normal crossing divisor relative to S* if the induced morphism $\widetilde{W}^t \to S$ is flat and for each $p \in \widetilde{W}^t$ there exists an open neighborhood $U \subset \widetilde{\mathbf{C}_0^{n+1} \times S}$ of p and a map ϕ,

$$\begin{array}{ccc} U & \xrightarrow{\ \phi\ } & \mathbf{C}_0^{n+1} \times S \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

biholomorphic onto its image, such that $\widetilde{W}^t \cap U$ is defined by the ideal $\phi^\star I$, where $I = \left(y_1^{a_1} \cdots y_{n+1}^{a_{n+1}}\right)$, $y_1, \ldots y_{n+1}$ is a coordinate system at o in $\mathbf{C}_0^{n+1}$ and the a_i are non-negative integers. If $p \in \widetilde{W}^s$, we require that $a_{n+1} = 1$ and that $\widetilde{W}^s \cap U$ be defined by the ideal $\phi^\star I'$, where $I' = (y_{n+1})$.

DEFINITION 4. We will say ϕ is a *simultaneous embedded resolution* if, in the above notation, the morphism $\widetilde{W}^s \to W$ is a very weak simultaneous resolution and $\widetilde{W}^t$ is a normal crossing divisor relative to S.

Let us recall that W is defined by

$$F(x,s) := f(x) + \sum_{i=1}^{l} h_i(s) g_i(x)$$

where $h_i \in \mathcal{O}_m^s$, $m \geq 1$, and $g_i \in \mathcal{O}_{n+1}^x$ such that $h_i(o) = g_i(o) = 0$.

Let $\varepsilon > 0$ (resp. $\varepsilon' > 0$) be small enough so that $f, g_1, \ldots, g_l$ (resp. $h_1, \ldots, h_l$) are defined in the open ball $B_\varepsilon(o) \subset \mathbf{C}^{n+1}$ (resp. $B_{\varepsilon'}(o) \subset \mathbf{C}^m$), and the singular locus of W is $\{o\} \times B_{\varepsilon'}(o)$. We will say that the deformation of W is *topologically trivial* if, in addition, there exists a homeomorphism ξ that commutes with the projection

$$pr_2 : B_\varepsilon(o) \times B_{\varepsilon'}(o) \to B_{\varepsilon'}(o) :$$

$$\begin{array}{ccc} B_\varepsilon(o) \times B_{\varepsilon'}(o) & \xrightarrow{\ \xi\ } & B_\varepsilon(o) \times B_{\varepsilon'}(o) \\ & {\scriptstyle pr_2}\searrow \quad \swarrow{\scriptstyle pr_2} & \\ & B_{\varepsilon'}(o) & \end{array}$$

such that $\xi(W) = V' \times B_{\varepsilon'}(o)$, where $V' := \xi(V)$, that is to say, ξ trivializes W. The following Proposition relates Simultaneous Embedded Resolutions, topologically trivial deformations and μ-constant deformations.

PROPOSITION II.E. *Let V and W be as above. Assume that W admits a simultaneous embedded resolution. Then:*

(1) The deformation W is topologically trivial.
(2) The deformation W is μ-constant.

PROOF. The Milnor number μ is a topological invariant, hence (1) implies (2). So, let us prove (1). As W admits a simultaneous embedded resolution, there exists a proper bimeromorphic morphism $\varphi : \widetilde{\mathbf{C}_0^{n+1} \times S} \to \mathbf{C}_0^{n+1} \times S$ such that $\widetilde{\mathbf{C}_0^{n+1} \times S}$ is formally smooth over S and W^t is a normal crossing divisor relative to S. In the topological context this translates into the existence of a proper bimeromorphic morphism $\varphi : \widetilde{B_\varepsilon(o) \times B_{\varepsilon'}(o)} \to B_\varepsilon(o) \times B_{\varepsilon'}(o)$ such that for all $p \in \varphi^{-1}(o)$ there exists $\varepsilon'' > 0$, and a diffeomorphism

$$\phi_p : B_{\varepsilon''}(p) \subset \widetilde{B_\varepsilon(o) \times B_{\varepsilon'}(o)} \to \widetilde{B_{\varepsilon''}(o) \times B_{\varepsilon'}(o)}$$

that trivializes $W^t \cap B_{\varepsilon''}(p)$. Using partitions of unity and the projection ξ, we obtain the desired trivialization. □

On the main result of this section

Keep the notation of the previous paragraphs. Recall that W is a deformation of V over $S := \mathbf{C}_o^m$ given by F. In the article [**127**] the author proves that if W is a non-degenerate μ-constant deformation of V that induces a negligible truncation of the Newton boundary then W admits a very weak simultaneous resolution. However if the method of proof used is observed with detail, what is really proved is that W admits a simultaneous embedded resolution in the special case when $n = 2$, $l = m = 1$, $h_1(s) = s$ and $g_1(x)$ is a monomial in x. Intuitively one might think that the condition that W admit a simultaneous embedded resolution is more restrictive than the condition that W is a μ-constant deformation. However, this intuition is wrong at least in the case of Newton non-degenerate μ-constant deformations. More precisely, in [**93**] we proved the following result:

THEOREM II.F. *(Leyton-Alvarez, Mourtada, Spivakovsky) Assume that W is a Newton non-degenerate deformation. Then the deformation W is μ-constant if and only if W admits a simultaneous embedded resolution.*

Observe that if W admits a simultaneous embedded resolution it follows directly from Proposition II.E that W is a μ-constant deformation. The converse of this is what needs to be proved.

From the above theorem and Proposition II.E we obtain the following corollary.

COROLLARY II.G. *Let W be a Newton non-degenerate μ-constant deformation. Then W is topologically trivial.*

The result of the corollary is already known (see [1]). In the general case, for $n \neq 2$ it is known that if W is a μ-constant deformation, then the deformation W is topologically trivial, (see [94]). The case $n = 2$ is a conjecture (the Lê–Ramanujan conjecture).

The theorem has an interesting implication to spaces of m-jets. Let $\mathbf{K}$ be a field and Y a scheme over $\mathbf{K}$. We denote by $Y - \mathcal{S}ch$ (resp. $\mathcal{S}et$) the category of schemes over Y (resp. sets), and let X be a Y-scheme. It is known that the functor $Y - \mathcal{S}ch \to \mathcal{S}et : Z \mapsto \mathrm{Hom}_Y(Z \times_{\mathbf{K}} \mathrm{Spec}\,\mathbf{K}[t]/(t^{m+1}), X)$, $m \geq 1$, is representable. More precisely, there exists a Y-scheme, denoted by $\mathrm{X(Y)}_m$, such that $\mathrm{Hom}_Y(Z \times_{\mathbf{K}} \mathrm{Spec}\,\mathbf{K}[t]/(t^{m+1}), X) \cong \mathrm{Hom}_Y(Z, \mathrm{X(Y)}_m)$ for all Z in $Y - \mathcal{S}ch$. The scheme $\mathrm{X(Y)}_m$ is called the *space of m-jets of X relative to Y*. For more details see [163] or [92]. Let us assume that Y is a reduced $\mathbf{K}$-scheme, and let Z be a Y-scheme. We denote also by Z the reduced Y-scheme associated to Z.

COROLLARY II.H. *Let $S = \boldsymbol{C}_0$ and let W be a Newton non-degenerate μ-constant deformation. The structure morphism $(\mathrm{W(S)}_m) \to S$ is flat for all $m \geq 1$.*

PROOF. By the previous theorem W admits an embedded simultaneous resolution. Hence the corollary is an immediate consequence of Theorem 3.4 of [92]. □

The main result of this section initiates a new approach to the Lê-Ramanujam conjecture. To wit, in characteristic 0 every singularity can be embedded in a higher dimensional affine space in such a way that it is either Newton non-degenerate or Schön (this is due to Tevelev, answering a question of Teissier, see [150], [154] and [106]). Note that Schön is the notion that generalizes Newton non-degenerate singularities to higher codimensions and guarantees the existence of embedded toric resolutions for singularities having this property. The idea is to prove a generalization of the main theorem of this article for an adapted embedding and then to apply the first part of Proposition II.E.

4. A graph encoding the irreducible components of the jet schemes

The study of the irreducible components of the the jet schemes is part of problematic (2) which as explained above concerns the divisorial valuations that are significant for embedded resolutions of singularities. But apart from this point of view, this problem is in our opinion an interesting and difficult problem. It is interesting because the jet schemes contains a lot of information ([49–51, 71, 76, 105, 117–119] etc...) but which comes in bulk. And one of the reasons why this is difficult is that while the motivic integration theory (or the geometry behind it) can say something about the irreducible components of maximal dimensions [119], it is much less powerful in understanding the other components which often contain the deep information about the singularities. Many questions arise in relation with these irreducible components:

What is the "structure" of the irreducible components of the jet schemes of a singular variety X?

While one can be interested in the irreducible components of the $m-$th jet scheme of X for a given $m \in \mathbf{N}$, these components come naturally in projective systems and their study becomes more exciting when we consider the variation of their geometry in these projective systems. Below we will give a meaning to the word "structure" in the question; this structure is still mysterious and very little studied, and we understand it in very few cases [**31**,**79**,**105**,**107**].

What is the relation between the geometry of the jet schemes of X and the geometry of the singular variety X ?

The arc space and the jet schemes of X are rather complicated compared to X : the arc space is in general infinite dimensional; the jet schemes have in general many irreducible components of different dimensions; they are in general not "reduced"... but the philosophy of which we want to convince the reader of this memoir is that difficult questions concerning X and its singularities can be translated to simple questions concerning the arc space and the jet schemes of X. One can formulate this philosophy as follows: Jet schemes transform a difficult problem concerning a relatively simple object into a simple problem concerning a difficult object.

Finally, finding explicit relations between the local geometry of the singularities and some resolution of singularities remains a central problem in singularity theory. In this memoir, jet schemes stay somehow in the middle: an answer to the second question above allows to relate the geometry of the jet schemes to the geometry of singularities and the geometric approach to resolution of singularities links the valuations which arise from the irreducible components of the jet schemes to resolution of singularities. Apart from this approach, it is now well known that there are deep relations between resolution of singularities and jet schemes, e.g. [**47**,**51**,**71**,**90**,**118**,**119**], but these relations are far from being completely explored; the Nash problem can be thought as one of these relations. We (partially) answer the first question and "completely" the second question for quasi-ordinary and toric surface singularities. Before saying a word about these answers and why we study that type of singularities, let us introduce the jet components graph which will encode the structure of the inverse system of irreducible components of the jet schemes.

DEFINITION 5. (Mourtada) The jet-components graph of and algebraic variety S (here it will be a surface with quasi-ordinary or toric singularities) is the leveled weighted graph Γ obtained by

- representing every irreducible components of the jet scheme S_m^0, $m \geq 1$, at $0 \in S$ by a vertex $v_{i,m}$, where the sub-index m is the level of the vertex;

- joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if the morphism $\pi_{m+1,m}$ induces a morphism between the corresponding irreducible components;
- weighting each vertex by the dimension of the corresponding irreducible component.

Recall that the morphism $\pi_{m+1,m} : S_{m+1} \longrightarrow S_m$ is the truncation morphism which is induced by the algebraic morphism $\mathbf{K}[t]/(t^{m+1}) \longrightarrow \mathbf{K}[t]/(t^m)$.

This graph was introduced in [**105**] and was refined in [**31**, **107**]. Sometimes, we also weight the irreducible components by their embedding dimensions; this can be necessary to recover the geometry of the singularity.

Let us present (very) briefly the singularities that we study here :

Quasi-ordinary singularities of dimension d are those singularities which (locally) can be projected to an affine space $(\mathbf{A}^d, 0)$ such that the discriminant locus is a normal crossing divisor; they are particularly important in Jung's point of view on resolution of singularities and in equisingularity theory [**97**]; they play an important role in this memoir. More about this type of singularities is explained in 6. We are concerned with quasi-ordinary hypersurface singularities (over a field of characteristic 0) which are defined (locally) by an element in the ring $\mathbf{K}[[x_1, \ldots, x_d]][z]$ that we see as a polynomial in the variable z. Thanks to the Abhyankar-Jung theorem[**4**, **78**], we know many properties of the roots of such a polynomial (in particular they can be represented as generalized Puiseux series) and one can use these properties to introduce invariants (characteristic pairs, semigroup, Lattices) of the singularity [**61**, **82**, **95**, **96**]; these are very powerful invariants that actually determine and are determined by the topological type of the singularity [**57**]. **In [31], we were able to determine in terms of these invariants the irreducible components of the jet schemes of a quasi-ordinary hypersurface singularity of dimension** 2. **We determined the geometries and the dimensions of open dense subsets of these irreducible components, which happen to be isomorphic to affine spaces or to trivial fibrations over some (non-normal) toric varieties which encode deeply the geometry of quasi-ordinary singularities defined by the approximate roots of our singularity; in particular they encode the geometry of the singularity itself.** Note that approximate roots are roughly speaking equisingular to suitable truncations of a root of a polynomial defining a quasi-ordinary singularity. We were able to prove:

THEOREM II.I. *(Cobo, Mourtada)*
A subgraph of the jet components graph of an irreducible quasi-ordinary hypersurface singularity $(S, 0)$ of dimension two determines and is determined by the embedded topological type of $(S, 0)$.

Note that the subgraph mentioned in the theorem is completely characterized by the intrinsic structure of the jet component graph. We show in figure 4 a part of the subgraph that appears in the theorem for a singularity whose singular locus has two irreducible components, a plane branch and a line. We do not put the weights here in order not to charge the picture. Here the arrows represent an infinite projective system of irreducible components.

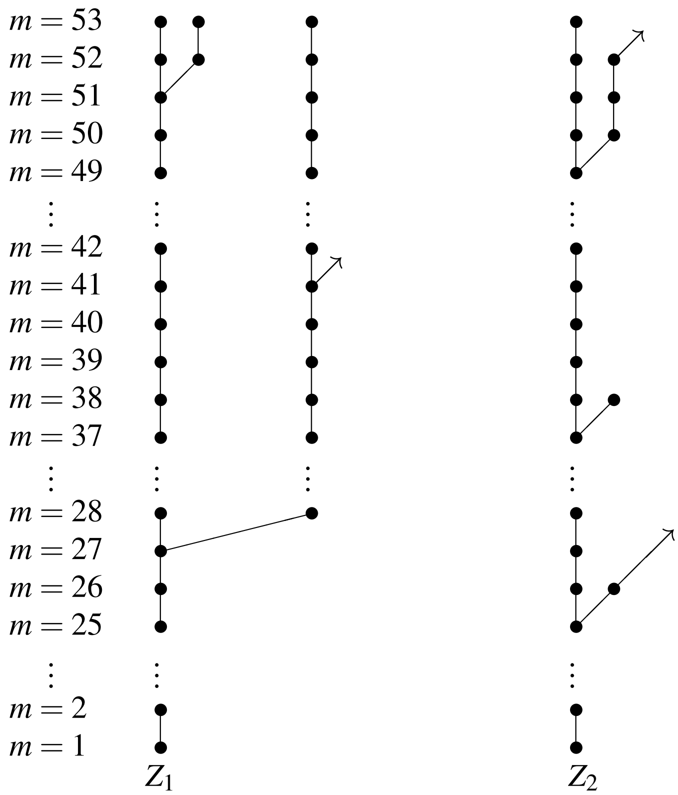

FIGURE 1. The jet components graph of the surface singularity defined by $f = ((z^2 - x_1^3)^2 - x_1^7 x_2^3)^2 - x_1^{11} x_2^5 (z^2 - x_1^3)$. Z_1 and Z_2 are the two irreducible components of the singular locus.

Here we would like to stress the fact that we are able to determine the topological type of the singularity by studying only how the geometry of the irreducible components varies in projective systems. Theorem II.I contains two very delicate results: the determination of the irreducible components and the subgraph of the jet components graph on one hand, and the fact the this subgraph determines the embedded topological type of the singularity; this is to compare with the motivic invariants which do not determine it [62].

The theorem, as we said before, partially answers the first question above and completely answers the second question. It only partially answers the first question because we don't determine all the edges in the jet components graph. This is related with and gives different and new insight on the generalized Nash problem [51, 74] which is an exciting problem and a subject of actual and future researches.

We also gave in [31] examples of quasi-ordinary surface singularities embedded in $\mathbf{A}^3$ whose log canonical threshold (this is an important invariant of singularities of pairs which is contributed by a divisorial valuation on a log resolution) is not obtained by a monomial valuation in any coordinates (For plane curves, the log canonical threshold is always contributed by a monomial valuation, up to change of coordinates).

An important observation that one can made about the geometry of the irreducible components of the jet schemes of a quasi-ordinary surface singularity is the following: the graded algebra of a divisorial valuation centered at $0 \in \mathbf{A}^3$ and which is associated with such an irreducible component can be, after forgetting the grading, embedded in a Noetherian algebra which is somehow universal for all the divisorial valuations obtained in this way. This statement will be made sharper and clearer in section 7.

Normal toric surface singularities are the simplest normal toric singularities. Such a singularity is simply given by the data of two coprime numbers, its embedding dimension can be as high as one wishes and hence it can be defined by a very large number of equations; moreover, apart from the case of the A_n singularities (which are hypersurfaces in $\mathbf{A}^3$) they are never locally complete intersections: this latter hypothesis is essential for many theorems involoving jet schemes [**52**, **118**]. The structure of the jet schemes of toric singularities or of their irreducible components are not known in general [**117**] and determining this structure seems to be a difficult problem. We think that our work on toric surfaces singularities is a significant advance in this direction. For instance, we have determined the irreducible components of the jet schemes of these singularities and as in the case of quasi-ordinary surface singularities in $\mathbf{A}^3$, we determined a subgraph of the jet components graph which encodes almost completely the singularity:

THEOREM II.J. *(Mourtada) The jet components graph determines the analytical type of a normal toric surface singularity in the following sense: two normal toric surface singularities are isomorphic if and only if they have the same jet components graph.*

It is worth noting here that Motivic type invariants do not catch the analytic type ([**91**, **123**]).

The proof of theorem II.J uses heavily the description of the defining equations of the embedding $S \subset \mathbf{A}^e$ of the surface singularity $(S,0)$ ([**141**],[**148**]), and some syzygies of these equations that we describe and that are ad hoc to the problem. It also uses known results on the arc space of a toric variety [**89**],[**75**],[**72**] and we reason by induction on m (the level of the jet scheme) and on the embedding dimension e. In particular we use a kind of approximation of the toric surface S by toric surfaces with smaller embedding dimensions. The irreducible components of the jet schemes of toric surface singularities were discovered in my Ph.D. thesis [**110**] but the complete understanding of their structure and its presentation were only completed in 2017, see [**107**].

5. Embedded Nash problem

This section somehow goes beyond problematics (1) and (2) since we assume here that we have a torific embedding; we actually consider singularities which are Newton non-degenerate and we are wondering whether one can recover an embedded resolution of singularities from the irreducible components of the jet schemes and whether such a resolution satisfies some minimality properties. All the singularities here are hypersurface singularities of dimension 2 and can be resolved by a toric morphism. An important family of toric resolutions of such a singularity $X = \{f = 0\} \subset \mathbf{A}^3$ is obtained from a regular subdivision of its Newton dual fan $D\Gamma(f)$ [**104**, **126**, **160**]; in other words, the Newton dual fan defines a toric threefold $Z_{D\Gamma(f)}$, of which any toric resolution of singularities gives an embedded resolution of our surface singularity. Infinitely many embedded resolutions of X are obtained in this way. But, one should note here, that in contrast to the abstract resolution case, there is no notion of (universal) minimal resolution of singularities. Still something similar exists: by [**10**, **26**], any three dimensional toric singularity can be resolved (abstractly) by a toric morphism such that the irreducible components of its exceptional locus correspond to essential divisorial valuations (valuations whose center on every resolution of singularities is an irreducible component). Such a resolution of a toric variety is called a G-desingularization ; in higher dimension, this is not true anymore [**26**, **27**]. In [**108**, **111**], we have defined the notion of a minimal toric embedded resolution of X :

DEFINITION 6. A minimal toric embedded resolution of X is a G-desingularization of $Z_{D\Gamma(f)}$.

Such a minimal resolution is not unique; one can pass from a minimal resolution to another one by flips and flops [**103**, **139**]; but the divisorial valuations associated with the components of the exceptional divisors are the same in every minimal toric embedded resolution. Note also that this notion of minimality exists only for hypersurfaces in $\mathbf{A}^3$ but one can define the notion of a toric embedded essential divisor in the same way for any non-degenerate singularity. So a minimal toric embedded resolution of X is a resolution where every irreducible component of the exceptional divisor corresponds to a toric embedded essential divisor over $\mathbf{A}^3$. As in the case of the classical Nash problem, we can search for the common information to all the embedded resolutions of singularities in the jet schemes. We have proved:

THEOREM II.K. *(Mourtada, Plénat,* [**108**, **111**]*) For a rational double point singularity X, the irreducible components of the jets schemes centered at the singularity and whose associated valuations are monomial valuations, are in a bijective correspondence with the divisors that appears on a minimal embedded toric resolution.*

The notion of minimality in the theorem is the one given in definition 6. In general, such a statement is hopeless: indeed, even for an irreducible plane curve singularity (say, the cusp $\{y^2 - x^3 = 0\} \subset \mathbb{A}^2$), the irreducible components of the jet schemes centered at the origin give divisorial valuations which do not appear on the minimal embedded resolution of the curve singularity (in that case, the minimal embedded resolution makes sense and is unique). The right question can be fixed by considering some convex geometry that we do not discuss at this moment. But the moral of the story is that we have "non-useful" divisorial valuations that come from the irreducible components of the jet schemes but that all the useful ones come from the jet schemes. So, in [79] we ask the following question:

($\star$) *Can we construct an embedded resolution of singularities of $X \subset \mathbb{A}^n$ from the irreducible components of the spaces X_m^{Sing} of jets centered at the singular locus of $X \subset \mathbb{A}^n$?*

The answer to ($\star$) is no in general. Indeed, consider the three-dimensional variety defined by

$$X = \{x^2 + y^2 + z^3 + w^5 = 0\} \subset \mathbb{A}^4.$$

It has a unique singularity at the origin 0. On one hand, with a direct computation, we see that the jet schemes X_m^0 centered at 0 are irreducible for every $m \geq 1$. On the other hand, we have two exceptional (irreducible) divisors that appear on every embedded resolution of the singularity (at least those which are an isomorphism outside of the singular locus) of X; these are the divisors associated with the monomial valuations on $k[x,y,z,w]$ defined by the vectors $(1,1,1,1)$ and $(2,2,2,1)$. The valuation associated with the vector $(2,2,2,1)$ does not correspond to any of the schemes X_m^0, $m \geq 1$. Note that this example is one of the counterexamples to the Nash problem given in [77]; note also that the Nash correspondence is bijective in dimension 2 [41, 46]. This suggests that a reasonable frame to study the question ($\star$) is the case of surface singularities.

We studied the question ($\star$) for a family of hypersurface singularities whose normalizations are rational triple point singularities (RTP-singularities, for short). These hypersurfaces were discovered in [13] and are called the non-isolated forms of RTP-singularities. We prove that, for such singularities, the answer to ($\star$) is **yes we can**. When X is of that type, we determined again a natural family of irreducible components of X_m^{Sing}, $m \geq 1$ whose associated divisorial valuations are monomial, hence defined by some vectors in $\mathbf{N}^3$. For all of the non-isolated forms of RTP-singularities except when X is of type $B_{k-1,2l-1}$, we showed that these vectors give a regular subdivision Σ of the dual Newton fan of X and hence a nonsingular toric variety Z_Σ; since our singularities are Newton non-degenerate, this gives a birational toric morphism $Z_\Sigma \longrightarrow \mathbb{A}^3$ which is an embedded resolution of $X \subset \mathbf{A}^3$; the irreducible components of the exceptional divisor correspond to the natural set of irreducible components of X_m^{Sing}. When X is of type $B_{k-1,2l-1}$, we again build a toric embedded resolution from the irreducible components of the jet schemes which does not factor through the toric morphism associated with the dual Newton fan. Note that such resolutions were considered by Leyton-Alvarez in his study for embedded resolution in family. This again shows mysteriously that the jet schemes tell something about the "minimality" of the embedded resolution, as in the case of rational double point singularities. One can summarize this in the following theorem; see [79].

THEOREM II.L. *(Karadeniz, Mourtada, Plénat, Tosun) The irreducible components of the jet schemes of non-isolated forms of rational triple point singularities give a resolution of singularities.*

6. On the notion of quasi-ordinary singularity in positive characteristics

It is more involved to explain how the work reported in this section fits in the general framework of problematics (1) and (2). It goes somewhat in a path parallel to what appeared before in this memoir. This will be probably clearer in the next section on valuations, but a rough explanation would be that the role played by jet schemes in the search for torific embeddings is now played by polyhedra. The issue is that, as we have seen in the section 5, the irreducible components of the jet schemes do not catch in general all the common information to all embedded resolutions of singularities. Polyhedra catch information about constructible subsets of the jet schemes. Here the word polyhedra refers to Newton polyhedra:

The Newton polyhedron is a convex body associated with the geometry of the defining equations of a singularity (i.e. using the exponents of the monomials appearing in the defining equations of a subvariety of the affine space). Despite their extrinsic nature, cleverly used, they encode subtle information about a subvariety or a singularity. Introduced by Newton [**121**], *Hironaka was a pioneer in their use to resolve singularities* [**67**].

The important issue here is the use of higher Newton polyhedra or Newton polyhedra after embedding in an affine space of higher dimension. We show how this idea allows to characterize quasi-ordinary singularities in characteristic 0 (see below) and to introduce a highly interesting counterpart in positive characteristics, Teissier singularities.

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0. A Weierstrass polynomial $f \in \mathbf{K}[[x_1,\dots,x_d]][z]$, satisfying $f(0) = 0$, is called quasi-ordinary if its discriminant with respect to z is a unit times a monomial in $\mathbf{K}[[x_1,\dots,x_d]]$ (we refer to this condition as the discriminant condition). Note that we have a finite map $(\{f = 0\},0) \longrightarrow (\mathbf{K}^d,0)$ (the projection on the first d coordinates) and that its ramification locus is the zero locus of the discriminant. If $\{f = 0\}$ is singular at the origin, we say that $0 \in \{f = 0\}$ is a quasi-ordinary hypersurface singularity. Quasi-ordinary singularities appear in Jung's method of resolution of singularities in char 0 [**78**]: for any $f \in \mathbf{K}[[x_1,\dots,x_d]][z]$, this method is recursive on the dimension and consists in using embedded resolution of singularities $\phi : Z \longrightarrow \mathbf{K}^d$ in dimension $d-1$, to transform the discriminant of f into a normal crossing divisor (locally a unit times a monomial). The pull back of $\{f = 0\}$ by ϕ will then have only quasi-ordinary singularities and the resolution problem is reduced to the problem of resolution of quasi-ordinary singularities and then patching these local resolutions. The first complete proof of resolution of singularities for algebraic surfaces in characteristic 0 was based on this approach ; see [**164**].
We first give a characterization of quasi-ordinary singularities in terms of an invariant of f, which we denote by $\kappa(f)$ (see [**113**]) and which we construct using a weighted version of Hironaka's characteristic polyhedron and successive embeddings of the singularity defined by f in affine spaces of higher dimensions; this invariant is inspired on one hand by resolution

invariants in char 0 (and p) [24, 85] and on the other hand by Teissier's conjecture on resolution of singularities by changing the embedding [153]. Note that Hironaka's characteristic polyhedron is a projection of the classical Newton polyhedron, but it has some intrinsic properties thanks to the minimizing process explained in [67]. **We actually have introduced a weighted version of an iterated Hironaka's characteristic polyhedron and proved its existence, which is subtler than the original proof of Hironaka**. The invariant $\kappa(f)$ is a string whose components are all in $\mathbf{Q}_+^d$, except of the last one which is either -1 or ∞. The size of $\kappa(f)$ depends on f. Let us give a feeling of this invariant: we expand f as follows

$$f = z^n + a_1 z^{n-1} + \cdots + a_n = z^n + \sum C_{A,b}\mathbf{x}^A z^b,$$

where $\mathbf{x} = (x_1, \ldots, x_d)$. With an irreducible f and the variables $(x_1, \ldots, x_d, z)$, one associates the polyhedron

$$\mathcal{H}_{(x_1,\ldots,x_d,z)} = \text{Convex hull}\left\{\frac{A}{n-b} + \mathbf{R}_+^d \mid C_{A,b} \neq 0\right\}. \tag{6.1}$$

By a theorem of Hironaka [67], there exists a change of variables of the type

- $z \longrightarrow z + s(\mathbf{x}), s(\mathbf{x}) \in \mathbf{K}[[\mathbf{x}]]$,
- $x_i \longrightarrow x_i + h_i(\mathbf{x}), h_i(\mathbf{x}) \in \mathbf{K}[[\mathbf{x}]]$,

which minimizes $\mathcal{H}_{(x_1,\ldots,x_n,z)}(f)$ with respect to the inclusion among all polyhedra obtained after such changes of variables (these changes of variables are compatible with the projection $(\mathbf{x}, z) \longrightarrow (\mathbf{x})$). The minimal polyhedron is then denoted by $\mathcal{H}_0$ and called the characteristic polyhedron.

We define the first component κ_1 of $\kappa(f)$ depending on the following three possible cases:

$$\kappa_1 := \begin{cases} v_1, & \text{if } \mathcal{H}_0 \text{ is the positive orthant with vertex } v_1, \\ \infty, & \text{if } \mathcal{H}_0 = \varnothing \text{ is empty}, \\ -1, & \text{else.} \end{cases}$$

In the last two cases, $\kappa(f)$ has only one component κ_1. In the first case, we define the first component of $\kappa(f)$ by $\kappa_1 := v_1$ and we consider the initial form of f with respect to v_1

$$\text{In}_{v_1}(f) := z^n + \sum_{\frac{A}{n-b}=v_1} C_{A,b}\mathbf{x}^A z^b.$$

The initial form satisfies $\text{In}_{v_1}(f) = (z^{n_1} - c_1\mathbf{x}^{A_1})^{e_1}$. The fact that $\text{In}_{v_1}(f)$ is a product of binomials follows from the fact that its support is a segment; its reduced form has only one factor $(z_1^n - c_1\mathbf{x}^{A_1})$ because we assumed that $f \in \mathbf{K}[[\mathbf{x}]][z]$ is irreducible; see [58]. Note that $n_1 > 1$, otherwise we will have a contradiction with the minimality of $\mathcal{H}_1$ since after a change of variables $\tilde{z} = z^{n_1} - c_1\mathbf{x}^{A_1}$ we obtain a "smaller polyhedron"; this implies $e_1 < n$. We introduce a new variable u_1 and we modify f by substituting any n_1-th power of z (i.e. z^{n_1}) by $u_1+c_1\mathbf{x}^{A_1}$: the obtained polynomial f_1 can be thought as a representative of the class of f in

$$\frac{\mathbf{K}[[\mathbf{x}]][z,u_1]}{(u_1-(z^{n_1}-c_1\mathbf{x}^{A_1}))}.$$

The polynomial f_1 has the shape

$$f_1=u_1^{e_1}+\sum C_{A,b,c}\mathbf{x}^A z^b u_1^c.$$

We associate with it a new "weighted" characteristic polyhedron as follows:

$$\mathcal{H}_{(x_1,\dots,x_n,z,u_1)}(f_1)=\text{Convex hull}\left\{\frac{A+bv_1}{e_1-c}+\mathbf{R}_+^d \mid C_{A,b,c}\neq 0\right\}.$$

It is very subtle and carefully explained in [**115**] how to minimize this weighted characteristic polyhedron; we call this minimum $\mathcal{H}_1$; it generalizes Hironaka's characteristic polyhedron. Again, we define the second component κ_2 of $\kappa(f)$ depending on the following three possible cases:

$$\kappa_2:=\begin{cases} v_2, & \text{if } \mathcal{H}_1 \text{ is orthant with vertex } v_2 \text{ (there is also a condition on } \mathrm{In}_{v_2}(f_1)\text{, see below)},\\ \infty, & \text{if } \mathcal{H}_1=\varnothing \text{ is empty},\\ -1, & \text{else.}\end{cases}$$

In the last two cases, $\kappa(f)$ has only two components (v_1,v_2). In the first case, we consider the initial form of f_1 with respect to v_2

$$\mathrm{In}_{v_2}(f_1):=u_1^{e_1}+\sum_{\frac{A+bv_1}{e_1-c}=v_2} C_{A,b,c}\mathbf{x}^A z^b.$$

This latter is not in general a power of a binomial, but we can prove that it is so modulo the equation $u_1-(z_1^n-c_1\mathbf{x}^{A_1})=0$. If its reduced form has more than one factor, then we put $\kappa(f)=(v_1,-1)$. Otherwise $\mathrm{In}_{v_2}(f_1)$ is of the form

$$(u_1^{n_2}-c_2\mathbf{x}^{A_2}z^{b_2})^{e_2};$$

in which case the first two components of $\kappa(f)$ are respectively v_1 and v_2; we continue playing the same game by introducing a new variable u_2 and modifying f_1 by substituting any n_2-th power of u_1 (i.e. $u_1^{n_2}$) by $u_2+c_2\mathbf{x}^{A_2}z^{b_2}$: the obtained polynomial f_2 can be thought as a representative of the class of f in

$$\frac{\mathbf{K}[[\mathbf{x}]][z,u_1,u_2]}{(u_1-(z_1^n-c_1\mathbf{x}^{A_1}),u_2-(u_1^{n_2}-c_2\mathbf{x}^{A_2}z^{b_2}))}.$$

Again, the minimality of the polyhedron $\mathcal{H}_2$ implies that $n_2>1$ and hence $e_2<e_1$. Thus, we are constructing a strictly decreasing positive integers sequence $n>e_1>e_2>\cdots$ which is then finite and which allows to determine $\kappa(f)$. In [113] we prove the following theorem:

THEOREM II.M. *(Mourtada, Schober) Let f be as above. The singularity $(\{f=0\},0)$ is quasi-ordinary with respect to the projection $(\{f=0\},0) \longrightarrow (\boldsymbol{K}^d,0)$ if and only if the last component of $\kappa(f)$ is ∞.*

It is worth mentioning that when $\mathbf{K}=\mathbf{C}$, we prove that the invariant $\kappa(f)$ is a complete invariant of the embedded topological type of $(\{f=0\},0) \subset (\mathbf{C}^{d+1},0)$.

On the one hand, while we know how to resolve quasi-ordinary singularities (in char 0), in positive characteristics, the singularities which satisfy the discriminant condition can be extremely wicked; e.g., hundreds of pages of of Cossart-Piltant's proof of resolution in dimension 3 are dedicated to this type of singularity (see [32] for the arithmetical case); the study of these singularities by Abhyankar has led to the celebrated Abhyankar conjecture in modular Galois theory [3, 64, 136, 144]. So, in positive characteristics, the reduction of the resolution of singularities problem to the singularities satisfying the discriminant condition cannot be compared with Jung's approach in characteristic 0. On the other hand, while in characteristic 0, the last component of $\kappa(f)$ being ∞ is equivalent to f being quasi-ordinary, in characteristic p, the invariant $\kappa(f)$ is still meaningful but the condition on its last component gives rise to a different condition than the one given by the discriminant. This leads us to define the following class of singularities [115]:

DEFINITION 7. (Mourtada, Schober) Let $\mathbf{K}$ be an algebraically closed field of characteristic $p>0$. Let $f \in \mathbf{K}[[x_1,\ldots,x_d]][z]$ satisfying $f(0)=0$. The hypersurface singularity $(X,0)=\{f=0\}$ is a **Teissier singularity** if the last component of $\kappa(f)$ is ∞.

The name "Teissier singularities" was suggested by the fact that Teissier proved that along an Abhyankar rational valuation, any hypersurface singularity can be embedded in a higher dimension affine space with a special type of equations that define an "overweight deformation" whose generic fiber is isomorphic to the singularity and whose special fiber is the toric variety associated with the graded algebra of the valuation [150]. For a Teissier singularity, all the valuations which extend rational monomial valuations on $\mathbf{K}[[x_1,\ldots,x_d]]$ to $\mathbf{K}[[x_1,\ldots,x_d]][z]/(f)$ induce the "same overweight deformation" and this property characterizes them [39]. Teissier singularities do not satisfy the discriminant condition in general, and a singularity satisfying the discriminant condition is not Teissier in general. But these singularities provide a very good positive characteristic counterpart of quasi-ordinary singularities thanks to the following result [115], where $O_{\mathbf{C}_p}$ is the valuation ring of the completed algebraic closure of $\mathbf{Q}_p$:

THEOREM II.N. *(Mourtada, Schober) A Teissier singularity $(X,0)$ sits in an equisingular family $\mathcal{X}$ over $\mathrm{Spec}(O_{\mathbf{C}_p})$ as a special fiber, and the generic fiber of $\mathcal{X}$ has only quasi-ordinary singularities.*

Note that the generic fiber is defined over a field of characteristic 0. Here, equisingular means that we have a simultaneous resolution of $\mathcal{X}$. Although Teissier singularities are complicated in general, we can resolve their singularities thanks to the understanding of the neighboring quasi-ordinary singularities. Note also that any quasi-ordinary singularity in characteristic 0 gives rise to a Teissier singularity.

Teissier singularities are candidates to play important roles in three directions: first, an alternative to Jung's approach to resolution of singularities where the role of quasi-ordinary singularities is played by Teissier singularities; for this, the major research direction is how to transform via birational proper maps any singularity into a space with only Teissier singularities: this looks like a combinatorial problem where we expect that the equisingular deformations that we have considered will contribute. Second, as it may be clearer after the section on valuations, along some "important" strata (their number is conjecturally finite) of the space of valuations, any singularity "is" Teissier; since these latter come with their torific embeddings, they give an approach of the torific embedding problem parallel to the one using jet schemes. Third, what about the "topology" of these singularities (their fundamental groups), in the continuity of Abhyankar's conjectures [64].

7. Valuations, defect and local uniformization

Let $\mathbf{F}$ be field and Φ an abelian totally ordered group. A valuation ν is a map $\nu : \mathbf{F} \longrightarrow \Phi \cup \infty$ satisfying the following properties for all $x, y \in \mathbf{F}$:

- $\nu(x) = \infty$ if and only if $x = 0$.
- $\nu(xy) = \nu(x) + \nu(y)$.
- $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$, with equality if $\nu(x) \neq \nu(y)$.

We will be mainly concerned with the case where $\mathbf{F}$ is a function field over an algebraically closed field $\mathbf{K}$ and where the valuation ν is trivial on $\mathbf{K}$ i.e., $\nu(\mathbf{K}^*) = \{0\}$. The valuation ring of ν is the local ring (R_ν, m_ν) whose elements are those $x \in \mathbf{F}$ having non-negative values (i.e. $\nu(x) \geq 0$). The residue field of ν is by definition the quotient R_ν/m_ν; we will denote it by k_ν. We say that ν dominates an algebraic local ring (R, m) if $R \subset R_\nu$ and $m_\nu \cap R = m$. In what follows we assume that the quotient field of R is $\mathbf{F}$. If X is a variety whose function field is $\mathbf{F}$, we say that ν has a center $p \in X$ if ν dominates the local ring $O_{X,p}$. Intuitively, a valuation with a center on X and value group $\mathbf{Q}$ can be thought as the order of contact with a "super" transcendental curve traced on X; "super" transcendental here means that no function $f \in O_{X,p}$ vanishes on this curve. Foundational results on valuations can be found in [2, 54, 86, 140, 159, 167].

Beside considering a super local version of torification, in this section we try to tell about meeting points between two famous problems in valuation theory; everybody knows that they meet (see [87]) even though these meetings and their circumstances are still vague and secret.

- The first one is the problem of extension of valuations and their ramification theory (mainly the defect problem) which finds its origin in algebraic number theory and in irreducibility questions like Eisenstein type irreducibility criteria [101, 128].

- The second is local uniformization and is closer to algebraic geometry:

  Local uniformization holds in dimension m if for every algebraic function field $\mathbf{F}$ of dimension (transcendence degree) m over an algebraically closed field $\mathbf{K}$ and for every valuation ν of $\mathbf{F}$ which is trivial over $\mathbf{K}$, there exists an algebraic local ring R of $\mathbf{F}$ which is regular and dominated by ν.
  Zariski found a clever patching argument [168] (which has been extended to positive characteristic and to other situations by Abhyankar [5] and Piltant [129]) which proves that local uniformization in dimension ≤ 3 implies resolution of singularities in dimension ≤ 3. However, there still is not a direct proof (even in characteristic zero) that a set of local uniformizations can be birationally modified so that they patch together to form a global (proper) resolution of singularities, unless you start out with such a strong version of resolution of singularities that patching becomes unnecessary.
  Local uniformization has been proven in all dimensions over characteristic zero ground fields $\mathbf{K}$ by Zariski [166] and has been proven in dimension ≤ 3 over ground fields $\mathbf{K}$ of characteristic $p > 0$ by Abyankar [5] and Cossart-Piltant [32]. A reasonably short proof of Abhyankar's result can be found in [37].

We will be interested in zero dimensional valuations (also called rational valuations when we work over an algebraically closed field $\mathbf{K}$); these are the valuations whose centers on any projective algebraic variety whose function field is $\mathbf{F}$ is a closed point (i.e. a $\mathbf{K}-$point, hence the name rational). These are the important valuations for the problem of local uniformization, see [169]. Most of the time we will be considering a rank one valuation ν (the rank of a valuation is the Krull dimension of its valuation ring or equivalently the cardinality of the set of isolated subgroups of its group); depending on the viewpoint, rank one valuations may be seen as the more relevant valuations for the problem of local uniformization, see [125]. Since $\mathbf{K}$ is assumed to be algebraically closed, $\mathbf{F}$ is a primitive extension of a rational function field , so we can assume that there is a hypersurface singularity whose local ring has $\mathbf{F}$ as a quotient field and is dominated by ν. In relation with section 6 as it will become apparent later, we begin by considering the case where this local ring is complete (see [43] and the references therein to estimate how strong is this hypothesis); we will drop this hypothesis later. Using the Noether normalization lemma, we can place ourselves in the following situation:

$$\mathbf{K}[[\mathbf{x}]] \longrightarrow \frac{\mathbf{K}[[\mathbf{x}]][z]}{(f)},$$

where ν dominates the local ring at the origin of $B := \frac{\mathbf{K}[[\mathbf{x}]][z]}{(f)}$ and extends a valuation ν_0 dominating $A := \mathbf{K}[[\mathbf{x}]]$; recall the notation $\mathbf{x} = (x_1, \ldots, x_d)$. We assume that $\nu_0 = \nu_\omega$ is a monomial valuation as in 1.1; here $\omega \in \mathbf{R}_+^d$ and its components are rationally independent. We are interested in the local uniformization of ν and in a "geometric" criterion to detect if ν_0 has a unique extension to B. It is shown in [**150**] that the local uniformization of Abhyankar valuations reduces to the local uniformization of such valuations ν which are extensions of monomial valuations. Note here that an Abhyankar valuation ν on $\mathbf{F}$ is a valuation for which the equality holds in Abhyankar's inequality:

$$\text{rrank}(\nu) + \text{trdeg}_{\mathbf{K}} k_\nu \leq \text{trdeg}_{\mathbf{K}} \mathbf{F};$$

rrank is the rational rank (which by definition is $\dim_{\mathbf{Q}}(\mathbf{Q} \otimes_{\mathbf{Z}} \Phi)$) and trdeg is the transcendence degree; Abhyankar's valuations play important roles in valuation theory and the study of valuation spaces. In [**150**], local uniformization for such a valuation ν is proved by determining the structure of $\text{gr}_\nu B$ which gives a "torification along the valuation" (other approaches to local uniformization of Abhyaknar valuations can be found in [**35, 83**]); this is done by analyzing the structure of the semigroup of $\nu(B)$. Here, we explain (very briefly) a different (more constructive) approach which is based on a local version of the invariant defined in 6; this will be later generalized to a wider class of valuations. First, let us go back to the definition of $\text{gr}_\nu B$ which is similar to the one that we gave in the preamble (section 1) in the case of divisorial valuations:

$$gr_\nu B = \bigoplus_{\phi \in \Phi_+} \frac{\mathcal{P}_\phi}{\mathcal{P}_\phi^+},$$

where

$$\mathcal{P}_\phi = \{h \in B \mid \nu(h) \geq \phi\},$$
$$\mathcal{P}_\phi^+ = \{h \in B \mid \nu(h) > \phi\}.$$

Determining the structure of $gr_\nu B$ means here determining its structure as a graded algebra over $gr_{\nu_0} A$ which is a graded polynomial ring with d variables whose weights are given by the components of ω. This structure is encoded in a generating series of ν (see definition 1). Using the Weierstrass preparation theorem, we can write (up to a multiplication by a unit)

$$f = z^n + a_1 z^{n-1} + \cdots + a_n = z^n + \sum C_{A,b} \mathbf{x}^A z^b.$$

It follows from [**150**] that we can assume, modulo changing the projection, that the characteristic of the field $\mathbf{K}$ does not divide n. We say then that the projection on the affine (it is better to say affinoid) space with coordinates $(x_1, \ldots, x_d)$ is tame. We consider as in section 6 Hironaka's characteristic polyhedron $\mathcal{H}_{(x_1,\ldots,x_d,z)}$ and the action of the linear function defined by ω on this polyhedron. Since ω has rationally independent components and the vertices of $\mathcal{H}_{(x_1,\ldots,x_d,z)}$ have integer components, the minimum

$$m=\min\{<\omega,A>;A\in\mathcal{H}_{(x_1,\ldots,x_d,z)}\}$$

where $<,>$ is the scalar product, is attained at a unique vertex of $\mathcal{H}_{(x_1,\ldots,x_d,z)}\}$ that we call V_1. Thinking of ω as a function on $\mathbf{R}^d$ we will write $\omega(A):=<\omega,A>$. We define the initial form of f with respect to ω to be

$$\mathrm{In}_{\omega}(f)=z^n+\sum_{\omega(\frac{A}{n-b})=m}C_{A,b}\mathbf{x}^Az^b=\mathrm{In}_{V_1}(f).$$

This latter is a product of binomials

$$\mathrm{In}_{\omega}(f)=z^r\prod_{i=1}^{\alpha_1}(z^{n_1}-c_{1,i}\mathbf{x}^{A_1})^{e_{1,i}},$$

where $V_1=\frac{A_1}{n_1},A_1\in\mathbf{N}^d,n_1,e_{1,i}\in\mathbf{N}$, and where the $c_{1,i}\in\mathbf{K}$ are distinct. There is a minimization of the polyhedron with respect to ω which ensures that $n_1>1$.

DEFINITION 8. We say that f is $0-\omega$ irreducible if the reduced form of $\mathrm{In}_{\omega}(f)$ has only one factor.

Assume that f is $0-\omega$ irreducible (hence $\mathrm{In}_{\omega}(f)=(z^{n_1}-c_1\mathbf{x}^{A_1})^{e_1}$). We introduce a new variable u_1 and as in section 6 we determine a representative f_1 of f in

$$\frac{A[z,u_1]}{(u_1-(z^{n_1}-c_1\mathbf{x}^{A_1}))}.$$

Note that the degree of f_1 as a polynomial in u_1 is $e_1<n$. We repeat the same game with f_1 by associating to z the value $\omega(V_1)$. The minimum of ω on the (weighted-) minimized polyhedron (which is also included in $\mathbf{R}^d_+$ and defined in a similar way to the one in section 6 apart from the fact that the minimization is more local with respect to ν_0) of f_1 is attained at a vertex V_2. Modulo the equation $u_1-(z^{n_1}-c_1\mathbf{x}^{A_1})=0$, we have that $\mathrm{In}_{V_2}(f_1)$ is a product of binomials and we say that f is $1-\omega$ irreducible if the reduced form of $\mathrm{In}_{V_2}(f_1)$ has only one factor, i.e. $\mathrm{In}_{V_2}(f_1)=(u_1^{n_2}-c_2\mathbf{x}^{A_2}z^{b_{20}})^{e_2}$. Noticing that the sequence of degrees, degree of f, the degree of f_1, the degree of f_2 (that we define along the same idea if f is $1-\omega$ irreducible), $\cdots$ is a strictly decreasing sequence, our procedure stops either if after g iterations we have that $\mathrm{In}_{V_g}(f_{g-1})$ has more then one factor, or if we get $n=n_1n_2\cdots n_g$. **In this last case, we say that f is $\infty-\omega$ irreducible**. In [**39**], we prove the following (we complete the explanation of the notations in the theorem immediately after its statement):

THEOREM II.O. *(Cutkosky, Mourtada, Teissier) We have the following:*

(1) The valuation $\nu_0=\nu_{\omega}$ defined on $A=\boldsymbol{K}[[\mathbf{x}]]$ has a unique extension ν to $B=\frac{\boldsymbol{K}[[\mathbf{x}]][z]}{(f)}$ if and only if f is $\infty-\omega$ irreducible.

(2) *The system* $(\mathbf{x}, z, u_1, \ldots, u_{g-1})$, *seen as elements in B after substituting the* u_i'*s by their expressions in the* $\mathbf{x}$ *and z, is a generating series of* ν.

(3) *The ideal generated by*

$$u_1 - (z^{n_1} - c_1\mathbf{x}^{A_1}) + \sum_{\star} c_{1\star}\mathbf{x}^{A_{1\star}}z^{b_{1\star}},$$

$$u_2 - (u_1^{n_2} - c_2\mathbf{x}^{A_2}z^{b_2}) + \sum_{\star} c_{2\star}\mathbf{x}^{A_{2\star}}z^{b_{2\star}}u_1^{d_{2\star}},$$

$$\vdots$$

$$u_{g-1} - (u_{g-2}^{n_{g-1}} - c_{g-1}\mathbf{x}^{A_{g-1}}z^{b_{g-1}}\boldsymbol{u}_{\leq g-3}^{\boldsymbol{b}_{g-1}}) + \sum_{\star} c_{(g-1)\star}\mathbf{x}^{A_{(g-1)\star}}z^{d_{(g-1)\star}}\boldsymbol{u}_{\leq g-2}^{\boldsymbol{d}_{(g-1)\star}},$$

$$(u_{g-1}^{n_g} - c_{g-1}\mathbf{x}^{A_g}z^{b_g}\boldsymbol{u}_{\leq g-2}^{\boldsymbol{b}_g}) + \sum_{\star} c_{g\star}\mathbf{x}^{A_{(g)\star}}z^{d_{g\star}}\boldsymbol{u}_{\leq g-1}^{\boldsymbol{d}_{g\star}}$$

in $\boldsymbol{K}[[\mathbf{x}]][z, u_1, \ldots, u_{g-1}]$ *determines an embedding* $\mathrm{Spec}(B) \longrightarrow \boldsymbol{A}^{d+g}$. *This embedding is a "torification" along* ν, *i.e., there exists a toric structure on* $\boldsymbol{A}^{d+g}$ *and a birational proper toric morphism* $Z \longrightarrow \boldsymbol{A}^{d+g}$ *which gives an embedded local uniformization of* ν.

(4) *We have*

$$gr_\nu B = \frac{\boldsymbol{K}[[\boldsymbol{X}]][Z, U_1, \ldots, U_{g-1}]}{(Z^{n_1} - c_1\boldsymbol{X}^{A_1}, U_1^{n_2} - c_2\boldsymbol{X}^{A_2}Z^{b_2}, \ldots, U_{g-1}^{n_g} - c_{g-1}\boldsymbol{X}^{A_g}Z^{b_g}\boldsymbol{U}_{\leq g-2}^{\boldsymbol{b}_g})}$$

which is a graded $\boldsymbol{K}$*−algebra and whose grading is induced by the weights of the variables; weight(*X_i*)* $= \nu(x_i) = \omega_i$ *the i−th component of* ω; *weight(*Z*)* $= \nu(z) = \omega(V_1)$, *and weight(*U_i*)* $= \nu(u_i) = \omega(V_{i+1})$.

In the theorem, all the coefficients which are denoted by c (with an index) are in $\mathbf{K}$; the symbol $\mathbf{u}_{\leq i}^{\mathbf{e}}$ de notes a monomial of the form $u_1^{e_1} \cdots u_i^{e_i}$. The power series denoted by

$$\sum_{\star} c_{i\star}\mathbf{x}^{A_{i\star}}z^{d_{i\star}}\mathbf{u}_{\leq i}^{\mathbf{d}_{i\star}},$$

appear when minimizing the polyhedra that we considered. The first assertion of this theorem can be seen in the continuity of the irreducibility criteria like Eisenstein criterion; indeed the uniqueness of the extension of ν_0 is equivalent to the irreducibility of f in $R_{\nu_0}^h[z]$ where $R_{\nu_0}^h$ is the henselization of the valuation ring R_{ν_0} of ν_0. The important novelty in this theorem is that it gives a constructive approach to determine in this setting a generating sequence which is a system of polynomials. This is to compare with MacLane approach which studies extensions of valuations from a field $\mathbf{F}$ to $\mathbf{F}[z]$ (were z may be algebraic over $\mathbf{F}$); the notion of key polynomials (introduced by MacLane and then generalized by Vaquié, Spivakovsky ... [48, 100, 157]) allows to determine generators for $gr_\nu\mathbf{F}[z]$, which will not be honest polynomials but rational functions. Both approaches meet when we have only one variable i.e. $d = 1$. Before mentioning a generalization of this algorithm in [**40**], let us go back for a second to quasi-ordinary singularities (see section 4 or 6 for the definition) that we can now characterize by a valuative criterion [**39**]. First assume that ν_ω has a unique extension to $\frac{\mathbf{K}[[\mathbf{x}]][z]}{(f)}$ denoted by μ_ω. We have:

THEOREM II.P. *(Cutkosky, Mourtada, Teissier) Let* $\boldsymbol{K}$ *be a field of characteristic* 0 *and* f *an irreducible Weierstrass polynomial in* $A[z]$ *such that* $f(0) = 0$. *The singularity at the origin of* $\{f = 0\}$ *is quasi-ordinary with respect to the projection induced by* $A \longrightarrow B = \frac{A[z]}{(f)}$ *if and only if for every* ω*, the valuation* ν_ω *has a unique extension* μ_ω *and* $gr_{\mu_\omega} B$ *is independent of* ω *in* $\boldsymbol{R}_+^d$ *(always with rationally independent components) if we forget the grading, and the grading varies linearly as a function on* $\boldsymbol{R}_+^d$.

Note that the graded algebra $gr_{\mu_\omega} B$ in the theorem is Noetherian and that it is a quotient of a polynomial ring (see part (4) of theorem II.O); its grading is induced by a weight function $W : \mathbf{R}_+^d \longrightarrow \mathbf{R}_+^{d+g}$. Theorem II.P insures that this function is linear (and continuous) on $\mathbf{R}_+^d$. Recall that in theorem II.O, we have the assumption that the projection is tame. This, as we mentioned before, when considering one valuation μ_ω can be achieved by changing the projection [**150**]. Theorem II.P works in the same way for Teissier singularities in the tame case and we think that it is true in the wild (non-tame) case; the reason why we do not have a complete answer in the wild case is that our algorithm to determine a generating series does not work in the same way. This is a current research theme. For the moment, we know how to obtain a generating series after a blowing up.

We now come to the link with the geometric approach to resolution of singularities. Roughly speaking, in section 6, we have determined a torification of Teissier singularities. Theorem II.P and the same assertion conjectured for Teissier singularities in positive characteristic, about which we are optimistic, tell us that this torification is also given by the torification along an extension of ν_ω for any ω as above in $\mathbf{R}_+^d$. In general, for a non-necessarily Teissier singularity, $gr_{\mu_\omega} B$ will vary with ω but we expect that there will be a finite number of sectors in $\mathbf{R}_+^d$ (depending on the discriminant) and that on each of these sectors, $gr_{\mu_\omega} B$ is independent of ω in the sector if we forget the grading, and the grading varies linearly as a function on the sector. The torification along μ_ω is independent of the choice of ω in a sector. The idea is that finding a torification can be reduced to finding a torification along μ_ω for an ω in each sector. This is in line with the generalization of Jung's approach to resolution of singularities that we mentioned at the end of section 6.

The two different ways towards torification, the one explained here and the one explained in the preamble 1 give similar results in the case of quasi-ordinary singularities. Indeed, one possible reading of the part on quasi-ordinary singularities in section 4, is that the geometry of the irreducible components of the jet schemes (which determine divisorial valuations) is determined by the approximate roots of the quasi-ordinary polynomial; moreover section 2 explains how one can obtain form the geometry of such an irreducible component (or more precisely from its equations) the generating series of the associated divisorial valuations; it follows that the generating series of any such divisorial valuation is determined by the approximate roots. This is a counterpart of theorem II.O for divisorial valuations.

Concerning the problem of determining a generating series of a valuation, we have found an algorithm which actually is a "realization" of Maclane's algorithm and which generalizes the algorithm of theorem II.O; it "recursively" determines in many situations a generating sequence of a valuation; the term recursively will be clearer after the theorem. We do not give the details of the algorithm here, see [**40**].

THEOREM II.Q. *(Cutkosky, Mourtada, Teissier) Let (A,m) be a local domain which contains an algebraically closed field $\boldsymbol{K}$ such A/m is isomorphic to $\boldsymbol{K}$. Let $\boldsymbol{Q}$ be the quotient field of A and suppose that ν_0 is a valuation of $\boldsymbol{Q}$ which dominates A, such that its residue field k_{ν_0} is $\boldsymbol{K}$. Suppose that $f(z)$ is unitary and irreducible, that there is a unique extension of ν_0 to a valuation ν of $\boldsymbol{Q}[z]/(f(z))$ and the characteristic p of $\boldsymbol{K}$ does not divide the degree of f in z. Then there exists a realization of Maclane's Key polynomials algorithm which gives a generating series of ν.*

Moreover, the shape of the generating series is exactly as in theorem II.O. Before this theorem there have been very few results in this direction, apart the dimension two case (see [**80**]).

Now we go back to an invariant, which as we mentioned before, seems to capture the main difficulty for the problem of local uniformization: the defect. For simplicity, we only define this invariant in the unique extension case. Let $\mathbf{Q}$ be a field and let ν_0 be a valuation on $\mathbf{Q}$. Let $\mathbf{F}$ be a finite extension of $\mathbf{Q}$ and assume that ν is the unique extension of ν_0 to $\mathbf{F}$. We denote by $f(\nu/\nu_0)$ the residue degree $[k_\nu : k_{\nu_0}]$ and by $e(\nu/\nu_0)$ the ramification index $[\Phi_\nu : \Phi_{\nu_0}]$ where Φ_ν and Φ_{ν_0} are the value groups respectively of ν and ν_0. We have Ostrowski's formula

$$[\mathbf{F}:\mathbf{Q}] = f(\nu/\nu_0)e(\nu/\nu_0)p^{\delta(\nu/\nu_0)}.$$

The integer number $\delta(\nu/\nu_0)$ is by definition the defect. We say that ν/ν_0 is a defectless extension if $\delta(\nu/\nu_0) = 0$. Note that this invariant appears only in positive characteristics. In [**31**], we have proved:

THEOREM II.R. *(Cutkosky, Mourtada) The reduction of the multiplicity of a characteristic $p > 0$ hypersurface singularity along a valuation is possible if there is a finite linear projection which is defectless.*

This gives a simple proof of the fact that the only obstruction to local uniformization in positive characteristic is defect in finite projections of singularities. A different approach to theorem II.R can be found in [**143**]. It has been shown in [**158**] how one can detect the defect using key polynomials. It is more difficult to see the defect in the graded algebras over rings. In [**40**], we prove:

THEOREM II.S. *(Cutkosky, Mourtada, Teissier) Suppose that (A, m_A) is an excellent local domain which contains an algebraically closed field $\boldsymbol{K}$ such that $A/m_A = \boldsymbol{K}$. Let $\boldsymbol{Q}$ be the quotient field of A and suppose that ν_0 is a rank 1 valuation of $\boldsymbol{Q}$ which dominates A and such that the residue field of the valuation ring of ν_0 is $\boldsymbol{K}$. Suppose that $f(z) \in A[z]$ is unitary, irreducible and separable and ν a valuation of $\boldsymbol{K}[z]/(f)$ which extends ν_0. Then ν is defectless over ν_0 if and only if there exists a normal birational extension A_1 of A which is dominated by ν and such that $gr_\nu A_1[z]$ is a finitely generated and presented $gr_{\nu_0}(A_1)-$module.*

It is one of my current research themes to determine how to use graded algebras in order to quantify the defect on the level of the rings instead of fields. This would give us more intuition on how to overcome the defect in the local uniformization problem.

8. Arc spaces and integer partitions

This line of research again finds its origin in the study of singularities ([108]) as we will show later but is now making its way into the world of combinatorics and classical number theory, so let us begin there. The following identity

$$1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{\vdots}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2} \right) e^{\frac{2\pi}{5}} \tag{8.1}$$

was imagined by Ramanujan and sent to Hardy who says in the article "The Indian Mathematician Ramanujan" (Amer. Math. Monthly 44 (1937), p. 144), see also [14]:

"[These formulas] defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."

Some years later, Ramanujan gave a proof of this formula by considering the following $q-$difference equation

$$F(x) = F(xq) + xqF(xq^2), \tag{8.2}$$

where $q \in \mathbf{C}^*$, and $F(x) = \sum a_n(q)x^n$ is an analytic function satisfying $F(0) = 1$.

If we define $c(x,q) := \frac{F(x)}{F(xq)}$, notice that we have

$$c(x,q) = 1 + \frac{xq}{c(xq,q)} = 1 + \cfrac{xq}{1 + \cfrac{xq^2}{c(xq^2,q)}}.$$

Iterating this last identity we obtain that the left member of the identity (8.1) is equal to $c(1,e^{-2\pi})$. Now if we plug $F(x)=\sum a_n(q)x^n$ in the equation (8.2), by comparing the coefficients of x^n we get

$$a_n(q)=\frac{q^{n^2}}{(q)_n}=\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

The miracle arrives in the following identity

$$1+\sum_{n\geq 1}\frac{q^{n^2}}{(q)_n}\overset{\text{Miracle}}{=}\prod_{i\equiv\ 1,4\ \mathrm{mod}(5)}\frac{1}{1-q^i}. \tag{8.3}$$

The left hand side in the identity 8.3 is $F(1)$. There is another miracle which is that $F(q)$ is also an infinite product and hence $c(1,q)$ is. And we may then deduce Ramanujan's continued fraction 8.1 by an appeal to the theory of elliptic theta functions.

The "miracles" above are called the Rogers-Ramanujan identities; they have appeared "in many different situations": in statistical mechanics, number theory, representation theory ... and we came to them first with Clemens Bruschek and Jan Schepers via Arc spaces. Before telling the story, let us state another version of the first Rogers-Ramanujan identity (8.3).

DEFINITION 9. A partition of a positive integer n is a decreasing sequence $\Lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r)$ such that $\lambda_1+\cdots+\lambda_r=n$. The λ_i's are called the parts of this partition and r is its size.

The identity (8.3) can be stated as follows:

THEOREM. *(Rogers, Ramanujan)*
The number of partitions of n with neither consecutive parts nor equal parts (of first type) is equal to the number of partitions of n whose parts are congruent to 1 *or* 4 *modulo* 5 *(of second type).*

The generating series of the cardinals of the partitions of the first type is the left hand member in the identity 8.3 and the generating series of the cardinals of the partitions of the second type is the right hand member in 8.3, i.e. the infinite product. Now we go back to algebraic geometry and to arc spaces. Let $(X,0)$ be a singularity defined over a field $\mathbf{K}$ which is assumed for simplicity to be of characteristic 0 for (0 being a closed point that after a change of coordinates may be chosen to be the origin of an affine space containing $(X,0)$). Let $X_\infty^0=\mathrm{Spec}A_\infty^0$ be the space of arcs centered at the point 0. It has a natural cone structure which induces a grading on A_∞^0 (i.e., $A_\infty^0=\oplus_{h\in\mathbf{N}}A_{\infty,h}^0$) and one can consider its Hilbert-Poincaré series that we call the Arc-Hilbert-Poincaré series of the singularity:

$$\mathrm{AHP}_{X,0}=\sum_{h\in\mathbf{N}}\dim_{\mathbf{K}}A_{\infty,h}^0\,q^h.$$

It is not difficult to see that this is an invariant of singularities (it detects regularity) and it contains different ingredients which motivate its study from the viewpoint of singularity theory: First, if $X \subset \mathbf{A}^e$ and considering the jet schemes $X_m \subset \mathbf{A}^e_m = \mathbf{A}^{e(m+1)}$ and $X^0_m \subset (\mathbf{A}^e)^0_m = \mathbf{A}^{em}$;

(1) one notices on examples that the defining ideal of X^0_m in $\mathbf{A}^{em}$ is independent of some of the variables of the polynomial ring which is the ring of global sections of $\mathbf{A}^{em}$ and that the number of variables needed to define this ideal (modulo a linear change of variables) depends on how singular X is at 0. The more X is singular, the less variables we need for a given m; such an invariant was actually defined by Hironaka as a resolution invariant, see for instance [22], but this is another story.

(2) The data of an $m-$jet determines its coordinates in $\mathbf{A}^{em}$ and as mentioned in the item (1), there are no constraints on some of these coordinates; but there are constraints on these "free coordinates" for the jet to be liftable to an arc and these constraints come from the equations defining X_l for $l \geq m$; the smallest l such that the equations defining X_l catch all the constraints on all the $m-$jets for them to be liftable is related to the Artin-Greenberg function which is another invariant of singularities [66, 142]: roughly speaking, Greenberg's theorem states that if a tuple $\gamma(t)$ of power series in $\mathbf{K}[[t]]$ is very close to being an arc on X (which means that γ defines an $m-$jet for m large enough) then there is an actual arc γ' on X which is close (in the $t-$ adic topology) to γ; closeness here is in the sense of $t-$adic topology; the Artin-Greenberg function measures how much you need to be close to be an arc to be actually close to an arc; again, roughly speaking, the larger the function is, the nastier the singularity $(X,0)$ is.

The Arc Hilbert Poincaré series is related in spirit to these two types of invariants: Heuristically, the more we have free variables at the level m, the larger will be the dimension of the homogeneous components of A^0_∞ of weight less than or equal to m will be (note that the homogeneous components of weight less than or equal to m are the same as those of the ring of global sections of X^0_m) but also the larger is the Artin-Greenberg function. But this invariant is very difficult to compute, because of the complicated homological properties of A^0_∞ in general, even though sometimes for mild singularities this is possible, [28]:

THEOREM II.T. *(Bruschek, Mourtada,Schepers) Let X be a normal hypersurface in $\mathbf{A}^n$ with a canonical singularity of multiplicity $n-1$ at the origin. Then*

$$\mathrm{AHP}_{X,0}(q) = \left(\prod_{i=1}^{n-2} \frac{1}{1-q^i}\right)^n \left(\prod_{i\geq n-1} \frac{1}{1-q^i}\right)^{n-1}.$$

This generalizes a theorem that was obtained in [108] for rational double point surface singularities. Some research is still ongoing to reveal the secrets of this invariant of singularities but let us go back now to partitions and to a beautiful link with the Arc-Hilbert-Poincaré series [29]:

THEOREM II.U. *(Bruschek, Mourtada, Schepers)*

$$\textit{For } X = Spec\frac{\boldsymbol{K}[x]}{(x^2)},\ \mathrm{AHP}_{X,0}(q) = \prod_{i\equiv\ 1,4\ mod(5)} \frac{1}{1-q^i}.$$

Notice that the power series in the theorem is the right side of the first Rogers Ramanujan identity. The proof uses the differential structure of A^0_∞ which for $X = \mathrm{Spec}\frac{\mathbf{K}[x]}{(x^2)}$ is given by

$$A^0_\infty = \frac{[x_i, i \in \mathbf{N}_{>0}]}{[x_1^2]},$$

where $[x_1^2]$ is the differential ideal generated by x_1^2 and its iterated derivatives with respect to the derivation D which is determined by $D(x_i) = x_{i+1}$. So

$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_1x_3 + 2x_2^2, \dots) \tag{8.4}$$

The grading of A^0_∞ is induced from the weights given to the variables, x_i being of weight i. We order the monomials using an "adapted" monomial ordering, the weighted reverse lexicographical ordering; Now, it is well known that the Hilbert Poincaré series of the quotient ring by an ideal I is equal to the Hilbert Poincaré series of the quotient ring by the leading ideal (relative to a monomial ordering which respects the weight) of I. This latter is generated by the leading monomials of the elements of a Groebner basis of I. In general, it is very complicated to find a Groebner basis theoretically, even when we consider, let us say, the ideal generated by the first 5 generators of $I := [x_1^2]$, we should add many polynomials to obtain a Groebner basis [19]; the miracle is that the generators in (8.4) give a Groebner basis with respect the the weighted reverse lexicographical ordering. The proof shows actually that any S-polynomial (this is a notion used in Buchberger algorithm for computing a Groebner basis) is not relevant and it comes out, after determining its weight w, from the $(w-4)-$th derivative (by D) of the equation

$$2x_2(x_1^2) - x_1(2x_1x_2) = 0.$$

We deduce that

$$\mathrm{AHP}_{X,0}(q) = \mathrm{HP}(\frac{[x_i, i \in \mathbf{N}]}{(x_i^2, x_ix_{i+1}; i \in \mathbf{N}_{>0})}),$$

where HP stands for the Hilbert-Poincaré series and where the ideal $(x_i^2, x_ix_{i+1}; i \in \mathbf{N}_{>0})$ is the leading ideal of $[x_1^2]$. Now after a short reasoning, one sees that $\mathrm{HP}(\frac{[x_i, i\in\mathbf{N}]}{(x_i^2, x_ix_{i+1}; i\in\mathbf{N}_{>0})})$ is exactly the generating series of the number of partitions of n with neither consecutive nor equal parts. Using the first Rogers-Ramanujan identity we get the formula in the theorem.

Moreover, with very simple commutative algebra applied to HP$(\frac{[x_i, i\in \mathbf{N}]}{(x_i^2, x_ix_{i+1}; i\in \mathbf{N}_{>0})})$ we find that there is a sequence of power series in the variable q which converges in the $q-$adic topology to both sides of the Rogers-Ramanujan identities giving a commutative algebra approach to these identities; this sequence was stated in an empirical way in [15].

This theorem was greatly generalized in [28]:

THEOREM II.V. *(Bruschek, Mourtada, Schepers)*

$$\text{For } X = Spec\frac{\boldsymbol{K}[x]}{(x^n)},\ AHP_{X,0}(q) = \prod_{i\not\equiv\ 0,n,n+1\ mod(2n+1)} \frac{1}{1-q^i}.$$

The proof uses similar ideas but the differential calculus is much more involved. This latter theorem is related to Gordon's identities which are partition identities generalizing the Rogers-Ramanujan identities. A commutative algebra proof of Gordon's identities was found in the thesis of my Ph.D student (at that time) Pooneh Afsharijoo [9].

Now recall that in the proof of theorem II.T, we considered the Groebner basis of the ideal $[x_1^2]$ with respect to the weighted reverse lexicographical ordering; the heuristic reason of the choice of this ordering is that this allows to see first (i.e., as leading monomials) the monomials which concern the larger neighborhoods from the point of view of Taylor series: for instance for the polynomial $x_2^2+x_1x_3$, the leading term with respect to the reverse lexicographical ordering is x_2^2 which concerns an approximation of order 2 while x_1x_3 concerns an approximation of order 3. But as mentioned before, the Hilbert series of the quotient by the ideal $[x_1^2]$ is equal to the Hilbert series of the quotient by its leading monomial ideal with respect to any monomial ordering respecting the weight. With Pooneh Afsharijoo, we considered the weighted lexicographical ordering and we knew that if we catch the leading monomial ideal of $[x_1^2]$ with respect to this ordering, its Hilbert series will be equal to the generating series of the number of partitions appearing in the Rogers-Ramanujan identities, but potentially it counts partitions with different properties. The problem is that while the Groebner basis of $[x_1^2]$ with respect to the weighted reverse lexicographical ordering is differentially finite (i.e. it is obtained from a finite number of polynomials -here only one polynomial- and all their derivatives), we were able to prove that with respect to the weighted lexicographical ordering, there is no Groebner basis of $[x_1^2]$ which is differentially finite [10]; A Groebner basis is then very difficult to determine; but using Groebner basis theory computations, we were able to conjecture what is the leading monomial ideal of $[x_1^2]$; this remains a conjecture but we were able to prove that the Hilbert series of the quotient by this monomial ideal is equal to the series appearing in the Rogers-Ramanujan identities. By taking a variation of the ideal $[x_1^2]$, we have been let to the following partition identities [10]:

THEOREM II.W. *(Afsharijoo, Mourtada) Let $n \geq k$ be a positive integer. The number of partitions of n with parts larger or equal to k and size less than or equal to (the smallest part* *minus $k-1$) is equal to the number of partitions of n with parts larger or equal to k and without neither consecutive nor equal parts.*

For $k=1$, this gives another member of Rogers-Ramanujan identities: Let $n \geq 1$ be a positive integer. The number of partitions of n with size less than or equal to the smallest part is equal to the number of partitions of n without consecutive nor equal parts.

It is playful to see this last identity on the partitions of 4 but let us first call the partitions of n with size less than or equal to the smallest part, partitions of third type; partitions of first and second type were defined in theorem 8. The partitions of 4 are

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

The partitions of 4 which are of the first type are the first and the second partitions.
The partitions of 4 which are of the second type are the first and the fifth partitions.
The partitions of 4 which are of the third type are the first and the third partitions. And as the theorem predicts, the number of these partitions, two, is the same for the three types.

Using a similar idea to the one used to guess theorem II.W, Pooneh Afsharijoo has conjectured in her thesis new identities which add new members to Gordon's identities [**8**, **9**]; she proved this conjecture in a particular case and very recently with Pooneh Afsharijoo, Jehanne Dousse and Frédéric Jouhet, we proved these very exciting identities in general, this is the content of an article in preparation [**11**].

These theorems are small steps (walking steps towards another planet) in studying what I would like to call **Ramanujan Hilbert scheme**, which parametrizes the schemes with a cone structure and whose Hilbert series is equal to $F(1)$.

There are various generalizations of these theorems or these line of thoughts which are in progress. I can mention for instance a theorem on partitions of two colours in [**10**].

CHAPTER III

Original contributions that one can find in this memoir

In this chapter we gather some of the ideas and contributions of this memoir. There is no order in the mentioned contributions.

- Refining the jet components graph which is a leveled graph that I have introduced in my PhD thesis and which encodes the structure of the jet schemes; studying the jet schemes and determining this graph for several classes of singularities. This graph encodes deep information of the singularities, for instance it catches the embedded topological type of a quasi-ordinary hypersurface singularity and the analytical type of a toric surface singularity. ([**31**, **107**])
- Giving a new geometric approach to embedded resolution of singularities which finds its origin in a reverse Nash problem and in Teissier's conjecture on toric resolution of singularities. Giving two alternatives of this approach, one using jet schemes and another using extensions of monomial valuations. Proving that this approach is successful for classes of singularities like curves or Teissier singularities ([**39**, **42**, **90**, **106**, **115**]). This text (HDR) tries to explain the idea of this approach.
- Introducing a polyhedral invariant which detects quasi-ordinary hypersurface singularities in characteristic zero and which is a complete invariant of the topological type of such singularities (this includes introducing and proving the existence of a weighted version of Hironaka's characteristic polyhedron). Giving a valuative criterion to determine whether a singularity is quasi-ordinary. [**39**, **113**]
- Introducing the notion of Teissier singularities and showing that, in positive characteristics, this notion is a very good counterpart of the notion of quasi-ordinary singularities: Any Teissier singularity is a special fiber of an equisingular family over a germ of curve of mixed characteristic, whose generic fiber has quasi-ordinary singularities. Here, equisingular means that we have an embedded simultaneous resolution of the family; in particular giving an embedded resolution of Teissier singularities. This leads to an approach similar to Jung's approach to resolution of singularities in positive characteristics. ([**114**, **115**])
- Proving a formula for the completion of the arc space at a schematic point which is associated with a divisorial valuation, in terms of the Mather log discrepancy of the divisor. ([**112**])
- Posing an Embedded Nash problem which is about constructing embedded resolutions of singularities from the data of the irreducible components of the jet schemes.

Settling this problem for classes of surface singularities and precising the frame to study it. ([79, 108, 111])

- Proving that $\mu-$constant Newton non-degenerate deformations admit a simultaneous embedded resolution of singularities. Giving a new approach to the $\mu-$constant problem which is an old open problem in singularity theory. Giving a complete answer to a question of Arnold on the monotonicity of Newton numbers in the case of convenient Newton polyhedra.([93])
- Studying the Motivic Igusa Zeta function and the monodromy conjecture in families and for ideals (not only hypersurfaces). Using the jet components graph for the computation of the motivic Igusa Zeta function.([102, 116])
- Discovering a beautiful relation between an invariant of singularities, the Arc Hilbert Poincaré series, and very famous identities in the theory of partitions, Rogers-Ramanujan identities. Extending this relation to more general identities; this led to new proofs of these identities, to discover and prove new partition identities. This bridge between arc spaces and partitions (and generalized partitions) gave beautiful theorems and seems to be very promising. ([10, 11, 28, 29])
- Giving an algorithm to determine generating series of some divisorial valuations from the equations of their representations as order of contact along some semi-algebraic sets of the jet schemes. ([106])
- Giving a polyhedral criterion for uniqueness of extension of monomial valuations. This also gives an algorithm to determine the graded algebra of an extension (when it is unique) of a monomial valuation.([39])
- Giving an algorithm which determines in many instances generating series of zero-dimensional valuations. ([40])
- Giving a simple demonstration that the defect is the only obstruction to the local uniformization problem. ([31]
- Characterizing the defect of extensions of valuations using the graded algebras of the valuations (at the level of rings). ([40])

CHAPTER IV

A glimpse on the work of my Ph.D. students

- Pooneh Afsharijoo (co-adviser: Marc Chardin) started in October 2015 and defended her PhD. thesis in May 2019. She worked on the bridge between arc spaces and partition identities. She wrote an article where she conjectures a family of partition identities (indexed by an integer number $r \geq 2$) and which complements Gordon's identities. In the same article she proved the conjecture for $r = 2, 3$. Recently, with her coauthors (including me), she has proved this conjecture for all integers $r \geq 2$. Very recently, using the same bridge, she has found and proved new partition identities which add new members to Rogers-Ramanujan identities.
- Andrei Bengus-Lasnier started his PhD thesis in September 2017. He is working in valuation theory, on the notions of key polynomials and truncated valuations. When considering an extension ν of a valuation ν_0, defined on a field $\mathbf{K}$, to $\mathbf{K}[x]$, truncated valuations approximate the extension ν. They somehow encode the extension and the usual way to determine them uses key polynomials. Many key polynomials may determine the same truncated valuation. Andrei wrote an article about objects called diskoids which are more "geometric" and which are candidate to encode extension of valuations and which are in bijection with truncated valuations. He proved that this is true for valuations of rank 1 or when $\mathbf{K}$ is Henselian. This notion of diskoids generalizes somehow the notion of maximal divisorial sets for divisorial valuations [74].
- Zahraa Mohsen (Co-adviser: Amine El Sahili) started in November 2019 and is working on Syzygies in the arc space.

CHAPTER V

References

[1] Ould M. Abderrahmane, *On deformation with constant Milnor number and Newton polyhedron*, Math. Z. **284** (2016), no. 1-2, 167–174.

[2] Shreeram Abhyankar, *Ramification theoretic methods in algebraic geometry*, Annals of Mathematics Studies, no. 43, Princeton University Press, Princeton, N.J., 1959.

[3] Shreeram S. Abhyankar, *Resolution of singularities and modular Galois theory*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 131–169.

[4] Shreeram Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math. **77** (1955), 575–592.

[5] S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[6] Shreeram S. Abhyankar, *Irreducibility criterion for germs of analytic functions of two complex variables*, Adv. Math. **74** (1989), no. 2, 190–257.

[7] Dan Abramovich, Michael Temkin, and Jaroslaw Wlodarczyk, *Functorial embedded resolution via weighted blowings up* (2019), available at 1906.07106.

[8] Pooneh Afsharijoo, *Looking for a new version of Gordon's identities, from algebraic geometry to combinatorics through partitions*, thèse de doctorat, Université de Paris (2019).

[9] ______, *Looking for a new version of Gordon's identities*, Submitted.

[10] Pooneh Afsharijoo and Hussein Mourtada, *Partition Identities and Application to Infinite Dimensional Groebner Basis and Vice Versa*, Arc schemes and singularities (2020).

[11] Pooneh Afsharijoo, Jehanne Dousse, Frédéric Jouhet, and Hussein Mourtada, *New members of Gordon's identities*, in preparation.

[12] Stefano Aguzzoli and Daniele Mundici, *An algorithmic desingularization of* 3*-dimensional toric varieties*, Tohoku Math. J. (2) **46** (1994), no. 4, 557–572.

[13] A. Altıntaş Sharland, G. Çevik, and M. Tosun, *Nonisolated forms of rational triple point singularities of surfaces and their resolutions*, Rocky Mountain J. Math. **46** (2016), no. 2, 357–388.

[14] George E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[15] George E. Andrews and R. J. Baxter, *A motivated proof of the Rogers-Ramanujan identities*, Amer. Math. Monthly **96** (1989), no. 5, 401–409.

[16] Fuensanta Aroca and Mirna Gómez-Morales and Khurram Shabbir, *Torical modification of Newton nondegenerate ideals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **107** (2013), no. 1, 221–239.

[17] Michael Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.

[18] Abdallah Assi, *Irreducibility criterion for quasi-ordinary polynomials*, J. Singul. **4** (2012), 23–34.

[19] Yuzhe Bai, Eugene Gorsky, and Oscar Kivinen, *Quadratic ideals and Rogers-Ramanujan recursions*, Ramanujan J. **52** (2020), no. 1, 67–89.

[20] Angélica Benito and Orlando E. Villamayor U., *Techniques for the study of singularities with applications to resolution of 2-dimensional schemes*, Math. Ann. **353** (2012), no. 3, 1037–1068.

[21] ———, *Monoidal transforms and invariants of singularities in positive characteristic*, Compos. Math. **149** (2013), no. 8, 1267–1311.

[22] Jérémy Berthomieu, Pascal Hivert, and Hussein Mourtada, *Computing Hironaka's invariants: ridge and directrix*, Arithmetic, geometry, cryptography and coding theory 2009, Contemp. Math., vol. 521, Amer. Math. Soc., Providence, RI, 2010, pp. 9–20.

[23] Bhargav Bhatt, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461.

[24] Edward Bierstone and Pierre D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[25] David Bourqui, Kevin Langlois, and Hussein Mourtada, *Arcs and torus actions*, work in progress.

[26] Catherine Bouvier and Gérard Gonzalez-Sprinberg, *Système générateur minimal, diviseurs essentiels et G-désingularisations de variétés toriques*, Tohoku Math. J. (2) **47** (1995), no. 1, 125–149.

[27] Catherine Bouvier, *Diviseurs essentiels, composantes essentielles des variétés toriques singulières*, Duke Math. J. **91** (1998), no. 3, 609–620 (French).

[28] Clemens Bruschek, Hussein Mourtada, and Jan Schepers, *Arc spaces and the Rogers-Ramanujan identities*, Ramanujan J. **30** (2013), no. 1, 9–38.

[29] ———, *Arc spaces and Rogers-Ramanujan identities*, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 211–220 (English, with English, French and Spanish summaries).

[30] Nero Budur, Pedro D. González-Pérez, and Manuel González Villa, *Log canonical thresholds of quasi-ordinary hypersurface singularities*, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4075–4083.

[31] Helena Cobo and Hussein Mourtada, *Jet schemes of quasi-ordinary surface singularities*, Nagoya Math. J. (to appear), 88 pages.

[32] Vincent Cossart and Olivier Piltant, *Resolution of singularities of arithmetical threefolds*, J. Algebra **529** (2019), 268–535.

[33] Vincent Cossart and Olivier Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** (2009), no. 7, 1836–1976.

[34] David A. Cox and John B. Little and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[35] Steven Dale Cutkosky, *Local Uniformization of Abhyankar Valuations*, Preprint.

[36] ———, *Resolution of singularities*, Graduate Studies in Mathematics, vol. 63, American Mathematical Society, Providence, RI, 2004.

[37] ———, *Resolution of singularities for 3-folds in positive characteristic*, Amer. J. Math. **131** (2009), no. 1, 59–127.

[38] Steven Dale Cutkosky and Hussein Mourtada, *Defect and local uniformization*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2019), 16 pages.

[39] Steven Dale Cutkosky, Hussein Mourtada, and Bernard Teissier, *On uniqueness of finite extensions of monomial valuations and their uniformization*.

[40] ———, *On the construction of valuations and generating sequences on hypersurface singularities*, 43 pages.

[41] Javier Fernández de Bobadilla and María Pe Pereira, *The Nash problem for surfaces*, Ann. of Math. (2) **176** (2012), no. 3, 2003–2029.

[42] Ana Bélen de Felipe, Pedro González Pérez, and Hussein Mourtada, *Resolving singularities of reducible curves with one toric morphism*, Preprint.

[43] Ana Bélen de Felipe and Bernard Teissier, *Valuations and Henselization*, Math. Annalen (2020).

[44] Tommaso de Fernex, *Three-dimensional counter-examples to the Nash problem*, Compos. Math. **149** (2013), no. 9, 1519–1534.

[45] Tommaso de Fernex and Roi Docampo, *Differentials on the arc space*, Duke Math. J. **169** (2020), no. 2, 353–396.

[46] ———, *Terminal valuations and the Nash problem*, Invent. Math. **203** (2016), no. 1, 303–331.

[47] Tommaso de Fernex, Lawrence Ein, and Shihoko Ishii, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 425–448.

[48] J. Decaup, W. Mahboub, and M. Spivakovsky, *Abstract key polynomials and comparison theorems with the key polynomials of Mac Lane–Vaquié*, Illinois J. Math. **62** (2018), no. 1-4, 253–270.

[49] Jan Denef and François Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232.

[50] ———, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[51] Lawrence Ein, Robert Lazarsfeld, and Mircea Mustaţă, *Contact loci in arc spaces*, Compos. Math. **140** (2004), no. 5, 1229–1244.

[52] Lawrence Ein and Mircea Mustaţă, *Inversion of adjunction for local complete intersection varieties*, Amer. J. Math. **126** (2004), no. 6, 1355–1365.

[53] Manfred Einsiedler and Mikhail Kapranov and Douglas Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.

[54] Otto Endler, *Valuation theory*, Springer-Verlag, New York-Heidelberg, 1972. To the memory of Wolfgang Krull (26 August 1899–12 April 1971); Universitext.

[55] Charles Favre and Mattias Jonsson, *The valuative tree*, Lecture Notes in Mathematics, vol. 1853, Springer-Verlag, Berlin, 2004.

[56] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[57] Yih-Nan Gau, *Embedded topological classification of quasi-ordinary singularities*, Mem. Amer. Math. Soc. **74** (1988), no. 388, 109–129. With an appendix by Joseph Lipman.

[58] E. R. García Barroso and P. D. González-Pérez, *Decomposition in bunches of the critical locus of a quasi-ordinary map*, Compos. Math. **141** (2005), no. 2, 461–486.

[59] Rebecca Goldin and Bernard Teissier, *Resolving singularities of plane analytic branches with one toric morphism*, Resolution of singularities (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 315–340.

[60] Pedro D. González Pérez, *Toric embedded resolutions of quasi-ordinary hypersurface singularities*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 6, 1819–1881 (English, with English and French summaries).

[61] ———, *The semigroup of a quasi-ordinary hypersurface*, J. Inst. Math. Jussieu **2** (2003), no. 3, 383–399.

[62] Pedro D. González Pérez and Manuel González Villa, *Motivic Milnor fiber of a quasi-ordinary hypersurface*, J. Reine Angew. Math. **687** (2014), 159–205.

[63] Gérard Gonzalez-Sprinberg and Monique Lejeune-Jalabert, *Modèles canoniques plongés. I*, Kodai Math. J. **14** (1991), no. 2, 194–209 (English, with French summary).

[64] David Harbater, Andrew Obus, Rachel Pries, and Katherine Stevenson, *Abhyankar's conjectures in Galois theory: current status and future directions*, Bull. Amer. Math. Soc. (N.S.) **55** (2018), no. 2, 239–287.

[65] Herwig Hauser and Stefan Perlega, *Cycles of singularities appearing in the resolution problem in positive characteristic*, J. Algebraic Geom. **28** (2019), no. 2, 391–403.

[66] M. Hickel, *Calcul de la fonction d'Artin-Greenberg d'une branche plane*, Pacific J. Math. **213** (2004), no. 1, 37–47 (French, with English summary).

[67] Heisuke Hironaka, *Characteristic polyhedra of singularities*, J. Math. Kyoto Univ. **7** (1967), 251–293.

[68] ———, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326.

[69] Jun-ichi Igusa, *Complex powers and asymptotic expansions. I. Functions of certain types*, J. Reine Angew. Math. **268(269)** (1974), 110–130.

[70] ______, *Complex powers and asymptotic expansions. II. Asymptotic expansions*, J. Reine Angew. Math. **278(279)** (1975), 307–321.
[71] Shihoko Ishii, *Smoothness and jet schemes*, Singularities—Niigata–Toyama 2007, Adv. Stud. Pure Math., vol. 56, Math. Soc. Japan, Tokyo, 2009, pp. 187–199.
[72] ______, *The arc space of a toric variety*, J. Algebra **278** (2004), no. 2, 666–683.
[73] ______, *Introduction to arc spaces and the Nash problem [translation of MR2796384]*, Sugaku Expositions **25** (2012), no. 2, 221–242.
[74] ______, *Maximal divisorial sets in arc spaces*, Algebraic geometry in East Asia—Hanoi 2005, Adv. Stud. Pure Math., vol. 50, Math. Soc. Japan, Tokyo, 2008, pp. 237–249.
[75] Shihoko Ishii and János Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** (2003), no. 3, 601–620.
[76] Shihoko Ishii and Ana J. Reguera, *Singularities in arbitrary characteristic via jet schemes*, Hodge theory and L^2-analysis, Adv. Lect. Math. (ALM), vol. 39, Int. Press, Somerville, MA, 2017.
[77] Jennifer M. Johnson and János Kollár, *Arc spaces of cA-type singularities*, J. Singul. **7** (2013), 238–252.
[78] Heinrich W. E. Jung, *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen x, y in der Umgebung einer Stelle $x = a,\ y = b$*, J. Reine Angew. Math. **133** (1908), 289–314 (German).
[79] Busra Karadeniz, Hussein Mourtada, Camille Plénat, and Meral Tosun, *The embedded Nash problem of birational models of rational triple point singularities*, Journal of singularities, to appear.
[80] Olga Kashcheyeva, *Constructing examples of semigroups of valuations*, J. Pure Appl. Algebra **220** (2016), no. 12, 3826–3860.
[81] Hiraku Kawanoue and Kenji Matsuki, *A new strategy for resolution of singularities in the monomial case in positive characteristic*, Rev. Mat. Iberoam. **34** (2018), no. 3, 1229–1276.
[82] K. Kiyek and M. Micus, *Semigroup of a quasiordinary singularity*, Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw, 1990, pp. 149–156.
[83] Hagen Knaf and Franz-Viktor Kuhlmann, *Abhyankar places admit local uniformization in any characteristic*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 6, 833–846 (English, with English and French summaries).
[84] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.
[85] ______, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.
[86] Franz-Viktor Kuhlmann, *Book on valuation theory*, https://math.usask.ca/ fvk/Fvkbook.htm.
[87] ______, *Valuation theoretic and model theoretic aspects of local uniformization*, Resolution of singularities (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 381–456.
[88] Monique Lejeune-Jalabert, *Courbes tracées sur un germe d'hypersurface*, Amer. J. Math. **112** (1990), no. 4, 525–568 (French).
[89] ______, *Arcs analytiques et résolution minimale des singularités des surfaces quasi-homogÃÍnes*, Séminaire sur les Singularités des Surfaces (Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976), Lecture Notes in Mathematics, vol. 777, Springer, 1980, pp. 303–332.
[90] Monique Lejeune-Jalabert, Hussein Mourtada, and Ana Reguera, *Jet schemes and minimal embedded desingularization of plane branches*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **107** (2013), no. 1, 145–157.
[91] Monique Lejeune-Jalabert and Ana J. Reguera, *The Denef-Loeser series for toric surface singularities*, Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001), 2003, pp. 581–612.

[92] Maximiliano Leyton-Álvarez, *Deforming spaces of m-jets of hypersurfaces singularities*, J. Algebra **508** (2018), 81–97.

[93] Maximiliano Leyton-Alvarez, Hussein Mourtada, and Mark Spivakovsky, *Newton non-degenerate mu-constant deformations admit simultaneous embedded resolutions*, Submitted.

[94] Lê Dũng Tráng and C. P. Ramanujam, *The invariance of Milnor's number implies the invariance of the topological type*, Amer. J. Math. **98** (1976), no. 1, 67–78.

[95] Joseph Lipman, *Quasi-ordinary singularities of embedded surfaces*, ProQuest LLC, Ann Arbor, MI, 1965. Thesis (Ph.D.)–Harvard University.

[96] ______, *Topological invariants of quasi-ordinary singularities*, Mem. Amer. Math. Soc. **74** (1988), no. 388, 1–107.

[97] ______, *Equisingularity and simultaneous resolution of singularities*, Resolution of singularities (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 485–505.

[98] F. Loeser, *Fonctions d'Igusa p-adiques et polynômes de Bernstein*, Amer. J. Math. **110** (1988), no. 1, 1–21 (French).

[99] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015.

[100] Saunders MacLane, *A construction for absolute values in polynomial rings*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 363–395.

[101] ______, *The Schönemann-Eisenstein irreducibility criteria in terms of prime ideals*, Trans. Amer. Math. Soc. **43** (1938), no. 2, 226–239.

[102] Jorge Martin-Morales, Hussein Mourtada, Wim Veys, and Lena Vos, *Note on the monodromy conjecture for a space monomial curve with a plane semigroup*, to appear in C. R. Math. Acad. Sci. Paris.

[103] Kenji Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.

[104] Michel Merle, *Polyèdre de Newton, eventail et desingularisation d'après A. N. Varchenko*, Séminaire sur les Singularités des Surfaces (Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976), Lecture Notes in Mathematics, vol. 777, Springer, 1980, pp. 289–294.

[105] Hussein Mourtada, *Jet schemes of complex plane branches and equisingularity*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 6, 2313–2336 (2012).

[106] ______, *Jet schemes and generating sequences of divisorial valuations in dimension two*, Michigan Math. J. **66** (2017), no. 1, 155–174.

[107] ______, *Jet schemes of normal toric surfaces*, Bull. Soc. Math. France **145** (2017), no. 2, 237–266 (English, with English and French summaries).

[108] ______, *Jet schemes of rational double point singularities*, Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014, pp. 373–388.

[109] ______, *Jet schemes and generating sequences of $\mathbb{H}-$divisorial valuations*, work in progress.

[110] ______, *Jet schemes of toric surfaces*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 9-10, 563–566.

[111] Hussein Mourtada and Camille Plénat, *Jet schemes and minimal toric embedded resolutions of rational double point singularities*, Comm. Algebra **46** (2018), no. 3, 1314–1332.

[112] Hussein Mourtada and Ana J. Reguera, *Mather discrepancy as an embedding dimension in the space of arcs*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 1, 105–139.

[113] Hussein Mourtada and Bernd Schober, *A polyhedral characterization of quasi-ordinary singularities*, Mosc. Math. J. **18** (2018), no. 4, 755–785.

[114] ______, *Teissier singularities: a viewpoint on quasi-ordinary singularities in positive characteristics*, Oberwolfach Reports **6** (2019).

[115] ______, *On the notion of quasi-ordinary singularities in positive characteristics: Teissier singularities and their resolutions*, Preprint.

[116] Hussein Mourtada, Wim Veys, and Lena Vos, *The motivic Igusa zeta function of a space monomial curve with a plane semigroup*, Submitted.

[117] Mircea Mustaţă, *The dimension of jet schemes of singular varieties*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, 2014, pp. 673–693.

[118] ______, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. **145** (2001), no. 3, 397–424. With an appendix by David Eisenbud and Edward Frenkel.

[119] Mircea Mustaţă, *Singularities of pairs via jet schemes*, J. Amer. Math. Soc. **15** (2002), no. 3, 599–615.

[120] John F. Nash Jr., *Arc structure of singularities*, Duke Math. J. **81** (1995), no. 1, 31–38 (1996). A celebration of John F. Nash, Jr.

[121] Isaac Newton, *Analysis per quantitatum series, fluxiones, ac differentias: cum enumeratione linearum tertii ordinis*, SAEM "Thales", Seville; Real Sociedad Matemática Española, Madrid, 2003 (Latin). Reprint of the 1711 original; With a preface by W. Jones.

[122] Duc Tam Nguyen, *Combinatorics of jet schemes and its applications*, University of Tokyo, 2016 (PhD Thesis).

[123] Johannes Nicaise, *Motivic generating series for toric surface singularities*, Math. Proc. Cambridge Philos. Soc. **138** (2005), no. 3, 383–400.

[124] Johannes Nicaise and Sam Payne, *A tropical motivic Fubini theorem with applications to Donaldson-Thomas theory*, Duke Math. J. **168** (2019), no. 10, 1843–1886.

[125] Josnei Novacoski and Mark Spivakovsky, *Reduction of local uniformization to the case of rank one valuations for rings with zero divisors*, Michigan Math. J. **66** (2017), no. 2, 277–293.

[126] Mutsuo Oka, *Non-degenerate complete intersection singularity*, Actualités Mathématiques. [Current Mathematical Topics], Hermann, Paris, 1997.

[127] ______, *On the weak simultaneous resolution of a negligible truncation of the Newton boundary*, Singularities (Iowa City, IA, 1986), Contemp. Math., vol. 90, Amer. Math. Soc., Providence, RI, 1989, pp. 199–210, DOI 10.1090/conm/090/1000603. MR1000603

[128] Oskar Perron, *Über eine Anwendung der Idealtheorie auf die Frage nach der Irreduzibilität algebraischer Gleichungen*, Math. Ann. **60** (1905), no. 3, 448–458 (German).

[129] Olivier Piltant, *An axiomatic version of Zariski's patching theorem*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **107** (2013), no. 1, 91–121.

[130] Olivier Piltant and Ana J. Reguera, *Local uniformization and arc spaces*, J. Pure Appl. Algebra **222** (2018), no. 7, 1898–1905.

[131] Camille Plénat and Mark Spivakovsky, *The Nash problem and its solution: a survey*, J. Singul. **13** (2015), 229–244.

[132] Patrick Popescu-Pampu, *Introduction to Jung's method of resolution of singularities*, Topology of algebraic varieties and singularities, Contemp. Math., vol. 538, Amer. Math. Soc., Providence, RI, 2011, pp. 401–432.

[133] ______, *Approximate roots*, Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Commun., vol. 33, Amer. Math. Soc., Providence, RI, 2003, pp. 285–321.

[134] ______, *On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity*, Duke Math. J. **124** (2004), no. 1, 67–104.

[135] Patrick Popescu-Pampu and Dmitry Stepanov, *Local tropicalization*, Algebraic and combinatorial aspects of tropical geometry, Contemp. Math., vol. 589, Amer. Math. Soc., Providence, RI, 2013, pp. 253–316.

[136] M. Raynaud, *Revêtements de la droite affine en caractéristique* $p > 0$ *et conjecture d'Abhyankar*, Invent. Math. **116** (1994), no. 1-3, 425–462 (French).

[137] Ana J. Reguera, *Towards the singular locus of the space of arcs*, Amer. J. Math. **131** (2009), no. 2, 313–350.

[138] ______, *A curve selection lemma in spaces of arcs and the image of the Nash map*, Compos. Math. **142** (2006), no. 1, 119–130.

[139] Miles Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 395–418.
[140] Paulo Ribenboim, *The theory of classical valuations*, Springer Monographs in Mathematics, Springer-Verlag, New York, 1999.
[141] Oswald Riemenschneider, *Zweidimensionale Quotientensingularitäten: Gleichungen und Syzygien*, Arch. Math. (Basel) **37** (1981), no. 5, 406–417 (German).
[142] Guillaume Rond, *Artin approximation*, J. Singul. **17** (2018), 108–192.
[143] Jean-Christophe San Saturnino, *Defect of an extension, key polynomials and local uniformization*, J. Algebra **481** (2017), 91–119.
[144] Jean-Pierre Serre, *Revêtements de courbes algébriques*, Astérisque **206** (1992), Exp. No. 749, 3, 167–182 (French, with French summary). Séminaire Bourbaki, Vol. 1991/92.
[145] ———, *Corps locaux*, Hermann, Paris, 1968 (French). Deuxième édition; Publications de l'Université de Nancago, No. VIII.
[146] Peter Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathematics, vol. 815, Springer, Berlin, 1980.
[147] Mark Spivakovsky, *Valuations in function fields of surfaces*, Amer. J. Math. **112** (1990), no. 1, 107–156.
[148] Jan Stevens, *Deformations of singularities*, Lecture Notes in Mathematics, vol. 1811, Springer-Verlag, Berlin, 2003.
[149] Bernard Teissier, *The hunting of invariants in the geometry of discriminants*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), 1977, pp. 565–678.
[150] ———, *Overweight deformations of affine toric varieties and local uniformization*, Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014, pp. 474–565.
[151] ———, *Appendix to Zariski's book*, The moduli problem for plane branches, 1973.
[152] ———, *Cycles évanescents, sections planes et conditions de Whitney*, Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), Soc. Math. France, Paris, 1973, pp. 285–362. Astérisque, Nos. 7 et 8 (French).
[153] Jenia Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (2007), no. 4, 1087–1104.
[154] ———, *On a question of B. Teissier*, Collect. Math. **65** (2014), no. 1, 61–66.
[155] G. N. Tjurina, *Absolute isolation of rational singularities, and triple rational points*, Funkcional. Anal. i Priložen. **2** (1968), no. 4, 70–81 (Russian).
[156] Lise Van Proeyen and Willem Veys, *The monodromy conjecture for zeta functions associated to ideals in dimension two*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 4, 1347–1362 (English, with English and French summaries).
[157] Michel Vaquié, *Extension d'une valuation*, Trans. Amer. Math. Soc. **359** (2007), no. 7, 3439–3481 (French, with English summary).
[158] ———, *Famille admissible de valuations et défaut d'une extension*, J. Algebra **311** (2007), no. 2, 859–876 (French, with French summary).
[159] ———, *Valuations and local uniformization*, Singularity theory and its applications, Adv. Stud. Pure Math., vol. 43, Math. Soc. Japan, Tokyo, 2006, pp. 477–527.
[160] A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), no. 3, 253–262.
[161] J.-L. Verdier, *Spécialisation de faisceaux et monodromie modérée*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 332–364 (French).
[162] Willem Veys and W. A. Zúñiga-Galindo, *Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 2205–2227.
[163] Paul Vojta, *Jets via Hasse-Schmidt derivations*, Diophantine geometry, CRM Series, vol. 4, Ed. Norm., Pisa, 2007, pp. 335–361.

[164] Robert J. Walker, *Reduction of the singularities of an algebraic surface*, Ann. of Math. (2) **36** (1935), no. 2, 336–365.

[165] André Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.

[166] Oscar Zariski, *The moduli problem for plane branches*, University Lecture Series, vol. 39, American Mathematical Society, Providence, RI, 2006. With an appendix by Bernard Teissier; Translated from the 1973 French original by Ben Lichtin.

[167] Oscar Zariski and Pierre Samuel, *Commutative algebra. Vol. II*, Springer-Verlag, New York-Heidelberg, 1975. Reprint of the 1960 edition; Graduate Texts in Mathematics, Vol. 29.

[168] Oscar Zariski, *Reduction of the singularities of algebraic three dimensional varieties*, Ann. of Math. (2) **45** (1944), 472–542.

[169] ———, *Local uniformization on algebraic varieties*, Ann. of Math. (2) **41** (1940), 852–896.

CHAPTER VI

Curriculum Vitae

Contents

Curriculum Vitae

Personal data

Hussein MOURTADA
Born on September 18, 1982, Baalbeck, Lebanon
Nationality: French, Lebanese
Married, two daughters

Personal address:
10 rue Guichard,
94230 Cachan, France.
Téléphone: (33) (0) 6 14 04 81 68

Professional address:
Institut de Mathématiques de Jussieu-Paris Rive Gauche
Université de Paris (Campus Diderot)
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75205 Paris Cedex 13,
France
Téléphone: (33) (0) 1 57 27 91 02

Email:
`hussein.mourtada@imj-prg.fr`

Web page:
`http://webusers.imj-prg.fr/~hussein.mourtada/`

Actual position

Maître de conférences at University Paris 7, since September 2011.

Research area

My research area is algebraic geometry and its interactions with commutative algebra and combinatorics. I am interested in various invariants that can be attached to singularities of algebraic varieties, coming from the geometry of their jet schemes and arc space, and resolution of singularities. Many objects, concepts and techniques come in the picture: Singularities of curves and surfaces, toric geometry, tropical geometry, rational singularities, Hilbert- Poincaré series, Groebner basis, combinatorial commutative algebra, partitions, Rogers-Ramanujan identities, valuations, Hironaka invariants, Nash and embedded Nash problem, motivic integration.

Education

2006-2010 : **Thèse de doctorat en mathématiques (PHD),** Université de Versailles Saint-Quentin (UVSQ).
Defended on June 23, 2010 at Laboratoire de Mathématiques de Versailles.

Title : Sur la géométrie des espaces de jets de quelques variétés algébriques singulières.

Thesis advisor : Monique Lejeune-Jalabert.

Grade : Très honorable.

2005-2006 : **Master 2: Algèbre et Géométrie** Université Paris 6-Pierre et Marie Curie.

Thesis : Spécialisation de faisceaux et monodromie, d'après Verdier.

Master thesis Advisor : Fouad Elzein.

2001-2005 : **Maîtrise en Mathématiques :** Lebanese University, Beirut - Lebanon.

Positions held

Since 2011 : Maître de Conférences at Paris Diderot University, Paris 7.

2009-2011 : ATER, Université de Versailles - UVSQ.

2008-2009 : Teaching assistant, Université de Versailles - UVSQ.

2007-2008 : Teaching assistant, Université Paris 9 - Dauphine.

Visiting positions

- April 2019, 1 month, Visiting associate professor, University of Chicago.
- April 2017, 1 month, Visiting Assistant professor, University of Missouri.

Research stays

- Oberwolfach, Germany, 2 weeks, July 2020.
- Talca University, Chili, 2 weeks, March 2020.
- Jiao Tong University, Shanghai, China, 1 week, November 2019.
- Galatasaray University, Istanbul, Turkey, 1 week, July 2019.
- University of Chicago, 1 month, April 2019.
- University of Hanover, Germany, 1 week, March 2018.
- University of Talca, Chili, 3 weeks, March 2018.
- Universidad Autonoma de Madrid, 1 week, January 2018.
- University of Missouri, 1 month, April 2017.
- Leuven University, Belgium, 1 week, March 2017.
- Mainz University, Germany, 1 week, June 2015.
- City University of New York, 1 week, May 2015.
- Ann Arbor, Michigan university, 3 weeks, April 2015.
- Instituto de Matematicos, Cuernavaca, Mexico, 1 week, October 2012.
- Valladolid University, 1 week, January 2012.
- University of Vienna, 1 week, October 2010.

Languages

English, French and Arabic.

Basic Spanish.

Computer skills

Computer algebra : Singular, Sage, Maple.

Programming : C++.

Other activities

Foot : Founder and player of the football club "Dabké Football Club Paris", 2009-.

Music : Oud player.

Purciens : Founding member of a "maths club" at the Lebanese University in 2004. The name "Purciens" is inspired from pure mathematics.

Other interests

Politics, Poetry, Cinema ...

Publications

Publications

- **Jet schemes of quasi-ordinary surface singularities** (with Helena Cobo), to appear in Nagoya Journal of Mathematics, 88 pages. DOI: https://doi.org/10.1017/nmj.2019.26
- **The embedded Nash problem of birational models of rational triple point singularities,** (with B. Karadeniz, C. Plénat, M. Tosun), Journal of Singularities volume 22 (2020), 337-372.
- **The motivic zeta function of a space monomial curve with a plane semigroup,** (with Wim Veys and Lena Vos), 32 pages, To appear in Advances of Geometry.
- **Note on the monodromy conjecture for a space monomial curve with a plane semigroup** (with Jorge Martin-Morales, Wim Veys, Lena Vos), C. R. Math. Acad. Sci. Paris 358 (2020), no. 2, 177–187.
- **Partitions identities and application to infinite dimensional Groebner basis and viceversa** (with Pooneh Afsharijoo), Arc Schemes and Singularities, World Scientific Publishing, pp. 145-161 (2020).
- **Defect and Local Uniformization** (with Steven Dale Cutkosky), Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 113 (2019), no. 4, 4211-4226.
- **Teissier singularities: a viewpoint on quasi-ordinary singularities in positive characteristics** (with Bernd Schober), Oberwolfach Reports. Report 6 (2019).
- **A polyhedral characterization of quasi-ordinary singularities** (with Bernd Schober), Moscow Math. J. 18 (2018), no. 4, 755-785.
- **Embedding dimension of the arc space at a stable point and Mather log discrepancy** (with Ana Reguera), Publ. Res. Inst. Math. Sci. 54 (2018), no. 1, 105-139.
- **Jet schemes and minimal embedded toric resolution of rational double point singularities** (with Camille Plénat), Comm. Algebra 46 (2018), no. 3, 1314-1332.
- **Jet schemes of normal toric surfaces**, Bull. Soc. Math. France 145 (2017), no. 2, 237-266.
- **Jet schemes and generating sequences of divisorial valuations in dimension two**, Michigan Math. J., Volume 66, Issue 1 (2017), 155-174.

- **Jet schemes of rational double point surface singularities** Valuation Theory in Interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Sept. 2014, pp: 373-388.
- **Jet schemes and minimal embedded desingularization of plane branches** (with Monique Lejeune-Jalabert and Ana Reguera), Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math., special issue dedicated to Professor H. Hironaka.
- **Arc spaces and Rogers-Ramanujan Identities** (with Clemens Bruschek and Jan Schepers) The Ramanujan Journal, January 2013, Volume 30, Issue 1, pp 9-38.
- **Jet schemes of complex plane branches and equisingularity** Annales de l'Institut Fourier, Tome 61, numéro 6 (2011), p. 2313-2336.
- **Jet schemes of toric surfaces** C. R. Math. Acad. Sci. Paris 349 (2011), no. 9-10, 563-566.
- **Computing Hironaka's invariants: ridge and directrix** (with Jérémy Berthomieu and Pascal Hivert), Contemporary Mathematics, vol. 521, Amer. Math. Soc., Providence, RI, 2010, pp. 9-20.
- **Arc spaces and Rogers-Ramanujan Identities** (with Clemens Bruschek and Jan Schepers), Discrete Mathematics and Theoretical Computer Science Proceedings, FPSAC (2011), 211-220.

Preprints

- **Generating sequences of valuations of finite extensions,** (with Steven Dale Cutkosky and Bernard Teissier), 43 pages, submitted.
- **Newton non-degenerate mu-constant deformations admit simultaneous embedded resolutions,** (with Maximiliano Leyton-Alvarez and Mark Spivakovsky), submitted.

A book as an editor

- **Algebraic Geometry and Number Theory**. Lecture notes from the CIMPA Summer School. Edited by Hussein Mourtada, Celal Cem Sarioglu, Christophe Soulé and Ayberk Zeytin. Progress in Mathematics, 321. Birkhauser, 2017.

Other scientific activities

Talks and invitations

- **Talca, Chili, March 2020:** Talk about Teissier singularities, in the Colloquium of Talca University.
- **Sendai, Japan, February 2020:** Talk in the conference Arithmetics and Singularities, Tohoku University.
- **Shanghai, China, November 2020:** Talk about Arcs and integer partitions, in the Algebraic Geometry seminar, Jiao-Tong University.
- **Lille, France, June 2019:** Teissier singularities, talk in a meeting on geometry of singularities.
- **Rennes, France, June 2019:** Jets, arcs and the minimal model program, talk in a meeting around the singularities of the minimal model program.
- **Dijon, France, May 2019:** Arc spaces and integer partitions, talk at Université de Dijon.
- **Seville, Spain, May 2019:** On the notion of quasi-ordinary singularities in positive characteristics, talk in a meeting at University of Seville.
- **Chicago, USA, April 2019:** Talk in the Algebraic Geometry seminar, University of Chicago.
- **Toulouse, France, February 2019:** Talk in the Colloquium of the department of mathematics, Université Paul Sabatier.
- **Oberwolfach, Germany, February 2019:** Talk entitled "A viewpoint on quasi-ordinary singularities in char p: Teissier singularities" in the conference "Singularities and Homological Aspects of Commutative Algebra".
- **Nha Trang, Vietnam, September 2018:** Talk in the he 6th Franco-Japanese-Vietnamese Symposium on Singularites, Arcs and integer partitions.
- **Hanover, Germany, March 2018:** Talk in the University of Hanover, Arcs and integer partitions.
- **Talca, Chile, March 2018:** Talk in the University of Talca, Arcs and integer partitions.
- **Beirut, Lebanon, February 2018:** Talk in Université Libanaise, Arcs and integer partitions.

- **Madrid, spain, January 2018:** Talk in Universidad Complutense de Madrid, Other Rogers Ramanujan type identities and an infinite dimensional Groebner basis.
- **Madrid, spain, January 2018:** Talk in Universidad Autonoma de Madrid, A geometric approach to resolution of singularities.
- **Marseille, France, November 2017:** Talk in the conference "Lipschitz Geometry".
- **Lille, France, May 2017:** Talk in the conference "Geometric aspects of singularities".
- **Columbia, United-States, April 2017:** Talk at the seminar of Algebra at University of Missouri.
- **Columbia, United-States, April 2017:** Talk in the Colloquium of the department of Mathematics, University of Missouri.
- **Leuven, Belgium, March 2017:** Talk at the seminar of Algebra of Leuven.
- **Rennes, France, November 2016:** Talk in the conference "Arc schemes and singularities".
- **Istanbul, Turkey, May 2016:** Talk in conference on Singularities in Topology and Geometry.
- **Nice, France, March 2016:** Talk in the conference "Singularities and Topology".
- **Marseille, France, January 2016:** Course at "la réunion des singularistes de Chambéry-Marseille-Nice".
- **Mainz, Germany, June 2015:** Talk at the Colloquium of Mainz University.
- **Leuven, Belgium, May 2015:** Talk at the seminar of Algebra of Leuven.
- **New York, United-States, May 2015:** Talk at The City university of New York.
- **Michigan, United-States, May 2015:** Talk in the seminar of Algebraic Geometry of Ann Arbor.
- **Marseille, France, March 2015:** Talk in the conference Artin approximation and infinite dimensional geometry.
- **Marseille, France, February 2015:** Talk in the conference Applications of Artin approximation in singularity theory.

- **Tehran, Iran, September 2014:** A course on jet schemes and resolution of singularities in the summer school Algebraic Geometry and Commutative Algebra in Tehran.

- **Sapporo, Japan, August 2014:** Talk in a conference on singularities at Hokkaido University.

- **Valladolid, Spain, February 2014:** Talk at the seminar of Algebra and Topology.

- **Lille,France, January 2014:** Talk at the meeting of the ANR Surface Singularities.

- **Marseille, France, May 2013:** Talk at the meeting bilipschitz, of the ANR Surface Singularities.

- **Santiago de Compostela, Spain, January 2013:** Talk at the session Singularities at the annual meeting of the royal Spanish mathematical society.

- **Cuernavaca, Mexico, October 2012:** Talk at the algebra seminar of the UNAM, Cuernavaca.

- **Séville, Spain, Juillet 2012:** Talk in the conference Singularities and applications .

- **Lille, France, May 2012:** Talk in the conference "Aspects of singularities".

- **Valladolid, Spain, January 2012:** Universidad de Valladolid, Talk at the Algebra and Geometry seminar.

- **Vienna, Austria, Nov. 2011:** The Erwin Schrodinger Institute, Talk at the workshop Arcs and Artin approximations.

- **Segovia, SpainJuly 2011:** Course on arc spaces and valuations at the second international conference and workshop on valuation theory.

- **CIRM MarseilleJan. 2011:** Talk at the meeting "Multiplier ideals in commutative algebra and singularities", annual meeting of the GDR Singularités et applications.

- **Beirut, Lebanon, Jan. 2010:** Talk at the first meeting of the Lebanese society of mathematical sciences.

- **Valladolid, Spain, June 2009:** University of Valladolid, talk at the seminar of Algebra and Geometry.

I also gave many talks in the following French institutions :

- Université d'Angers.
- Université de Bordeaux 1.
- Université de Bourgogne.
- Université de Caen.
- Université de Lille.
- Université de Provence, Marseille.
- Institut de Mathématiques de Luminy, Marseille.
- Université de Savoie.
- École polytechnique, Palaiseau.
- Institut de Mathématiques de Jussieu-Paris Rive Gauche (Séminaire sur les Singularités, Séminaire de Géométrie Algébrique).
- Université de Versailles.

Supervision of students

PHD students

- Zahraa Mohsen, Espaces d'arcs des points multiples, théorie des graphes et Partitions (Co-direction with Amine El-Sahili), beginning from September 2019.
- Andrei Bengus-Lasnier, Toric resolution of surface singularities in equal and mixed characteristics, beginning from September 2017.
- Pooneh Afsharjoo (Co-direction with Marc Chardin), Looking for a new version of Gordon's identities, from algebraic geometry to combinatorics through partitions. Defended on May 10, 2019.

Master students

- Lyes Lamri, Théorème de Bernstein-Kouchnirenko, March-September 2020.
- Lina El Ayoubi, Graphs , Simplicial Complexes and Monomial ideals, April-September 2018.
- Andrei Bengus-Lasnier, Resolution of singularities and local uniformization, during winter and spring 2017.
- Stéphanie Magonara, A topological proof of Abhyankar-Moh embedding line theorem (after Rudolph), during winter and spring 2017.
- Robin Michaud, Morse Homology, 2015.
- Mickal Montessinos, Tropical geometry and Legendre transformation, 2015.
- Dorian Chanfi, Valuation theory, 2015.
- Pooneh Afsharijoo, Toric varieties 2014.
- Dekens Leonard, Groebner Basis, 2014.
- Theo Marty, Tropical Geometry, 2014.

Undergraduate students

- Sebastien Son, Jacobi-Perron Algorithm, 2015.

Organizing of conferences

1. Co-organizer of the annual meeting of the GDR Singularités et applications, 23 - 27 November 2020, Paris.
2. Co-organizer of the conference Hilbert schemes, Mckay correspondence and singularities, 16 - 18 December 2019, Paris.
3. Co-organizer of the conference Arc schemes and algebraic group actions, 2 - 4 December 2019, Paris.
4. Co-organizer of the conference Free divisors and Hyperplane arrangements, 17 - 19 December 2018, Paris.
5. Co-organizer of the conference Lipschitz Geometry of Singularities, LISA, 23 - 25 Mai 2018, Paris.
6. Co-organizer of a conference on Deformations and singularities, 11 - 13 December 2017, Paris.
7. Co-organizer of a conference on the theory of valuations, 4 - 6 December 2017, Paris.
8. Co-organizer of a conference on Resolution of foliations, 17 - 19 October 2016, Paris.
9. Co-organizer of a workshop on resolution of singularities at Mittag Leffler institute, 23-27 May 2016.
10. Member of the scientific committee of the workshop Young researchers in Singularity theory, Nice 25-30 April 2016.
11. Co-organizer of the meeting Tropical varieties and T-varieties in Paris, 5-7 October 2015.
12. Co-organizer of the meeting Singularities and Tropical Geometry in Paris, September 2014.
13. Co-organizer of the CIMPA school: Algebraic geometry and number theory 2-10 june 2014, Istanbul.
14. Co-Organizer of the meeting, Metric and variational structures in singular varieties, 23-27 September 2013, Chambéry.

Organizing of seminars

1. Co-organizer of séminaire sur les singularités at institut de Mathématiques de Jussieu - Paris Rive Gauche, since September 2012.
2. Co-organizer of the seminar of PHD Students of Insitut de Mathématiques de Jussieu-Paris Rive Gauche (Coaching young researchers before their talks).
3. Co-organizer of a working group of commutative algebra (https://webusers.imj-prg.fr/ hussein.mourtada/GDTAC.html).
4. Organizer of a working group about singularities of algebraic varieties (http://www.math.jussieu.fr/ mourtada/GDT.html).

PhD thesis juries

1. Member of the jury of the PhD thesis of Isaac Konan, December 2020, Paris: Identités de type Rogers-Ramanujan: preuves bijectives et approche à la théorie de Lie; Supervised by Jeremy Lovejoy.
2. Member of the jury of the PhD thesis of Octave Kurmi, June 2019, Lille: Topology of smoothings of non-isolated singularities of complex surfaces; supervised by Patrick Popescu-Pampu.
3. Reviewer and Member of the jury of the PhD thesis of Maria de la Paz Tirado Hernandez, May 2019, Seville: Leaps of the chain of m-integrable derivations in the sense of Hasse-Schmidt; supervised by Luis Narvaes.
4. Reviewer and Member of the jury of the PhD thesis of B. Pascual, January 2018, Madrid: Algorithmic Resolution of Singularities and Nash multiplicity sequences; supervised by Ana Bravo and Santiago Encinas.
5. Member of the PhD thesis jury of A. Abbas, September 2017, Angers: Combinatoire des singularités de certaines courbes et hypersurfaces; supervised by Abdallah Assi.

Board of examiners committees

1. Member of the board of examiners of a mutation of a Maître de conférences position in Operator Algebras (position number MCF0573 - 4393, Paris Diderot university).
2. Member of the board of examiners of the Maître de conférences position in Algebra, Topology and Geometry (position number 1905, Paris Diderot university).

3. Member of the board of examiners of the Maître de conférences position in Geometry and representation theory (position number 1903, Paris Diderot university).

Other responsibilities

1. Member of the committee of the Faculty of Mathematics of Paris Diderot University.

2. Member of the scientific committee of the library MIR (Mathématique-Informatique Recherche).

3. Member of the committee responsible of PHD students in the group Geometry and Dynamics at institut de Mathématiques de Jussieu- Paris Rive Gauche.

4. Correspondent in Paris of "Groupe de Recherche International sur les Singularités", a cooperation France-Japan-Vietnam around singularities.

5. Correspondent of the "GDR Singularités et Applications" at institut de Math-ématiques de Jussieu-Paris Rive Gauche.

6. Tutor of two starting assistant professors in University of Paris.

7. Reviewer for the journals :

   - Journal of Algebraic Geometry.
   - Indiana journal of Mathematics.
   - Mathematica Zeitschrift.
   - Algebra and Number Theory.
   - International Journal of Mathematics.
   - Journal of Algebra.
   - Revista Matematica Complutense.
   - Journal of the Korean mathematical society.
   - Journal of pure and applied algebra.
   - Journal of Singularities.
   - Communications in Algebra.
   - Manuscripta mathematica.
   - Contemporary mathematics.
   - Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas.
   - Proceedings of the arc schemes conference.

- Proceedings of Némethi 60 conference.

8. Reviewer for Mathscinet and Zentralblatt.

9. Reviewer for the Chilean National Commission for Scientific and Technological Research.

10. Reviewer for the Belgium research foundation FWO.

11. Reviewer of an associate professorship promotion in USA.

Awards and grants

- Prime d'excellence scientifique 2018-2022.
- Miller Scholarship, University of Missouri, Columbia, United States of America (April 2017).
- University of Chicago FACCTS grant "Conormal and arc spaces in the deformation theory of singularities" with Antoni Rangachev and Bernard Teissier, 2018-2020.
- Member of the research group (ANR LISA),2018-2022.
- Member of the research group Surface Singularities (ANR SUSI),2012-2016.
- Member of the research group More Invariants From Arc Schemes (MIAS), a research project between France and Spain.

Teaching (in French)

2020-2021

- **Outils mathémtiques pour la Chimie**, Licence, 2ème année.
- **Algèbre et Analyse élémentaire**, première année.
- **Oraux blancs**, Agrégation interne.
- **Théorie des groupes**, 3ème année.
- **Algèbre**, 2ème année.

2019-2020

- **Outils mathémtiques pour la Chimie**, Licence, 2ème année.
- **Algèbre et Analyse élémentaire**, première année.
- **Oraux blancs**, Agrégation interne.
- Accueil en délégation CNRS.

2018-2019

- **Outils mathémtiques pour la Chimie**, Licence, 2ème année.
- **Logique**, Licence, 2ème année.
- **Arithmtique**, Licence, 2ème année.
- **Algèbre et Analyse élémentaire**, première année.
- **Oraux blancs**, Agrégation interne.

2017-2018

- **Topologie différentielle**, Master 2.
- **Logique**, Licence, 2ème année.
- **Arithmtique**, Licence, 2ème année.
- **Algèbre linéaire**, Agrégation externe.
- **Algèbre et Analyse élémentaire**, première année.
- **Oraux blancs**, Agrégation interne.
- **Algebra, Geometry and Combinatorics**, Master 2, Lebanese University.

2016-2017

- **Topologie différentielle**, Master 2.
- **Logique**, Licence, 2ème année.
- **Horizons Mathématiques**, première année.

2015-2016

- **Topologie différentielle**. (Master 2)
- **Théorie de Morse**. (Master 2)
- **Differential equations for biologists**. (Deuxième année)
- Cours **Compléments de mathématiques**. (Première année)

2014-2015

- Chargé de TD du cours de **Topologie différentielle**.(Master 2)
- Chargé de TD du cours de **Théorie de Morse**. (Master 2)
- Chargé de TD du cours de **Géométrie Différentielle**.(Master 1)

- Chargé de TD du cours de **Raisonnement mathématique**.(Première année)

- **Compléménts de mathématiques**. (Première année)

<u>2013-2014</u>

- **Homologie, Fibrés et Classes Caractéristiques**

  Public concerné: Etudiants en M2.

  Volume horaire : 39 heures de travaux dirigés.

  Lieu: UFR de Mathématiques, Université Paris 7.

- **Remise à Niveau**

  Public concerné: Etudiants en L1.

  Volume horaire : 41 heures de travaux dirigés.

  Lieu: UFR de Mathématiques, Université Paris 7.

- **Topologie Algébrique**

  Public concerné: Etudiants en M1.

  Volume horaire : 78 heures de travaux dirigés.

  Lieu: UFR de Mathématiques, Université Paris 7.

- **Géométrie différentielle**

  Public concerné: Etudiants en M1 de l'ENS Cachan.

  Volume horaire : 39 heures de travaux dirigés.

  Lieu: UFR de Mathématiques, Université Paris 7.

- **Projets**
  Public concerné: Etudiants en L1 et L2.
  Volume horaire : 20 heures de travaux dirigés.
  Lieu: UFR de Mathématiques, Université Paris 7.

<u>2012-2013</u>

- **Topologie Algébrique**
  Public concerné: Etudiants en M1.
  Volume horaire : 78 heures de travaux dirigés.
  Lieu: UFR de Mathématiques, Université Paris 7.

- **Géométrie différentielle**
  Public concerné: Etudiants en M1 de l'ENS Cachan.
  Volume horaire : 39 heures de travaux dirigés.
  Lieu: UFR de Mathématiques, Université Paris 7.

- **Projets**
  Public concerné: Etudiants en L1 et L2.
  Volume horaire : 20 heures de travaux dirigés.
  Lieu: UFR de Mathématiques, Université Paris 7.

<u>2011-2012</u>

- **Topologie Algébrique**
  Public concerné: Etudiants en M1.
  Volume horaire : 78 heures de travaux dirigés.
  Lieu: UFR de Mathématiques, Université Paris 7.

- **Géométrie différentielle**

  Public concerné: Etudiants en M1 de l'ENS Cachan.

  Volume horaire : 39 heures de travaux dirigés.

  Lieu: UFR de Mathématiques, Université Paris 7.

- **Projets**

  Public concerné: Etudiants en L1 et L2.

  Volume horaire : 20 heures de travaux dirigés.

  Lieu: UFR de Mathématiques, Université Paris 7.

2010-2011

- **Introduction à la Géométrie Algébrique**

  Public concerné: Etudiants en M1.

  Contenu : Espaces topologiques, Topologie de Zariski de $\mathbb{K}^n$, Correspondance idéaux *vs* fermés algébriques, Equivalence de la Catégorie des $\mathbb{K}-$Algèbres et la Catégorie des variétés affines, Anneaux de fractions, Extensions entières, Théorèmes de "going up" et "going down" de Cohen-Seidenberg, lemme de normalisation et Nullstellensatz, Dimension et degré de transcendance.

  Volume horaire : 27 heures de travaux dirigés.

  Lieu: UFR de Sciences, Versailles.

- **Mathématiques Générales 2**

  Public concerné: Etudiants en première année.

  Contenu : Formule des accroissements finis et applications, Formules de Taylor, Développements limités, Calcul intégral : Sommes de Darboux, Sommes de Riemann, Calcul de primitives : intégration par parties, changement de variables, décomposition en éléments simples, Séries infinies : rappels sur les suites ; liaisons avec les séries, Séries à termes positifs : critères de D'Alembert et de Cauchy, séries de Riemann, Convergence absolue, séries alternées, Espaces vectoriels, applications linéaires, Noyaux, image, théorème du rang ; matrices, matrices invertibles, Changement de base, matrices de passage.

Volume horaire : 54 heures de travaux dirigés.

Lieu: UFR de Sciences, Versailles.

- **Projet**

  Public concerné: Deux Etudiants en Licence 3.

  Contenu : Direction de projet sur la théorie des partitions des entiers et le théorème du nombre pentagonal d'Euler.

  Volume horaire officel: 3 heures de travaux dirigés.

  Lieu: UFR de Sciences, Versailles.

- **Mathématiques Générales 3**

  Public concerné: Etudiants en deuxième année.

  Contenu : Séries Entières, Convergence normale , comparaison avec une suite numérique majorante, Rayon de convergence, Dérivation et intégration terme à terme d'une série entière, Suites de Fonctions, séries de Fonctions, Convergence simple, uniforme, normale, Intégrales dépendant d'un paramètre, Continuité et dérivabilité.

  Volume horaire : 2 groupes, 2 x 36 heures de travaux dirigés.

  Lieu: UFR de Sciences, Versailles.

- **Mathématiques 3, Suites Matricielles et Optimisation**

  Public concerné: Etudiants en deuxième année.

  Contenu : *Algèbre Linéaire :* Compléments d'analyse matricielle, Valeurs propres et vecteurs propres, Suites matricielles, Oscillateur de Samuelson. *Optimisation :* Méthode du simplexe, Formes quadratiques, Optimisation non linéaire sans contrainte, Optimisation non linéaire sous contrainte.

  Volume horaire : 3 groupes, 3 x 12 heures de travaux dirigés.

  Lieu: UFR de Sciences sociales et humaines, Guyancourt.

2009-2010

- **Chimie et théorie des groupes**

  Public concerné: Etudiants en deuxième année.

  Contenu: Révisions d'algèbre linéaire, Isométries linéaires de $\mathbb{R}^2$ et $\mathbb{R}^3$, Groupes (définitions, premières propriétés et exemples), Représentations linéaires d'un groupe fini (théorie des caractères).

  Volume horaire : 36 heures de travaux dirigés.

  Lieu: UFR des Sciences, Versailles.

- **Algèbre**

  Public concerné: Etudiants en deuxième année, cycle préparatoire intégré de l'école d'ingénieur ISTY.

  Contenu: Déterminants : Déterminant d'une matrice et d'un endomorphisme, Formules de Cramer. Diagonalisation : Valeurs propres, vecteurs propres, polynme caractéristique. Puissance d'une matrice diagonalisable, application aux suites à récurrence linéaire, intért géométrique de la diagonalisation. Théorème admis : toute matrice symétrique réelle est diagonalisable dans une base orthonormée.

  Volume horaire : 18 heures de cours et 18 heures de travaux dirigés..

  Lieu: Ecole d'ingénieur ISTY, Versailles.

- **Préparation aux concours ENSI**

  Public concerné: Etudiants en deuxième année.

  Contenu: Les sujets de mathématiques des concours ENSI des années précédentes.

  Volume horaire : 27 heures de travaux dirigés.

  Lieu: UFR des Sciences, Versailles.

  2008-2009

- **Géométrie différentielle élémentaire : courbes et surfaces**

  Public concerné: Etudiants en deuxième année.

  Contenu: Etude et tracé des courbes paramétrées planes, Fonctions de $\mathbb{R}^2$ dans $\mathbb{R}^3$, Intégrales multiples, Intégrales curvilignes, Intégrales de surface.

  Volume horaire : 36 heures de travaux dirigés.

  Lieu: UFR des Sciences, Versailles.

- **Mathématiques Générales 3**

  Public concerné: Etudiants en deuxième année.

  Contenu: Voir ci-dessus, année 2010-2011.

  Volume horaire : 36 heures de travaux dirigés.

  Lieu: UFR des Sciences, Versailles.

2007-2008

- **Algèbre 1**

  Public concerné: Etudiants en première année.

  Contenu: Théorie des ensembles, Logique, Quantificateurs. Nombres Complexes. Polynmes. Matrices et systèmes linéaires.

  Volume horaire : deux groupes, 2x40 heures de travaux dirigés.

  Lieu: Université Paris 9, Dauphine.

- **Algèbre 2**

  Public concerné: Etudiants en première année.

  Contenu: Structures de $\mathbb{R}^d$, déterminant, valeurs propres, diagonalisation, trigonalisation.

  Volume horaire : deux groupes, 2x40 heures de travaux dirigés.

  Lieu: Université Paris 9, Dauphine.