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Jet schemes and minimal embedded desingularization of plane branches

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To Professor Heisuke Hironaka on the occasion of his 80th birthday

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Abstract For a plane branch C with g Puiseux pairs, we determine the irreducible components of its jet schemes which correspond to the star (or rupture) and end divisors that appear on the dual graph of the minimal embedded desingularization of C. We exploit these informations to construct a Teissier type resolution of C embedded in $\mathbb{C}^{g+1}$, which is special in the sense that its restriction to the strict tranform of the plane induces the minimal embedded desingularization of C.

Keywords Jet schemes · Resolution of singularities · Toric Geometry

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1 Introduction

This article has two sources of motivations :

On one hand, Teissier's approach to resolution of singularities, which roughly speaking consists in re-embedding the variety, in such a way that in the new

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coordinates, the variety is non-degenerate with respect to its Newton polyhedron in the sense of Khovansky-Kouchnirenko, so that it can be desingularized by one toric morphism. In particular in [GT], the authors, by considering the specialization of a plane branch C to the monomial curve whose semigroup is the one of C, gave such a desingularization for plane branches.

On the other hand, Nash's approach to detect the intrinsic data in various abstract resolution of singularities of a variety, from its arc space. While the arc space contains information about abstract resolution of singularities, jet schemes, as pointed out in [ELM], encode information about embedded resolution of singularities (See also [dFEI]).

In the case of a plane branch $C \subset \mathbb{C}^2$, we will mix the two points of view as follows: First we will use some information about irreducible components of the jet schemes in order to detect interesting divisors that appear on the minimal embedded resolution of C, namely the root divisor, the end divisors and the star divisors (see definition 25). Then we will associate with these irreducible components a combinatorial data, that we will exploit to give an embedded resolution of the branch $C \subset \mathbb{C}^{g+1}$. This last resolution is special between those given in [GT], in the sense that its restriction to the strict transform of the plane $\mathbb{C}^2$ is the minimal embedded resolution of $C \subset \mathbb{C}^2$. This also gives a jet-theoretical interpretation of the notion of maximal contact in [L].

2 Jet schemes and dual graph of a plane branch

We begin by recalling the definitions of jet scheme and by giving some notations.

Let $\mathbb{K}$ be an algebraically closed field. Let X be a $\mathbb{K}$-scheme and let $m \in \mathbb{N}$. The functor $F_m : \mathbb{K}-Schemes \longrightarrow Sets$ which to an affine scheme defined by a $\mathbb{K}-$algebra A associates

$$F_m(Spec(A)) = Hom_{\mathbb{K}}(SpecA[t]/(t^{m+1}), X)$$

is representable by a $\mathbb{K}-$scheme X_m [V]; X_m is called the *$m-$th jet scheme of X*, and F_m is isomorphic to its functor of points. In particular the closed points of X_m are in bijection with the $\mathbb{K}[t]/(t^{m+1})$ points of X.

For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$. This yields an inverse system whose limit X_∞ is a $\mathbb{K}-$scheme called *the arc space of X*. Note that $X_0 = X$. We denote the canonical projections $X_m \longrightarrow X_0$ by π_m and $X_\infty \longrightarrow X_m$ by Ψ_m.

Example 1 *The m-th jet scheme of the affine space $\mathbb{A}^n_{\mathbb{K}} = Spec\ \mathbb{K}[x_0, \dots x_{n-1}]$ is $(\mathbb{A}^n_{\mathbb{K}})_m = Spec\ \mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(m)}]$ where, for $j \geq 0$, $\underline{x}^{(j)} = (x_0^{(j)}, \cdots, x_{n-1}^{(j)})$ is an n-uplet of indeterminates. Hence, $(\mathbb{A}^n_{\mathbb{K}})_m$ is isomorphic to $\mathbb{A}^{(m+1)n}_{\mathbb{K}}$ and* *the projection* $\pi_{m,m-1} : (\mathbb{A}^n_{\mathbb{K}})_m \longrightarrow (\mathbb{A}^n_{\mathbb{K}})_{m-1}$ *is the map that forgets the last* n *coordinates.*

For $f \in \mathbb{K}[x_0, \ldots, x_n]$*, and* $j \geq 0$*, let* $F^{(j)}(\underline{x}^{(0)}, \cdots, \underline{x}^{(j)}) \in \mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(j)}]$ *be defined by the Taylor expansion as follows:*

$$f(\sum_j \underline{x}^{(j)} t^j) = \sum_{j=0}^{m} F^{(j)}(\underline{x}^{(0)}, \ldots, \underline{x}^{(j)})\ t^j.$$

Now, let $X = Spec\ \frac{\mathbb{K}[x_0,\cdots,x_n]}{(f_1,\cdots,f_r)}$ *be an affine* $\mathbb{K}-$*scheme. Then*

$$X_m = Spec\ \frac{\mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(m)}]}{(F_l^{(j)})_{l=1,\cdots,r}^{j=0,\cdots,m}}$$

Indeed, for a $\mathbb{K}$*-algebra* A*, to give an* A*-point of* X_m *is equivalent to give a* $\mathbb{K}$*-algebra homomorphism*

$$\varphi : \frac{\mathbb{K}[x_0, \cdots, x_n]}{(f_1, \cdots, f_r)} \longrightarrow A[t]/(t^{m+1}).$$

The map φ *is completely determined by the image of* $x_i, i = 0, \cdots, n$*, that is*

$$x_i \longmapsto \varphi(x_i) = \mathrm{x}_i^{(0)} + \mathrm{x}_i^{(1)} t + \ldots + \mathrm{x}_i^{(m)} t^m \ \in\ A[t]/(t^{m+1})$$

such that $f_l(\varphi(x_0), \ldots, \varphi(x_n)) \in (t^{m+1})$*,* $l = 1, \ldots, r$*. This is equivalent to determine* $\underline{\mathrm{x}}^{(j)} = (\mathrm{x}_0^{(j)}, \ldots, \mathrm{x}_{n-1}^{(j)}) \in A^n, j = 0, \ldots, m$*, which satisfy*

$$F_l^{(j)}(\underline{\mathrm{x}}^{(0)}, \ldots, \underline{\mathrm{x}}^{(j)}) = 0$$

where $l = 1, \cdots, r$ *and* $j = 0, \cdots, m$*.*

From now on, in this section, $\mathbb{K}$ is an algebraically closed field of characteristic 0. Let f be a nonzero polynomial of $\mathbb{K}[x_0, x_1]$ and assume that $f(0,0) = 0$ and that f is irreducible in $\mathbb{K}[[x_0, x_1]]$, i.e. the curve defined by f has one branch at $O = (0,0)$. We denote by C this branch. By possibly a change of variables, we may assume that $x_0 = 0$ is transversal to C, and that $x_1 = 0$ has the maximal contact with C in the sense of [L]. By the Newton-Puiseux theorem, there exists a parametrization of C of the form

$$x_0(t) = t^{\beta_0}$$

$$x_1(t) = \sum_{i > \beta_0} a_i t^i$$

where $gcd(\beta_0, \{i \ / \ a_i \neq 0\}) = 1$. Let $\beta_1, \cdots, \beta_g$ be the sequence of Puiseux exponents of C, that is, the β_i's are defined recursively by

$$\beta_i = min\{i, a_i \neq 0,\ gcd(\beta_0, \cdots, \beta_{i-1}) \text{ is not a divisor of } i\}.$$

Let $e_0 = \beta_0$ and $e_i = gcd(e_{i-1}, \beta_i), i \geq 1$. The sequence of positive integers $e_0 > e_1 > \cdots > e_i > \cdots$ is strictly decreasing, and there exists $g \in \mathbb{N}$, such that $e_g = 1$. We set $n_i := \frac{e_{i-1}}{e_i}, i = 1, \cdots, g$ and by convention, we set $\beta_{g+1} = +\infty$ and $n_{g+1} = 1$.

On the other hand, let v_C be the divisorial valuation defined by C, that is for $h \in \mathbb{K}[[x_0, x_1]]$, $v_C(h)$ is the intersection number

$$(f,h)_0 := dim_{\mathbb{K}} \frac{\mathbb{K}[[x_0, x_1]]}{(f,h)} = ord_t\ h(x_0(t), x_1(t)).$$

Let $\Gamma(C)$ be the semigroup of v_C i.e $\Gamma(C) = \{(f,h)_0 \in \mathbb{N}, h \not\equiv 0\ mod(f)\}$. Then, the minimal system of generators of $\Gamma(C)$ is $\bar{\beta}_0, \cdots, \bar{\beta}_g$ where the $\bar{\beta}_i$'s are determined by $\bar{\beta}_0 = \beta_0$, $\bar{\beta}_1 = \beta_1$ and $\bar{\beta}_i = n_{i-1}\overline{\beta_{i-1}} + \beta_i - \beta_{i-1}$ for $1 \leq i \leq g$. Note that

$$e_i = gcd(\bar{\beta}_0, \cdots, \bar{\beta}_i),\ 0 \leq i \leq g,$$

and that, for $1 \leq i \leq g$, there exists a unique system of nonnegative integers b_{ij}, $0 \leq j < i$ such that $b_{ij} < n_j$ for $1 \leq j < i$ and

$$n_i \bar{\beta}_i = \sum_{0 \leq j < i} b_{ij} \bar{\beta}_j \quad 1 \leq i \leq g.$$

Let $\{x_0, x_1, x_2, \ldots, x_{g+1}\}$ be a minimal generating sequence for the divisorial valuation v_C. In fact, one can choose $x_{g+1} = f$ and x_i, $2 \leq i \leq g$, such that they satisfy identities of the form

$$x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\eta = (\eta_0, \cdots, \eta_i)} c_{i,\eta} x_0^{\eta_0} \cdots x_i^{\eta_i}, \quad 1 \leq i \leq g \quad (\star)$$

with, $0 \leq \eta_j < n_j$, for $1 \leq j \leq i$, and $\Sigma_j \eta_j \bar{\beta}_j > n_i \bar{\beta}_i$ and with $c_{i,\eta}, c_i \in \mathbb{K}$ and $c_i \neq 0$. These last equations $(\star)$ let us realize C as a complete intersection in $\mathbb{K}^{g+2} = Spec\ \mathbb{K}\ [[X_0, \cdots, X_g, X_{g+1}]]$ defined by the equations $X_{g+1} = 0$ and

$$f_i = X_{i+1} - (X_i^{n_i} - c_i X_0^{b_{i0}} \cdots X_{i-1}^{b_{i(i-1)}} - \sum_{\eta = (\eta_0, \cdots, \eta_i)} c_{i,\eta} X_0^{\eta_0} \cdots X_i^{\eta_i})$$

for $1 \leq i \leq g$.

In [Mo1], we have described the irreducible components of $C_m^0 := \pi_m^{-1}(0)$, (recall that $\pi_m : C_m \longrightarrow C$ is the canonical morphism) as follows: For $e \in \mathbb{N}$, set

$$Cont^e(x_0)_m \quad (resp.\ Cont^{>e}(x_0)_m) := \{\gamma \in C_m \mid ord_t x_0 \circ \gamma = e \quad (resp. > e)\}.$$

Theorem 21 *([Mo1], cor. 4.4 and th. 4.9) Let C be a plane branch with g Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leq m < n_1\bar{\beta}_1 + e_1$, $C_m^0 = Cont^{>0}(x_0)_m$ is irreducible. For $qn_1\bar{\beta}_1 + e_1 \leq m < (q+1)n_1\bar{\beta}_1 + e_1$, with $q \geq 1$ in $\mathbb{N}$, the irreducible components of C_m^0 are:*

(i) the infinite components:

$$C_{m\kappa I} = \overline{Cont^{\kappa\bar{\beta}_0}(x_0)_m} \quad \text{for } 1 \leq \kappa \text{ and } \kappa\bar{\beta}_0\bar{\beta}_1 + e_1 \leq m,$$

(ii) the vanishing components:

$$C^j_{m\kappa v} = \overline{Cont^{\frac{\kappa\bar{\beta}_0}{n_j\cdots n_g}}(x_0)_m} \quad \text{for } j = 2, \cdots, g,\ 1 \leq \kappa \text{ and } \kappa \not\equiv 0 \text{ mod } n_j,$$

$$\kappa n_1 \cdots n_{j-1}\bar{\beta}_1 + e_1 \leq m < \kappa\bar{\beta}_j$$

and

(iii) the big component:

$$B_m = Cont^{>n_1 q}(x_0)_m.$$

Moreover, the restrictions of the morphisms $\pi_{m+1,m}$ define projective systems of three types: the first type

$$\ldots \to B_{m+1} \to B_m \to \cdots \to B_1$$

the second

$$\ldots \to C_{(m+1)\kappa I} \to C_{m\kappa I} \to \cdots \to C_{(\kappa\bar{\beta}_0\bar{\beta}_1+e_1)\kappa I} \to B_{\kappa\bar{\beta}_0\bar{\beta}_1+e_1-1} \cdots \to B_{\kappa\bar{\beta}_0\bar{\beta}_1}$$

and the third, for $2 \leq j \leq g$ and $\kappa \not\equiv 0 \text{ mod}(n_j)$,

$$C^j_{(\kappa\bar{\beta}_j-1)\kappa v} \longrightarrow C^j_{(\kappa\bar{\beta}_j-2)\kappa v} \cdots \to C^j_{(\kappa n_1\cdots n_{j-1}\bar{\beta}_1+e_1)\kappa v} \to B_{\kappa n_1\cdots n_{j-1}\bar{\beta}_1+e_1-1} \cdots$$

$$\to B_{\kappa n_1\cdots n_{j-1}\bar{\beta}_1}.$$

To the irreducible component appearing in the second and the third type of projective systems, we associate the invariant κ that we will call *index of speciality*. The components appearing at the left hand side of the finite projective systems of the third type, will be called the *end components*. Later, we will be interested in the end components of index of speciality equal to 1, these are $C^j_{(\bar{\beta}_j-1)1v}$, $j = 2, \ldots, g$. Note that for $m < n_1\bar{\beta}_1 + e_1$, C_m^0 is irreducible, in particular $C^0_{\bar{\beta}_1-1}$ is irreducible and we call it the *first end component*. Let

$$m_0 := \min\{m \in \mathbb{N}, m \geq 1 \mid \text{codim}(C^0_{m+1}, (\mathbb{A}^2_{\mathbb{K}})_{m+1}) > \text{codim}(C^0_m, (\mathbb{A}^2_{\mathbb{K}})_m)\}.$$

It follows from proposition 4.1 in [Mo1], that $m_0 = \bar{\beta}_0 - 1$. We have that $C^0_{\bar{\beta}_0-1}$ is irreducible, and we call it the *root component*.

Given $m \geq 1$ and an irreducible component H_m of C_m^0, given $h \in \mathcal{O}_C$ we set

$$\mathrm{ord}_h H_m := \mathrm{ord}_t \gamma^\sharp(h)$$

where $\gamma : \mathrm{Spec}\ \kappa(H_m)[[t]]/(t)^{m+1} \to C$ is the generic point of H_m and $\gamma^\sharp$ is its induced morphism of rings $\gamma^\sharp : \mathcal{O}_C \to \kappa(H_m)[[t]]/(t)^{m+1}$.
For $i = 1, \ldots, g$, let

$$v^i(H_m) := (\mathrm{ord}_{x_0}(H_m), \mathrm{ord}_{x_1}(H_m), \ldots, \mathrm{ord}_{x_i}(H_m)) \in \mathbb{N}^{i+1},$$

and let $\mu_i \in \mathbb{N} \cup \infty$ be defined as follows :

$$\begin{aligned} \mu_i := \min\{m \geq 1 \mid \text{there exists a pair of irreducible components}(H_m, H_{m+1}) \\ \text{verifying: } i)\ \ \pi_{m+1,m}(H_{m+1}) \subset H_m, \\ ii)\ \ \mathrm{codim}(H_m, (\mathbb{A}^2_{\mathbb{K}})_m) < \mathrm{codim}(H_{m+1}, (\mathbb{A}^2_{\mathbb{K}})_{m+1}) \\ iii)\ \ v^i(H_m) = v^i(H_{m+1})\ \ \}. \end{aligned}$$

Proposition-definition 22 *(1) For $i = 1, \ldots, g$, we have that $\mu_i = n_i\overline{\beta}_i - 1$.*

(2) The pair (H_{μ_i}, H_{μ_i+1}) with the above property is unique, and we have that
$H_{\mu_1} = C^0_{\mu_1}$.
$H_{\mu_2} = C_{\mu_2 1I}$ (resp. $C^3_{\mu_2 1v}$) for $g = 2$ (resp. $g > 2$) and $\bar{\beta}_2 - n_1\bar{\beta}_1 \neq e_2$, otherwise if $g \geq 2$ and $\bar{\beta}_2 - n_1\bar{\beta}_1 = e_2$ then $H_{\mu_2} = B_{\mu_2}$.
$H_{\mu_i} = C^{i+1}_{\mu_i 1v}$ (resp. $H_{\mu_g} = C_{\mu_g 1I}$) for $3 \leq i \leq g-1$ (resp. $3 \leq i = g$).
We call the components H_{μ_i} the **rupture components***.*

(3) For $i = 1, \ldots, g$, $v^i(H_{\mu_i}) = \frac{1}{e_i}(\bar{\beta}_0, \ldots, \bar{\beta}_i)$.

Proof: For $i = 1$ the proof is a direct consequence of proposition 4.1 of [Mo1]. We now prove the case $2 \leq i \leq g-1$, the case $i = g$ goes along the same lines (The only difference lies in the notation). For $i \geq 2$, it follows from the conditions i, ii and iii on the pair (H_m, H_{m+1}) that H_{m+1} and H_m both belong to one of the projective systems of the second or the third type. Indeed, if H_{m+1} belongs to

$$B_{(k+1)n_1\bar{\beta}_1 - 1} \to \cdots \to B_{kn_1\bar{\beta}_1 + e_1}, \quad k \in \mathbb{N}$$

we have by corollary 4.2 in [Mo1] that when the codimension changes v^1 changes, and always by the same corollary $v^2(B_{(k+1)n_1\bar{\beta}_1 - 1}) \neq v^2(B_{(k+1)n_1\bar{\beta}_1})$.

It follows from the definition of the components H_m appearing in a projective system of the second (resp. third) type, from condition iii and corollary 4.2 in [Mo1] that $\mathrm{ord}_{x_0}(H_m) = kn_1 = \kappa n_1 \ldots n_g$(resp. $\kappa n_1 \ldots n_{j-1}$.) There also exists $l \geq 2$ such that $\kappa n_{l-1} \cdots n_{j-1}\bar{\beta}_{l-1} \leq m+1 < \kappa n_l \cdots n_{j-1}\overline{\beta}_l$. If moreover we have $m+1 < n_i\overline{\beta}_i$, then $l \leq i$ and by combining proposition 4.5, proposition 4.7 (see also the formula which appears after proposition 4.7) and again corollary 4.2 in *loc. cit.*, we observe that $\mathrm{codim}(H_{m+1}) > \mathrm{codim}(H_m)$ implies that $\mathrm{ord}_{x_l}(H_{m+1}) > \mathrm{ord}_{x_l}(H_m)$.

Since by proposition 4.7 in *loc. cit.* we have that

$$\operatorname{codim}(C^{i+1}_{(n_i\bar{\beta}_i-1)1v},(\mathbb{A}^2_{\mathbb{K}})_m)<\operatorname{codim}(C^{i+1}_{(n_i\bar{\beta}_i)1v},(\mathbb{A}^2_{\mathbb{K}})_{m+1}),$$

it remains to prove $v^i(C^{i+1}_{(n_i\bar{\beta}_i-1)1v})=v^i(C^{i+1}_{(n_i\bar{\beta}_i)1v})=\frac{1}{e_i}(\bar{\beta}_0,\ldots,\bar{\beta}_i)$. By applying lemma 4.6 in *loc. cit.* for $j=i,i+1$ we have that

$$\operatorname{ord}_{x_0}(C^{i+1}_{(n_i\bar{\beta}_i-1)1v})=\operatorname{ord}_{x_0}(C^{i+1}_{(n_i\bar{\beta}_i)1v})=\frac{\bar{\beta}_0}{e_i},$$

$\operatorname{ord}_{x_1}(C^{i+1}_{(n_i\bar{\beta}_i-1)1v})=\operatorname{ord}_{x_1}(C^{i+1}_{(n_i\bar{\beta}_i)1v})=\frac{\bar{\beta}_1}{e_i}$, and for $2\leq l\leq i$, we have that $\operatorname{ord}_{x_l}(C^{i+1}_{(n_i\bar{\beta}_i-1)1v})\geq\frac{\bar{\beta}_l}{e_i}$, $\operatorname{ord}_{x_l}(C^{i+1}_{(n_i\bar{\beta}_i)1v})\geq\frac{\bar{\beta}_l}{e_i}$. Moreover, by the lemma *loc. cit.* we have that the generic point of $C^{i+1}_{(n_i\bar{\beta}_i-1)1v}$ satisfies the equations

$$x_l^{(\frac{\bar{\beta}_l}{e_i})^{n_l}}-c_lx_0^{(\frac{\bar{\beta}_0}{e_i})^{b_{l0}}}\ldots x_{l-1}^{(\frac{\bar{\beta}_{l-1}}{e_i})^{b_{l(l-1)}}},\quad l=2,\ldots,i-1.$$

The generic point of $C^{i+1}_{(n_i\bar{\beta}_i)1v}$ satisfies beside the above equations, the following

$$x_i^{(\frac{\bar{\beta}_i}{e_i})^{n_i}}-c_lx_0^{(\frac{\bar{\beta}_0}{e_i})^{b_{i0}}}\ldots x_{i-1}^{(\frac{\bar{\beta}_{l-1}}{e_i})^{b_{i(i-1)}}}.$$

Since at the generic point of $C^{i+1}_{(n_i\bar{\beta}_i)1v}$, we have that $x_0^{(\frac{\bar{\beta}_0}{e_i})}$ and $x_1^{(\frac{\bar{\beta}_1}{e_i})}$ are both different from zero, it follows by induction on l, using the above equations that, at the generic point of $C^{i+1}_{(n_i\bar{\beta}_i)1v}$, $x_l^{(\frac{\bar{\beta}_l}{e_i})}\neq 0$ for $l=2,\ldots,i$. Whence the proposition.

Corollary 23 *For $i=1,\ldots,g$ we have that $v^g(H_{\mu_i})=\frac{1}{e_i}\delta_i$ where*

$$\delta_i:=(\overline{\beta}_0,\ldots,\overline{\beta}_i,n_i\overline{\beta}_i,\ldots,n_i\ldots n_{g-1}\overline{\beta}_i)\in\mathbb{Z}^{g+1}_{\geq 0}.\quad(\star\star)$$

Proof: The case $i=g$ is proved in proposition-definition 2.2 (3). Let us consider the case $i=1,\ldots,g-1$.. By applying proposition 4.5 in *loc. cit.* with i replaced successively by $i+1,\ldots,g-1$ and j by $i+1$, we deduce that for $l=i+1,\ldots,g$, $ord_{x_l}(H_{\mu_i})\geq\frac{1}{e_i}n_i\ldots n_{l-1}\overline{\beta}_i$, the equality follows because the codimension grows.

The vectors δ_i and the cones $\sigma_{i,j},i=1,\ldots,g;\ j=1,2$ that we will introduce in the following remark, will be of particular importance in the next section.

Remark 24 *Let $\varepsilon_i\in\mathbb{N}^{g+1}$ be the vector whose $i-th$ component is* 1, *and its other components are* 0. *Let δ_0 be defined as in* $(\star\star)$ *where we set $n_0:=0$, i.e.* $\delta_0=(\overline{\beta}_0,0,\ldots,0)$. *For $0\leq i\leq g$, we define the cones $\sigma_{i,1}:=<\delta_{i-1},\delta_i>$ and*

$\sigma_{i,2} :=< \varepsilon_i, \delta_i >$, $1 \leq i \leq g$.
We consider the irreducible components H_m in the following inverse systems

$$B_{(k+1)n_1\bar{\beta}_1-1} \to \cdots \to B_{kn_1\bar{\beta}_1+e_1}, \quad k \in \mathbb{N}$$

and which are at the end position or verify codim$(H_{m+1}) >$ codim(H_m), where H_{m+1} is the consecutive element in the inverse system. We have that the vectors v^g of such irreducible components belong to $\sigma_{1,1} \cup \sigma_{2,2}$. For $i = 2, \ldots, g$, we consider the irreducible components H_m in the following inverse systems

$$C^i_{(\overline{\beta}_i-1)1v} \longrightarrow C^i_{(\overline{\beta}_i-2)1v} \cdots \longrightarrow C^i_{(n_{i-1}\overline{\beta_{i-1}})1v}$$

$$C^{i+1}_{(n_i\overline{\beta}_i-1)1v} \longrightarrow C^{i+1}_{(n_i\overline{\beta}_i-2)1v} \cdots \longrightarrow C^{i+1}_{(n_i n_{i-1}\overline{\beta_{i-1}})1v}$$

$$(\textit{resp.}\quad C_{(n_g\bar{\beta}_g-1)1I} \longrightarrow C_{(n_g\bar{\beta}_g-2)1I} \longrightarrow C_{(n_g n_{g-1}\overline{\beta_{g-1}}-2)1I} \quad \textit{if } i = g)$$

and which are at the end position or verify codim$(H_{m+1}) >$ codim(H_m), where H_{m+1} is the consecutive element in the inverse system. By reasoning as in the above proposition, we can prove that the vectors v^g of such irreducible components belong to $\sigma_{i,1} \cup \sigma_{i,2}$.

We now will associate with a rupture component a divisorial valuation over $\mathbb{A}^2_{\mathbb{K}}$. Let $\pi : X \longrightarrow \mathbb{A}^2_{\mathbb{K}}$ be the minimal embedded resolution of $C \subset \mathbb{A}^2_{\mathbb{K}}$, which is a composition of a finite number t of point blowing ups. Since C is an hypersurface in $\mathbb{A}^2_{\mathbb{K}}$, π is a log resolution. Let E_i, $1 \leq i \leq t$, be the strict transform on X of the exceptional locus of the i-th point blowing up. The curves $\{E_i\}_{i=1}^t$ will be called *exceptional divisors* and the exceptional divisor E_1, which is defined by the first blowing up, will be called *root divisor*. Let $E = \sum_{i=1}^t r_i E_i$ be defined by

$$f.O_X = O_X(-\sum_{i=1}^t r_i E_i).$$

For $m \in \mathbb{N}$, let $\psi^a_m : \mathbb{A}^2_\infty \longrightarrow \mathbb{A}^2_m$ be the canonical morphism, here the exponent a stands for ambient. For $p \in \mathbb{N}$, we now consider the following cylinder in the arc space

$$\mathcal{C}ont^p(f) = \{\gamma \in \mathbb{A}^2_\infty; \mathrm{ord}_t f \circ \gamma = p\}.$$

Note that this notation is different from the notation "$Cont$" that we have introduced before, here we are considering arcs in the ambient space. From example 1, we know that ψ^a_m is a trivial fibration, therefore for a rupture component H_{μ_i}, we have that

$$\psi^{a\;-1}_{\mu_i}(H_{\mu_i}) \cap \mathcal{C}ont^{\mu_i+1}(f)$$

is an irreducible component of $\mathcal{C}ont^{\mu_i+1}(f)$. Note the fact that by definition of rupture components the codimension of H_{μ_i+1} jumps, implies that

${\psi^{a}_{\mu_i}}^{-1}(H_{\mu_i}) \cap \mathcal{C}ont^{\mu_i+1}(f) \neq \emptyset$. We associate with H_{μ_i} a discrete valuation $\nu_{H_{\mu_i}}$ as follows: let γ be the generic point of ${\psi^{a}_{m}}^{-1}(H_{\mu_i}) \cap \mathcal{C}ont^{\mu_i+1}(f)$, then for every $h \in \mathbb{K}[x_0, x_1]$, we set

$$\nu_{H_{\mu_i}}(h) = \mathrm{ord}_t h \circ \gamma.$$

It follows from corollary 2.6 in [ELM], that $\nu_{H_{\mu_i}}$ is a divisorial valuation (see also [R], prop. 3.7 (vii) applied to ${\psi^{a}_{\mu_i}}^{-1}(H_{\mu_i})$). In the same manner, we associate with the end components a divisorial valuation.

Let us consider the dual graph associated with the configuration of the exceptional divisors.

Definition 25 *A star divisor is either an exceptional divisor whose corresponding vertex on the dual graph has valence equal to* 3 *or the exceptional divisor which intersects the strict transform of the branch. An end divisor is an exceptional divisor whose corresponding vertex has valence equal to* 1, *and which is not the root divisor (see figures* 3 *and* 2).

Then we can state the following theorem :

Theorem 26 *1. The divisorial valuations associated with the rupture components are the valuations defined by the star divisors.*
2. The end components of index of speciality one correspond to the end divisors and the root component corresponds to the root divisor.

Proof: We prove the first assertion, the second one follows in the same way. Let $E_{i_j}, j = 1, \ldots, g$ be a star divisor locally defined by $g_{i_j} = 0$, we consider the set

$$\mathcal{C}ont^1(E_{i_j}) = \{\gamma \in X_\infty; \mathrm{ord}_t g_{i_j} \circ \gamma = 1\}.$$

Let $\pi_\infty : X_\infty \longrightarrow \mathbb{A}^2_\infty$ be the canonical morphism induced by π. Then by corollary 2.6 of [ELM], we need to prove that $\pi_\infty(\mathcal{C}ont^1(E_{i_j}))$ is dense in ${\psi^{a}_{n_j\bar{\beta}_j-1}}^{-1}(H_{\mu_i}) \cap \mathcal{C}ont^{n_j\bar{\beta}_j}(f)$. First, by [C],[G], [L] we know that a projection of a curvette i.e. an element in $\mathcal{C}ont^1(E_{i_j})$ has intersection multiplicity whith C which is equal to $n_j\bar{\beta}_j$ and intersection multiplicity with x_0 which is equal to $n_1 \cdots n_j$, therefore we have the inclusion

$$\pi_\infty(\mathcal{C}ont^1(E_{i_j})) \subset {\psi^{a}_{n_j\bar{\beta}_j-1}}^{-1}(H_{\mu_i}) \cap \mathcal{C}ont^{n_j\bar{\beta}_j}(f).$$

On the other hand, knowing the numerical data of the minimal embedded resolution of the branch (see e.g. [C],[G]), we apply theorem 2.1 of [ELM] to find the codimension of $\pi_\infty(\mathcal{C}ont^1(E_{i_j})) \subset {\psi^{a}_{n_j\bar{\beta}_j-1}}^{-1}(H_{\mu_i})$ in $\mathbb{A}^2_\infty$ (codimension in the sense of [ELM],) and which is equal by proposition 4.7 of [Mo1] to the codimension of ${\psi^{a}_{n_j\bar{\beta}_j-1}}^{-1}(H_{\mu_i}) \cap \mathcal{C}ont^{n_j\bar{\beta}_j}(f)$ in $\mathbb{A}^2_\infty$. Since they are irreducible, we conclude that their closures are equal, and therefore they define the same valuation, hence the theorem.

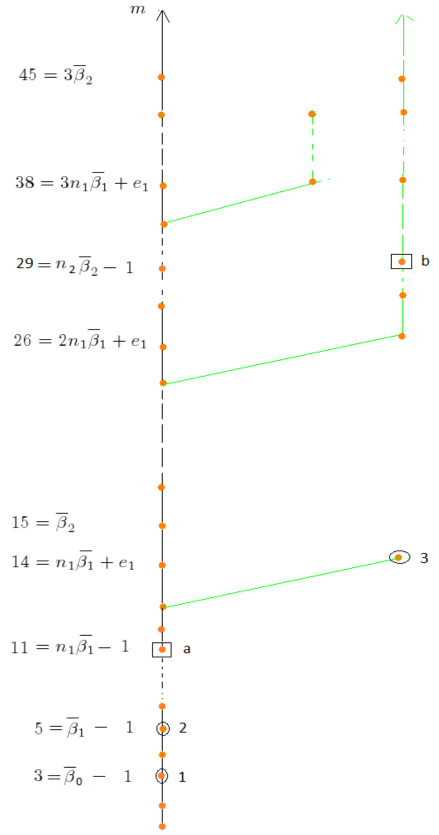

Figure 1

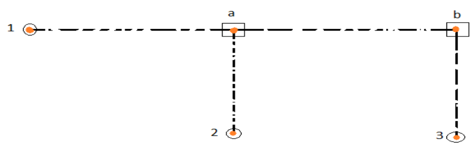

Figure 2

We get a tree $T_{C,0}$ by representing each irreducible component of $C_m^0, m \geq 1$, by a vertex $v_{i,m}, 1 \leq i \leq N(m)$, and by joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if $\pi_{m+1,m}$ induces one of the maps appearing in the three type of the projective systems between the corresponding irreducible components.
We represent in figure 1 below the tree for the branch defined by $f(x,y) = (y^2-x^3)^2 - 4x^6y - x^9 = 0$, whose semigroup is $< \bar{\beta}_0 = 4, \bar{\beta}_1 = 6, \bar{\beta}_2 = 15) >$, and for which we have $e_1 = 2,\ e_2 = 1$ and $n_1 = n_2 = 2$.

We represent in figure 2 the dual graph of the same branch.

The theorem 26 determines a corrspondance between on one side, the irreducible components denoted in figure 1 by 1,2,3 (the root component and the end components) and those denoted by a,b (rupture components), and on the other side the vertices on the dual graph which are denoted in figure 2, by the same numbers (resp. letters).

3 Minimal desingularization

Recall that $f \in \mathbb{K}[x_0, x_1]$ is a nonzero polynomial such that $f(0,0) = 0$ and f is irreducible in $\mathbb{K}[[x_0, x_1]]$, and C is the plane branch defined by f at $O = (0,0)$. Recall also that $x_0, x_1, x_2, \ldots, x_{g+1} = f$ is a minimal system of generators for v_C. We consider the embedding of the formal neighborhood $\widehat{\mathcal{X}_0} = \text{Spec } \mathbb{K}[[x_0, x_1]]$ of O in $\mathbb{K}^2$ into the formal neighborhood $\widehat{\mathcal{Z}_0} = Spec\ k[[X_0, \ldots, X_g]]$ of O in $\mathcal{Z}_0 = \mathbb{K}^{g+1}$, given by sending X_i to x_i. Let $C^\Gamma \subseteq \widehat{\mathcal{Z}_0}$ be the monomial curve parametrized by $X_i = t^{\overline{\beta}_i}$.

Recall from corollary 23 that $\delta_i := (\overline{\beta}_0, \ldots, \overline{\beta}_i, n_i\overline{\beta}_i, \ldots, n_i \ldots n_{g-1}\overline{\beta}_i) \in \mathbb{Z}_{\geq 0}^{g+1}$, $0 \leq i \leq g$, and that we consider the cones $\sigma_{i,1} :=< \delta_{i-1}, \delta_i >$ and $\sigma_{i,2} :=< \varepsilon_i, \delta_i >$ for $1 \leq i \leq g$, where ε_i is the unit vector on the X_i-axis. Let $\Sigma_{\mathcal{N}}$ be the Newton fan of the g functions defining $\widehat{\mathcal{X}_0}$ in $\widehat{\mathcal{Z}_0}$ and let $\mathcal{Z}_{\Sigma_{\mathcal{N}}}$ be the toric variety defined by $\Sigma_{\mathcal{N}}$. Then the cones $\sigma \in \Sigma_{\mathcal{N}}$ whose orbit $\mathbb{O}_\sigma$ in $\mathcal{Z}_{\Sigma_{\mathcal{N}}}$ intersect the strict transform $\widehat{\mathcal{X}}_{\Sigma_{\mathcal{N}}}$ of $\widehat{\mathcal{X}_0}$ in $\mathcal{Z}_{\Sigma_{\mathcal{N}}}$ are $\{\sigma_{ij}\}_{1 \leq i \leq g, j=1,2}$ and their faces, and $\widehat{\mathcal{X}}_{\Sigma_{\mathcal{N}}}$ has $2g$ toric singularities which are its intersection points with the orbits $\mathbb{O}_{\sigma_{ij}}$, $1 \leq i \leq g$, $j = 1,2$ (see [LR], prop. 1.3). Next we will show a canonical way to obtain a regular subdivision Σ_C of $\Sigma_{\mathcal{N}}$ inducing the minimal regular subdivision in each σ_{ij}. In particular, we will have that $\mathbb{R}_{\geq 0}\delta_g$ is a cone in Σ_C. The reason why we are interested in such a subdivision is the following:

Lemma 31 *Let Σ be a regular subdivision of $\Delta := \mathbb{R}_{\geq 0}^{g+1}$, $\mathcal{Z}_\Sigma$ the toric variety defined by Σ, and $\widehat{\mathcal{X}}_\Sigma$ the strict transform of $\widehat{\mathcal{X}_0}$ by the equivariant morphism $\pi_\Sigma : \mathcal{Z}_\Sigma \to \mathcal{Z}_0$. The following conditions are equivalent:*

(i) $\mathbb{R}_{\geq 0}\delta_g$ is a cone of Σ
(ii) the map $\widehat{\mathcal{X}}_\Sigma \to \widehat{\mathcal{X}_0}$ is an embedded desingularization of C.

Besides, if the above conditions hold then π_Σ desingularizes the monomial curve C^Γ.

Proof: Let $\sigma =< \underline{a}_0, ..., \underline{a}_g >\in \Sigma$ where $\underline{a}_i = (a_{0i}, ..., a_{gi})$ and let V_σ be the affine toric variety defined by σ. Since σ is a regular cone, the map

$$V_\sigma = Spec\ k[U_0, ..., U_g] \to \mathcal{Z}_0$$

is given by $X_i = U_0^{a_{0i}} ... U_g^{a_{gi}}$, $0 \leq i \leq g$, and the orbit $\mathbb{O}_{\underline{a}_i}$ of $\mathbb{R}_{\geq 0}\underline{a}_i$ is defined in V_σ by $U_i = 0$, $U_j \neq 0$, $j \neq i$. Let $\{X_i(t)\}_{0\leq i\leq g}$ be a parametrization of C. If the strict transform C' (resp. $(C^\Gamma)'$) of C (resp. C^Γ) in $\mathcal{Z}_\Sigma$ intersects V_σ and is given by $\{U_i(t)\}_{i=0}^g$, then $\delta_g = (ord_t\ X_i(t))_{0\leq i\leq g} = (ord_t\ t^{\overline{\beta}_i})_{0\leq i\leq g}$ is a combination of the $\underline{a}_i$'s with coefficients $\lambda_i = ord_t\ U_i(t) \in \mathbb{Z}_{\geq 0}$, thus it belongs to σ, and besides, $U_i(t) = t^{\lambda_i} w_i(t)$ where $\{w_i(t)\}_{i=0}^g$ are the unique solutions in $(k[[t]]^*)^{g+1}$ of the system $w_0(t)^{a_{0i}} ... w_g(t)^{a_{gi}} = X_i(t)/t^{\overline{\beta}_i} \in k[[t]]^*$ (resp. $w_i(t) = 1,\ 0 \leq i \leq g$).

If $\mathbb{R}_{\geq 0}\delta_g \in \Sigma$ then we may assume $\lambda_0 = 1, \lambda_i = 0$ for $i \neq 0$ and hence both C' and $(C^\Gamma)'$ are smooth and C' intersects $\mathbb{O}_{\delta_g} \cap \widehat{\mathcal{X}}_\Sigma$ transversally and does not intersect $\mathbb{O}_\tau$ for $\tau \in \Sigma$, $\tau \neq \mathbb{R}_{\geq 0}\delta_g$. This proves $(i) \Rightarrow (ii)$ and also the last statement in the lemma. For $(ii) \Rightarrow (i)$, if $\mathbb{R}_{\geq 0}\delta_g \notin \Sigma$, then we may assume $\lambda_0, \lambda_1 > 0$ and hence $\mathbb{O}_{\underline{a}_0} \cap \mathbb{O}_{\underline{a}_1} \cap C' \neq \emptyset$, i.e. $\widehat{\mathcal{X}}_\Sigma \to \widehat{\mathcal{X}}_0$ is not an embedded desingularization.

Let $N_{\sigma_{ij}}$ be the lattice obtained intersecting $\mathbb{Z}^{g+1}$ with the real space spanned by σ_{ij}, and let γ_i, λ_i and ρ_i be the primitive vectors on the half lines through δ_{i-1}, $\delta_i - \delta_{i-1}$ and $\delta_i - (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\ \varepsilon_i$ respectively. Then $\{\gamma_i, \lambda_i\}$ (resp. $\{\rho_i, \varepsilon_i\}$) is a $\mathbb{Z}$-basis of $N_{\sigma_{i,1}}$ (resp. $N_{\sigma_{i,2}}$) and we have $\sigma_{i,1} =< \gamma_i,\ e_{i-1}\gamma_i + (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\ \lambda_i >$ and $\sigma_{i,2} = < e_{i-1}\rho_i + (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\ \varepsilon_i,\ \varepsilon_i >$. Thus, if we set $\widetilde{\delta}_i := (e_{i-1},\ \overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1}) \in \mathbb{Z}^2_{\geq 0}$, and $\widetilde{\sigma}_{i,1} =< (1,0), \widetilde{\delta}_i >$, $\widetilde{\sigma}_{i,2} =< \widetilde{\delta}_i, (0,1) >$ are the two-dimensional cones in the subdivision of $\mathbb{R}^2_{\geq 0}$ by $\mathbb{R}^2_{\geq 0}\widetilde{\delta}_i$, then the pair $(\sigma_{i,j}, N_{\sigma_{i,j}})$ is isomorphic to $(\widetilde{\sigma}_{i,j}, \mathbb{Z}^2)$ for $1 \leq i \leq g$, $j = 1, 2$. From this it follows that, if $O_{ij} := \mathbb{O}_{\sigma_{i,j}} \cap \widehat{\mathcal{X}}_{\Sigma_\mathcal{N}}$ and $\widetilde{O}_{i,j}$ is the unique closed orbit in the toric surface $\mathcal{Z}_{\widetilde{\sigma}_{i,j}}$, then the germs $(\widehat{\mathcal{X}}_{\Sigma_\mathcal{N}}, O_{i,j})$ and $(\mathcal{Z}_{\widetilde{\sigma}_{i,j}}, \widetilde{O}_{ij})$ are analytically isomorphic.

Therefore, for any regular subdivision Σ of Δ inducing the minimal regular subdivision on each σ_{ij}, the dual graph $\mathcal{G}$ of the morphism $\widehat{\mathcal{X}}_\Sigma \to \widehat{\mathcal{X}}_0$ is

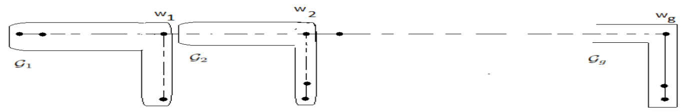

Figure 3

where if we denote by Θ_i the minimal regular subdivision of $\mathbb{R}^2_{\geq 0}$ by $\mathbb{R}_{\geq 0}\widetilde{\delta}_i$, then $\mathcal{G}_i$ is obtained from $Proj\ \Theta_i$ by erasing its ends $Proj\ (\mathbb{R}_{\geq 0}(1,0))$ and $Proj\ (\mathbb{R}_{\geq 0}(0,1))$, and the star w_i is $Proj\ (\mathbb{R}_{\geq 0}\,\widetilde{\delta}_i)$. But from lemma 3.1 and prop. 1.3 in [LR] it follows that $\mathcal{G}$ is the dual graph of the minimal embedded desingularization of C in $\widehat{\mathcal{X}}_0$. Thus the equivariant morphism $\mathcal{Z}_\Sigma \to \mathcal{Z}_0$ induces on $\widehat{\mathcal{X}}_0$ the minimal embedded desingularization of C.

The isomorphism between $(\sigma_{i,j}, N_{\sigma_{i,j}})$ and $(\widetilde{\sigma}_{i,j}, \mathbb{Z}^2)$ allows us to describe the minimal system of generators $G_{\sigma_{i,j}}$ of $\sigma_{i,j}$ from the minimal system of generators of $\widetilde{\sigma}_{i,j}$. In fact, given $n \geq 1$ and independent variables $u_1, \ldots, u_n$, we define a sequence of polynomials $\{P_s(u_1, \ldots, u_s)\}_{s=0}^n$, $P_s \in \mathbb{N}[u_1, \ldots, u_s]$, by $P_0 = 1$ and

$$P_s(u_1, \ldots, u_s) = u_1\ P_{s-1}(u_2, \ldots, u_s) + P_{s-2}(u_3, \ldots, u_s)$$

where we set $P_{-1} = 0$. Note that $P_s(u_1, \ldots, u_s)\ /\ P_{s-1}(u_2, \ldots, u_s) = [u_1, \ldots, u_s]$ where

$$[u_1, \ldots, u_s] := u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \cfrac{1}{\ddots + \cfrac{1}{u_s}}}}$$

is the continued fraction expansion. Now let $[a_1^{(i)}, \ldots, a_{s_i}^{(i)}]$ be the expression of $(\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\ /\ e_{i-1}$ as continued fraction (this notation does not coincide with the one in [Sp]). We define $\Omega := \{(i,s,a)\ /\ 1 \leq i \leq g,\ 1 \leq s \leq s_i,\ 1 \leq a \leq a_s^{(i)}\} \cup \{(i,0,0),(i,1,0)\ /\ 1 \leq i \leq g\}$ and, for $(i,s,a) \in \Omega$, we set

$$P(i,s,a) := P_s(a_1^{(i)}, \ldots, a_{s-1}^{(i)}, a) \qquad \overline{P}(i,s,a) := P_{s-1}(a_2^{(i)}, \ldots, a_{s-1}^{(i)}, a)$$

and

$$v_{(i,s,a)} := \begin{cases} \overline{P}(i,s,a)\ \gamma_i + P(i,s,a)\ \lambda_i & \text{if } s \text{ odd} \\ \overline{P}(i,s,a)\ \rho_i + P(i,s,a)\ \varepsilon_i & \text{if } s \text{ even} \end{cases}$$

(note that $v_{(i,0,0)} = \varepsilon_i$, $v_{(i,1,0)} = \gamma_i$ and $v_{(i,s_i,a_{s_i}^{(i)})} = \gamma_{i+1}$). Then we have

$$G_{\sigma_{i,1}} = \{v_{(i,s,a)}\ /\ (i,s,a) \in \Omega,\ s \text{ odd }\} \cup \{\gamma_{i+1}\}$$

$$G_{\sigma_{i,2}} = \{v_{(i,s,a)} \ / \ (i,s,a) \in \Omega, \ s \text{ even}\} \cup \{\gamma_{i+1}\}.$$

Let us describe some properties of the vectors $\{v_\alpha\}_{\alpha\in\Omega}$. Let us consider the lexicographic order in Ω. For $\alpha = (i,s,a) \in \Omega$, $\alpha \neq (i,0,0),(i,1,0)$, there exists a unique element in Ω strictly smaller than α of the type (i,s',a') with s' odd (resp. even) and maximal with this property, let us denote it by α_o (resp. α_e). Note that the unique element $\alpha-1$ of Ω strictly smaller than α and maximal with this property is one of α_o, α_e. With this notation we have:

Lemma 32 *i) Let $\alpha = (i,s,a) \in \Omega$, $\alpha \neq (i,0,0),(i,1,0)$. Then v_α belongs to the interior of the cone $< v_{\alpha_o}, v_{\alpha_e}, \varepsilon_{i+1}, \dots, \varepsilon_g >$.*

ii) Given $\alpha, \alpha' \in \Omega$, $v_\alpha = v_{\alpha'}$ iff $\alpha = \alpha'$ or else there exists i, $1 \leq i \leq g-1$, such that $\{\alpha,\alpha'\} = \{(i, s_i, a^{(i)}_{s_i}),\ (i+1,1,0)\}$.

iii) If $\alpha, \alpha' \neq (i,0,0),(i,1,0)$ for $1 \leq i \leq g$, then $v_\alpha < v_{\alpha'}$ (where we consider the usual order in $\mathbb{R}^{g+1}$) iff $\alpha < \alpha'$.

Proof: It follows from explicit calculus.

3.3. Given a regular fan Σ_0 in the lattice $\mathbb{Z}^{g+1}_{\geq 0}$ and a primitive vector $\delta \in \mathbb{Z}^{g+1}_{\geq 0}$ in the support of Σ_0, let us show a canonical algorithm to obtain a regular subdivision Σ of Σ_0 which contains $\mathbb{R}_{\geq 0}\delta$:

If $\mathbb{R}_{\geq 0}\delta$ is a cone in Σ_0, we set $\Sigma = \Sigma_0$ and finish. If it is not, then there exists a unique cone σ_0 in Σ_0 such that δ belongs to the interior σ_0° of σ_0. The cone σ_0 is regular, i.e. its extremal vectors $\{\underline{a}_0, \dots, \underline{a}_{r_0}\}$ form part of a basis of the lattice. Let $\omega_1 := \sum_{0\leq k\leq r_0} \underline{a}_k$ and let us consider the minimal subdivision Σ_1 of Σ_0 containing $\mathbb{R}_{\geq 0}\omega_1$. The fan Σ_1 is regular. If $\omega_1 = \delta$ then Σ_1 contains $\mathbb{R}_{\geq 0}\delta$. In this case, we set $\Sigma = \Sigma_1$ and we stop the algorithm. In the other case, $\omega_1 < \delta$ (where we consider the usual order in $\mathbb{R}^{g+1}$), and the unique cone σ_1 in Σ_1 such that $\delta \in \sigma_1^\circ$ has ω_1 as extremal vector. We repeat the procedure and obtain ω_2 such that $\omega_1 < \omega_2 \leq \delta$ and a regular fan Σ_2 which is the minimal subdivision of Σ_1 containing $\mathbb{R}_{\geq 0}\omega_2$. We go on in this way.

The preceding process stops after a finite number of steps since, if $\omega_i \in \mathbb{Z}^{g+1}_{\geq 0}$ is the vector appearing in the i-th step, then $\omega_1 < \ldots < \omega_{i-1} < \omega_i \leq \delta$. It determines a canonical way of obtaining a regular subdivision of Σ_0 containinig $\mathbb{R}_{\geq 0}\delta$.

3.4. Let us now describe the algorithm to obtain Σ_C: First note that Δ contains the half-lines defined by $v_{(1,1,0)} = \varepsilon_0$, $v_{(i,0,0)} = \varepsilon_i$, $1 \leq i \leq g$. Let us consider $v_{(1,1,1)}$, which belongs to *Supp* Δ, then we can apply the algorithm in 3.3 and obtain a regular subdivision $\Sigma_{(1,1,1)}$ of Δ which contains $\mathbb{R}_{\geq 0}v_{(1,1,1)}$. Now, let $\alpha = (i,s,a) \in \Omega$, $\alpha \neq (i,0,0),(i,1,0)$, and suppose we have obtained a regular subdivision $\Sigma_{\alpha-1}$ of Δ containing $\{\mathbb{R}_{\geq 0}v_{\alpha'} \ / \ \alpha' \leq \alpha - 1\}$. In order to define Σ_α we observe that the primitive vector v_α belongs to *Supp* $\Sigma_{\alpha-1}$, in fact it belongs to the interior of $< v_{\alpha_0}, v_{\alpha_e}, \varepsilon_{i+1}, \dots, \varepsilon_g >$. We apply the algorithm 3.3 and obtain a regular subdivision Σ_α of $\Sigma_{\alpha-1}$ which contains $\{\mathbb{R}_{\geq 0}v_{\alpha'} \ / \ \alpha' \leq \alpha\}$. We continue this process until we obtain regular fans Σ_α for all $\alpha \in \Omega$, $\alpha \neq (i,0,0),(i,1,0)$. We define Σ_C to be $\Sigma_{(g,s_g,a^{(g)}_{s_g})}$.

Theorem 33 *The equivariant morphism $\mathcal{Z}_{\Sigma_C} \to \mathcal{Z}_0$ desingularizes both curves C and C^Γ and besides it induces on $\widehat{\mathcal{X}}_0$ the minimal embedded desingularization of C.*

Proof: It is clear that Σ_C contains all half-lines $\mathbb{R}_{\geq 0}v_\alpha$ for $\alpha \in \Omega$, i.e. all half-lines defined by the vectors in the minimal regular subdivision of each σ_{ij}. In order to prove the theorem it suffices to show that these are the unique 1-dimensional cones of Σ_C whose support is in some σ_{ij}. In fact, when we determine Σ_α from $\Sigma_{\alpha-1}$ applying 3.3, the new half-lines that appear are defined by a finite number of vectors $\omega_s \in \mathbb{Z}^{g+1}_{\geq 0}$ which are smaller or equal than v_α and contained in the interior of $< v_{\alpha_o}, v_{\alpha_e}, \varepsilon_{i+1}, \ldots, \varepsilon_g >$. Since, for $\alpha' < \alpha$, $v_{\alpha'}$ does not belong to the interior of that cone, from lemma 3.1 *ii*), *iii*) it follows that the unique vector ω_s appearing which is contained in some σ_{ij} is v_α.

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