PARTITION IDENTITIES AND APPLICATION TO INFINITE DIMENSIONAL GROEBNER BASIS AND VICEVERSA

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ABSTRACT. In the first part of this article, we consider a Groebner basis of the differential ideal $[x_1^2]$ with respect to "the" weighted lexicographical monomial order and show that its computation is related with an identity involving the partitions that appear in the first Rogers-Ramanujan identity. We then prove that a Groebner basis of this ideal is not differentially finite in contrary with the case of "the" weighted reverse lexicographical order. In the second part, we give a simple and direct proof of a theorem of Nguyen Duc Tam about the Groebner basis of the differential ideal $[x_1y_1]$; we then obtain identities involving partitions with 2 colors.

INTODUCTION

An integer partition of a positive integer number n is a decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l),$$

such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. The λ_i's are called the parts of λ and l is its size. The book [A] is a classic in the theory of integer partitions. A famous identity related to partitions and which plays an important role in this paper is the **First Rogers-Ramanujan Identity:**

The number of partitions of n with no consecutive parts, neither equal parts is equal to the number of of partitions of n whose parts are congruent to 1 or 4 modulo 5.

In [BMS] (see also [BMS1]) we got to this identity by considering the space of arcs of the double point $X = \mathrm{Spec}(\mathbf{K}[x]/x^2)$ centred at the origin; denote it by X_∞^0. The coordinate ring A of X_∞^0 is naturally graded and we associate with it its Hilbert-Poincaré series that we call the Arc Hilbert-Poincaré series; we denote it by $AHP_{X,0}(t)$. Note that this is an invariant of singularities of algebraic varieties [M, BMS]. We prove in [BMS] that $AHP_{X,0}(t)$ is equal to the generating sequence of the number of partitions appearing in the Rogers-Ramanujan identities. The proof uses a Groebner basis computation associated with a monomial ordering (reverse lexicographical). Note that we have

$$A = \frac{\mathbf{K}[x_i, i \in \mathbf{Z}_{>0}]}{[x_1^2]},$$

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where $[x_1^2]$ is the differential ideal generated by x_1^2 and its iterated derivative by the derivation D which is determined by $D(x_i) = x_{i+1}$. So

$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_1x_3 + 2x_2^2, \ldots).$$

The grading of A is determined by giving to x_i the weight i.

In the frist part of this article, we consider a different monomial ordering (lexicographical) and we find that the Groebner basis computation with respect to this monomial ordering is related with an other identity involving the partitions that appear in the first Rogers-Ramanujan identity. We then use this other member in the Rogers-Ramanujan identitiy to carry the computations of the Groebner-basis in small degrees (usual degree) but in all weights. This leads us to prove, in contrast with the case of the reverse lexicographical ordering, that a Groebner basis of our ideal with respect to the lexicographical ordering is not differentially finite.

In the other part of this article, we will give a direct and simpler proof of a theorem by Nguyen Duc Tam [N] where he computes a Groebner basis of the arc space of $X = \mathrm{Spec}(\mathbf{K}[x, y]/(xy))$ or simply of the differential ideal generated by xy. We then use this theorem to obtain identities of partitions with two colors.

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1. Hilbert series and integer partitions

In this section, we will begin by considering the Hilbert-Poincaré series of some graded algebras that are inspired by Groebner basis computations; we then interpret these series as generating sequences of partitions having special properties. At the end of the section, we will give more explanations on how we have found these graded algebras.

We will denote by $\mathbf{K}$ an algebraically closed field of characteristic 0. Recall that for a graded $\mathbf{K}-$algebras, $A = \oplus_{i\in\mathbf{N}} A_i$ (we assume that $\dim_{\mathbf{K}}(A_i) < \infty$) the Hilbert-Poincaré series of A, that we denote by $HP(A)$, is by definition

$$HP(A) = \sum_{i\in\mathbf{N}} \dim_{\mathbf{K}}(A_i) q^i,$$

where q is a variable. For more about Hilbert-Poincaré series, see the appendix in [BMS] and the references there.

Let $n, l \geq 1$ be integer numbers. We consider the graded algebra $\mathbf{K}[x_l, x_{l+1}, \ldots]$ where we give x_i the weight i, for $i \geq l$. This grading induces a grading on the $\mathbf{K}$-algebra $\frac{\mathbf{K}[x_l, x_{l+1}, \ldots]}{(x_{i_1} \cdots x_{i_n},\ i_j \geq l)}$.

We consider the Hilbert-Poincaré series
$$H_n^l = HP\left(\frac{\mathbf{K}[x_l, x_{l+1}, \ldots]}{(x_{i_1}\cdots x_{i_n},\ i_j \geq l)}\right).$$

Lemma 1.1. *We have*
$$H_n^l = q^l H_{n-1}^l + H_n^{l+1}.$$

Proof. Using corollary 6.2 in [BMS], we have
$$H_n^l = HP\left(\frac{\mathbf{K}[x_l, x_{l+1}, \ldots]}{(x_{i_1}\cdots x_{i_n},\ i_j \geq l)}\right) =$$
$$HP\left(\frac{\mathbf{K}[x_l, x_{l+1}, \ldots]}{(x_l, x_{i_1}\cdots x_{i_n},\ i_j \geq l)}\right) + q^l HP\left(\frac{\mathbf{K}[x_l, x_{l+1}, \ldots]}{((x_{i_1}\cdots x_{i_n},\ i_j \geq l) : x_l)}\right)$$
The last term is exactly $H_n^{l+1} + q^l H_{n-1}^l$.

□

The above lemma allows us to determine H_n^l.

Proposition 1.2. *We have*
$$H_n^l = 1 + \frac{q^l}{1-q} + \frac{q^{2l}}{(1-q)(1-q^2)} + \cdots + \frac{q^{(n-1)l}}{(1-q)(1-q^2)\cdots(1-q^{n-1})}.$$

Proof. The proof is by induction on the integer n. Notice that for $n = 1$, $H_1^l = HP(\mathbf{K}) = 1$. For $n = 2$, the weighted-homogeneous components of
$$\frac{\mathbf{K}[x_l, x_{l+1}, \ldots]}{(x_{i_1}x_{i_2},\ i_j \geq l)}$$
are generated by 1 in degree 0 and x_i in degree i for $i \geq l$; for $i = 1, \ldots l-1$, the weighted-homogeneous components of degree i is the null vector space. Let us assume that the formula is true for $H_j^l, j \leq n-1$ and prove it for H_n^l. Using lemma 1.1 repetitively, we obtain
$$H_n^l = q^l H_{n-1}^l + H_n^{l+1} = q^l H_{n-1}^l + q^{l+1} H_{n-1}^{l+1} + H_n^{l+2} = \cdots =$$
$$q^l H_{n-1}^l + q^{l+1} H_{n-1}^{l+1} + \cdots + q^m H_{n-1}^m + H_n^{m+1}.$$
But we have
$$\lim_{m\to\infty} H_n^m = 1,$$
where the limit is considered for the q-adic topology in $\mathbf{C}[[q]]$; hence we can write
$$H_n^l = 1 + q^l H_{n-1}^l + q^{l+1} H_{n-1}^{l+1} + \cdots + q^m H_{n-1}^m + q^{m+1} H_{n-1}^{m+1} + \cdots =$$
$$1 + \sum_{m\geq l} q^m H_{n-1}^m.$$
By the induction hypothesis we obtain
$$H_n^l = 1+$$
$$q^l + \frac{q^{2l}}{1-q} + \frac{q^{3l}}{(1-q)(1-q^2)} + \cdots + \frac{q^{(n-1)l}}{(1-q)(1-q^2)\cdots(1-q^{n-2})} +$$

$$q^{l+1}+\frac{q^{2(l+1)}}{1-q}+\frac{q^{3(l+1)}}{(1-q)(1-q^2)}+\cdots+\frac{q^{(n-1)(l+1)}}{(1-q)(1-q^2)\cdots(1-q^{n-2})}+\cdots=$$

(summing by columns)

$$1+(q^l+q^{l+1}+\cdots)+(\frac{q^{2l}}{1-q}+\frac{q^{2(l+1)}}{1-q}+\cdots)+\cdots+$$

$$(\frac{q^{(n-1)l}}{(1-q)(1-q^2)\cdots(1-q^{n-2})}+\frac{q^{(n-1)(l+1)}}{(1-q)(1-q^2)\cdots(1-q^{n-2})}+\cdots).$$

The formula for H_n^l follows from the formula

$$q^{jl}+q^{j(l+1)}+q^{j(l+2)}+\cdots=\frac{q^{jl}}{1-q^j},\ j=1,\ldots,n-1.$$

□

Corollary 1.3. *Let $l,n\geq 1$ be integers. The generating series of the integer partitions with parts greater or equal to l and size (number of parts) less or equal to $n-1$ is*

$$1+\frac{q^l}{1-q}+\frac{q^{2l}}{(1-q)(1-q^2)}+\cdots+\frac{q^{(n-1)l}}{(1-q)(1-q^2)\cdots(1-q^{n-1})}.$$

Proof. This is just the combinatorial interpretation of proposition 1.2: A basis of the $i-$th weighted-homogeneous component of $\frac{\mathbf{K}[x_l,x_{l+1},\ldots]}{(x_{i_1}\cdots x_{i_n},\ i_j\geq l)}$ is given by the monomials $x_{h_1}\cdots x_{h_r}$ such that $h_1+\cdots+h_r=i, h_j\geq l$ and $x_{h_1}\cdots x_{h_r}\notin(x_{i_1}\cdots x_{i_n},\ i_j\geq l)$; this corresponds to the partitions of i with parts greater or equal to l and size (number of parts) less or equal to $n-1$. □

We now look at the Hilbert series that makes the link to the first Rogers-Ramanujan identity. For $j\geq 1$, let

$$HP\left(\frac{\mathbf{K}[x_j,x_{j+1},\ldots]}{(x_{i_1}\cdots x_{i_k}x_k,\ i_1\geq i_2\geq\ldots\geq i_k\geq k\geq j)}\right). \tag{1}$$

Lemma 1.4. *We have*

$$H_j=q^jH_j^j+H_{j+1}.$$

Proof. Using corollary 6.2 in [BMS], we have

$$H_j=HP\left(\frac{\mathbf{K}[x_j,x_{j+1},\ldots]}{(x_{i_1}\cdots x_{i_k}x_k,\ i_1\geq i_2\geq\ldots\geq i_k\geq k\geq j)}\right)=$$

$$HP\left(\frac{\mathbf{K}[x_j,x_{j+1},\ldots]}{(x_j,x_{i_1}\cdots x_{i_k}x_k,\ i_1\geq i_2\geq\ldots\geq i_k\geq k\geq j)}\right)+$$

$$q^jHP\left(\frac{\mathbf{K}[x_j,x_{j+1},\ldots]}{((x_{i_1}\cdots x_{i_k}x_k,\ i_1\geq i_2\geq\ldots\geq i_k\geq k\geq j):x_j)}\right).$$

The last term is exactly $H_{j+1}+q^jH_j^j$.

□

Theorem 1.5. *We have*

$$H_1 = 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots =$$

$$1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Proof. Applying lemma 1.4 for $j = 1$, we obtain that $H_1 = qH_1^1 + H_2$; applying the same lemma for $j = 2$ we have that $H_1 = qH_1^1 + q^2H_2^2 + H_3$; Applying repetitively and in the same way lemma 1.4, we obtain that for $m \geq 2$,

$$H_1 = qH_1^1 + q^2H_2^2 + \cdots + q^mH_m^m + H_{m+1}.$$

Noticing that

$$\lim_{m\to\infty} H_m = 1$$

(where the limit is considered for the q-adic topology in $\mathbf{C}[[q]]$, $\mathbf{C}$ being the field of complex numbers) we can write

$$H_1 = 1 + qH_1^1 + q^2H_2^2 + \cdots + q^mH_m^m + \cdots .$$

Using proposition 1.2, we obtain that H_1 is equal to

$$\begin{array}{lllll}
1 & +q & & & \\
 & +q^2 & +\frac{q^4}{1-q} & & \\
 & +q^3 & +\frac{q^6}{1-q} & +\frac{q^9}{(1-q)(1-q^2)} & \\
 & +q^4 & +\frac{q^8}{1-q} & +\frac{q^{12}}{(1-q)(1-q^2)} & + \cdots
\end{array}$$

Summing by columns we obtain the result. □

An equivalent statement of the theorem the following.

Theorem 1.6. *Let $n \geq 1$ be a positive integer. The number of partitions of n with size less than or equal to the smallest part is equal to the number of partitions of n without consecutive nor equal parts.*

Proof. This follows from the known fact ([A, BMS]) that the series obtained in theorem 1.5 is also the generating sequence of the partitions without consecutive nor equal parts. □

It was pointed to us by Jan Schepers that the theorem [OEIS] is mentioned in [OEIS] but without a clear reference. We have rediscovered this theorem from Groebner basis computations (see the explanations after theorem 1.7); our point of view allows us to obtain the following family of identities (indexed by an integer number k) which are generalizations of theorem 1.6.

Theorem 1.7. *Let $n \geq k$ be a positive integer. The number of partitions of n with parts larger or equal to k and size less than or equal to (the smallest part minus $k-1$) is equal to the number of partitions of n with parts larger or equal to k and without consecutive nor equal parts.*

Proof. We denote by F_k the Hilbert series of

$$\frac{\mathbf{K}[x_k, x_{k+1}, \ldots]}{(x_{i_1} \cdots x_{i_{j-k+1}} x_j,\ i_1 \geq i_2 \geq \ldots \geq i_{j-k+1} \geq j \geq k)}.$$

It follows from similar computations to those in the proof of the theorem 1.5 that

$$F_k = 1 + q^k H_1^k + q^{k+1} H_2^{k+1} + \cdots + q^{k+m} H_{m+1}^{k+m} + \cdots.$$

By proposition 1.2 we have

$$\begin{array}{llll} F_k = 1 & +q^k & & \\ & +q^{k+1} & +\frac{q^{2k+2}}{1-q} & \\ & +q^{k+2} & +\frac{q^{2k+4}}{1-q} & +\frac{q^{3k+6}}{(1-q)(1-q^2)} \\ & +q^{k+3} & +\frac{q^{2k+6}}{1-q} & +\frac{q^{3k+9}}{(1-q)(1-q^2)} \quad + \cdots. \end{array}$$

Summing by columns we obtain that

$$F_k = 1 + \sum_{n \geq 1} \frac{q^{n(n+k-1)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

This implies that

$$F_{k+1} + q^k F_{k+2} =$$

$$\begin{array}{lllll} 1 & +\frac{q^{k+1}}{1-q} & +\frac{q^{2k+4}}{(1-q)(1-q^2)} & +\frac{q^{3k+9}}{(1-q)(1-q^2)(1-q^3)} & + \cdots \\ & +q^k & +\frac{q^{2k+2}}{1-q} & +\frac{q^{3k+6}}{(1-q)(1-q^2)} & + \cdots. \end{array}$$

Again summing by columns we obtain that

$$F_{k+1} + q^k F_{k+2} = 1 + \sum_{n\geq 1} \frac{q^{n(n+k-1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = F_k.$$

It follows from proposition 5.8 in [BMS] (with a shift in the indices) that for $k \geq 1$ there exist $A_j, B_j \in \mathbf{C}[[q]]$ such that

$$F_k = A_{k+i}F_{k+i} + B_{k+i+1}F_{k+i+1},$$

$A_{k+1} = 1, B_{k+2} = q^k$, and for all $i \geq 2$ we have

$$A_{k+i} = A_{k+i-1} + B_{k+i},$$

$$B_{k+i+1} = q^{k+i-1}A_{k+i-1}.$$

Denote now by H'_k the Hilbert series of

$$\frac{\mathbf{K}[x_k, x_{k+1}, \ldots]}{(x_i^2, x_{i+1}^2, i \geq k)}.$$

By equation (7) in [BMS], H'_k satisfies the same recursion formula

$$H'_k = A_{k+i}H'_{k+i} + B_{k+i+1}H'_{k+i+1}.$$

Since $\lim B_j = 0$ and $\lim F_j = \lim H'_j = 1$ (in the (q)-adic topology) we have

$$F_k = \lim\left(A_{k+i}F_{k+i} + B_{k+i+1}F_{k+i+1}\right) = \lim A_{k+i} =$$
$$\lim\left(A_{k+i}H'_{k+i} + B_{k+i+1}H'_{k+i+1}\right) = H'_k.$$

Noticing that F_k is the generating series of the number of partitions with parts larger or equal to k and size less than or equal to (the smallest part minus $k-1$) and that H'_k is the generating series of the number of partitions with parts larger or equal to k and without consecutive nor equal parts, we obtain the result in the theorem.

□

Theorem 1.6 is inspired from a Groebner basis computation of the differential ideal $[x_1^2]$: By [BMS], the initial ideal of $[x_1^2]$ with respect to the reverse lexicographical ordering is $(x_i^2, x_i x_{i+1}, i \geq 1)$, while we can guess that its initial ideal with respect to the lexicographical ordering is $((x_{i_1}\cdots x_{i_k}x_k,\ i_1 \geq i_2 \geq \ldots \geq i_k \geq k \geq 1)$ (we make use of this guess in the next section). Hence the Hilbert series of the quotient rings of $\mathbf{K}[x_1, x_2, \ldots]$ by these ideals are equal. The Hilbert series of the quotient by $(x_i^2, x_i x_{i+1}, i \geq 1)$, is the generating series of the partitions without consecutive nor equal parts; The Hilbert series of the quotient by $((x_{i_1}\cdots x_{i_k}x_k,\ i_1 \geq i_2 \geq \ldots \geq i_k \geq k \geq 1)$ is the generating series of the partitions with size less or equal to the smallest part. Theorem 1.7 can be guessed in the same way by considering the ideal $[x_k^2]$ in $\mathbf{K}[x_k, x_{k+1}, \ldots]$.

2. On the lex Groebner basis of $[x_1^2]$

Again, let $\mathbf{K}$ be a field of characteristic 0, and consider the graded ring $\mathbf{K}[x_1, x_2, \dots]$, where the weight of x_i is i. So the weight of the monomial $x^\alpha := x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}$, is equal to $\sum_{j=1}^{n} i_j\alpha_j$, and its (usual) degree is equal to n.
Let $f_2 = x_1^2$ and for $i \geq 3, f_i = D^{i-2}(f_2) := D(f_{i-1})$, where D is the derivation determined by $D(x_i) = x_{i+1}$; then $I = [f_2] := (f_2, f_3, \dots) \subset \mathbf{K}[x_1, x_2, \dots]$, is the defining ideal (up to isomorphism) of the space of arcs centred at the origin of $X = \mathrm{Spec}(\mathbf{K}[x]/(x^2))$. The ideal I is a differential ideal, i.e. we have $D(I) \subset I$. We are interested in the possibility that I have a differentially finite Groebner basis with respect to a monomial ordering ; see the following definition.

Definition 2.1. *Let $J \subset \boldsymbol{K}[x_1, x_2, \dots]$ be a differential ideal with respect to D (i.e. $D(J) \subset J$). Let " $<$ " be a total monomial order defined on $\boldsymbol{K}[x_1, x_2, \dots]$. We say that J has a differentially finite Groebner basis with respect to " $<$ ", if there exist a finite number of polynomials $h_1, \dots, h_r \in \boldsymbol{K}[x_1, x_2, \dots]$ such that $J = [h_1, \dots, h_r]$ and the initial ideal $In_<(J)$ of J with respect to " $<$ " satisfies*

$$In_<(J) = (In_<(D^i(h_j)), j = 1, \dots, r; i \geq 0),$$

where D^i denotes the $i-$th iterated derivative and D^0 is the identity.

Note that the notation $[h_1, \dots, h_r]$ in the definition denotes the differential ideal generated by the $h_i, i = 1, \dots, r$ and by all their iterated derivatives. . Note that there might be different notions of differential Groebner basis, see [CF], [O] and their bibliography.

In this section, we prove that no Groebner basis of I with respect to the weighted lexicographical order is differentially finite. Note that, in contrary with this case, it follows for [BMS] that in the case of the weighted reverse lexicographical order I has a differentially finite Groebner basis.

We denote the n-th derivative of a polynomial f_i by $f_i^{(n)}$, so we have
$f_n = f_2^{(n-2)} = (x_1^2)^{(n-2)} = \sum_{i=0}^{n-2} \binom{n-2}{i} x_1^{(i)} x_1^{(n-2-i)} = \sum_{i=0}^{n-2} \binom{n-2}{i} x_{1+i} x_{n-i-1}.$
Denote the leading term of f_n with respect to the weighted lexicographical order by $LT(f_n)$. So $LT(f_n) = 2x_1x_{n-1}$ for all $n \geq 2$.

Recall that the S-polynomial of $f, g \in \mathbf{K}[x_1, x_2, \dots]$ is by definition

$$S(f, g) := \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g,$$

where x^γ is the least common multiple of the leading monomials of f and g. A possible reference about S-polynomials and Groebner basis is [GP].

A direct computation of the S-polynomial of f_3 and f_4 gives

$$S(f_3, f_4) = x_2^3.$$

We set $F_{x_2^3} := S(f_3, f_4)$. For $k > 2$, we recursively define

$$F_{x_2x_k^2} := S(F^{(2)}_{x_2x_{k-1}^2}, S(f_{k+1}, f_{k+2})).$$

We then have the following lemma.

Lemma 2.1. *With respect to the weighted lexicographic order, for $k > 2$, the leading monomial of $F_{x_2x_k^2}$ is $x_2x_k^2$.*

Proof. The proof is by induction on the integer k. Notice that for $k = 3$ we have $F^{(1)}_{x_2^3} = 3x_2^2x_3$, $F^{(2)}_{x_2^3} = 3x_2^2x_4 + 6x_2x_3^2$ and $S(f_4, f_5) = x_2^2x_4 - 3x_2x_3^2$. So we have

$$S(F^{(2)}_{x_2^3}, S(f_4, f_5)) = 5x_2x_3^2 := F_{x_2x_3^2}.$$

For $k = 4$ we have $F^{(1)}_{x_2x_3^2} = 10x_2x_3x_4 + 5x_3^3$, $F^{(2)}_{x_2x_3^2} = 10x_2x_3x_5 + 10x_2x_4^2 + 25x_3^2x_4$ and $S(f_5, f_6) = 3x_2x_3x_5 - 4x_2x_4^2 - 3x_3^2x_4$. So

$$S(F^{(2)}_{x_2x_3^2}, S(f_5, f_6)) = \frac{7}{3}x_2x_4^2 + \frac{7}{2}x_3^2x_4 := F_{x_2x_4^2}.$$

Now assume that claim holds for $k - 1 \geq 4$. This means that for $k - 1 \geq 4$ we have

$$F_{x_2x_{k-1}^2} := S(F^{(2)}_{x_2x_{k-2}^2}, S(f_k, f_{k+1})).$$

Since the leading monomial of $F_{x_2x_{k-1}^2}$ is $x_2x_{k-1}^2$, we can assume that $F_{x_2x_{k-1}^2} = ax_2x_{k-1}^2 + g_3$, for some rational number a and some polynomial g_3 with the monomials of the form $x_{i_1}x_{i_2}x_{i_3}$ such that $3 \leq i_1 \leq i_2 \leq i_3$. So, on one hand, the second derivative of $F_{x_2x_{k-1}^2}$ will be as follow

$$F^{(2)}_{x_2x_{k-1}^2} = 2ax_2x_{k-1}x_{k+1} + 2ax_2x_k^2 + h_3.$$

Where $h_3 = 4ax_3x_{k-1}x_k + ax_4x_{k-1}^2 + g_3^{(2)}$. On the other hand, we have

$$S(f_{k+1}, f_{k+2}) = S(\sum_{i=0}^{k-1} \tbinom{k-1}{i} x_{1+i}x_{k-i}, \sum_{i=0}^{k} \tbinom{k}{i} x_{1+i}x_{k+1-i})$$
$$= \tfrac{1}{2}\sum_{i=0}^{k-1} \tbinom{k-1}{i} x_{1+i}x_{k-i}x_{k+1} - \tfrac{1}{2}\sum_{i=0}^{k} \tbinom{k}{i} x_{1+i}x_{k+1-i}x_k$$
$$= \tfrac{1}{2}\sum_{i=1}^{k-2} \tbinom{k-1}{i} x_{1+i}x_{k-i}x_{k+1} - \tfrac{1}{2}\sum_{i=1}^{k-1} \tbinom{k}{i} x_{1+i}x_{k+1-i}x_k.$$

So by the above equation we obtain that $LT(S(f_{k+1}, f_{k+2})) = (k-1)x_2x_{k-1}x_{k+1}$. Now we can compute $S(F^{(2)}_{x_2x_{k-1}^2}, S(f_{k+1}, f_{k+2}))$.

$$S(F^{(2)}_{x_2x_{k-1}^2}, S(f_{k+1}, f_{k+2}))$$
$$= \tfrac{1}{2a}(2ax_2x_{k-1}x_{k+1} + 2ax_2x_k^2 + h_3) - \tfrac{1}{(k-1)}(\tfrac{1}{2}\sum_{i=1}^{k-2} \tbinom{k-1}{i} x_{1+i}x_{k-i}x_{k+1} - \tfrac{1}{2}\sum_{i=1}^{k-1} \tbinom{k}{i} x_{1+i}x_{k+1-i}x_k)$$
$$= x_2x_k^2 + \tfrac{1}{2a}h_3 - \tfrac{1}{(k-1)}(\tfrac{1}{2}\sum_{i=2}^{k-3} \tbinom{k-1}{i} x_{1+i}x_{k-i}x_{k+1} - \tfrac{1}{2}\sum_{i=1}^{k-1} \tbinom{k}{i} x_{1+i}x_{k+1-i}x_k)$$

$= x_2x_k^2 + \frac{1}{2a}h_3 + \frac{k}{k-1}x_2x_k^2 - \frac{1}{(k-1)}(\frac{1}{2}\sum_{i=2}^{k-3}\binom{k-1}{i}x_{1+i}x_{k-i}x_{k+1} - \frac{1}{2}\sum_{i=2}^{k-2}\binom{k}{i}x_{1+i}x_{k+1-i}x_k)$

$= \frac{2k-1}{k-1}x_2x_k^2 + \frac{1}{2a}h_3 - \frac{1}{(k-1)}(\frac{1}{2}\sum_{i=2}^{k-3}\binom{k-1}{i}x_{1+i}x_{k-i}x_{k+1} - \frac{1}{2}\sum_{i=2}^{k-2}\binom{k}{i}x_{1+i}x_{k+1-i}x_k).$

In the first sum, $2 \leq i \leq k-3$ and in the second one $2 \leq i \leq k-2$. So each monomial that appears in $S(F^{(2)}_{x_2x_{k-1}^2}, S(f_{k+1}, f_{k+2}))$ is of the form $x_3x_{i_1}x_{i_2}x_{i_3}$ such that $3 \leq i_1 \leq i_2 \leq i_3$, except $x_2x_k^2$ and hence

$$LT(S(F^{(2)}_{x_2x_{k-1}^2}, S(f_{k+1}, f_{k+2}))) = \frac{2k-1}{k-1}x_2x_k^2.$$

□

Theorem 2.2. *A Groebner basis of the ideal I, with respect to the weighted lexicographic order, is not differentially finite.*

Proof. For proving this fact, we will use the idea of Buchberger's algorithm (mainly that any cancellation of initial monomials comes from an $S-$ polynomial [CLO]) to construct a part of a Groebner basis of the ideal I with respect to the weighted lexicographic order, which is differentially infinite.

By lemma 2.1, we have that for every integer $n \geq 3$, the initial monomial of the polynomial $F_{x_2x_n^2}$ is included in the initial ideal of I.

Let $G = \{f_i, F_{x_2x_n^2}, F^{(m)}_{x_2x_n^2} | i \geq 2, m \geq 1, n \geq 3\}$. By Buchberger's algorithm G may be a part of a Groebner basis of the ideal I but it is not a Groebner basis of I because

$$S(F_{x_2x_3^2}, F^{(1)}_{x_2x_3^2}) = S(F_{x_2x_3^2}, F_{x_2x_3x_4}) = 5x_3^4.$$

But the monomial x_3^4 which is a member of the ideal I is not divisible by the leading terms of any element of G.

Note that the (usual) degree of the $S-$polynomial of two polynomials is at least equal to the maximum of degrees of these two polynomials. On the other hand, the derivative of a polynomial has the same degree as itself. Hence the degree of the f_i's is equal to two, and other elements of G have degree strictly bigger than 2.
This means that the monomials of degree two that appear as the leading terms of elements of Groebner basis, are of the form x_1x_i for $i \geq 1$.
So we do not have any polynomial in a Groebner basis whose leading monomial is x_2x_n for some $n \geq 2$, and so a polynomial having the same initial monomial of $F_{x_2x_n^2}$ should be included in this Groebner basis for each integer $n \geq 3$. Since the initial monomial of the polynomial $F_{x_2x_n^2}$ is not the initial of the derivative of any other element of G, the Groebner basis of the ideal I with respect to the weighted lexicographic order will not be differentially finite: it should contain polynomials whose initial monomials are the initials of $F_{x_2x_n^2}, n \geq 3$ and no one of these initial monomials is the derivative of an other initial monomial of an element in G.

□

3. Two colors partitions and the node

Let $S := \mathbf{K}[x_1, x_2, \dots, y_1, y_2, \dots]$ be the graded polynomial ring where x_i, y_i have the weight i for every $i \geq 1$; the order of appearance of the variables is important since we will use below a reverse lexicographical ordering. We consider the derivation on S defined by $D(x_i) = x_{i+1}$ and $D(y_i) = y_{i+1}$. Let $f_2 = x_1 y_1$, and let

$$I = [f_2] = (x_1 y_1, x_2 y_1 + x_1 y_2, \dots)$$

be the ideal generated by xy and its iterated derivatives $f_i, i \geq 3$ by D : for $i \geq 3, f_i = D(f_{i-1})$. Note that the scheme defined by I is the space of arcs centred at the origin of the node $X = \{xy = 0\} \subset \mathbf{A}^2$.
In this section, we are interested in determining a Groebner basis of I with respect to the weighted reverse lexicographical order and then to apply this result to integer partitions. Note that the Groebner basis below was found by Nguyen Duc Tam [N]; he has a beautiful but very long and difficult proof that this is actually a Groebner basis. Below we give a simpler and very short proof.
We will begin by defining elements of I, and we will show later that these elements give the Groebner basis cited above.

Definition 3.1. *([N]) For $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k$ and for $k \geq 2$, we set*

$$G_{i_1, i_2+1, i_3+2, \dots, i_k+k-1} := \det \begin{bmatrix} x_{i_1-k+2} & x_{i_1-k+3} & \cdots & x_{i_1} & f_{i_1+1} \\ x_{i_2-k+3} & x_{i_2-k+4} & \cdots & x_{i_2+1} & f_{i_2+2} \\ x_{i_3-k+4} & x_{i_3-k+5} & \cdots & x_{i_3+2} & f_{i_3+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i_k+1} & x_{i_k+2} & \cdots & x_{i_k+k-1} & f_{i_k+k} \end{bmatrix}$$

where det stands for determinant.

Expanding the determinant with respect to the last column, we see that these are elements of I. A direct computation using the definition of the f_i gives the following:

Lemma 3.1. [N] *The leading term of $G_{i_1, i_2+1, i_3+2, \dots, i_k+k-1}$ with respect to weighted reverse lexicographic order, is $x_{i_1} x_{i_2} x_{i_3} \dots x_{i_k} y_k$.*

We denote by $\mathbb{G}$ the set whose elements are the $G_{i_1, i_2+1, i_3+2, \dots, i_k+k-1}$ and the f_i. It follows from lemma 3.1 that the ideal generated by the initials of the elements of $\mathbb{G}$ is

$$J := (x_{i_1} x_{i_2} \dots x_{i_k} y_k | \ i_j, k \geq 1).$$

First, we are interested in computing the Hilbert-Poincaré series of S/J. For that we introduce for $n \geq 1$ the Hilbert-Poincaré series

$$HP_n = HP\left(\frac{\mathbf{K}[x_i, y_j | i \geq 1, j \geq n]}{(x_{i_1} x_{i_2} \dots x_{i_k} y_k | \ i_j \geq 1, k \geq n)}\right).$$

So $HP(S/J) = HP_1$. We will use the following form of H_n^1 from section 1:

Lemma 3.2. *For any $n \geq 2$ we have*

$$H_n^1 = \frac{1}{(1-q)(1-q^2)\dots(1-q^{n-1})}.$$

Proof. By proposition 1.2 we have

$$H_n^1 = 1 + \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \cdots + \frac{q^{n-1}}{(1-q)\ldots(1-q^{n-1})}$$

We prove the expression in the lemma by induction on the integer n. For $n = 2$

$$H_2^1 = 1 + \frac{q}{1-q} = \frac{1}{1-q}$$

Assume that $H_n^1 = \frac{1}{(1-q)\ldots(1-q^{n-1})}$. Now we have

$$\begin{aligned} H_{n+1}^1 &= 1 + \frac{q}{1-q} + \cdots + \frac{q^{n-1}}{(1-q)\ldots(1-q^{n-1})} + \frac{q^n}{(1-q)\ldots(1-q^n)} \\ &= H_n^1 + \frac{q^n}{(1-q)\ldots(1-q^n)} \end{aligned}$$

By induction hypothesis we obtain

$$H_{n+1}^1 = \frac{1}{(1-q)\ldots(1-q^{n-1})} + \frac{q^n}{(1-q)\ldots(1-q^n)} = \frac{1}{(1-q)\ldots(1-q^n)}$$

□

Lemma 3.3.

$$HP_n = HP_{n+1} + q^n \prod_{i\geq 1} \frac{1}{1-q^i}.$$

Proof. Using corollary 6.2 in [BMS], we have

$$HP_n = HP\left(\frac{\mathbf{K}[x_i, y_j | i \geq 1, j \geq n]}{(x_{i_1}x_{i_2}\ldots x_{i_k}y_k |\ i_j \geq 1, k \geq n)}\right) =$$

$$HP\left(\frac{\mathbf{K}[x_i, y_j | i \geq 1, j \geq n+1]}{(x_{i_1}x_{i_2}\ldots x_{i_k}y_k |\ i_j \geq 1, k \geq n)}\right) + q^n HP\left(\frac{\mathbf{K}[x_i, y_j | i \geq 1, j \geq n]}{(x_{i_1}x_{i_2}\ldots x_{i_n} |\ i_j \geq 1)}\right) =$$

$$HP_{n+1} + q^n HP\left(\frac{\mathbf{K}[x_1, x_2, \cdots]}{(x_{i_1}x_{i_2}\ldots x_{i_n} |\ i_j \geq 1)}\right)\ HP\left(\mathbf{K}[y_n, y_{n+1}, \cdots]\right) =$$

$$HP_{n+1} + q^n H_n^1 \prod_{i\geq n} \frac{1}{1-q^i};$$

by lemma 3.2 this is equal to

$$HP_{n+1} + \frac{q^n}{(1-q)\cdots(1-q^{n-1})} \prod_{i\geq n} \frac{1}{1-q^i} =$$

$$HP_{n+1} + q^n \prod_{i\geq 1} \frac{1}{1-q^i}.$$

□

Proposition 3.4. *We have*

$$HP(S/J) = HP_1 = \frac{1}{1-q} \prod_{i\geq 1} \frac{1}{1-q^i}.$$

Proof. Using lemma 3.3 repetitively we obtain that, for $m \geq 2$

$$HP_1 = q\prod_{i\geq 1}\frac{1}{1-q^i} + q^2\prod_{i\geq 1}\frac{1}{1-q^i} + \cdots q^m\prod_{i\geq 1}\frac{1}{1-q^i} + HP_{m+1}.$$

On the other hand

$$\lim_{m\to\infty} HP_m = \prod_{i\geq 1}\frac{1}{1-q^i},$$

where the limit is considered for the q-adic topology in $\mathbf{C}[[q]]$; so we have,

$$\begin{aligned} HP_1 &= \prod_{i\geq 1}\frac{1}{1-q^i} + q\prod_{i\geq 1}\frac{1}{1-q^i} + q^2\prod_{i\geq 1}\frac{1}{1-q^i} + \cdots \\ &= (1+q+q^2+\cdots)\prod_{i\geq 1}\frac{1}{1-q^i} \\ &= \frac{1}{1-q}\prod_{i\geq 1}\frac{1}{1-q^i}. \end{aligned}$$

□

We now are ready to prove:

Theorem 3.5. *([N]) We have that $\mathbb{G}$ is a Groebner basis of I.*

Proof. Let $In(I)$ be the initial ideal of I with respect to the weighted reverse lexicographical order. Since all the elements of $\mathbb{G}$ are also in I, we have that $J \subset In(I)$; to prove that $\mathbb{G}$ is a Groebner basis of I, we need to prove that $J = In(I)$.
Noticing that $(f_2, f_3, \ldots)$ is a regular sequence [GS] (Note that this is rarely the case [M1]) and that f_i is of weight i, we deduce that

$$HP(S/I) = \frac{1}{1-q}\prod_{i\geq 1}\frac{1}{1-q^i},$$

which is equal by proposition 3.4 to $HP(S/J)$. But since we have a flat deformation with generic fibre S/I and special fibre $S/In(I)$, we have that $HP(S/I) = HP(S/In(I))$, hence $HP(S/In(I)) = HP(S/J)$. We deduce that the homogeneous components of the same weight of $S/(In(I)$ and S/J have the same (finite) dimension, and since we have an inclusion in one sense because $J \subset In(I)$, they are equal. Hence $J = In(I)$.

□

We will interpret the above results in terms of two colors partitions: consider that we have two copies of each positive integer number m, one is blue and the other is red; we denote these copies by m_b and m_r. We define an order between the colored integers by $m_b > m_r$ (so that we do not count in a partition $m_b + m_r$ and $m_r + m_b$ as different); if $m > k$, we say $m_c > k_{c'}$ for $c, c' \in \{b, r\}$.

An integer partition of a positive integer number n is a decreasing sequence (with respect to the order that we have just defined) of positive integers of one color or an other

$$\lambda = (\lambda_{1,c_1} \geq \lambda_{2,c_2} \geq \ldots \geq \lambda_{l,c_l}),$$

where $c_i \in \{b, r\}$ and such that $\lambda_{1,c_1} + \lambda_{2,c_2} + \cdots + \lambda_{l,c_l} = n$. For example, the two colors integer partitions of 2 are:

$$\begin{gathered} 2_b \\ 2_r \\ 1_b + 1_b \\ 1_r + 1_r \\ 1_b + 1_r. \end{gathered}$$

Colored partitions has already appeared in the work of Andrews and Agarwal [APS].

On one hand, we can interpret the series

$$\frac{1}{1-q} \prod_{i \geq 1} \frac{1}{1-q^i}$$

as the generating series of the partitions with 2 colors of 1 and only the red color of any other positive integer. So the partitions of 2 of this type are all the partitions appearing in the above example except the first one.

On the other hand, the monomials in S/J of weight n are in bijection with the partitions with 2 colors of n whose number of blue parts is strictly less than its smallest red part (if this latter exists). In the above example of partions of 2, all the partitions but the last one are of this type. The Hilbert-Poincaré series $HP(S/J)$ is then the generating sequence of this type of partitions. Hence proposition 3.4 gives:

Theorem 3.6. *The number of partitions of n with 2 colors of 1 and only the red color of any other positive integer is equal to the number of partitions with 2 colors of n whose number of blue parts is strictly less than its smallest red part (if this latter exists).*

Playing the same game with the ideal $[x_j y_j]$ instead of $[x_1 y_1]$ we can prove the following generalization of theorem 3.6:

Theorem 3.7. *Let j be a positive integer number. The number of partitions of n with 2 colors of $1, \ldots, 2j-1$ and only the red color of any other positive integer is equal to the number of partitions with 2 colors of n whose number of blue parts is strictly less than its smallest red part (if this latter exists) minus $(j-1)$.*

We recover theorem 3.6 by putting $j = 1$.

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