THE NASH PROBLEM FOR TORUS ACTIONS OF COMPLEXITY ONE

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ABSTRACT. We solve the equivariant generalized Nash problem for any non-rational normal variety with torus action of complexity one. Namely, we give an explicit combinatorial description of the Nash order on the set of equivariant divisorial valuations on any such variety. Using this description, we positively solve the classical Nash problem in this setting, showing that every essential valuation is a Nash valuation. We also describe terminal valuations and use our results to answer negatively a question of de Fernex and Docampo by constructing examples of Nash valuations which are neither minimal nor terminal, thus illustrating a striking new feature of the class of singularities under consideration.

1. INTRODUCTION

By Hironaka's theorem (1964), an algebraic variety X defined over a field of characteristic zero has infinitely many resolutions of singularities. In his celebrated paper [Nas95] (written in 1968), Nash tried to capture some common information to all these resolutions using the arc space associated with X. He defined an injection from the set of irreducible components of the space of arcs centered at the singular locus of X to the set of essential divisors (or divisorial valuations) of X. Here, a divisorial valuation ν centered on the singular locus of an algebraic variety X is said to be essential if it appears on every resolution of singularities (more precisely, if the center of ν on every resolution of singularities of X is an irreducible component of the exceptional locus). Nash asked whether this injection was surjective. Since then, a tremendous amount of work has been dedicated to this question, hereafter also designated by the classical Nash problem. Numerous works dealing with the case of surfaces (see *e.g.* [Reg95, LJRL99, Plé05, PPP06, Mor08, PP13]) culminated in [FdBP12], where Bobadilla and Pereira showed that the answer was always positive in dimension two. On the other hand, in [IK03], Ishii and Kollar gave a 4-dimensional example for which the answer was negative; 3-dimensional counter-examples to the classical Nash problem were exhibited thereafter in [dF13] and [JK13].

These counter-examples prompted the challenge to determine stronger and geometrically meaningful conditions on essential valuations that guarantee, in any dimension, that they are Nash valuations. The ultimate goal

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would be to establish a nice general and geometric characterization of Nash valuations. We hereafter designate this question by the extended Nash problem. A remarkable step in this direction was accomplished in [dFD16], where de Fernex and Docampo showed that any terminal valuation is Nash, recovering in particular the result of Bobadilla and Pereira. In [LJR12], Lejeune and Reguera showed that any valuation determined by a non-uniruled exceptional divisor is Nash.

A useful way towards a better apprehension of a general characterization of Nash valuations is to determine significantly large families of higher dimensional singularities for which the answer to Nash's question is positive. The first example is again due to Ishii and Kollar in [IK03]: they showed that the classical Nash problem holds for toric singularities; subsequent refinements of this result are to be found in [Ish05]. Nowadays, in dimension $\geqslant 3$, the classical Nash problem is known to hold for the following families: some quasi-rational hypersurface singularities ([LA11]), some families of 3-dimensional hypersurfaces ([LA16]), possibly reducible quasi-ordinary hypersurface singularities ([GP07]), a certain class of normal isolated singularities of arbitrary dimension, ([PPP08]), Schubert varieties ([DN17]).

Being given two divisorial valuations ν, ν' on an algebraic variety, the condition: "any arc with order ν' is a limit of arcs with order ν" defines a poset structure on the set of divisorial valuations, the Nash order, and the Nash valuations may be understood as the minimal elements for the Nash order of the set of valuations centered at the singular locus. Thus a better understanding of the nature of the Nash valuations could be achieved via the study of the Nash order. The generalized Nash problem asks for a meaningful description of the Nash order. Few instances of a satisfactory solution to this problem seem to be known. It has been solved for equivariant valuations on toric varieties ([Ish04, Ish08]) and determinantal varieties ([Doc13]); in [DN17], some partial results are obtained for Grassmanians. In its full generality, it seems to be a very difficult problem, even for valuations on the affine plane (see [FdBPPPP17]).

1.1. **Results on the generalized and the classical Nash problems.** In the present paper, we study the classical Nash problem and its generalized version for a normal variety endowed with a complexity one torus action. Assuming that our variety is non-rational, in other words that the rational quotient by the torus action is a curve of positive genus, and using the semi-combinatoric description of its geometry due to Altmann-Hausen and Timashev, we solve the generalized Nash problem for equivariant valuations.

Theorem 1.1. *Let k be an algebraically closed field of characteristic zero. Consider a divisorial fan $\mathscr{E}$ over a smooth projective curve of positive genus, and the associated normal k-variety X, which comes equipped with a complexity one torus action. Then the Nash order on the set of equivariant valuations of X can be explicitly described in terms of an "hypercombinatorial" order on the set of integral points of the book of valuations associated* *with $\mathscr{E}$, which on each page of the book coincide with the combinatorial order used in the toric case by Ishii and Kollar in* [Ish08]*.*

See Subsection 5.1 for a precise description of the hypercombinatorial order alluded to in the statement.

Note that despite the terminology, though a solution to the generalized Nash problem in particular allows to explicitly describe the set of Nash valuations, it does not provide a solution to the classical Nash problem for free, for the latter requires in addition a sufficiently fine understanding of the set of essential valuations.

Under the same hypotheses as Theorem 1.1, we study carefully resolutions of singularities in order to describe the set of essential valuations; this description coupled with Theorem 1.1 allows us to give a positive answer to the classical Nash problem.

Theorem 1.2. *We keep the setting of Theorem 1.1. Then any essential valuation of X is a Nash valuation. Equivalently, the Nash map is bijective.*

In fact our arguments allow to establish Theorems 1.1 and 1.2 also for any normal variety endowed with a complexity one torus action and which is toroidal. Note however that in this case, the bijectivity of the Nash map can be deduced directly from Ishii and Kollar's result (see Remark 5.11).

In an ongoing work, we plan to get back to the case where the rational quotient is a curve of genus zero, which seems much more challenging (see Subsection 1.5 below, as well as Section 8).

1.2. **Terminal valuations and an answer to a question of de Fernex and Docampo.** Under the assumptions of Theorem 1.1, we also give a description of the set of terminal valuations (which is a subset of the set of Nash valuations by [dFD16]); see Theorem 6.16. As pointed out in [dFD16, §6.3, *in fine*] in the case of the two main families for which the Nash map is known to be bijective, the Nash valuations are either all of them minimal (the case of toric varieties) or all of them terminal (the case of surface singularities), and in general no examples of Nash valuations which are neither minimal nor terminal are known. This led the authors of *op.cit.* to suggest that the set of Nash valuations might be in full generality the (non-disjoint) union of the set of minimal valuations and the set of terminal valuations. As pointed out in the discussion after Corollary 6.16 in [dF15], if this happens to be the case, this would give a complete solution to the extended Nash problem.

Using Theorem 1.1 and our description of the terminal valuations, we obtain the following result, which shows in some sense that the extended Nash problem remains widely open, and illustrates a striking and completely new feature of the class of singularities that we are considering.

Theorem 1.3. *Keep the setting of Theorem 1.1. Then there are examples of such varieties X possessing Nash valuations which are neither minimal nor terminal.*

See §7.3 for a description of such examples.

1.3. **An application.** Trinomial hypersurfaces are classical examples of normal affine varieties with torus action of complexity one. Using the work of Kruglov (cf. [Kru19]), which makes the Altmann-Hausen description explicit for trinomial hypersurfaces, our main result on the classical Nash problem implies the following.

Theorem 1.4. *Consider the hypersurface*

$$X = \mathbb{V}(t_1^{n_1} + t_2^{n_2} + t_3^{n_3}) \subset \mathbb{A}^n_{\mathbb{C}} \text{ with } t_i^{n_i} = \prod_{j=1}^{r_i} t_{i,j}^{n_{i,j}} \text{ for } i = 1, 2, 3,$$

where $r_i, n_{i,j} \in \mathbb{Z}_{>0}$ *and* $n = r_1+r_2+r_3$. *Let us write* $u_i := \gcd(n_{i,1}, \ldots, n_{i,r_i})$, $d = \gcd(u_1, u_2, u_3)$, $d_1 = \gcd(u_2/d, u_3/d)$, $d_2 = \gcd(u_1/d, u_3/d)$, $d_3 = \gcd(u_1/d, u_2/d)$, *and* $u = dd_1d_2d_3$. *If*

$$u - d_1 - d_2 - d_3 \geqslant 0,$$

then the classical Nash problem is valid for the variety X, *i.e. the Nash map is bijective.*

Note that Theorem 1.4 is sharp in the sense that there are examples of trinomial hypersurfaces with condition $u - d_1 - d_2 - d_3 < 0$ for which the Nash map is not bijective. For instance, the hypersurface of Johnson-Kollar $t_0t_1 + t_2^2 + t_3^5 = 0$ (see [JK13] and Section 8) is such an example.

1.4. **Strategy of proof.** We now describe our strategy for proving Theorems 1.1 and 1.2. For any normal variety X equipped with a complexity one torus action, there exists a natural equivariant proper morphism $\widetilde{X} \to X$, called the toroidification, where $\widetilde{X}$ has toroidal singularities (see subsection 3.6).

In case X is non-rational, and drawing our inspiration from the proof of the main result of [LJR12] we show that this morphism has the following crucial property: any wedge on X not contained in the singular locus lifts to $\widetilde{X}$ (see Subsection 5.2).

The fact that $\widetilde{X}$ has toroidal singularities allows to describe explicitly the Nash order on the set of its equivariant valuations, starting from the known case of toric varieties, and using elementary properties of étale morphisms and a consequence of Reguera's curve selection lemma (see Subsection 5.1). Then one deduces from the lifting property along the toroidification the sought for description of the Nash order on X (see subsection 5.3); again Reguera's curve selection lemma is a crucial tool.

In order to establish the bijectivity of the Nash map, we now have to locate the essential valuations of X. The toroidification morphism is a partial equivariant desingularization of X, in the sense that any equivariant desingularization of X factors through it (see Subsection 4.3); it is a consequence of Luna's slice Theorem. And since $\tilde{X}$ is toroidal, the location of its essential valuations is provided by Ishii and Kollar's argument in the toric case. Using the description of the exceptional locus of the toroidification, this already gives information on the essential valuations of X, but not sharp enough to conclude. Due in particular to the fact that in general not every essential valuation of $\tilde{X}$ is an essential valuation of X, some extra work is needed in order to obtain a sufficiently accurate description of a finite set of equivariant valuations containing all the essential valuations of our non-toroidal variety X; the arguments remain purely combinatorial, but one has to take into account the degree of the polyhedral divisor defining X (see Subsection 4.5). By our description of the Nash order, the finite set we obtain is contained in the set of Nash valuations, and we are done, since any Nash valuation is essential.

1.5. **The rational case.** In case X is a rational normal variety equipped with a complexity one torus action, the above arguments completely fail. Even in dimension 2, where the classical Nash problem is known to have a positive solution, it is no longer true that the toroidification is a partial equivariant desingularization of X, nor that wedges lift along the toroidification. The 3-dimensional counter-example to the classical Nash problem of Kollar and Johnson, which, as already pointed out, can be equipped with a complexity one torus action, may also be seen as an illustration of the issues encountered when the rational quotient of X is the projective line. We strongly believe that the situation deserves deeper investigation. Though the toroidification seems no longer useful to understand the essential valuations of X, we might still be able to exploit the semi-combinatorial description of the equivariant resolutions to obtain a meaningful interpretation of essential valuations in terms of hypercones. Understanding the Nash order in terms of the semi-combinatorial data is also an interesting challenge, already in dimension 2. As a first step in this direction, using the toric embedding defined by Ilten and Manon in [IM19], we give a combinatoric description of the pointwise order (which is finer than the Nash order) on the set of equivariant valuations (see Proposition 8.4). Let us stress that in view of Johnson-Kollar's threefold, one may hope that a finer understanding of the issues in the rational case should allow to produce systematic families of counter-examples to the classical Nash problem among varieties equipped with a torus action, which in turn would be useful to get a better comprehension of why essential valuations fail in general to be Nash.

1.6. **Organization of the paper.** We briefly describe the content of each section of the paper.

Section 2 contains the necessary background on essential and Nash valuations, maximal divisorial sets and Reguera's curve selection lemma. Special attention is paid to the toric case, which is an important ingredient of the proof of our results.

In Section 3, we recall some useful facts about the geometry of normal varieties equipped with a complexity-one torus action. In the last subsection, some technical lemmas about extensions of valuation along étale morphisms are proved.

In Section 4, we obtain information about the equivariant resolutions and the location of the essential valuations of a non-rational variety equipped with a complexity-one torus action.

In Section 5, we define a poset structure of combinatorial nature on the set of equivariant valuations of a variety equipped with a complexity-one torus action. In the non-rational case, we show that wedges lift to the toroidification, allowing us to deduce the main results of the paper.

In Section 6, we give a combinatorial descriptions of the terminal valuations of a non-rational variety equipped with a complexity-one torus action.

Section 7 describes some examples constructed from polyhedral divisors and illustrating our results. In particular examples of Nash valuations which are neither terminal nor minimal are given.

Section 8 discusses the rational case, pointing out the extra difficulties in comparison with the non-rational case, and giving a combinatorial description of the pointwise order on the set of equivariant valuations.

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1.8. **General notation.** Let $(E, \prec)$ be a poset. For any subset F of E, we denote by $\operatorname{Min}(F, \prec)$ the set of elements of F which are minimal for the induced poset structure. If $\prec'$ is another order on E, we denote by $\prec' \Rightarrow \prec$ the fact that $\prec'$ is finer than $\prec$, *i.e.* for any $x_1, x_2 \in E$, $x_1 \prec' x_2 \Rightarrow x_1 \prec x_2$.

Throughout the whole paper, we fix an algebraically closed field of characteristic zero, denoted by k. By an algebraic variety X over k, we mean an integral scheme separated and of finite type over k. We denote by X^{sing} the singular locus of X and by X^{sm} its smooth locus. In case X is affine, we denote by $k[X]$ its algebra of global regular functions.

2. Essential and Nash valuations and the Nash order

2.1. **Essential valuations.** By a *(k)-valuation* of an extension K of k, we always mean a discrete valuation on K trivial on k and with group of value contained in $\mathbb{Z}$, that is, a map $\nu\colon K \to \mathbb{Z} \cup \{+\infty\}$ such that:

(1) $\nu(k) = \{0\}$;

(2) $\forall f, g \in K, \quad \nu(fg) = \nu(f) + \nu(g)$;
(3) $\forall f, g \in K, \quad \nu(f+g) \geqslant \inf(\nu(f), \nu(g))$;
(4) $\nu^{-1}(\{+\infty\}) = \{0\}$.

The associated valuation ring is $\mathcal{O}_\nu := \{f \in K, \nu(f) \geqslant 0\}$. The valuation ideal $\mathcal{M}_\nu := \{f \in K, \nu(f) > 0\}$ is then a prime ideal of $\mathcal{O}_\nu$. A $\mathbb{Q}$-*valuation* is a map $\nu\colon K \to \mathbb{Q} \cup \{+\infty\}$ such that a positive multiple of ν is a valuation in the previous sense.

By an algebraic variety X over k, we mean an integral scheme separated and of finite type over k. We denote by X^{sing} the singular locus of X and by X^{sm} its smooth locus. In case X is affine, we denote by $k[X]$ its algebra of global regular functions.

Let X be an algebraic variety X over k and $K := k(X)$ be the function field of X. A valuation ν on K is said to be *centered on* X if there exists a non-empty open affine subset U of X such that $k[U] \subset \mathcal{O}_\nu$. Then the prime ideal $\mathcal{M}_\nu \cap k[U]$ defines a schematic point $\mathrm{cent}_X(\nu)$ of X which does not depend on the choice of U and is called the *center of* ν *on* X. In the literature, the center is often rather defined as the closed subset of X whose $\mathrm{cent}_X(\nu)$ is the generic point. We adopt the present definition since it will prove convenient to use the schematic language of generic points. We denote by $\mathrm{Val}(X)$ the set of valuations on $k(X)$ which are centered on X. In case X is normal and E is a prime divisor on X, the valuation ord_E defined by the local ring of X at E (which is a discrete valuation ring) and its multiples are prototypical examples of elements of $\mathrm{Val}(X)$.

A *divisorial valuation* of X is an element $\nu \in \mathrm{Val}(X)$ such that there exists a normal k-variety Z which is k-birational to X and such that ν is centered on Z, and $\mathrm{Adh}(\mathrm{cent}_Z(\nu))$ is a prime divisor of Z. Equivalently, there exist a prime divisor E on Z and a positive integer ℓ such that $\nu = \ell \cdot \mathrm{ord}_E$. We say that the divisorial valuation ν is of multiplicity one if one can take $\ell = 1$. Equivalently, $\nu(k(X)^\star) = \mathbb{Z}$.

Let $\mathrm{DV}(X) \subset \mathrm{Val}(X)$ be the set of divisorial valuations on X. For the sake of convenience, the trivial valuation (which sends $K \setminus \{0\}$ to 0) is considered as a divisorial valuation. We set

$$\mathrm{DV}(X)^{\mathrm{sing}} := \{\nu \in \mathrm{DV}(X), \quad \mathrm{cent}_X(\nu) \in X^{\mathrm{sing}}\}.$$

A *resolution of singularities of* X is a a proper birational morphism $f\colon Z \to X$ with Z a smooth variety and where f induces an isomorphism $f^{-1}(X^{\mathrm{sm}}) \xrightarrow{\sim} X^{\mathrm{sm}}$. Its *exceptional locus* $\mathrm{Exc}(f) \subset Z$ is the closed subset of Z where f is not an isomorphism, in other words $z \in \mathrm{Exc}(f)$ if and only if the induced morphism of local rings $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ is not an isomorphism. We say that a divisorial valuation $\nu \in \mathrm{DV}(X)$ is f-*exceptional* if $\mathrm{cent}_Z(\nu)$ is the generic point of an irreducible component of $\mathrm{Exc}(f)$.

A *divisorial resolution of the singularities of* X is a resolution of singularities $f\colon Y \to X$ such that the exceptional locus $\mathrm{Exc}(f)$ is of pure codimension 1.

A divisorial valuation $\nu \in \mathrm{DV}(X)$ of multiplicity one is said to be a (*divisorially*) *essential valuations* of X if for every (divisorial) resolution $f\colon Z \to X$ of the singularities of X, ν is f-exceptional. Denote by $\mathrm{Ess}(X)$ (resp. $\mathrm{DivEss}(X)$) the set of essential (resp. divisorially essential) valuations on X. Note that by the very definitions one has $\mathrm{Ess}(X) \subset \mathrm{DivEss}(X) \subset \mathrm{DV}(X)^{\mathrm{sing}}$.

We now assume that the variety X is equipped with an action of an algebraic group G. We denote by $\mathrm{Val}(X)_G$ the set of elements of $\mathrm{Val}(X)$ which are G-equivariant (*i.e.* invariant under the natural action of $G(k)$ on $k(X)$) ; we also use the self-explanatory notation $\mathrm{DV}(X)_G$ and so on.

A divisorial valuation $\nu \in \mathrm{DV}(X)$ is said to be a (*divisorially*) *G-essential valuations* of X if for every (divisorial) G-equivariant resolution $f\colon Z \to X$ of the singularities of X, ν is f-exceptional. Note that by [Kol07, Proposition 3.9.1], there exist G-equivariant divisorial resolutions of the singularities of X. Let $G - \mathrm{Ess}(X)$ (resp. $G - \mathrm{DivEss}(X)$) be the set of G-essential (resp. divisorially G-essential) valuations of X.

The following diagram of inclusions follows directly from the definitions.

$$\begin{array}{ccccc} \mathrm{DivEss}(X) & \subset & G-\mathrm{DivEss}(X) & \subset & \mathrm{DV}(X)_G^{\mathrm{sing}} \\ \cup & & \cup & & \\ \mathrm{Ess}(X) & \subset & G-\mathrm{Ess}(X) & & \end{array}$$

2.2. **Arc spaces, fat arcs and the pointwise order.** We denote by $\mathscr{L}_\infty(X)$ the space of arcs associated with a k-algebraic variety X. For more information and references on the properties of arc spaces, see *e.g.* [CLNS18]. Recall in particular that $\mathscr{L}_\infty(X)$ is a k-scheme such that for any k-extension K one has a functorial bijection between the set $\mathscr{L}_\infty(X)(K)$ of K-points of $\mathscr{L}_\infty(X)$ and the set $X(K[[t]])$ of $K[[t]]$-points of X. An element of $\mathscr{L}_\infty(X)(K)$ will be called a K-arc on X. If α is a K-arc on X, the natural k-algebra morphism $K[[t]] \hookrightarrow K((t))$ (resp. $K[[t]] \to K$, $t \mapsto 0$) induces an element of $X(K((t))$ (resp. $X(K)$) respectively, whose image in X is called the generic point (resp. the origin or the special point) of the K-arc α. Note that any $\alpha \in \mathscr{L}_\infty(X)$, with residue field $\kappa(\alpha)$, naturally induces a $\kappa(\alpha)$-arc on X (often still denoted by α). Recall also that there exists a natural k-morphism $\pi_X\colon \mathscr{L}_\infty(X) \to X$ mapping any $\alpha \in \mathscr{L}_\infty(X)$ to its origin $\alpha(0)$. For any subset $A \subset \mathscr{L}_\infty(X)$ we denote by $\mathscr{L}_\infty(X)^A$ the set $\pi_X^{-1}(A)$. Recall also that any morphism $f\colon Z \to X$ of algebraic varieties naturally induces a morphism of k-schemes $\mathscr{L}_\infty(f)\colon \mathscr{L}_\infty(Z) \to \mathscr{L}_\infty(X)$ such that for any k-extension K, the map $\mathscr{L}_\infty(Z)(K) \to \mathscr{L}_\infty(X)(K)$ induced by $\mathscr{L}_\infty(f)$ coincides with the map $Z(K[[t]]) \to X(K[[t]])$ induced by f.

Let $\alpha \in \mathscr{L}_\infty(X)$ and U be an open affine subset of X containing the origin of α. Then α induces a k-algebra morphism $\alpha_U^*\colon k[U] \to \kappa(\alpha)[[t]]$. Following [Ish05], we say that α is a *fat arc* if α_U^* is injective. The condition does not depend on the choice of U, and is equivalent to the fact that the generic point of α is the generic point of X (Recall that X is assumed to be irreducible.). If α is fat, it defines a valuation $\mathrm{ord}_\alpha \in \mathrm{Val}(X)$ as follows: for any $f \in k[U] \setminus \{0\}$, one sets $\mathrm{ord}_\alpha(f) := \mathrm{ord}_t(\alpha^* f) \in \mathbb{N}$. Note that $\mathrm{cent}_X(\mathrm{ord}_\alpha) = \alpha(0)$. We denote by $\mathscr{L}_\infty(X)^{\mathrm{fat}}$ the set of fat arcs on X.

Let $\nu \in \mathrm{Val}(X)$. We denote by $\mathscr{L}_\infty(X)^{\mathrm{ord}=\nu}$ the set of fat arcs on X such that $\mathrm{ord}_\alpha = \nu$. Note that $\mathscr{L}_\infty(X)^{\mathrm{ord}=\nu}$ is non-empty (see *e.g.* [Mor09, Proposition 3.12]).

Remark 2.1. Let $f\colon Y \to X$ be a proper birational morphism. By the valuative criterion of properness, f induces a bijective morphism

$$\mathscr{L}_\infty(f)\colon \mathscr{L}_\infty(Y)^{\mathrm{fat}} \to \mathscr{L}_\infty(X)^{\mathrm{fat}}.$$

Moreover $\mathrm{Val}(X)$ naturally identifies with $\mathrm{Val}(Y)$ and the diagram

$$\begin{array}{ccc} \mathscr{L}_\infty(Y)^{\mathrm{fat}} & \xrightarrow{\ \mathrm{ord}\ } & \mathrm{Val}(Y) = \mathrm{Val}(X) \\ {\scriptstyle \mathscr{L}_\infty(f)}\big\downarrow & \nearrow_{\mathrm{ord}} & \\ \mathscr{L}_\infty(X)^{\mathrm{fat}} & & \end{array}$$

is commutative. In particular $\mathscr{L}_\infty(f)$ sends $\mathscr{L}_\infty(Y)^{\mathrm{ord}=\nu}$ bijectively to $\mathscr{L}_\infty(X)^{\mathrm{ord}=\nu}$.

There is a natural poset structure on $\mathrm{Val}(X)$, the "pointwise" poset structure, introduced by Ishii in [Ish08].

Definition 2.2. One defines a poset structure $\leqslant_X$ on $\mathrm{Val}(X)$ as follows: let $\nu_1, \nu_2 \in \mathrm{Val}(X)$; then $\nu_1 \leqslant_X \nu_2$ if one of the following equivalent conditions hold.

(1) there exists a non-empty open affine subset U of X such that $\mathrm{cent}_{\nu_1}(X) \in U$ and for any $f \in k[U]$ one has $\nu_1(f) \leqslant \nu_2(f)$;
(2) for any $f \in \mathscr{O}_{X,\mathrm{cent}_{\nu_1}(X)}$, one has $\nu_1(f) \leqslant \nu_2(f)$;
(3) for any non-empty open affine subset U of X such that $\mathrm{cent}_{\nu_1}(X) \in U$ and for any $f \in k[U]$ one has $\nu_1(f) \leqslant \nu_2(f)$.

For $\nu \in \mathrm{Val}(X)$, one sets

$$\mathscr{L}_\infty(X)^{\mathrm{ord}\leqslant_X \nu} := \{\alpha \in \mathscr{L}_\infty(X), \quad \mathrm{ord}_\alpha \leqslant_X \nu\}$$
$$\text{and} \quad \mathscr{L}_\infty(X)^{\mathrm{ord}\geqslant_X \nu} := \{\alpha \in \mathscr{L}_\infty(X), \quad \mathrm{ord}_\alpha \geqslant_X \nu\}.$$

Remark 2.3. In case X is affine, for any $\nu_1, \nu_2 \in \mathrm{Val}(X)$, one has $\nu_1 \leqslant_X \nu_2$ if and only if for any $f \in k[X]$ one has $\nu_1(f) \leqslant \nu_2(f)$.

This poset structure behaves well with respect to the topological order on arcs, thanks to the following fundamental property, which is a straightforward consequence of [Ish05, Proposition 2.7].

Proposition 2.4. *Let* $\alpha_1, \alpha_2 \in \mathscr{L}_\infty(X)^{fat}$. *Assume that* $\alpha_2 \in \mathrm{Adh}(\alpha_1)$. *Then* $\mathrm{ord}_{\alpha_1} \leqslant_X \mathrm{ord}_{\alpha_2}$.

2.3. **Maximal divisorial sets and the Nash order.** Let $\nu \in \mathrm{DV}(X)$. Following [Ish08, Definition 2.8], one defines $\mathcal{C}_X(\nu)$ as the Zariski closure in $\mathscr{L}_\infty(X)$ of the set $\mathscr{L}_\infty(X)^{\mathrm{ord}=\nu}$. One calls $\mathcal{C}_X(\nu)$ the *maximal divisorial set (in short mds) associated with* ν. By *op.cit.*, $\mathcal{C}_X(\nu)$ is irreducible and its generic point $\eta_{X,\nu}$ is fat and satisfies $\mathrm{ord}_{\eta_{X,\nu}} = \nu$.

Remark 2.5. Let $\nu \in \mathrm{DV}(X)$ and U be any open subset of X containing $\mathrm{cent}_X(\nu)$. Then $\mathcal{C}_X(\nu) \cap \mathscr{L}_\infty(U) = \mathcal{C}_U(\nu)$, and $\eta_{X,\nu} = \eta_{U,\nu}$.

Remark 2.6. Retain the notation of Remark 2.1. Then, since $\mathscr{L}_\infty(f)$ is continuous, one obtains the inclusion $\mathscr{L}_\infty(f)(\mathcal{C}_Y(\nu)) \subset \mathcal{C}_X(\nu)$ and the equality $\mathrm{Adh}(\mathscr{L}_\infty(f)(\mathcal{C}_Y(\nu))) = \mathcal{C}_X(\nu)$. Here, Adh stands for the Zariski closure in $\mathscr{L}_\infty(X)$.

Definition 2.7. We define a poset structure $\leqslant_{\mathrm{mds},X}$ (or $\leqslant_{\mathrm{mds}}$ when there is no risk of confusion) on $\mathrm{DV}(X)$ as follows: let $\nu_1, \nu_2 \in \mathrm{DV}(X)$; then $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$ if and only if $\mathcal{C}_X(\nu_2) \subset \mathcal{C}_X(\nu_1)$. We call $\leqslant_{\mathrm{mds},X}$ the Nash order (or mds order) on $\mathrm{DV}(X)$.

Remark 2.8. One has $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$ if and only if η_{X,ν_2} is a specialization of η_{X,ν_1}. In particular, if $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$, then $\mathrm{cent}_X(\nu_2)$ is a specialization of $\mathrm{cent}_X(\nu_1)$.

Moreover, one has $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$ if and only if there exists an open subset U of X containing $\mathrm{cent}_X(\nu_2)$ such that $\nu_1 \leqslant_{\mathrm{mds},U} \nu_2$.

Indeed by Remark 2.5, if $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$, then for any open subset U of X containing $\mathrm{cent}_X(\nu_2)$ one has $\nu_1 \leqslant_{\mathrm{mds},U} \nu_2$.

On the other hand, assume that there exists an open subset U of X containing $\mathrm{cent}_X(\nu_2)$ and $\mathrm{cent}_X(\nu_2)$ such that $\nu_1 \leqslant_{\mathrm{mds},U} \nu_2$. Then η_{U,ν_2} is a specialization of η_{U,ν_2} in $\mathscr{L}_\infty(U)$. But $\eta_{U,\nu_i} = \eta_{X,\nu_i}$ and $\mathrm{Adh}_{\mathscr{L}_\infty(U)}(\eta_{X,\nu_1}) \subset \mathrm{Adh}_{\mathscr{L}_\infty(X)}(\eta_{X,\nu_1})$.

Proposition 2.9. *Let $f\colon Y \to X$ be a proper birational morphism. Let $\nu_1, \nu_2 \in \mathrm{DV}(Y) = \mathrm{DV}(X)$, and assume that $\mathcal{C}_Y(\nu_2) \subset \mathcal{C}_Y(\nu_1)$. Then $\mathcal{C}_X(\nu_2) \subset \mathcal{C}_X(\nu_1)$. In other words, on $\mathrm{DV}(Y) = \mathrm{DV}(X)$, one has $\leqslant_{mds,Y} \Rightarrow \leqslant_{mds,X}$.*

Proof. Indeed, if $\mathcal{C}_Y(\nu_2) \subset \mathcal{C}_Y(\nu_1)$, since $\mathscr{L}_\infty(f)$ is continuous, then

$$\mathrm{Adh}(\mathscr{L}_\infty(f)(\mathcal{C}_Y(\nu_2))) \subset \mathrm{Adh}(\mathscr{L}_\infty(f)(\mathcal{C}_Y(\nu_1)))$$

and one concludes by Remark 2.6. □

Remark 2.10. If $f\colon Y \to X$ be a proper birational morphism and $\nu \in \mathrm{DV}(Y) = \mathrm{DV}(X)$, then $\mathscr{L}_\infty(f)(\eta_{Y,\nu}) = \eta_{X,\nu}$. Indeed

$$\mathrm{Adh}(\mathscr{L}_\infty(f)(\eta_{Y,\nu})) = \mathrm{Adh}(\mathscr{L}_\infty(f)(\mathrm{Adh}(\eta_{Y,\nu}))) = \mathrm{Adh}(\mathscr{L}_\infty(f)(\mathcal{C}_Y(\nu))) = \mathcal{C}_X(\nu)$$

thus $\mathscr{L}_\infty(f)(\eta_{Y,\nu})$ is the generic point of $\mathcal{C}_X(\nu)$.

2.4. **The Nash valuations and the Nash problem.**

Definition 2.11. The set of *Nash valuations* of X, denoted by $\mathrm{Nash}(X)$, is the set of minimal elements of $\mathrm{DV}(X)^{\mathrm{sing}}$ with respect to the Nash order $\leqslant_{\mathrm{mds},X}$:

$$\mathrm{Nash}(X) := \mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}, \leqslant_{\mathrm{mds},X}).$$

Remark 2.12. In particular, in the equivariant case, one has $\mathrm{Nash}(X) \subset \mathrm{DV}(X)_G^{\mathrm{sing}}$. In fact, since by Remark 2.13 every $\nu \in \mathrm{DV}(X)^{\mathrm{sing}}$ is $\geqslant_{\mathrm{mds}} \nu'$ for $\nu' \in \mathrm{Nash}(X)$, one even has $\mathrm{Nash}(X) = \mathrm{Min}(\mathrm{DV}(X)_G^{\mathrm{sing}}, \leqslant_{\mathrm{mds}})$.

Remark 2.13. Following [IK03], let us give the original definition of the Nash valuations by Nash in [Nas95], translated into a modern schematic language. With every irreducible component C of $\mathscr{L}_\infty(X)^{X^{\mathrm{sing}}}$ not contained in $\mathscr{L}_\infty(X^{\mathrm{sing}})$, one associates an essential valuation of X as follows: take $Y \to X$ a resolution of the singularities of X. Then by the valuative criterion of properness, the generic point α of C lifts to an element $\widetilde{\alpha}$ of $\mathscr{L}_\infty(Y)$. One then shows that the closure $\widetilde{\alpha}(0)$ defines an essential divisor, and that this construction defines an injective map (the Nash map) from the set of irreducible components of $\mathscr{L}_\infty(X)^{X^{\mathrm{sing}}}$ to the set $\mathrm{Ess}(X)$. The set $\mathrm{Nash}(X)$ of Nash valuations is defined as the image of the Nash map.

The latter definition is equivalent to definition 2.11. Indeed, by [dFEI08], we know that $\mathscr{L}_\infty(X)^{X^{\mathrm{sing}}}$ has a finite number of irreducible fat components (that is, their generic point is a fat arc), and that these fat components are maximal divisorial sets. Moreover by [IK03], $\mathscr{L}_\infty(X)^{X^{\mathrm{sing}}}$ has no irreducible component whose generic point is not fat (recall that the latter property is known to fail in nonzero characteristic). On the other hand, for any $\nu' \in \mathrm{DV}(X)^{\mathrm{sing}}$, $\mathcal{C}_X(\nu')$ is an irreducible closed subset of $\mathscr{L}_\infty(X)^{X^{\mathrm{sing}}}$. Summing up, the $\mathcal{C}_X(\nu)$ for $\nu \in \mathrm{Nash}(X)$ (in the sense of definition 2.11) are exactly the irreducible components of $\mathscr{L}_\infty(X)^{X^{\mathrm{sing}}}$.

From definition 2.11 and the above remark, one obtains:

Proposition 2.14 (Nash)**.** *One has* $\mathrm{Nash}(X) \subset \mathrm{Ess}(X)$.

The *Nash problem*, in its original formulation, asks whether this inclusion is an equality. Since [IK03], one knows that the equality does not hold in general, and a more general form of the Nash problem (already present in [Nas95]) would ask for a sensible geometric characterization of the elements of $\mathrm{Nash}(X)$ among the elements of $\mathrm{Ess}(X)$.

Following [Ish08, DN17, FdBPPPP17], what we call the *generalized Nash problem* asks for a meaningful interpretation of the Nash order on $\mathrm{DV}(X)$. The problem is well-understood for equivariant valuations on toric varieties ([Ish08], see also 2.28 below) and determinantal varieties ([Doc13]). In [DN17], some partials results are obtained concerning the generalized Nash problem for contact strata in Grassmanians (See also [Mou14, MP18, KMPT20] for a variant of this problem, namely the embedded Nash problem, for a class of surface singularities.). Theorem 5.9 below solves the generalized Nash problem for equivariant valuations on non-rational normal varieties equipped with a complexity one torus action.

Remark 2.15. For any non-empty open subset U of X, one has $\mathrm{Nash}(U) = \mathrm{Nash}(X) \cap \mathrm{DV}(U)$ and $\mathrm{Ess}(U) = \mathrm{Ess}(X) \cap \mathrm{DV}(U)$. Thus the Nash problem is a local problem.

The generalized Nash problem is also of local nature, in the following sense. Let $\nu_1, \nu_2 \in \mathrm{DV}(X)$; then the following are equivalent:

- $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$;
- there exists a non-empty open affine subset U of X such that $\mathrm{cent}_X(\nu_1) \in U$, $\mathrm{cent}_X(\nu_2) \in U$ and $\nu_1 \leqslant_{\mathrm{mds},U} \nu_2$;
- for any non-empty open subset U of X such that $\mathrm{cent}_X(\nu_1) \in U$ and $\mathrm{cent}_X(\nu_2) \in U$, one has $\nu_1 \leqslant_{\mathrm{mds},U} \nu_2$.

Also note that if the condition $\nu_1 \leqslant_{\mathrm{mds},X} \nu_2$ is fulfilled, then $\mathrm{cent}_X(\nu_2)$ is a specialization of $\mathrm{cent}_X(\nu_1)$; in particular, for any covering $X = \cup_{i\in I} U_i$ of X by affine open subsets U_i, there always exists $i \in I$ such that $\mathrm{cent}_X(\nu_1), \mathrm{cent}_X(\nu_2) \in U_i$.

The following proposition, due to Ishii ([Ish08, Lemma 3.11]), shows that the pointwise order is finer than the Nash order.

Proposition 2.16. *Let $\nu, \nu' \in \mathrm{DV}(X)$ such that $\nu \leqslant_{mds} \nu'$. Then $\nu \leqslant_X \nu'$.*

Corollary 2.17. *The set $\mathrm{Min}(\mathrm{DV}(X)^{sing}, \leqslant_X)$ is contained in $\mathrm{Nash}(X)$. Moreover for every $\nu \in \mathrm{DV}(X)^{sing}$, there exists $\nu' \in \mathrm{Min}(\mathrm{DV}(X)^{sing}, \leqslant_X)$ such that $\nu' \leqslant_X \nu$.*

Proof. The inclusion is a direct consequence of Proposition 2.16. We show the second assertion. By Remark 2.13, there exists $\nu_1 \in \mathrm{Nash}(X)$ such that $\nu_1 \leqslant_{\mathrm{mds}} \nu$ thus $\nu_1 \leqslant_X \nu$. If $\nu_1 \in \mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}, \leqslant_X)$ we are done. Otherwise, let $\nu_2 \in \mathrm{DV}(X)^{\mathrm{sing}}$ such that $\nu_2 \leqslant_X \nu_1$ and $\nu_2 \neq \nu_1$. Again by Remark 2.13, there exists $\nu_3 \in \mathrm{Nash}(X)$ such that $\nu_3 \leqslant_X \nu_2$. Since $\nu_3 \leqslant_X \nu_2 \leqslant_X \nu_1$ and $\nu_1 \neq \nu_2$, we must have $\nu_3 \neq \nu_1$. Since $\mathrm{Nash}(X)$ is finite, repeating the process a finite number of times will eventually produce $\nu_i \in \mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}, \leqslant_X)$ such that $\nu_i \leqslant_X \nu$. □

Definition 2.18. The elements of the set $\mathrm{MinVal}(X) := \mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}, \leqslant_X)$ are called the *minimal valuations* on X.

Remark 2.19. In the equivariant case, one has

$$\mathrm{MinVal}(X) = \mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}_G, \leqslant_X)$$

Indeed, let $\nu \in \mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}_G, \leqslant_X)$ and $\nu' \in \mathrm{DV}(X)^{\mathrm{sing}}$ such that $\nu' \leqslant_X \nu$. There exists $\nu'' \in \mathrm{DV}(X)^{\mathrm{sing}}_G$ such that $\nu'' \leqslant_{\mathrm{mds}} \nu'$. In particular $\nu'' \leqslant_X \nu' \leqslant_X \nu$ and $\nu = \nu'$.

2.5. **The toric case.** Thanks to the work of Ishii ([Ish04, Ish08]), the Nash order and its relation with the pointwise order are especially well understood for the toric valuations of a toric variety. We recall here the relevant definitions and results, some of which will be needed for our study of the Nash order on varieties equipped with a complexity one torus action. We in particular explain how to obtain the solution of the Nash problem for toric varieties; though our presentation does not feature substantial differences with the original argument of Ishii and Kollar in [IK03], it is slightly more direct, in particular thanks to the use of the results of Ishii's paper [Ish04].

We use standard toric notation, definitions and facts (see also Section 3 below; a standard reference on toric geometry is [CLS11]).

We assume that $X = X_\sigma = \operatorname{Spec}(k[\sigma^\vee \cap M])$ is a toric affine k-variety, where $M = \operatorname{Hom}(N, \mathbb{Z})$ is the dual of a lattice N and $\sigma^\vee$ is the dual of a strictly convex polyhedral cone σ of $N \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus X is equipped with an action of the torus $\mathbb{T} := \operatorname{Spec}(k[M])$. Let $n \in \sigma \cap N$. Let $f \in k[X]$, and write $f = \sum_{m \in M \cap \sigma^\vee} f_m \cdot \chi^m$ with $f_m \in k$ and where χ^m is character associated with $m \in M$. Set

$$\nu_n(f) := \operatorname*{Inf}_{\substack{m \in M \cap \sigma^\vee \\ f_m \neq 0}} \langle m \,,\, n \rangle \,.$$

Then $\nu_n \in \mathrm{DV}(X)_{\mathbb{T}}$ and the map $n \mapsto \nu_n$ is a bijection between $\mathrm{DV}(X)_{\mathbb{T}}$ and $\sigma \cap N$. Denote by σ_{sing} the union of the relative interiors of the non-smooth faces of σ. Modulo the above identification, one has

$$\mathrm{DV}(X)^{\mathrm{sing}}_{\mathbb{T}} = \sigma_{\mathrm{sing}} \cap N.$$

On $\mathrm{DV}(X)_{\mathbb{T}}$, besides $\leqslant_X$ and $\leqslant_{\mathrm{mds}}$, one defines, following [Ish04, Definition 4.6], a third natural poset structure of combinatorial nature.

Definition 2.20. For any $\nu, \nu' \in \mathrm{DV}(X)_{\mathbb{T}}$, set

$$\nu \leqslant_\sigma \nu' \text{ iff } \nu' \in \nu + \sigma.$$

The following proposition is a straightforward consequence of the definition.

Proposition 2.21. *On* $\mathrm{DV}(X)_{\mathbb{T}}$ *the poset structures defined by* $\leqslant_\sigma$ *and* $\leqslant_X$ *coincide.*

Using Remark 2.19, one then obtains:

Corollary 2.22. *The set* $\mathrm{MinVal}(X)$ *identifies with* $\mathrm{Min}(\sigma_{sing} \cap N, \leqslant_\sigma)$.

The resolution of the Nash problem in the toric case is then a direct consequence of the latter corollary and Proposition 2.25 below, the proof of which is purely combinatorial and contained in the proof of [IK03, Lemma 3.15]. Note that this proof relies on a construction due initially to Bouvier and Gonzalez-Sprinberg ([BGS95]). In order to state the proposition, and for the sake of convenience, we first introduce some notation and terminology which will also be useful later on. We will often use it in case the fan to be refined is the fan of the faces of a cone σ, identified by abuse of notation with σ itself.

Definition 2.23. Let Σ be a fan and Σ' be a fan refining Σ.

1. The fan Σ' is said to be a star refinement of Σ if Σ' may be obtained from Σ by a finite succession of star subdivisions (see [CLS11, §11.1]).
2. The fan Σ' is said to be a big refinement of Σ if the following holds: for any $\tau \in \Sigma$ such that $\tau \notin \Sigma$, there exists a ray of τ which is not a ray of Σ.
3. Assume that Σ' is smooth, *i.e.* every cone of Σ' is smooth; the fan Σ' is said to be a smooth economical refinement of Σ if every smooth cone of Σ is a cone of Σ'.

Remark 2.24. Σ' is a smooth economical refinement of Σ if and only if the induced equivariant proper birational morphism $X(\Sigma') \to X(\Sigma)$ is a resolution of singularities of $X(\Sigma)$. Indeed, the condition guarantees that the induced morphism is an isomorphism over the smooth locus of $X(\Sigma)$.

On the other hand Σ' is a big refinement of Σ if and only if the exceptional locus of $X(\Sigma') \to X(\Sigma)$ has pure codimension 1.

Proposition 2.25. *Let ν be a primitive element in $(\sigma_{sing} \cap N) \setminus \mathrm{Min}(\sigma_{sing} \cap N, \leqslant_\sigma)$. Then there exists a fan Σ which is a big and smooth economical star refinement of σ and such that the cone τ of Σ such that $\nu \in \mathrm{Relint}(\tau)$ has dimension $\geqslant 2$.*

In particular, there exists a divisorial equivariant resolution f of the singularities of X such that ν is not f-exceptional; in other words $\nu \notin \mathbb{T} - \mathrm{DivEss}(X)$.

Remark 2.26. This statement contains the well-known fact that there exists a big and smooth economical star refinement of any strictly convex polyhedral cone σ (see [CLS11, Theorem 11.1.9]).

Corollary 2.27 (The Nash problem in the toric case)**.** *Let X be an affine toric variety. Then $\mathbb{T} - \mathrm{DivEss}(X) = \mathbb{T} - \mathrm{Ess}(X) = \mathrm{DivEss}(X) = \mathrm{Ess}(X) = \mathrm{Nash}(X) = \mathrm{MinVal}(X)$. In particular, the Nash problem has a positive answer in the toric case.*

Proof. Proposition 2.25 shows that $\mathbb{T} - \mathrm{DivEss}(X) \subset \mathrm{MinVal}(X)$. Since the inclusions $\mathrm{MinVal}(X) \subset \mathrm{Nash}(X) \subset \mathrm{Ess}(X) \subset \mathbb{T} - \mathrm{Ess}(X) \subset \mathbb{T} - \mathrm{DivEss}(X)$ and $\mathrm{Ess}(X) \subset \mathrm{DivEss}(X) \subset \mathbb{T} - \mathrm{DivEss}(X)$ always hold, one gets the result. □

Now let us turn to the generalized Nash problem on toric varieties. The following proposition is a consequence of [Ish04, Proposition 4.8] and [Ish08, Example 2.10 & Lemma 3.11].

Proposition 2.28. *On $\mathrm{DV}(X)_{\mathbb{T}}$ the three poset structures defined by $\leqslant_\sigma$, $\leqslant_X$ and $\leqslant_{mds}$ coincide.*

2.6. **Stable points and Reguera's curve selection lemma.** The notion of stable points of the arc scheme of an algebraic variety was introduced by Reguera. One of the main feature of these points is that their formal neighborhood is noetherian, a fact allowing Reguera to obtain a version the curve selection lemma for arc spaces which turned out to be crucial in subsequent works on the Nash problem. We only recall here the definitions and properties which are relevant for our needs; see [Reg06, Reg21, Reg09, MR18, dFD20] for more information and results on stable points. We also state and prove a simple consequence of Reguera's curve selection lemma (Corollary 2.31) which we shall use later. Recall that a wedge on X is an arc on $\mathscr{L}_\infty(X)$, in other words a point of the scheme $\mathscr{L}_\infty(\mathscr{L}_\infty(X))$. The generic point (resp. the special point) of a wedge is called its generic arc (resp. its special arc). Note that the closure in $\mathscr{L}_\infty(X)$ of the generic arc of a wedge on X always contains the special arc of the wedge.

Definition 2.29. (See [Reg06, Reg21, Reg09] as well as [dFD20, §10]. Let X be an algebraic k-variety. A point $\alpha \in \mathscr{L}_\infty(X)$ is stable if it is not contained in $\mathscr{L}_\infty(X^{\mathrm{sing}})$ and it is the generic point of an irreducible constructible subset of $\mathscr{L}_\infty(X)$.

Theorem 2.30. (1) *Let $\nu \in \mathrm{DV}(X)$ and $\eta_{X,\nu}$ be the generic point of $\mathcal{C}_X(\nu)$. Then $\eta_{X,\nu}$ is a stable point of $\mathscr{L}_\infty(X)$.*

(2) *Let $\alpha \in \mathscr{L}_\infty(X)$ be a stable point. Then the following holds.*
  (a) *Every generization of α is again a stable point.*
  (b) *The Krull dimension of the local ring $\mathscr{O}_{\mathscr{L}_\infty(X),\alpha}$ is finite.*
  (c) *(the curve selection lemma for stable points) Let N be an irreducible closed subset of $\mathscr{L}_\infty(X)$, such that $\mathrm{Adh}(\alpha)$ is a proper subset of N. Then there exist an extension K of k and a K-wedge $\mathrm{Spec}(K[[t,u]]) \to X$ with special arc α and generic arc an element of $N \setminus \mathrm{Adh}(\alpha)$.*

Proof. Assertion 1 comes from [MR18, §2.4]. Assertion 2a is [Reg09, Proposition 3.7(vi)]; it is also a direct consequence of [dFD20, Proposition 10.5]. Assertion 2b is [Reg09, Proposition 3.7(iv)] whereas assertion 2c is a consequence of [Reg06, Corollary 4.8]. □

Corollary 2.31. *Let $\alpha, \alpha' \in \mathscr{L}_\infty(X)$ be two stable points such that α is a specialization of α'. Then there exist an extension K/k and a finite sequence of K-wedges $w_1, \ldots, w_r$ on X such that the special arc of w_1 is α, the generic arc of w_r is α' and for any $1 \leqslant i \leqslant r-1$ the generic arc of w_i is the special arc of w_{i+1}.*

Proof. The result is trivial if $\alpha = \alpha'$. Thus one may assume that $\mathrm{Adh}(\alpha)$ is a proper subset of $N' := \mathrm{Adh}(\alpha')$. By assertion 2c of Theorem 2.30, there exist an extension K/k and a K-wedge w_1 on X with special arc α and generic arc $\alpha_1 \in N' \setminus \mathrm{Adh}(\alpha)$. If $\alpha_1 = \alpha'$, we are done. Otherwise, note that α_1 is a (strict) generization of α', thus is a stable point of $\mathscr{L}_\infty(X)$, and $\mathrm{Adh}(\alpha_1)$ is a proper subset of N'. Thus we may apply the curve selection lemma again and (extending K if necessary) find a K-wedge w_2 on X with special arc α_1 and generic arc $\alpha_2 \in N' \setminus \mathrm{Adh}(\alpha_1)$. Since the local ring $\mathscr{O}_{\mathscr{L}_\infty(X),\alpha}$ has finite Krull dimension, there does not exist arbitrary long sequences $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_r$ of $\mathscr{L}_\infty(X)$ such that α_i is a strict generization of α_{i+1}. Thus after a finite number of steps of the above procedure, we end up with a wedge with generic arc α'. □

3. Algebraic torus actions of complexity one

Recall that if $\mathbb{T}$ is an algebraic torus, the *complexity* of an effective algebraic $\mathbb{T}$-action on a variety X is the number $\dim(X) - \dim(\mathbb{T})$. This section introduces preliminaries from Altmann-Hausen's theory [AH06, AHS08] for the classification of effective algebraic torus actions on normal varieties, limiting ourselves to the case of complexity one, though the theory deals with torus actions of arbitrary complexity. In the case of complexity zero, Altmann-Hausen's theory reduces to the classical setting of the combinatorial classification of normal toric varieties. Complexity-one normal $\mathbb{T}$-varieties are instances of $\mathbb{T}$-varieties where, similarly to the toric case, Altmann-Hausen's description is particularly explicit, and where one may use Timashev's language of hypercones [Tim08], which we will also recall.

3.1. **Cones and polyhedrons.** We start by fixing standard toric notation, trying to respect as much as possible the notation and terminology of the standard reference [CLS11]. Namely, $N \simeq \mathbb{Z}^d$ is a lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$ is the dual lattice and $M_\mathbb{Q}, N_\mathbb{Q}$ are respectively the associated $\mathbb{Q}$-vector spaces obtained from M, N by tensoring with $\mathbb{Q}$. We denote by

$$M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q},\ (m,n) \mapsto \langle m\,,\,n\rangle$$

the natural pairing deduced from the duality between M and N. The notation $\mathbb{T}$ stands for the algebraic torus $\mathbb{G}_m \otimes_\mathbb{Z} N \simeq \mathbb{G}_m^n$ whose character and one-parameter subgroup lattices are respectively M and N. We distinguish two notations: the lattice vector $m \in M$ and the character χ^m corresponding to m seen as regular function on the torus $\mathbb{T}$. For a polyhedral cone σ we denote by $\mathrm{Relint}(\sigma)$ its relative interior (i.e. the complement of the union of its proper faces) and by

$$\sigma^\vee := \{m \in M_\mathbb{Q} \mid \langle m\,,\,n\rangle \geqslant 0 \text{ for any } n \in \sigma\}$$

its dual cone. Recall that σ is said to be *strictly convex* if $\{0\}$ is a face of σ, and (N)-*smooth* if σ may be generated as a cone by a part of a $\mathbb{Z}$-basis of the lattice N.

Given any polyhedron $\mathcal{P} \subset N_\mathbb{Q}$ we set

$$\mathrm{Tail}(\mathcal{P}) := \{v \in N_\mathbb{Q} \mid v + \mathcal{P} \subset \mathcal{P}\},$$

which is a polyhedral cone of $N_\mathbb{Q}$.

3.2. **Polyhedral divisors.** Let us fix a polyhedral strictly convex cone $\sigma \subset N_{\mathbb{Q}}$. From the datum (N, σ) we define the semigroup

$$\mathrm{Pol}_{\mathbb{Q}}^{+}(N, \sigma) := \{\mathcal{P} \subset N_{\mathbb{Q}} \,|\, \mathcal{P} \text{ polyhedron with } \mathrm{Tail}(\mathcal{P}) = \sigma\}$$

whose addition is the Minkowski sum and neutral element is the cone σ. We also consider the extended semigroup $\mathrm{Pol}_{\mathbb{Q}}(N, \sigma) = \mathrm{Pol}_{\mathbb{Q}}^{+}(N, \sigma) \cup \{\varnothing\}$, where the element $\varnothing$ is an absorbing element, i.e. $\mathcal{P} + \varnothing = \varnothing$ for any $\mathcal{P} \in \mathrm{Pol}_{\mathbb{Q}}(N, \sigma)$.

Let Y be a smooth algebraic curve; hereafter we identify Y with its set of closed points. Write $\mathrm{Div}(Y)$ (resp. $\mathrm{Div}_{\geqslant 0}(Y)$) for the group of Cartier divisors (resp. the semigroup of effective Cartier divisors) on Y. By a *σ-tailed polyhedral divisor over* (Y, N) we mean an element

$$\mathcal{D} \in \mathrm{Pol}_{\mathbb{Q}}(N, \sigma) \otimes_{\mathbb{Z}_{\geqslant 0}} \mathrm{Div}_{\geqslant 0}(Y).$$

In particular, $\mathcal{D}$ has a decomposition as a formal sum

$$\mathcal{D} = \sum_{y \in Y} \mathcal{D}_y \cdot [y],$$

where $\mathcal{D}_y \in \mathrm{Pol}_{\mathbb{Q}}(N, \sigma)$ is equal to σ for all but finitely many $y \in Y$. The *tail* of $\mathcal{D}$ denoted by $\mathrm{Tail}(\mathcal{D})$ is the cone σ and the *locus* of $\mathcal{D}$ is the non-empty open set of Y defined by

$$\mathrm{Loc}(\mathcal{D}) := Y \setminus \bigcup_{y \in Y,\, \mathcal{D}_y = \varnothing} [y].$$

The *evaluation* is the piecewise linear map

$$m \in \sigma^{\vee} \mapsto \mathcal{D}(m) := \sum_{\mathcal{D}_y \neq \varnothing} \min \langle \mathcal{D}_y\, ,\, m \rangle \cdot y \in \mathrm{Div}_{\mathbb{Q}}(\mathrm{Loc}(\mathcal{D})),$$

where $\mathrm{Div}_{\mathbb{Q}}(\mathrm{Loc}(\mathcal{D}))$ is the vector space of $\mathbb{Q}$-Cartier divisors on $\mathrm{Loc}(\mathcal{D})$. The *support* of $\mathcal{D}$ is the finite set

$$\mathrm{Supp}(\mathcal{D}) = \{y \in Y \,|\, \mathcal{D}_y \notin \{\varnothing, \mathrm{Tail}(\mathcal{D})\}\}.$$

We define the *degree* of $\mathcal{D}$ as the Minkowski sum

$$\deg(\mathcal{D}) := \sum_{y \in Y} \mathcal{D}_y \in \mathrm{Pol}_{\mathbb{Q}}(N, \sigma)$$

when $\mathrm{Loc}(\mathcal{D})$ is complete, and in case $\mathrm{Loc}(\mathcal{D})$ affine, we set $\deg(\mathcal{D}) := \varnothing$.

Definition 3.1. Let Y be a smooth algebraic curve. A polyhedral divisor $\mathcal{D}$ over (Y, N) is *proper* (or a *p-divisor*) if one of the following conditions hold

1. the locus $\mathrm{Loc}(\mathcal{D})$ is affine;
2. the locus $\mathrm{Loc}(\mathcal{D})$ is complete, $\deg(\mathcal{D})$ is a proper subset of σ and for every $m \in \sigma^{\vee}$ satisfying $\deg(\mathcal{D}(m)) = 0$, the $\mathbb{Q}$-divisor $\mathcal{D}(m)$ has a principal multiple.

For any non-empty open subset Y_0 of the curve Y, consider the restriction of $\mathcal{D}$ to Y_0, that is to say the polyhedral divisor

$$\mathcal{D}_{|_{Y_0}} := \sum_{y \in Y_0} \mathcal{D}_y \cdot [y].$$

Note that if $\mathcal{D}$ is a p-divisor, then $\mathcal{D}_{|_{Y_0}}$ is also a p-divisor.

Each polyhedral divisor $\mathcal{D}$ has the property $\mathcal{D}(m)+\mathcal{D}(m') \leqslant \mathcal{D}(m+m')$ for all $m, m' \in \mathrm{Tail}(\mathcal{D})^{\vee}$. Hence the multiplication on the field $k(Y)$ naturally defines an M-graded $\mathcal{O}_{\mathrm{Loc}(\mathcal{D})}$-algebra

$$\mathcal{A}(\mathcal{D}) := \bigoplus_{m \in \mathrm{Tail}(\mathcal{D})^{\vee} \cap M} \mathcal{O}_{\mathrm{Loc}(\mathcal{D})}(\mathcal{D}(m))\chi^m.$$

We may therefore define two affine k-schemes

$$\widetilde{X}(\mathcal{D}) = \mathbf{Spec}_{\mathrm{Loc}\,\mathcal{D}}\, \mathcal{A}(\mathcal{D}) \text{ and } X(\mathcal{D}) = \mathrm{Spec}\, \Gamma(\mathrm{Loc}(\mathcal{D}), \mathcal{A}(\mathcal{D})).$$

Moreover the M-grading on $\mathcal{A}(\mathcal{D})$ naturally induces algebraic $\mathbb{T}$-actions on the schemes $\widetilde{X}(\mathcal{D})$ and $X(\mathcal{D})$. By the very construction, one has:

Lemma 3.2. *The structural morphism $q\colon \widetilde{X}(\mathcal{D}) \to \mathrm{Loc}(\mathcal{D})$ is affine.*

The following result is a particular case of Altmann-Hausen's classification result (see [AH06, Theorem 3.1, Theorem 3.4, Theorem 8.8] for more general statements in arbitrary complexity; see also [Lan15] for the complexity one case)

Theorem 3.3. *Let Y be a smooth algebraic curve and $\mathcal{D}$ be a p-divisor over (Y, N). Then $X(\mathcal{D})$ is a normal affine variety, and the natural action of $\mathbb{T}$ on $X(\mathcal{D})$ is effective and of complexity one.*

Conversely, for any normal affine variety X equipped with an effective algebraic action of $\mathbb{T}$ of complexity one, there exist a smooth algebraic curve Y and p-divisor $\mathcal{D}$ over (Y, N) such that X is equivariantly isomorphic to $X(\mathcal{D})$.

3.3. **Divisorial fans.** According to Sumihiro's Theorem any normal variety with torus action is a finite union of affine open subsets that are stable by the torus action. This leads to describe any normal variety with an effective algebraic torus action of complexity one by a finite collection of p-divisors satisfying similar conditions to those defining the notion of fan for toric varieties.

Definition 3.4. Let $\mathcal{D}, \mathcal{D}'$ be two polyhedral divisors over (Y, N). The *intersection* of $\mathcal{D}$ and $\mathcal{D}'$ is the polyhedral divisor over (Y, N) defined by the relation

$$\mathcal{D} \cap \mathcal{D}' := \sum_{y \in Y} \mathcal{D}_y \cap \mathcal{D}'_y \cdot [y]$$

Assume now that $\mathcal{D}$ and $\mathcal{D}'$ are two p-divisors. We say that $\mathcal{D}$ is a *face* of $\mathcal{D}'$ if for any $y \in Y$, $\mathcal{D}_y$ is a face of $\mathcal{D}'_y$ and $\deg(\mathcal{D}) = \deg(\mathcal{D}') \cap \mathrm{Tail}(\mathcal{D})$.

If $\mathcal{D}$ and $\mathcal{D}'$ are two p-divisors such that for any $y \in Y$ the polyhedron $\mathcal{D}_y$ is a face of $\mathcal{D}'_y$, then $\mathcal{D}$ is a face of $\mathcal{D}'$ if and only if the natural morphism $X(\mathcal{D}) \to X(\mathcal{D}')$ is an open immersion ([IS11, Lemma 1.4]).

Definition 3.5. A *divisorial fan* or an *f-divisor* over (Y, N) is a finite set $\mathscr{E}$ of p-divisors, stable by intersection and such that for all $\mathcal{D}, \mathcal{D}' \in \mathscr{E}$ the p-divisor $\mathcal{D} \cap \mathcal{D}'$ is a mutual face of $\mathcal{D}$ and $\mathcal{D}'$.

Any divisorial fan $\mathscr{E}$ over (Y, N) defines a k-scheme $X(\mathscr{E})$ with effective algebraic $\mathbb{T}$-action of complexity one. The scheme $X(\mathscr{E})$ is obtained by gluing the family of varieties $(X(\mathcal{D}))_{\mathcal{D}\in\mathscr{E}}$ in a such way that $X(\mathcal{D} \cap \mathcal{D}')$ is identified with $X(\mathcal{D}) \cap X(\mathcal{D}')$ for all $\mathcal{D}, \mathcal{D}' \in \mathscr{E}$ ([AHS08, Theorem 5.3]). By [AHS08, Remark 7.4], the k-scheme $X(\mathscr{E})$ is separated and is thus a normal k-variety. Conversely, any normal variety with an effective algebraic $\mathbb{T}$-action of complexity one comes from a divisorial fan ([AHS08, Theorem 5.6]).

3.4. **Hypercones and hyperfans.** We now discuss some combinatorial objects coming from the classification of Timashev of the algebraic torus actions of complexity one [Tim08]. We still consider a lattice N and a smooth algebraic curve Y. The associated *hyperspace* or *book* is the set $\mathcal{N}_{\mathbb{Q}}$ defined as the quotient set

$$Y \times N_{\mathbb{Q}} \times \mathbb{Q}_{\geqslant 0}/\sim,$$

where the equivalence relation $\sim$ is given by

$$(y, a, b) \sim (y', a', b') \text{ if and only if } (y = y', a = a', b = b') \text{ or } (a = a', b = b' = 0).$$

The image of $(y, a, b) \in Y \times N_{\mathbb{Q}} \times \mathbb{Q}_{\geqslant 0}$ in $\mathcal{N}_{\mathbb{Q}}$ will be denoted by $[y, a, b]$. In case $b = 0$, it is also denoted by $[\bullet, a, 0]$ since it does not depend on y.

Note that the natural map

$$N_{\mathbb{Q}} \to \mathcal{N}_{\mathbb{Q}},\ a \mapsto [\bullet, a, 0]$$

allows to identify $N_{\mathbb{Q}}$ with a subset of $\mathcal{N}_{\mathbb{Q}}$ called the *spine* $\mathcal{S}$ of $\mathcal{N}_{\mathbb{Q}}$. The set $\mathcal{N}$ of integral points of $\mathcal{N}_{\mathbb{Q}}$ is the image in $\mathcal{N}_{\mathbb{Q}}$ of $Y \times N \times \mathbb{N}$.

Let $y \in Y$. The associated *page* of the book $\mathcal{N}_{\mathbb{Q}}$ is the set

$$\mathcal{N}_{y,\mathbb{Q}} := \{[y, a, b] \,|\, (a, b) \in N_{\mathbb{Q}} \times \mathbb{Q}_{\geqslant 0}\}$$

Note that $\mathcal{N}_{y,\mathbb{Q}}$ contains $\mathcal{S}$ and may be identified with $N_{\mathbb{Q}} \times \mathbb{Q}_{\geqslant 0}$ in a way compatible with the identification of $\mathcal{S}$ with $N_{\mathbb{Q}} = N_{\mathbb{Q}} \times \{0\}$. For any two points $y \neq y'$ of Y one has $\mathcal{N}_{y,\mathbb{Q}} \cap \mathcal{N}_{y',\mathbb{Q}} = \mathcal{S}$.

Consider now a polyhedral divisor $\mathcal{D}$ over (Y, N) with tail σ. For any $y \in Y$, the associated *Cayley cone* is the cone $C_y(\mathcal{D}) \subseteq \mathcal{N}_{y,\mathbb{Q}} \subset N_{\mathbb{Q}} \times \mathbb{Q}$ generated by $(\sigma \times \{0\}) \cup (\mathcal{D}_y \times \{1\})$.

The *hypercone* associated with $\mathcal{D}$ is the subset of $\mathcal{N}_{\mathbb{Q}}$ defined by

$$HC(\mathcal{D}) := \bigcup_{y\in Y} C_y(\mathcal{D}).$$

In particular, one has $HC(\mathcal{D}) \cap \mathcal{S} = \sigma$ and for any $y \in Y$ one has $HC(\mathcal{D}) \cap \mathcal{N}_{y,\mathbb{Q}} = C_y(\mathcal{D})$.

Definition 3.6. For a p-divisor $\mathcal{D}$ over (Y, N) we say that a subset $\theta \subset \mathcal{N}_{\mathbb{Q}}$ is a *hyperface* of the hypercone $HC(\mathcal{D})$ if it satisfies one of the following conditions.

(i) We have $\theta = HC(\mathcal{D}')$, where $\mathcal{D}'$ is a p-divisor over (Y, N), with the property that $\theta \cap \deg(\mathcal{D}) \neq \varnothing$ and $C_y(\mathcal{D}')$ is a non-empty face of $C_y(\mathcal{D})$ for any $y \in Y$. In other words, $\mathcal{D}'$ has complete locus and is a face of $\mathcal{D}$.

(ii) We have $\theta \cap \deg(\mathcal{D}) = \varnothing$ and θ is a face of $C_y(\mathcal{D})$ for some $y \in Y$. In this case θ is a subset of $\mathcal{N}_{y,\mathbb{Q}}$.

Let us now consider a divisorial fan $\mathscr{E}$ over (Y, N). We call *hyperfan* of the divisorial fan $\mathscr{E}$ the set $H\Sigma(\mathscr{E}) := \{\theta$ hyperfaces of $HC(\mathcal{D})$ for some $\mathcal{D} \in \mathscr{E}\}$. For an element $\theta \in H\Sigma(\mathscr{E})$ we define its *relative interior* $\mathrm{Relint}(\theta)$ and its dimension $\dim(\theta)$ in a obvious way.

The viewpoint of hyperfans has the following geometric interpretation in terms of valuation theory (Remember our convention about valuations in Section 2.1.).

Let $X = X(\mathscr{E})$ be the normal variety with an effective algebraic $\mathbb{T}$-action of complexity one described by the divisorial fan $\mathscr{E}$.

We have a one-to-one correspondence $[y, a, b] \mapsto \mathrm{val}_{[y,a,b]}$ between $\mathcal{N}_{\mathbb{Q}}$ and the set of $\mathbb{T}$-invariant $\mathbb{Q}$-valuations on $k(X)$ [Tim08, §2, Lemma 1]. This correspondence can be described as follows.

First, remark that, since X is birationally equivalent to $Y \times \mathbb{T}$ (see [Tim08, §1, Corollary 3]), the field $k(X)$ is the fraction field of the semigroup algebra

$$k(Y)[M] = \bigoplus_{m \in M} k(Y) \cdot \chi^m.$$

Given any element

$$f = \sum_{m \in M} f_m \cdot \chi^m \in k(Y)[M],$$

with $f_m \in k(Y)^\times$ and every f_m but a finite number of them is zero, we define the corresponding valuation $\mathrm{val}_{[y,a,b]}(f)$ via the formula

$$\mathrm{val}_{[y,a,b]}(f) = \operatorname*{Inf}_{\substack{m \in M \\ f_m \neq 0}} \langle m\, ,\, a \rangle + b \cdot \mathrm{ord}_y(f_m),$$

where ord_y is the vanishing order at the point y.

Proposition 3.7. *The above one-to-one correspondence* $[y, a, b] \mapsto \mathrm{val}_{[y,a,b]}$ *between* $\mathcal{N}_{\mathbb{Q}}$ *and the set of* $\mathbb{T}$*-invariant* $\mathbb{Q}$*-valuations on* $k(X)$ *induces a one-to-one correspondence between the set* $\mathrm{DV}(X)_{\mathbb{T}}$ *of* $\mathbb{T}$*-invariant divisorial valuations on* X *and the set* $(\cup_{\mathcal{D} \in \mathscr{E}} HC(\mathcal{D})) \cap \mathcal{N}$*. Modulo this correspondence, for any* $\mathcal{D} \in \mathscr{E}$ *and* $\nu \in (\cup_{\mathcal{D} \in \mathscr{E}} HC(\mathcal{D})) \cap \mathcal{N}$*, one has* $\mathrm{cent}_X(\nu) \in X(\mathcal{D})$ *if and only if* $\nu \in HC(\mathcal{D})$*. Moreover, if* $[y, a, b] \in HC(\mathcal{D}) \cap \mathcal{N}$ *for* $\mathcal{D} \in \mathscr{E}$*, for* *any non-empty open subset* Y_0 *of* Y *such that* $y \in Y_0$, $\mathrm{val}_{y,a,b}$ *is centered at* $X(\mathcal{D}_{|Y_0})$.

Proof. The only non-obvious point is that the valuations induced by integral elements of the hypercone are divisorial. But this follows from [Tim11, Proposition 19.8]. □

3.5. **Prime invariant cycles and hyperfaces.** Let Y be a smooth algebraic curve, $\mathscr{E}$ be a divisorial fan over (Y, N) and $X = X(\mathscr{E})$ be the associated $\mathbb{T}$-variety. Call *prime* $\mathbb{T}$-*cycle* of the $\mathbb{T}$-variety X any $\mathbb{T}$-stable irreducible closed subset of X. Elements of the hyperfan $H\Sigma(\mathscr{E})$ bijectively correspond to prime $\mathbb{T}$-cycles of X. More precisely, the prime $\mathbb{T}$-cycle $\mathcal{Z}(\theta) \subset X$ (also denoted by $\mathcal{Z}(\mathscr{E}, \theta)$ if some confusion on the divisorial fan under consideration may occur) associated with $\theta \in H\Sigma(\mathscr{E})$ is the closure of the center in X of any $\mathbb{Q}$-valuation $\mathrm{val}_{[y,a,b]}$ where $[y, a, b]$ belongs to $\mathrm{Relint}(\theta)$ (see [Tim08, §4, Theorem 6] for more details; remember our convention on the center of a valuation in Subsection 2.1). Furthermore, the codimension of $\mathcal{Z}(\theta)$ is equal to $\dim(\theta)$, and the correspondence $\theta \mapsto \mathcal{Z}(\theta)$ respects the ordering, namely, $\mathcal{Z}(\theta_1) \subset \mathcal{Z}(\theta_2)$ if and only if θ_2 is a (hyper)face of θ_1. Note also the following: let $\mathcal{D} \in \mathscr{E}$, $\nu \in HC(\mathcal{D})$ and θ be the face of $HC(\mathcal{D})$ such that $\nu \in \mathrm{Relint}(\theta)$. Assume that $\theta \cap \deg(\mathcal{D}) \neq \varnothing$. Then $\mathrm{Adh}(\mathrm{cent}_X(\nu)) = \mathcal{Z}(HC(\mathcal{D}'))$ where $\mathcal{D}'$ is the unique face of $\mathcal{D}$ with tail $\theta \cap \mathcal{S}$.

3.6. **Toroidification.** Let Y be a smooth algebraic curve. If $\mathscr{E}$ is a divisorial fan over (Y, N), then the varieties $\widetilde{X}(\mathcal{D})$ for $\mathcal{D} \in \mathscr{E}$ glue together into a $\mathbb{T}$-variety $\widetilde{X}$, which may be described by the following divisorial fan over (Y, N): let $(U_i)_{i \in I}$ be any finite set of open sets of Y that cover Y; then $\widetilde{X}$ is isomorphic to $X(\widetilde{\mathscr{E}})$ where $\widetilde{\mathscr{E}}$ is the divisorial fan generated by $\{\mathcal{D}_{|U_i}\}_{\mathcal{D} \in \mathscr{E},\, i \in I}$.

Definition 3.8. Set $X = X(\mathscr{E})$. The morphism $\pi\colon \widetilde{X} \to X$ obtained by gluing the natural morphisms $\widetilde{X}(\mathcal{D}) \to X(\mathcal{D})$ for $\mathcal{D} \in \mathscr{E}$ is called the *toroidification* (or the *contraction map*) of X.

Note that the toroidification is always proper and birational [AH06, Theorem 3.1 (ii)]. Consider now the *rational quotient* $p\colon X \dashrightarrow Y$ induced by the inclusion $k(Y) = k(X)^{\mathbb{T}} \subset k(X)$. Let $X_0 \subseteq X$ be a Zariski dense open subset in which $p_{|X_0}$ is a morphism. We call *graph* of the rational map p the Zariski closure of the subset

$$\{(x, y) \in X_0 \times Y \mid y = p(x)\} \subset X \times Y.$$

The next result is an application of Zariski Main Theorem.

Proposition 3.9. [Vol10, §3, Lemma 1] *Let X be a normal variety with effective algebraic $\mathbb{T}$-action of complexity one. Assume that X is described by a divisorial fan $\mathscr{E}$ over (Y, N), where Y is a smooth projective curve. Then the total space of the toroidification $\widetilde{X}$ is equivariantly isomorphic to* *the normalization of the graph of the rational quotient* $p : X \dashrightarrow Y$*. Under this identification, the toroidification* $\pi\colon \widetilde{X} \to X$ *is induced by the natural projection on* X *of the graph of* p *and the global quotient* $q\colon \widetilde{X} \to Y$ *by the projection on* Y*.*

3.7. **Exceptional locus of a toroidal refinement.** Let Y be a smooth algebraic curve, $\mathcal{D}$ be a p-divisor over (Y, N) and $\mathscr{E}$ be a divisorial fan over (Y, N) refining $\mathcal{D}$. We assume that $\mathscr{E}$ is toroidal, in other words that for any $\mathcal{D}' \in E$, $\mathrm{Loc}(\mathcal{D}')$ is affine; equivalently, $X(\mathscr{E})$ is isomorphic to $X(\widetilde{\mathscr{E}})$, or the natural morphism $f\colon X(\mathscr{E}) \to X(\mathcal{D})$ factors through the toroidification morphism. Let $\mathscr{E}^{\mathrm{exc}}$ be the set of elements θ of $H\Sigma(\mathscr{E})$ such that $\theta \cap \deg(\mathcal{D}) \neq \varnothing$ or θ is not a hyperface of $HC(\mathcal{D})$.

Proposition 3.10. *The exceptional locus of the proper birational morphism* $f\colon X(\mathscr{E}) \to X(\mathcal{D})$ *is*

$$\mathrm{Exc}(f) = \cup_{\theta \in \mathscr{E}^{\mathrm{exc}}} \mathcal{Z}(\mathscr{E}, \theta).$$

As a particular case, we obtain a description of the exceptional locus of the toroidification morphism.

3.8. **Toric étale charts on the toroidification.** Let Y be a smooth algebraic curve, $y \in Y$ and ϖ_y be a uniformizer of the local ring $\mathcal{O}_{Y,y}$. Then the map

$$\varphi\colon U \to \mathbb{A}^1,\ u \mapsto \varpi_y(u)$$

is an étale morphism for some affine Zariski open subset $U \subset Y$ containing y. If $\varphi^{-1}(0) = \{y\}$, then we say that the pair (U, φ) is an *étale chart* around the point y. The following is a well known fact from the theory of toroidal embeddings [KKMSD73, Chapter 4]. Since the explicit construction of the involved étale morphism will be used later on, we include a short proof.

Lemma 3.11. *Let* $\mathcal{D}$ *be a* p*-divisor over* (Y, N) *and let* $y \in \mathrm{Loc}(\mathcal{D})$*. Let* (U, φ) *be an étale chart around* y *such that* $U \subset \mathrm{Loc}(\mathcal{D})$ *and* $U \cap \mathrm{Supp}(\mathcal{D}) \subset \{y\}$*. Consider the following* p*-divisor over* $(\mathbb{A}^1, N)$ *with locus* U*:*

$$\mathcal{D}_\varphi := \sum_{y \in U} \mathcal{D}_y \cdot [\varphi(y)].$$

Then there exists a $\mathbb{T}$*-equivariant isomorphism* $X(\mathcal{D}_{|U}) \simeq U \times_{\mathbb{A}^1} X(\mathcal{D}_\varphi)$*. Moreover, this isomorphism induces a* $\mathbb{T}$*-equivariant étale morphism between* $X(\mathcal{D}_{|U})$ *and the toric* $\mathbb{G}_m \times \mathbb{T}$*-variety* $X_{C_y(\mathcal{D})}$*.*

Proof. For any $m \in \mathrm{Tail}(\mathcal{D})^\vee$, write $\mathcal{D}(m)_{|U} = a_m \cdot [y]$ for some $a_m \in \mathbb{Q}$. Consider $t \in k[\mathbb{A}^1]$ so that $k[\mathbb{A}^1] = k[t]$ and $\varphi^\star(t) = \varpi_y$. Then, setting $V := \varphi(U)$, observe that

$$k[U] \otimes_{k[t]} \Gamma(V, \mathcal{D}_\varphi(m)) = k[U] \otimes_{k[t]} t^{-\lfloor a_m \rfloor} \cdot k[t] \text{ and } \Gamma(U, \mathcal{D}(m)) = \varpi_y^{-\lfloor a_m \rfloor} \cdot k[U].$$

So the k-algebra morphism

$$\delta : k[U] \otimes_{k[\mathbb{A}^1]} \mathcal{A}(\mathcal{D}_\varphi)(V) \to \mathcal{A}(\mathcal{D})(U),\ f \otimes \gamma \mapsto f \cdot \varphi^\star(\gamma)$$

is an isomorphism, proving that $X(\mathcal{D}_{|U}) \simeq U \times_{\mathbb{A}^1} X(\mathcal{D}_\varphi)$. Finally, after identifying $X(\mathcal{D}_{|U})$ with $U \times_{\mathbb{A}^1} X(\mathcal{D}_\varphi)$, we define an étale morphism by composing the natural projection $X(\mathcal{D}_{|U}) \to X(\mathcal{D}_\varphi)$ with the open immersion $X(\mathcal{D}_\varphi) \hookrightarrow X_{C_y(\mathcal{D})}$. This proves the lemma. □

3.9. **Extension of valuations and étale morphisms.** In this subsection we state and prove some technical lemmas about extensions of valuation along étale morphisms, which will be useful to study the Nash order in Section 5. The only direct connection between the present subsection and the rest of Section 3 is that one lemma involves the étale morphism of Lemma 3.11.

Lemma 3.12. *Let X and Z be affine algebraic k-varieties equipped with the action of an algebraic torus $\mathbb{T}$ and $\theta\colon X \to Z$ be a $\mathbb{T}$-equivariant étale morphism. Let $\mu \in \mathrm{Val}(Z)_{\mathbb{T}}$ and $\nu \in \mathrm{Val}(X)$ such that $\theta^*(\nu) = \mu$. Then ν is $\mathbb{T}$-equivariant.*

Proof. Since θ and $\mu = \theta^*(\nu)$ are $\mathbb{T}$-equivariant, for any $t \in \mathbb{T}(k)$, one has $\theta^*(t \cdot \nu) = \mu$. Since θ is étale, the set $\{t \cdot \nu,\, t \in \mathbb{T}(k)\}$ is thus finite. On the other hand, $\mathbb{T}(k)$ acts transitively on it. Since k is algebraically closed, $\mathbb{T}(k)$ has no subgroup of finite index. Thus one has $\{t \cdot \nu,\, t \in \mathbb{T}(k)\} = \{\nu\}$ and ν is $\mathbb{T}$-equivariant. □

Proposition 3.13. *Keep the notation and assumptions of Lemma 3.11. Denote by θ the $\mathbb{T}$-equivariant étale morphism $X := X(\mathcal{D}_{|U}) \to Z := X_{C_y(\mathcal{D})}$ of Lemma 3.11.*

Let $[y, a, b]$ be an element of $HC(\mathcal{D}) \cap \mathcal{N}$. In particular (a, b) defines an element of the Cayley cone $C_z(\mathcal{D})$. Let $\nu := \mathrm{val}_{[y,a,b]}$ be the induced $\mathbb{T}$-equivariant valuation on X.

1. *The valuation $\mu := \theta^*(\nu)$ is the toric valuation $\mathrm{val}_{a,b}$ on Z induced by (a, b). Moreover, upon shrinking U, ν is the only element of $\mathrm{Val}(X)_{\mathbb{T}}$ such that $\theta^*\nu = \mu$.*
2. *Upon shrinking U, one has $\mathscr{L}_\infty(\theta)^{-1}(\mathscr{L}_\infty(Z)^{\mathrm{ord}=\mu}) = \mathscr{L}_\infty(X)^{\mathrm{ord}=\nu}$, the map $\mathscr{L}_\infty(X)^{\mathrm{ord}=\nu} \to \mathscr{L}_\infty(Z)^{\mathrm{ord}=\mu}$ induced by $\mathscr{L}_\infty(\theta)$ is onto, and maps $\eta_{X,\nu}$ to $\eta_{Z,\mu}$.*

Proof. Identifying $k[Z]$ (resp. $k[X]$) with a subring of $k[\mathbb{Z} \times M]$ (resp. $k(U)[M]$), we may write

$$k[Z] = \bigoplus_{(r,m) \in C_z(\mathcal{D})^\vee \cap \mathbb{Z} \times M} k \cdot t^r \chi^m \quad \text{and} \quad k[X] = \bigoplus_{m \in \sigma^\vee \cap M} H^0(U, \lfloor \mathcal{D}_{|U}(m) \rfloor) \cdot \chi^m.$$

By the very construction, $\theta^*\colon k[Z] \to k[X]$ maps $t^r \chi^m$ to $\varpi_z^r \chi^m$. Now take

$$f = \sum_{(r,m) \in C_y(\mathcal{D})^\vee \cap \mathbb{Z} \times M} \alpha_{r,m} \cdot t^r \chi^m \in k[Z]$$

with $\alpha_{r,m} \in k$ for all r, m. For $m \in M \cap \sigma^\vee$ set

$$\theta^*(f)_m := \sum_{\substack{r\in\mathbb{N} \\ (r,m)\in C_z(\mathcal{D})^\vee \cap \mathbb{Z}\times M}} \alpha_{r,m}\varpi_z^r.$$

Note that $\operatorname{ord}_y(\theta^*(f)_m) = \operatorname{Inf}\{r \in \mathbb{N}, \quad (r,m) \in C_y(\mathcal{D})^\vee \cap \mathbb{Z}\times M \quad \text{and} \quad \alpha_{r,m} \neq 0\}$.

Then $\operatorname{val}_{[y,a,b]}(\theta^* f)$ equals

$$\operatorname*{Inf}_{\substack{m\in M\cap\sigma^\vee \\ \theta^*(f)_m\neq 0}} b\cdot \operatorname{ord}_y(\theta^*(f)_m) + \langle m\,,\, a\rangle$$

which by the above remark equals

$$\operatorname*{Inf}_{\substack{(\nu,m)\in C_y(\mathcal{D})^\vee\cap\mathbb{Z}\times M \\ \alpha_{r,m}\neq 0}} b\cdot r + \langle m\,,\, a\rangle$$

and indeed corresponds to the value at f of the toric valuation μ associated with (a, b) (see Subsection 2.5). This shows the first part of the first assertion.

Now, since there is only a finite number of closed points y' in U such that $\varpi_y(y') = 0$, upon shrinking U, one may assume that for any closed point $y' \neq y$ in U, one has $\operatorname{ord}_{y'}(\varpi_y) = 0$. Let ν' be an element of $\operatorname{Val}(X)_{\mathbb{T}}$ such that $\theta^*(\nu') = \mu$. Write $\nu' = [y', a', b']$ with $y' \in U$ and $(a', b') \in C_{z'}(\mathcal{D}) \cap \mathcal{N}$.

First assume $y' \neq y$. Since $\theta^*(\nu') = \mu$ and $\operatorname{ord}_{y'}(\varpi_y) = 0$, for any element $(m, r) \in C_y(\mathcal{D})^\vee \cap (M \times \mathbb{Z})$ one has

$$b\cdot r + \langle m\,,\, a\rangle = \langle m\,,\, a'\rangle$$

In particular, taking $r = 0$ and $m \in \sigma_M^\vee$, one infers that $a = a'$. Taking (m, r) with $r > 0$, one then obtains $b = 0$. Thus $\nu' = \operatorname{val}_{[y',a,0]} = \operatorname{val}_{[y,a,0]} = \nu$.

In case $y' = y$, one obtains for any element $(m, r) \in C_y(\mathcal{D})^\vee_{M\times\mathbb{Z}}$ the relation

$$b'\cdot r + \langle m\,,\, a\rangle = b\cdot r + \langle m\,,\, a'\rangle$$

which easily gives $a = a'$ and $b = b'$.

Let $\beta \in \mathscr{L}_\infty(Z)$ such that $\operatorname{ord}_\beta = \mu$. Let $\mathfrak{q} \subset k[Z]$ (resp. $\mathfrak{p} \subset k[X]$) be the prime ideal defining the center of μ in Z (resp. of ν in X). Since $\mu = \theta^*(\nu)$, one has $\mathfrak{p} \cap k[Z] = \mathfrak{q}$, thus there exists an extension K of $\kappa(\mathfrak{p})$ such that the K-point of Z defined by $\mathfrak{q}$ and the extension $K/\kappa(\mathfrak{q})$ lifts to $X(K)$. Since $X \to Z$ is étale, one infers that there exists $\alpha \in \mathscr{L}_\infty(X)$ such that $\mathscr{L}_\infty(\theta)(\alpha) = \beta$. Since $\theta^*(\operatorname{ord}_\alpha) = \mu$ and θ is $\mathbb{T}$-equivariant, by Lemma 3.12, $\operatorname{ord}_\alpha$ is $\mathbb{T}$-equivariant. By the first assertion, upon shrinking U, one may conclude that $\operatorname{ord}_\alpha = \nu$.

Let us show that $\mathscr{L}_\infty(\theta)(\eta_{X,\nu}) = \eta_{Z,\mu}$. Let $\alpha \in \mathscr{L}_\infty(X)^{\operatorname{ord}=\nu}$ such that $\mathscr{L}_\infty(\theta)(\alpha) = \eta_{Z,\mu}$. Since α is a specialization of $\eta_{X,\nu}$, $\eta_{Z,\mu}$ is a specialization of $\mathscr{L}_\infty(\theta)(\eta_{X,\nu})$. Since $\operatorname{ord}_{\mathscr{L}_\infty(\theta)(\eta_{X,\nu})} = \mu$, one infers that $\mathscr{L}_\infty(\theta)(\eta_{X,\nu}) = \eta_{Z,\mu}$.

□

Lemma 3.14. *Let X and Z be affine algebraic k-varieties and $\theta\colon X \to Z$ be an étale morphism. Let $\nu \in \mathrm{Val}(X)$ and $\mu := \theta^*(\nu) \in \mathrm{Val}(Z)$. Let $\beta \in \mathscr{L}_\infty(Z)$ such that $\mathrm{ord}_\beta = \mu$. Then there exists $\alpha \in \mathscr{L}_\infty(X)$ such that $\mathscr{L}_\infty(\theta)(\alpha) = \beta$ and ord_α has the same center on X as ν.*

Proof. Let $\mathfrak{p}$ be the prime ideal of $k[X]$ given by $\mathfrak{p} := \mathrm{cent}_X(\nu)$ and let $\mathfrak{q} := k[Z] \cap \mathfrak{p}$. Then $\mathfrak{q} := \mathrm{cent}_X(\mu)$. Let K be the residue field of β and $\beta^*\colon k[Z] \to K[[t]]$ be the induced morphism. Upon enlarging K, one may assume that the natural extension $\kappa(\mathfrak{q}) \to K$ factors through $\kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$ Then $\mathfrak{p}$ and the extension $\kappa(\mathfrak{p}) \to K$ define a K-point of X whose image by f is the K-point of Z defined by $(t \mapsto 0) \circ \beta^*$.

Since $\mathscr{L}_\infty(X) = \mathscr{L}_\infty(Z) \times_Z X$, the above data define a K-arc $\alpha^*\colon k[X] \to K[[t]]$ such that $\alpha^*\theta^* = \beta^*$ and the kernel of $(t \mapsto 0) \circ \alpha^*$ is $\mathfrak{p}$. This defines $\alpha \in \mathscr{L}_\infty(X)$ such that $\mathscr{L}_\infty(\theta)(\alpha) = \beta$ and the center of ord_α on X is $\mathfrak{p}$. ☐

Lemma 3.15. *Let $\theta\colon X \to Z$ be an étale morphism of affine algebraic k-varieties. Let K be an extension of k, and w be a K-wedge on Z, whose special arc lifts to an arc $\alpha \in \mathscr{L}_\infty(X)$. Then there exist an extension L of K and a L-wedge $\widetilde{w}$ on X lifting w and whose special arc is α.*

Proof. Since $\theta\colon X \to Z$ is étale, by [CLNS18, Proposition 3.7.1], for any extension L of k, one has a bijection $\mathscr{L}_\infty(X)(L[[t]]) = \mathscr{L}_\infty(Y)(L[[t]]) \times_{\mathscr{L}_\infty(Y)(L)} \mathscr{L}_\infty(X)(L)$ such that the natural map $\mathscr{L}_\infty(X)(L[[t]]) \to \mathscr{L}_\infty(X)(L)$ sending a L-wedge on X to its L-special arc is induced by the second projection. The result follows. ☐

4. Equivariant resolutions of $\mathbb{T}$-varieties of complexity one

In this section, by studying the equivariant desingularizations of a non-rational variety with complexity-one torus action, we obtain some information on the location of essential valuations on the hypercone.

4.1. **Luna's Slice Theorem.**

We recall a consequence of Luna's Slice Theorem.

Lemma 4.1. [Lun73, Page 98, III.1, Corollaire 2] *Let V be an affine variety with an algebraic action of a reductive group G. Assume that the ring of invariants $k[V]^G$ is k. Then V has a unique closed orbit $G \cdot x$ and one has a homogeneous fiber space decomposition $V = G \times^{G_x} W$, where G_x is the isotropy group at x and W is an affine G_x-variety having a unique closed orbit which is a fixed point y. If further W is smooth at y, then W is G_x-equivariantly isomorphic to T_yW.*

We now fix a smooth projective curve Y and denote by $\rho_g(Y)$ its genus.

Proposition 4.2. *Let $\mathcal{D}$ be a p-divisor over (Y, N) with complete locus. If $X(\mathcal{D})$ is a d-dimensional smooth variety, then $X(\mathcal{D})$ is isomorphic to $\mathbb{G}_m^r \times \mathbb{A}^{d-r}$ for $0 \leqslant r \leqslant d$. In particular, the curve Y is isomorphic to the projective line.*

Proof. In order to have $X(\mathcal{D}) \simeq \mathbb{G}_m^r \times \mathbb{A}^{d-r}$ for $0 \leqslant r \leqslant d$, we need to prove that $X(\mathcal{D})$ is a toric variety. Indeed, by Lemma 4.1 we have a homogeneous fiber space decomposition $X(\mathcal{D}) = \mathbb{T} \times^{\mathbb{T}_x} W$, where $\mathbb{T}_x$ acts with a unique closed orbit which is a fixed point y. Note that the natural projection $\mathbb{T} \times^{\mathbb{T}_x} W \to \mathbb{T}/\mathbb{T}_x$ is a locally trivial fibration for the étale topology. So W is smooth and therefore by *loc. cit.* we may identify W with the $\mathbb{T}_x$-variety T_yW. Let $\mathbb{D}$ be a maximal torus containing the image of $\mathbb{T}_x$ in $\mathrm{GL}(T_yW)$ by the linear action. Set $\mathbb{G} := \mathbb{T} \times \mathbb{D}$. Then the $\mathbb{G}$-action on $\mathbb{T} \times W$ given by $(g,h)\cdot(t,w) = (g\cdot t, h\cdot w)$ for all $g,t \in \mathbb{T}$, $h \in \mathbb{D}$ and $w \in W$ descends to a $\mathbb{G}$-action on $X(\mathcal{D})$ with an open orbit. This implies that $X(\mathcal{D})$ is toric. Finally, since $X(\mathcal{D})$ is a rational variety and the rational quotient $q : X(\mathcal{D}) \dashrightarrow Y$ is dominant, the curve Y is isomorphic to $\mathbb{P}^1$ by Lüroth's Theorem. This proves the proposition. ☐

Corollary 4.3. *Let $\mathscr{E}$ be a divisorial fan over (Y, N). Assume that $X = X(\mathscr{E})$ is smooth and that $\rho_g(Y) > 0$. Then for any $\mathcal{D} \in \mathscr{E}$ the curve* $\mathrm{Loc}(\mathcal{D})$ *is affine.*

Proof. If there is $\mathcal{D} \in \mathscr{E}$ with complete locus, then we would have $Y \simeq \mathbb{P}^1$ from Proposition 4.2, contradicting the assumption $\rho_g(Y) > 0$. ☐

4.2. **Singular locus.** We now fix a smooth projective curve Y of genus $\geqslant 1$. Our goal is to give a combinatorial description of the singular locus of $X(\mathscr{E})$ for $\mathscr{E}$ a divisorial fan over Y.

Definition 4.4. Let $\mathcal{D}$ be a p-divisor with locus a Zariski open dense subset of Y. We say that a hyperface θ of $HC(\mathcal{D})$ is of *orbit type* if θ satisfies Condition (i) of Definition 3.6 or θ satisfies Condition (ii) of *loc. cit.* with the extra property that $\theta \not\subset N_{\mathbb{Q}}$. Note that the associated $\mathbb{T}$-stable closed subset $\mathcal{Z}(\theta)$ is an orbit closure if and only if θ is of orbit type.

We denote by $HC(\mathcal{D})^{\star}_{\mathrm{sing}}$ the set of hyperfaces $\theta \subset HC(\mathcal{D})$ such that θ is a non-smooth cone whenever θ satisfies Condition (ii) of *loc. cit.*. Finally for any divisorial fan $\mathscr{E}$ over Y we set

$$HC(\mathscr{E})^{\star}_{\mathrm{sing}} = \bigcup_{\mathcal{D} \in \mathscr{E}} HC(\mathcal{D})^{\star}_{\mathrm{sing}}.$$

Remark 4.5. Assume that for any $\mathcal{D} \in \mathscr{E}$, $\mathrm{Loc}(\mathcal{D})$ is affine (Recall that in this case only Condition (ii) of Definition 3.6 can be satisfied.) Then $X = X(\mathscr{E})$ is toroidal and using Lemma 3.11 and the classical description of the singular locus of a toric variety, one sees that

$$X^{\mathrm{sing}} = \bigcup_{\theta \in HC(\mathscr{E})^{\star}_{\mathrm{sing}}} \mathcal{Z}(\theta).$$

Note that in this case $\theta \in HC(\mathscr{E})^{\star}_{\mathrm{sing}}$ if and only if θ is a non-smooth face of a Cayley cone of an element of $\mathscr{E}$. Note also that the above holds even if Y is a rational curve.

Proposition 4.6. *Let $\mathscr{E}$ be a divisorial fan over a smooth projective curve Y of genus $\geqslant 1$. Then the singular locus of $X = X(\mathscr{E})$ is given by the formula*

$$X^{sing} = \bigcup_{\theta \in HC(\mathscr{E})^{\star}_{sing}} \mathcal{Z}(\theta).$$

Proof. Let $\pi\colon \widetilde{X} \to X$ be the toroidification and let $\theta \in H\Sigma(\mathscr{E})$ be of orbit type. Consider the open orbit $O(\theta) \subset \mathcal{Z}(\theta)$. If θ satisfies Condition (ii) of Definition 3.6, then $O(\theta)$ identifies with an orbit of $\widetilde{X} \setminus E$, where E is the exceptional locus of π. By Remark 4.5, we deduce that $O(\theta) \subset X^{\rm sing}$ if and only if θ is a non-smooth cone. Now assume that θ satisfies Condition (i) of *loc. cit.* Then $\theta = HC(\mathcal{D}')$ for some p-divisor $\mathcal{D}'$ with locus Y. It follows, for instance, from [IS11, Lemma 1.4], that $X(\mathcal{D}')$ is an open subset of X. Moreover, the hyperface-orbit relations imply that $O(\theta)$ is the unique closed orbit of $X(\mathcal{D}')$. But Luna's Slice Theorem (see Proposition 4.2), Lurӧth's Theorem and the fact that the genus of Y is $\geqslant 1$ yield that $O(\theta) \subset X^{\rm sing}$, which establishes our statement. □

4.3. **Equivariant resolutions of singularities and toroidification.** We keep the same notation as in 4.1. Strictly speaking, the next proposition is not needed in order to establish the main result of this section. It has however its own interest, and explains in some sense why the set of essential valuations of a non-rational $\mathbb{T}$-variety of complexity one is rather close to the set of essential valuations of its toroidification. The situation is dramatically different in the rational case (see Section 8).

Proposition 4.7. *Let $\mathscr{E}$ be a divisorial fan over (Y, N) and let $X = X(\mathscr{E})$ be the associated $\mathbb{T}$-variety. Assume that $\rho_g(Y) > 0$. Let $\pi\colon \widetilde{X} \to X$ be the toroidification. Consider a smooth variety X' with an algebraic $\mathbb{T}$-action. Then for any $\mathbb{T}$-equivariant proper birational morphism $\psi\colon X' \to X$, there is a $\mathbb{T}$-equivariant proper birational morphism $\varsigma\colon X' \to \widetilde{X}$ such that $\psi = \pi\circ\varsigma$,*

Proof. Assume that there is a $\mathbb{T}$-equivariant proper birational morphism $\psi : X' \to X$. Note that this implies that the $\mathbb{T}$-action on X' is of complexity one. Let $\mathscr{E}'$ be a divisorial fan over (Y', N) such that $X' = X(\mathscr{E}')$, where Y' is a smooth projective curve. Since $\psi^{\star}$ maps isomorphically $k(X)^{\mathbb{T}} = k(Y)$ to $k(X')^{\mathbb{T}} = k(Y')$ we see that ψ induces an isomorphism $\gamma : Y' \to Y$. In particular, $\rho_g(Y') > 0$ and by Corollary 4.3 any element of $\mathscr{E}'$ has affine locus. So the natural invariant maps $X(\mathcal{D}) \to Y'$ for $\mathcal{D} \in \mathscr{E}'$ glue together into a global quotient $q' : X' \to Y'$. Set $q_0 = \gamma \circ q'$. Let $\mathcal{G} \subset X \times Y$ be the graph of the rational quotient $q : X \dashrightarrow Y$. Since $q \circ \psi = q_0$, the morphism

$$s : X' \to \mathcal{G},\ x \mapsto (\psi(x), q_0(x))$$

is well defined, $\mathbb{T}$-equivariant and birational (as ψ is birational). The composition of s with the projection from $\mathcal{G}$ to X is the proper morphism ψ. So s is proper. The universal property of the normalization applied to s and proposition 3.9 then give the existence of a $\mathbb{T}$-equivariant proper birational morphism $\varsigma : X' \to \widetilde{X}$ satisfying $\psi = \pi \circ \varsigma$. □

4.4. **Smooth refinements with respect to a polyhedron.** This section contains the technical tools needed to establish the analog of Proposition 2.25 (which locates the essential valuations in the toric case) in the case of $X(\mathcal{D})$ where $\mathcal{D}$ is a polyhedral divisor over a smooth projective curve of positive genus. Proposition 2.25 itself is a crucial ingredient, but more work is needed, in particular due to the fact that there may exist non-Nash valuations on X which become Nash on the toroidification and to which Proposition 2.25, in some sense, no longer applies (see Remark 4.14).

In the whole subsection, unless otherwise specified, we consider the following setting: let N be a lattice, $\mathcal{C} \subset N_{\mathbb{Q}}$ be a strictly convex polyhedral cone in $N_{\mathbb{Q}}$, σ be a face of $\mathcal{C}$ and $\mathcal{P} \subset \sigma$ be a σ-tailed polyhedron. Being given a fan Σ refining $\mathcal{C}$, the set of cones of Σ which are contained in σ is a fan refining σ, called the fan induced by Σ on σ and denoted by $\Sigma_{|\sigma}$.

Notation 4.8. Being given a fan Σ and ν an element of the support of Σ, we denote by $\Sigma(\nu)$ the unique cone of Σ whose relative interior contains ν.

One may compare the following definition with definition 2.23.

Definition 4.9. Let Σ be a smooth fan refining $\mathcal{C}$.

(1) We say that Σ is a *$\mathcal{P}$-economical refinement* of $\mathcal{C}$ if any smooth face of $\mathcal{C}$ which does not meet $\mathcal{P}$ is a cone of Σ.
(2) We say that Σ is a *$\mathcal{P}$-big refinement* of $\mathcal{C}$ if:
  - any cone of Σ which is not a face of $\mathcal{C}$ and does not meet $\mathcal{P}$ has a ray which is not a ray of $\mathcal{C}$;
  - any cone of Σ which meets $\mathcal{P}$ has a ray which meets $\mathcal{P}$ or is not a ray of $\mathcal{C}$.

Lemma 4.10. *Let Σ be a smooth refinement of $\mathcal{C}$. Assume that the fan $\Sigma_{|\sigma}$ induced by Σ on σ is a big (resp. $\mathcal{P}$-big) refinement of σ. Then there exists a refinement Σ' of Σ which is a smooth big (resp. $\mathcal{P}$-big) refinement of Σ. If moreover Σ is a smooth economical (resp. $\mathcal{P}$-economical) refinement of $\mathcal{C}$, then Σ' may be chosen as a smooth economical (resp. $\mathcal{P}$-economical) refinement of $\mathcal{C}$.*

Proof. Let us call special cone of Σ any cone τ of Σ which is not a face of $\mathcal{C}$ and such that every ray of τ is a ray of $\mathcal{C}$. If $\Sigma_{|\sigma}$ is a big refinement of σ, any special cone τ of Σ is not contained in σ. Thus the sum n_τ of the primitive generators of the ray of τ is not a ray of $\mathcal{C}$ and does not belong to σ. Therefore, the star subdivision $\mathrm{St}(\Sigma, n_\tau)$ of Σ with respect to n_τ is smooth, has strictly less special cones than Σ, and satisfies $\mathrm{St}(\Sigma, n_\tau)_{|\sigma} = \Sigma_{|\sigma}$. Moreover any smooth face of $\mathcal{C}$ which is a cone of Σ cannot have τ as a face, and is thus a cone of $\mathrm{St}(\Sigma, n_\tau)$. Thus we obtain a fan Σ' as in the statement after a finite number of such star subdivisions.

Let us now call $\mathcal{P}$-special cone of Σ any cone τ of Σ which is either a special cone of Σ which does not meet $\mathcal{P}$, or meets $\mathcal{P}$ and any ray of τ is either a ray of $\mathcal{C}$ or meets $\mathcal{P}$; note that in the latter case $\tau \cap \sigma$ has the same properties. Thus if $\Sigma_{|\sigma}$ is a $\mathcal{P}$-big refinement of σ, any $\mathcal{P}$-special cone τ of Σ is a special cone of Σ which is not contained in σ and does not meet $\mathcal{P}$. In particular, with the same notation as before, n_τ is not a ray of $\mathcal{C}$ and does not meet $\mathcal{P}$, and $\mathrm{St}(\Sigma, n_\tau)$ is a smooth refinement of Σ that has strictly less $\mathcal{P}$-special cones than Σ, and satisfies $\mathrm{St}(\Sigma, n_\tau)_{|\sigma} = \Sigma_{|\sigma}$. Moreover any smooth face of $\mathcal{C}$ which does not meet $\mathcal{P}$ and is a cone of Σ cannot have τ as a face, and is thus a cone of $\mathrm{St}(\Sigma, n_\tau)$. We conclude as before. □

Lemma 4.11. *Let Σ be a smooth $\mathcal{P}$-economical refinement of $\mathcal{C}$. Let $\nu \in \mathcal{C} \cap N$ and τ be the unique face of $\mathcal{C}$ whose relative interior contains ν.*

(1) *Assume that any cone of Σ which is not a face of $\mathcal{C}$ and does not meet $\mathcal{P}$ has a ray which is not a ray of $\mathcal{C}$.*
   (a) *Then there exists a smooth star refinement Σ' of Σ which is a $\mathcal{P}$-economical and $\mathcal{P}$-big refinement of $\mathcal{C}$, and such that every smooth cone of Σ which does not meet $\mathcal{P}$ is a cone of Σ'.*
   (b) *Assume moreover that* $\dim(\tau) \geqslant 2$, *and τ has a ray which meets $\mathcal{P}$ or is not a ray of $\mathcal{C}$. Then one has the same conclusion as in* (1a), *with the additional condition that the cone $\Sigma'(\nu)$ (see notation 4.8) has dimension $\geqslant 2$ and meets $\mathcal{P}$.*

(2) *Assume that* $\dim(\tau) \geqslant 2$, *τ has a ray which meets $\mathcal{P}$, and every face of τ is either a face of $\mathcal{C}$ or meets $\mathcal{P}$. Then one has the same conclusion as in* (1a), *with the additional condition that the cone $\Sigma'(\nu)$ has dimension $\geqslant 2$ and meets $\mathcal{P}$.*

Proof. Let us call 1-special cone of Σ any cone τ' of Σ which is not a face of $\mathcal{C}$, does not meet $\mathcal{P}$ and such that every ray of τ' is a ray of $\mathcal{C}$, and 2-special cone of Σ any cone τ' of Σ which meets $\mathcal{P}$ and such that no ray of τ' meets $\mathcal{P}$. In particular, if Σ has no 1-special nor 2-special cone, Σ is a big $\mathcal{P}$-refinement of $\mathcal{C}$.

Let us show (1a). By assumption, Σ is a $\mathcal{P}$-economical refinement of $\mathcal{C}$ and has no 1-special cone. Assume that Σ has a 2-special cone, and let τ' be a minimal 2-special cone of Σ. Consider the fan $\mathrm{St}(\Sigma, n_{\tau'})$ obtained from Σ by the star subdivision with respect to the sum $n_{\tau'}$ of the primitive generators of the rays of τ'.

By the minimality of τ', one has $\mathbb{Q}_{\geqslant 0} n_{\tau'} \cap \mathcal{P} \neq \varnothing$. Since $\mathbb{Q}_{\geqslant 0} n_{\tau'}$ is a ray of any cone of $\mathrm{St}(\Sigma, n_{\tau'}) \setminus \Sigma$ and $\tau' \notin \mathrm{St}(\Sigma, n_{\tau'})$, $\mathrm{St}(\Sigma, n_{\tau'})$ has strictly less 2-special cones than Σ, and has no 1-special cone. Moreover any cone of Σ which does not meet $\mathcal{P}$ cannot have τ' as a face, and is therefore an element of $\mathrm{St}(\Sigma, n_{\tau'})$.

Therefore, after a finite number of applications of such star subdivisions, we end up with a fan Σ' which is a smooth $\mathcal{P}$-economical and $\mathcal{P}$-big refinement of $\mathcal{C}$, and such that any cone Σ which does not meet $\mathcal{P}$ is a cone of Σ'.

Let us show (1b).

First assume that τ has a ray which meets $\mathcal{P}$. Let τ'' be the cone of the above described fan $\mathrm{St}(\Sigma, n_{\tau'})$ such that $\nu \in \mathrm{Relint}(\tau'')$. It suffices to show that τ'' has dimension $\geqslant 2$ and has a ray which meets $\mathcal{P}$. This is clear if τ' is not a face of τ, since in this case $\tau \in \mathrm{St}(\Sigma, n_{\tau'})$. Assume now that τ' is a face of τ. Since no ray of τ' meets $\mathcal{P}$, τ' is a proper face of τ. Therefore, since $\nu \in \mathrm{Relint}(\tau)$, $n_{\tau'}$ is a ray of τ'' and $\dim(\tau'') \geqslant 2$.

Assume now that τ has a ray ρ which is not a ray of $\mathcal{C}$. By the procedure described in the proof of (1a), one may assume that any minimal 2-special cone of Σ is a face of τ. If each of these cones has a ray which is not a ray of $\mathcal{C}$, then any 2-special cone of Σ has a ray which is not a ray of $\mathcal{C}$, thus Σ is a big $\mathcal{P}$-refinement of $\mathcal{C}$ and we are done. Otherwise, there exists a face τ' of τ which is a minimal 2-special cone of Σ and ρ is not a ray of τ'. Thus, with the same notation as before, one has $\dim(\tau'') \geqslant 2$ and ρ is a ray of τ'' which is not a ray of $\mathcal{C}$.

Let us now show (2). By (1b), it suffices to construct, using star subdivisions, a smooth refinement Σ_1 of Σ which has no 1-special 1-cone, is a $\mathcal{P}$-economical refinement of $\mathcal{C}$, and contains τ.

Let τ' be a 1-special cone of Σ and consider the fan $\mathrm{St}(\Sigma, n_{\tau'})$ obtained from Σ by the star subdivision with respect to the sum $n_{\tau'}$ of the primitive generators of the rays of τ'. Since $\dim(\tau') \geqslant 2$, $\mathbb{Q}_{\geqslant 0} n_{\tau'}$ is not a ray of $\mathcal{C}$. Since $\tau' \notin \Sigma_1$ and $\mathbb{Q}_{\geqslant 0} n_{\tau'}$ is a ray of any element of $\Sigma_1 \setminus \Sigma$, Σ_1 has strictly less 1-special cones than Σ. On the other hand, by the assumptions on τ and τ', τ' can not be a face of τ, thus $\tau \in \Sigma_1$. Similarly, any smooth face of $\mathcal{C}$ which does not intersect $\mathcal{P}$ is a cone of $\mathrm{St}(\Sigma, n_{\tau'})$. Therefore, after a finite number of applications of such star subdivisions, we end up with a fan Σ_1 with the desired properties. □

Lemma 4.12. *Let N be a two-dimensional lattice, γ be a polyhedral strictly convex full-dimensional cone of $N_{\mathbb{Q}}$, ν_0 and ν_1 be the primitive generators of the rays of γ. Assume that γ is* not *smooth. Let $\nu \in \mathrm{Min}(\mathrm{Relint}(\gamma), \leqslant_\gamma)$. Then $\nu \notin \nu_0 + \gamma$.*

Proof. One may find a a $\mathbb{Z}$-basis (e_1, e_2) of N such that $\nu_0 = e_2$ and $\nu_1 = de_1 - ke_2$, d, k are coprime positive integers with $k < d$ and $\gcd(d, k) = 1$. Thus $\gamma^\vee$ is generated by $e_1^\vee$ and $ke_1^\vee + de_2^\vee$ Hereafter, we use the description of the minimal resolution of singularities of a toric surface in terms of Hirzebruch-Jung continued fractions and the notation of [CLS11, Chapter 10.2]. Any element of $\mathrm{Min}(\mathrm{Relint}(\gamma), \leqslant_\gamma)$ may be written as $u_i = P_i e_1 - Q_i e_2$, $i \geqslant 1$. In order to show that $u_i \notin \nu_0 + \gamma$, it suffices to show that $kP_i - (d+1)Q_i < 0$. For any i, one checks that $k_i Q_{i+1} - k_{i+1} Q_i = k$. Since $\frac{P_i}{Q_i} - \frac{P_{i+1}}{Q_{i+1}} = \frac{1}{Q_i Q_{i+1}}$, one infers that $\frac{P_i}{Q_i} = \frac{d}{k} + \frac{k_i}{kQ_i}$ thus

$$\frac{P_i}{Q_i} = \frac{d}{k} + \frac{d}{kQ_i} + \frac{k_i - d}{kQ_i}$$

Since $k_i < k < d$, this shows that $kP_i - (d+1)Q_i < 0$. □

Proposition 4.13. *Let N be a lattice, $\mathcal{C} \subset N_{\mathbb{Q}}$ be a strictly convex polyhedral cone in $N_{\mathbb{Q}}$, σ be a face of $\mathcal{C}$ and $\mathcal{P} \subset \sigma$ be a σ-tailed polyhedron. Recall that we denote by $\mathcal{C}_{sing}$ the union of the relative interiors of the non-smooth faces of $\mathcal{C}$.*

Let $\mathcal{C}^\star$ be the reunion of $\mathcal{C}_{sing}$ with the union of the relative interiors of the faces of $\mathcal{C}$ which intersect $\mathcal{P}$.

Let ν be a primitive element of $(\mathcal{C}^\star \cap N) \setminus \operatorname{Min}(\mathcal{C}^\star \cap N, \leqslant_{\mathcal{C}})$

Then there exists a smooth fan Σ which is a smooth $\mathcal{P}$-economical and $\mathcal{P}$-big star refinement of $\mathcal{C}$ and such that $\Sigma(\nu)$ has dimension $\geqslant 2$ and either meets $\mathcal{P}$ or is not a cone of $\mathcal{C}$.

Remark 4.14. In the proof, we shall consider separately the three following cases:

(1) $\nu \in (\mathcal{C}_{\mathrm{sing}} \cap N) \setminus \operatorname{Min}(\mathcal{C}_{\mathrm{sing}} \cap N, \leqslant_{\mathcal{C}})$;
(2) $\nu \in (\mathcal{C}^\star \cap N) \setminus \mathcal{C}_{\mathrm{sing}}$;
(3) $\nu \in \operatorname{Min}(\mathcal{C}_{\mathrm{sing}} \cap N, \leqslant_{\mathcal{C}})$.

In the first case, one essentially reduces easily to the toric case, *i.e.* to Proposition 2.25 with the aid of the previous lemmas. The third case is the most challenging. It corresponds geometrically to the case of a p-divisor $\mathcal{D}$ with locus a smooth projective curve of positive genus such that there are Nash valuations on the toroidification $\widetilde{X}$ of $X(\mathcal{D})$ which are not Nash valuations on $X(\mathcal{D})$ (see Section 5). See Section 7.2 below for an explicit example.

Proof. First assume that $\nu \in (\mathcal{C}_{\mathrm{sing}} \cap N) \setminus \operatorname{Min}(\mathcal{C}_{\mathrm{sing}} \cap N, \leqslant_{\mathcal{C}})$. By Proposition 2.25, there exists a star refinement Σ of $\mathcal{C}$ such that:

- Σ is a smooth economical and big refinement of $\mathcal{C}$;
- the cone τ of Σ such that $\nu \in \operatorname{Relint}(\tau)$ is not a face of $\mathcal{C}$ and has dimension $\geqslant 2$.

Since Σ is a smooth economical and big refinement of $\mathcal{C}$, τ has a ray which is not a ray of $\mathcal{C}$, and we may apply lemma 4.11(1a) in case $\tau \cap \mathcal{P} = \varnothing$, and lemma 4.11(1b). in case $\tau \cap \mathcal{P} \neq \varnothing$.

Assume now that $\nu \in (\mathcal{C}^\star \cap N) \setminus \mathcal{C}_{\mathrm{sing}}$. Let τ be the unique face of $\mathcal{C}$ whose relative interior contains $\mathcal{C}$. Thus τ is a smooth face of $\mathcal{C}$ which meets $\mathcal{P}$. Since $\nu \notin \operatorname{Min}(\mathcal{C}^\star \cap N, \leqslant_{\mathcal{C}})$, one has $\dim(\tau) \geqslant 2$. Let τ' be a minimal face of τ meeting $\mathcal{P}$ and $n_{\tau'}$ be the sum of the primitive elements generating the rays of τ'. In particular $\mathbb{Q}_{\geqslant 0} n_{\tau'} \cap \mathcal{P} \neq \varnothing$ and since $\nu \notin \operatorname{Min}(\mathcal{C}^\star \cap N, \leqslant_{\mathcal{C}})$, one has $n_{\tau'} \neq \nu$. Let Σ_1 be the fan obtained from $\mathcal{C}$ by the star subdivision with respect to $n_{\tau'}$. Then any smooth cone of $\mathcal{C}$ which does not meet $\mathcal{P}$ is a cone of Σ_1. Moreover, $\Sigma_1(\nu)$ is a smooth cone of Σ_1 containing ν in its relative interior, has $\mathbb{Q}_{\geqslant 0} n_{\tau'}$ as a ray, and such that any face of $\Sigma_1(\nu)$ is either a face of $\mathcal{C}$ or contains $\mathbb{Q}_{\geqslant 0} n_{\tau'}$, thus meets $\mathcal{P}$. Let Σ be a smooth economical star refinement of Σ_1. In particular, Σ is a smooth $\mathcal{P}$-economical refinement of $\mathcal{C}$. Now one may apply lemma 4.11(2).

We now assume that $\nu \in \mathrm{Min}(\mathcal{C}_{\mathrm{sing}} \cap N, \leqslant_{\mathcal{C}})$. We are going to show that there exists a star subdivision of $\mathcal{C}$ such that the resulting fan Σ_0 is such that the cone $\Sigma_0(\nu)$ is smooth, has dimension $\geqslant 2$ and each of its face is either a face of $\mathcal{C}$ or meets $\mathcal{P}$. Take this for granted for the moment. Then, by Remark 2.26, there exists a fan Σ which is a smooth economical star refinement of Σ_0. In particular, the cone $\Sigma_0(\nu)$ and any smooth face of $\mathcal{C}$ which does not meet $\mathcal{P}$ are elements of Σ and one may apply lemma 4.11(2) again.

Let us now explain the construction of Σ_0. Let $\nu_0 \in \mathrm{Min}(\mathcal{C}^\star \cap N, \leqslant_{\mathcal{C}})$ such that $\nu_0 \leqslant_{\mathcal{C}} \nu$ and τ_0 be the face of $\mathcal{C}$ containing ν_0 in its relative interior. By assumption, ν_0 does not belong to $\mathcal{C}_{\mathrm{sing}} \cap N$. Thus τ_0 is a smooth face of σ which intersects $\mathcal{D}$. Since $\nu_0 \in \mathrm{Min}(\mathcal{C}^\star \cap N, \leqslant_{\mathcal{C}})$ and τ_0 is smooth, ν_0 is the sum of the primitive elements generating the rays of τ_0.

Let P be the vector plane generated by ν_0 and ν, and γ be the two-dimensional cone $\mathcal{C} \cap P$. Since τ_0 is a face of $\mathcal{C}$ not containing ν, $\tau_0 \cap P$ is a proper face of γ, thus ν_0 generates one of the rays of γ. Moreover, since $\nu_0 \leqslant_{\mathcal{C}} \nu$, one has $\nu_0 \leqslant_\gamma \nu$. In particular, ν lies in the relative interior of γ; otherwise ν and ν_0 would be collinear, but this would contradict $\nu \notin \tau_0$).

Note that any element ν' of $\mathrm{Relint}(\gamma) \cap N$ lies in $\mathcal{C}_{\mathrm{sing}} \cap N$. Otherwise, ν' would belong to a smooth face τ' of $\mathcal{C}$ such that $\tau' \cap P = \gamma$, but since $\nu \in \gamma \cap \mathcal{C}_{\mathrm{sing}} \cap N$, one has a contradiction.

Since $\nu \in \mathrm{Min}(\mathcal{C}_{\mathrm{sing}} \cap N, \leqslant_{\mathcal{C}})$ we infer that $\nu \in \mathrm{Min}(\mathrm{Relint}(\gamma), \leqslant_\gamma)$. Since $\nu_0 \leqslant_\gamma \nu$, lemma 4.12 shows that γ is a $P \cap N$-smooth cone, thus (since $N \cap P$ is a saturated submodule of N), also a N-smooth cone. Let ν_1 be the primitive generator of the other ray of γ. Note that since $\tau_0 \cap P = \mathbb{Q}_{\geqslant 0}\nu_0$, ν_1 does not belong to the vector space generated by τ_0.

Note also that since $\nu \in \mathrm{Min}(\mathrm{Relint}(\gamma), \leqslant_\gamma)$, and γ is smooth, one has $\nu = \nu_0 + \nu_1$. Let τ_1 be the face of $\mathcal{C}$ whose relative interior contains ν_1. Since $\nu_1 \leqslant_{\mathcal{C}} \nu$ and $\nu \in \mathrm{Min}(\mathcal{C}_{\mathrm{sing}} \cap N, \leqslant_{\mathcal{C}})$ τ_1 is smooth. Note that ν_0 does not belong to the vector space $\mathrm{Span}(tau_1)$ generated by τ_1. Otherwise, one would have $\nu_0 \in \mathrm{Span}(\tau_1) \cap \mathcal{C} = \tau_1$ thus $\nu \in \tau_1$, which would contradict $\nu \in \mathcal{C}_{\mathrm{sing}} \cap N$.

Let us show that the cone τ generated by τ_1 and ν_0 is smooth. Note that this is a simplicial cone, containing ν in its relative interior. Let η be the face of $\mathcal{C}$ containing ν in its relative interior. In particular η is not a smooth cone. Note that τ_1 and τ_0 are faces of the cone η, which therefore contains τ. Since $\mathrm{Relint}(\tau) \cap \mathrm{Relint}(\eta) \neq \varnothing$, one has $\mathrm{Relint}(\tau) \subset \mathrm{Relint}(\eta)$.

Denote by $e_1, \dots, e_r$ the primitive generators of the rays of τ_1 and assume that τ is not smooth. Then by [CLS11, Proposition 11.1.8] there exists $(\lambda_i)_{0 \leqslant i \leqslant r} \in]0,1] \in \mathbb{Q}^{r+1}$ such that

$$\nu_3 := \lambda_0 \nu_0 + \sum_{i=1}^{r} \lambda_i e_i \in \tau \cap N$$

and at least one λ_i is < 1. Since $\nu = \nu_0 + \nu_1 = \nu_0 + \sum_{i=1}^r \lambda_i' e_i$ with $\lambda_i' \geqslant 1$ for every i, one deduces that $\nu_3 \leqslant_{\mathcal{C}} \nu$ and $\nu_3 \neq \nu$. But $\nu_3 \in \mathrm{Relint}(\tau)$, thus $\nu_3 \in \mathrm{Relint}(\gamma)$. This contradicts the fact that $\nu \in \mathrm{Min}(\mathrm{Relint}(\gamma), \leqslant_\gamma)$. Thus τ is smooth.

Now let Σ_0 be the star subdivision of $\mathcal{C}$ with respect to ν_0. Note that τ is a smooth cone of Σ_0, containing ν in its relative interior, and $\mathbb{Q}_{\leqslant 0}\nu_0$ is a ray of τ which intersects $\mathcal{P}$. Moreover, since $\mathbb{Q}_{\leqslant}\nu_0 \cap \mathcal{P} \neq \varnothing$, any face of $\mathcal{C}$ which does not intersect $\mathcal{P}$ does not contain ν_0, thus is a cone of Σ_0.

On the other hand, by the construction of τ, each face of τ is either a face of τ_0, thus a face of $\mathcal{C}$, or contains ν_0, thus intersects $\mathcal{P}$.

□

4.5. **Location of essential valuations.** In the whole subsection, unless otherwise specified, we consider the following setting and notation. Let $\mathcal{D}$ be a σ-tailed p-divisor over (Y, N) where Y is smooth projective curve. Let $\{y_1, \dots, y_r\}$ be a finite set of points of Y such that $\mathrm{Supp}(\mathcal{D}) \subset \{y_1, \dots, y_r\}$. For $1 \leqslant i \leqslant r$, set $U_i := \mathrm{Loc}(\mathcal{D}) \setminus \{y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_r\}$. Let $\widetilde{\mathcal{D}}$ be the toroidal divisorial fan over (Y, N) generated by the p-divisors $\{\mathcal{D}_{|U_i}\}_{1 \leqslant i \leqslant r}$. In particular $X(\widetilde{\mathcal{D}})$ is the toroidification $\widetilde{X}$ of $X := X(\mathcal{D})$. Note that for any $1 \leqslant i \leqslant r$, if Σ is a smooth (resp. smooth economical, resp. big) refinement of $C_{y_i}(\mathcal{D})$, then $\Sigma_{|\sigma}$ is a smooth (resp. smooth economical, resp. big) refinement of σ.

Lemma 4.15. *Let $\Sigma_1, \dots, \Sigma_r$ be fans refining respectively $C_{y_1}(\mathcal{D}), \dots, C_{y_r}(\mathcal{D})$ and inducing the same fan $\Sigma(\sigma)$ on the tail σ.*

1. *There exists a toroidal divisorial fan $\mathscr{E} = \mathscr{E}(\Sigma_1, \dots, \Sigma_r)$ over (Y, N) which is a refinement of $\widetilde{\mathcal{D}}$ and induces the fan Σ_i on $C_{y_i}(\mathcal{D})$ for every $1 \leqslant i \leqslant r$. We denote by $f\colon X(\mathscr{E}) \to \widetilde{X}$ the induced proper birational equivariant morphism.*
2. *Consider the following sets of cones of $H\Sigma(\mathscr{E})$, ordered by inclusion:*

$$\Sigma_{i,\mathrm{exc}}(\widetilde{X}) = \{\tau \in \Sigma_i, \quad \tau \not\subset C_{y_i}(\mathcal{D})\}, \quad 1 \leqslant i \leqslant r,$$

$$\mathscr{E}_{\mathrm{exc}}(\widetilde{X}) := \cup_{i=1}^r \Sigma_{i,\mathrm{exc}}(\widetilde{X})$$

$$\mathscr{E}_{\mathrm{exc}}(X) := \mathscr{E}_{\mathrm{exc}}(\widetilde{X}) \cup \{\tau \in \Sigma(\sigma), \, \tau \cap \deg(\mathcal{D}) \neq \varnothing\}$$

   Let ν be a primitive element of $HC(\mathcal{D}) \cap \mathcal{N}$ and $\theta(\mathscr{E}, \nu)$ be the unique cone of $H\Sigma(\mathscr{E})$ whose relative interior contains ν. Then ν is f-exceptional (resp. $\pi \circ f$ exceptional) if and only if $\theta(\mathscr{E}, \nu)$ is a minimal element of $\mathscr{E}_{\mathrm{exc}}(\widetilde{X})$ (resp. of $\mathscr{E}_{\mathrm{exc}}(X)$))
3. *In particular, if each Σ_i is a smooth economical refinement of $C_{y_i}(\mathcal{D})$, then f is an equivariant resolution of singularities of $\widetilde{X}$. If in addition each Σ_i is a big refinement of $C_{y_i}(\mathcal{D})$, then f is a divisorial equivariant resolution of singularities of $\widetilde{X}$.*

(4) *Assume that* $\mathrm{Loc}(\mathcal{D}) = Y$ *and* Y *is a smooth projective curve of positive genus. If each* Σ_i *is a smooth* $\deg(\mathcal{D})$*-economical refinement of* $C_{y_i}(\mathcal{D})$*, then* $\pi \circ f$ *is an equivariant resolution of singularities of* X*. If in addition each* Σ_i *is a* $\deg(\mathcal{D})$*-big refinement of* $C_{y_i}(\mathcal{D})$*, then* $\pi \circ f$ *is a divisorial equivariant resolution of singularities of* X*.*

Proof. Let $(\tau_j)_{j\in J}$ be the maximal cones of the fan with support σ induced by the Σ_i's. For $1 \leqslant i \leqslant r$ and $j \in J$, let $\Sigma_i^{(j)}$ be the set of cones $\gamma \in \Sigma_i$ such that $\gamma \not\subset \sigma$ and τ_j is a face of γ. For any such cone γ, set

$$\gamma_{y_i} := \gamma \cap \{[y_i, a, 1]\}_{a\in N_{\mathbb{Q}}}.$$

Then one can take for $\mathscr{E}$ the divisorial fan generated by the following family of p-divisors:

$$\gamma_{y_i} \cdot [y_i] + \sum_{\substack{z\in U_i \\ z\neq y_i}} \tau_j \cdot [z], \quad 1 \leqslant i \leqslant r,\ j \in J,\ \gamma \in \Sigma_i^{(j)}.$$

The remainder of the proposition is a consequence of §3.5 and propositions 3.10 and 4.6. □

Lemma 4.16. *Let* Σ_1 *be a smooth fan which is a star refinement of* $C_{y_1}(\mathcal{D})$*.*

(1) *There exist smooth fans* $\Sigma_2, \ldots, \Sigma_r$ *refining* $C_{y_2}(\mathcal{D}), \ldots, C_{y_r}(\mathcal{D})$ *respectively and such that for* $2 \leqslant i \leqslant r$*, one has* $(\Sigma_i)_{|\sigma} = (\Sigma_1)_{|\sigma}$*.*
(2) *Assume that* Σ_1 *is a big (resp. economical, resp. big and economical) refinement of* $C_{y_1}(\mathcal{D})$*. Then for any* $2 \leqslant i \leqslant r$*,* Σ_i *may be chosen as a big (resp. economical, resp. big and economical) refinement of* $C_{y_i}(\mathcal{D})$*.*
(3) *Same statement as before with "economical" and "big" replaced respectively with "*$\deg(\mathcal{D})$*-economical" and "*$\deg(\mathcal{D})$*-big".*

Proof. Set $\Sigma(\sigma) := (\Sigma_1)_{|\sigma}$.

Note that if Σ_1 is a big (resp. smooth economical) refinement of $C_{y_1}(\mathcal{D})$, then $\Sigma(\sigma)$ is a a big (resp. smooth economical) refinement of σ.

Let $n_1, \ldots, n_s$ be a finite sequence of elements of $C_{y_1}(\mathcal{D})$ such that Σ_1 is obtained by applying successive star subdivisions at $n_1, \ldots, n_s$.

For $r \geqslant i \geqslant 2$, consider the fan Σ'_i refining $C_{y_i}(\mathcal{D})$ obtained from $C_{y_i}(\mathcal{D})$ by applying the following successive operations for $j \in \{1, \ldots, s\}$: if $n_j \in \sigma$, apply the star subdivision at n_j; if $n_j \notin \sigma$, do nothing.

By construction, the fan induced by Σ'_i on σ is $\Sigma(\sigma)$. If Σ'_i is not smooth, by Remark 2.26, there exist a smooth economical refinement Σ_i of Σ'_i. Since every cone of $\Sigma(\sigma)$ is smooth, the fan induced by Σ_i on σ is $\Sigma(\sigma)$.

Assume that Σ_1 is a smooth economical refinement of $C_{y_1}(\mathcal{D})$. Let τ be a smooth face of $C_{y_i}(\mathcal{D})$ Then, for any $j \in \{1, \ldots, s\}$ such that $n_j \in \sigma$, one has $n_j \notin \tau$; otherwise, n_j would lie in the relative interior of a common smooth face τ' of σ and τ, and τ' would not be a cone of Σ_1, contradicting the fact that Σ_1 is a smooth economical refinement of $C_{y_1}(\mathcal{D})$. Thus, by the construction of Σ'_i, one has $\tau \in \Sigma'_i$, thus $\tau \in \Sigma_i$ and Σ_i is a smooth economical refinement of $C_{y_i}(\mathcal{D})$. Likewise, if Σ_1 is a smooth $\deg(\mathcal{P})$-economical refinement of $C_{y_1}(\mathcal{D})$, we may conclude that Σ_i is a smooth $\deg(\mathcal{P})$-economical refinement of $C_{y_i}(\mathcal{D})$.

The remaining assertions are now consequences of lemma 4.10.

□

Theorem 4.17. *Let Y be a smooth projective curve. Let $\mathcal{D}$ be a p-divisor over (Y,N), $X := X(\mathcal{D})$ and $\widetilde{X}$ be the toroidification of X.*

1. *Let $\nu \in \mathrm{DV}(\widetilde{X})^{sing}_{\mathbb{T}}$ such that $\nu \notin \mathrm{Min}(\mathrm{DV}(\widetilde{X})^{sing}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$. Then there exists an equivariant divisorial resolution of singularities of $\widetilde{X}$ such that ν is not exceptional with respect to this resolution. In particular, the set $\mathbb{T}-\mathrm{DivEss}(\widetilde{X})$ of divisorially $\mathbb{T}$-essential valuations on $\widetilde{X}$ is contained in $\mathrm{Min}(\mathrm{DV}(\widetilde{X})^{sing}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$*
2. *Assume that $\mathrm{Loc}(\mathcal{D}) = Y$ and $\rho_g(Y) > 0$. Let $\nu \in \mathrm{DV}(X)^{sing}_{\mathbb{T}}$ be a primitive element such that $\nu \notin \mathrm{Min}(\mathrm{DV}(X)^{sing}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$. Then there exists an equivariant divisorial resolution of singularities of $\widetilde{X}$ such that ν is not exceptional with respect to this resolution. In particular, the set $\mathbb{T}-\mathrm{DivEss}(X)$ of divisorially $\mathbb{T}$-essential valuations on X is contained in $\mathrm{Min}(\mathrm{DV}(X)^{sing}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$.*

Proof. Let $\nu \in \mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}}$ be a primitive element such that

$$\nu \notin \mathrm{Min}(\mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}}, \leqslant_{\mathcal{D}}).$$

Let $z \in Y$ such that $\nu \in C_z(\mathcal{D}) =: \mathcal{C}$. Since $\nu \notin \mathrm{Min}(\mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$, by Remark 4.5 and the very definition of $\leqslant_{\mathcal{D}}$, one has $\nu \in \mathcal{C}_{\mathrm{sing}} \cap \mathcal{N} \setminus \mathrm{Min}(\mathcal{C}_{\mathrm{sing}} \cap \mathcal{N}, \leqslant_{\mathcal{C}})$. By Proposition 2.25, there exists a fan Σ which is a big smooth economical star refinement of $\mathcal{C}$ such that $\dim(\Sigma(\nu)) \geqslant 2$. Let $\{y_1, \ldots, y_r\}$ be a finite set of points of $\mathrm{Loc}(\mathcal{D})$ containing z and $\mathrm{Supp}(\mathcal{D})$. Using lemmas 4.16 and 4.15, one may then construct an equivariant divisorial resolution $f\colon X(\mathscr{E}) \to \widetilde{X}$ of the singularities of $\widetilde{X}$ such that ν is not f-exceptional.

Assume now that $\mathrm{Loc}(\mathcal{D}) = Y$ and $\rho_g(Y) > 0$, and consider a primitive element $\nu \in \mathrm{DV}(X)^{\mathrm{sing}}_{\mathbb{T}}$ such that $\nu \notin \mathrm{Min}(\mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$. Let $z \in Y$ such that $\nu \in C_z(\mathcal{D}) =: \mathcal{C}$. Since $\nu \notin \mathrm{Min}(\mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}}, \leqslant_{\mathcal{D}})$, by Proposition 4.6 and the very definition of $\leqslant_{\mathcal{D}}$, and using the notation of Proposition 4.13 with $\mathcal{P} := \deg(\mathcal{D})$ and $N := N \times \mathbb{Z}$, one has $(\mathcal{C}^{\star} \cap \mathcal{N}) \setminus \mathrm{Min}(\mathcal{C}^{\star} \cap \mathcal{N}, \leqslant_{\mathcal{C}})$. We now may conclude by applying proposition 4.13 and lemmas 4.16 and 4.15. □

5. The Nash order for torus actions of complexity one

5.1. **The hypercombinatorial order on the equivariant valuations of a $\mathbb{T}$-variety of complexity one.** Let X be a normal complexity one $\mathbb{T}$-variety. By Subsections 2.2 and 2.3, the set $\mathrm{DV}(X)$ carries two natural poset structures $\leqslant_{\mathrm{mds}}$ and $\leqslant_X$ such that $\leqslant_{\mathrm{mds}} \Rightarrow \leqslant_X$. In addition, one can define on the set $\mathrm{DV}(X)_{\mathbb{T}}$ of $\mathbb{T}$-equivariant divisorial valuations a third natural poset structure of combinatorial nature. We use the notation and terminology introduced in Section 3.

Definition 5.1. Let Y be a smooth algebraic curve and $\mathcal{D}$ be a p-divisor over (Y,N). Define a poset structure $\leqslant_{\mathcal{D}}$ on $HC(\mathcal{D})$ as follows: let ν_1 and ν_2 be elements of $HC(\mathcal{D})\cap\mathcal{N}$. Then one has $\nu_1 \leqslant_{\mathcal{D}} \nu_2$ if and only if there exists a page $\mathcal{N}_{y,\mathbb{Q}}$ containing ν_1 and ν_2 and one has $\nu_2\in\nu_1+C_y(\mathcal{D})$.

Now let $\mathscr{E}$ be a divisorial fan over (Y,N) and $X:=X(\mathscr{E})$ be the associated normal $\mathbb{T}$-variety of complexity one. Define an order $\leqslant_{\mathscr{E}}$ on $\mathrm{DV}(X)_{\mathbb{T}}$ as follows: let $\nu_1,\nu_2\in\mathrm{DV}(X)_{\mathbb{T}}$. Then $\nu_1\leqslant_{\mathscr{E}}\nu_2$ if and only if there exists a p-divisor $\mathcal{D}$ of $\mathscr{E}$ such that the centers of ν_1 and ν_2 lie on $X(\mathcal{D})$, and, identifying $\mathrm{DV}(X(\mathcal{D}))_{\mathbb{T}}$ with $HC(\mathcal{D})\cap\mathcal{N}$, one has $\nu_1\leqslant_{\mathcal{D}}\nu_2$.

Remark 5.2. The restriction of $\leqslant_{\mathcal{D}}$ to any Cayley cone $C_y(\mathcal{D})$ of $\mathcal{D}$ is the order $\leqslant_{C_y(\mathcal{D})}$ (see Definition 2.20).

Remark 5.3. Identifying $\mathrm{DV}(X(\mathscr{E}))_{\mathbb{T}}$ with $\cup_{\mathcal{D}\in\mathscr{E}}HC(\mathcal{D})\cap\mathcal{N}$, the order $\leqslant_{\mathscr{E}}$ may thus also be described as follows: let $\nu_1,\nu_2\in\mathrm{DV}(X)_{\mathbb{T}}$; then $\nu_1\leqslant_{\mathscr{E}}\nu_2$ if and only if there exists $\mathcal{D}\in\mathscr{E}$ such that $\nu_1,\nu_2\in HC(\mathcal{D})$ and $\nu_1\leqslant_{\mathcal{D}}\nu_2$ (see Proposition 3.7).

If ν_1,ν_2 are elements of $\mathrm{DV}(X)_{\mathbb{T}}$ such that $\nu_1\leqslant_{\mathscr{E}}\nu_2$, then for any p-divisor $\mathcal{D}$ of $\mathscr{E}$ such that $X(\mathcal{D})$ contains the centers of ν_1 and ν_2, one has $\nu_1\leqslant_{\mathcal{D}}\nu_2$.

In particular, in case $\mathscr{E}$ is the divisorial fan generated by a single p-divisor $\mathcal{D}$, on $\mathrm{DV}(X(\mathcal{D}))_{\mathbb{T}}$ one has $\leqslant_{\mathcal{D}}=\leqslant_{\mathscr{E}}$.

Proposition 5.4. *Let Y be a smooth algebraic curve, $\mathscr{E}$ be a divisorial fan over (Y,N) and $X:=X(\mathscr{E})$ be the associated $\mathbb{T}$-variety of complexity one.*

(1) *Let $\nu_1,\nu_2\in\mathrm{DV}(X)_{\mathbb{T}}$. Then one has: $\nu_1\leqslant_{\mathscr{E}}\nu_2$ if and and only if $\nu_1\leqslant_X\nu_2$ and $(\nu_1)_{|Y}\leqslant_Y(\nu_2)_{|Y}$. In particular, one always has $\leqslant_{\mathscr{E}}\Rightarrow\leqslant_X$.*
(2) *Assume that the locus $\mathrm{Loc}(\mathcal{D})$ is affine for any $\mathcal{D}\in\mathscr{E}$. Then the three poset structures $\leqslant_{\mathscr{E}}$, $\leqslant_{mds}$ and $\leqslant_X$ coincide on $\mathrm{DV}(X)_{\mathbb{T}}$. In particular, one has:*

$$\mathrm{MinVal}(X)=\mathrm{Nash}(X)=\mathrm{Min}(\mathrm{DV}(X)^{sing}_{\mathbb{T}},\leqslant_{\mathscr{E}}).$$

(3) *In general, on $\mathrm{DV}(X)_{\mathbb{T}}$, the Nash order is finer than the hypercombinatorial order, in other words one has*

$$\leqslant_{\mathscr{E}}\Rightarrow\leqslant_{mds}.$$

In particular, one always has the inclusion

$$\mathrm{Nash}(X)\subset\mathrm{Min}(\mathrm{DV}(X)^{sing}_{\mathbb{T}},\leqslant_{\mathscr{E}}).$$

Remark 5.5. Assume that $\mathscr{E}$ is toroidal or Y is a smooth projective curve of positive genus. Then using the description $\mathrm{DV}(X)^{\mathrm{sing}}_{\mathbb{T}}$ deduced from Proposition 4.6 one sees that any element $\mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}_{\mathbb{T}},\leqslant_{\mathscr{E}})$ lies either on the spine or on some non-trivial Cayley cone associated with a polyhedral divisor in $\mathscr{E}$. In particular $\mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_{\mathscr{E}})$ is contained in the reunion of a finite number of Cayley cones.

Proof. First let us show the "only if" part of the first assertion. By the very definitions of the involved poset structures, one may assume that $X = X(\mathcal{D})$ is affine. Let $\nu_1, \nu_2 \in \mathrm{DV}(X)_{\mathbb{T}}$ such that $\nu_1 \leqslant_{\mathcal{D}} \nu_2$. Recall that if $f = f_m \cdot \chi^m$ is a semi-invariant function and $\nu = [y, n, \ell] \in \mathrm{DV}(X)_{\mathbb{T}}$ then $\nu(f) = \ell \operatorname{ord}_y(f_m) + \langle n\,,\, m\rangle$. By the very definition of $\leqslant_{\mathcal{D}}$, there exists $y \in Y$ such that $\nu_1 = [y, n_1, \ell_1]$ and $\nu_2 = [y, n_2, \ell_2]$ and $(n_2, \ell_2) - (n_1, \ell_1) \in C_y(\mathcal{D})$. In particular one has $\ell_1 \leqslant \ell_2$. Thus one has $\nu_1(f) \leqslant_X \nu_2(f)$ for every semi-invariant element $f \in k[X]$, hence $\nu_1 \leqslant_X \nu_2$. Moreover, since $(\nu_i)_{|Y} = \ell_i \operatorname{ord}_y$ and $\ell_1 \leqslant \ell_2$, for every non-empty affine open subset Y_0 of the Y containing y, $(\nu_1)_{|Y}$ and $(\nu_2)_{|Y}$ are centered at Y_0 and $(\nu_1)_{|Y} \leqslant_{Y_0} (\nu_2)_{|Y}$. Thus $(\nu_1)_{|Y} \leqslant_Y (\nu_2)_{|Y}$.

Now assume $\nu_1, \nu_2 \in \mathrm{DV}(X)_{\mathbb{T}}$ satisfy $\nu_1 \leqslant_X \nu_2$ and $(\nu_1)_{|Y} \leqslant_Y (\nu_2)_{|Y}$. Let $\mathcal{D} \in \mathscr{E}$ such that $\mathrm{cent}_X(\nu_1) \in X(\mathcal{D})$. Write $\nu_1 = [y_1, \ell_1, n_1]$ and $\nu_2 = [y_2, \ell_2, n_2]$ with $y_1, y_2 \in \mathrm{Loc}(\mathcal{D})$, $[\ell_i, n_i] \in C_{y_i}(\mathcal{D}) \cap \mathcal{N}$.

Since $(\nu_1)_{|Y} \leqslant_Y (\nu_2)_{|Y}$, there exists a non-empty affine open subset Y_0 of the Y such that $(\nu_1)_{|Y}$ and $(\nu_2)_{|Y}$ are centered at Y_0 and $(\nu_1)_{|Y} \leqslant_{Y_0} (\nu_2)_{|Y}$. Since $(\nu_i)_{|Y} = \ell_i \operatorname{ord}_{y_i}$, the latter condition implies that $\ell_1 = 0$ or $y_1 = y_2$ and $\ell_1 \leqslant \ell_2$. In particular one may assume that $y_1 = y_2 =: y$.

Now write $k[X(\mathcal{D})] = k[f_i \cdot \chi^{m_i}]_{i \in I}$ where $f_i \cdot \chi^{m_i}$ are semi-invariant regular functions and I is finite. Then for every $i \in I$, one has $\nu_1(f_i) \leqslant \nu_2(f_i)$, thus

$$(\ell_2 - \ell_1) \operatorname{ord}_y(f_i) + \langle m_i\,,\, n_2 - n_1\rangle \geqslant 0.$$

This and the above condition $\ell_1 \leqslant \ell_2$ exactly says that $[y, \ell_2 - \ell_1, n_2 - n_1] \in C_y(\mathcal{D})$. Thus $\nu_1 \leqslant_{\mathcal{D}} \nu_2$. This completes the proof of the first assertion.

Assume that the locus $\mathrm{Loc}(\mathcal{D})$ is affine for any $\mathcal{D} \in \mathscr{E}$ and let us show the second assertion. In this case, for any $\mathcal{D} \in \mathscr{E}$, $k[\mathrm{Loc}(\mathcal{D})]$ is a subring of $k[X(\mathcal{D})]$. Thus for any $\nu_1, \nu_2 \in \mathrm{DV}(X)_{\mathbb{T}}$ such that $\nu_1 \leqslant_X \nu_2$, one has $(\nu_1)_{|Y} \leqslant_Y (\nu_2)_{|Y}$. By the first assertion, one has $\leqslant_{\mathscr{E}} = \leqslant_X$.

Thus by Proposition 2.16, it remains to show that that $\leqslant_{\mathscr{E}} \Rightarrow \leqslant_{\mathrm{mds}}$. Let $\nu_1, \nu_2 \in \mathrm{DV}(X)_{\mathbb{T}}$ such that $\nu_1 \leqslant_{\mathscr{E}} \nu_2$ and let us show that $\nu_1 \leqslant_{\mathrm{mds}} \nu_2$. By the definition of $\leqslant_{\mathscr{E}}$ and Remark 2.8, one may assume that $X = X(\mathcal{D})$ is affine and $\nu_1 \leqslant_{\mathcal{D}} \nu_2$. There exist a closed point y of $\mathrm{Loc}(\mathcal{D})$ and elements $(a_1, b_1), (a_2, b_2) \in C_y(\mathcal{D}) \cap \mathcal{N}$ such that $\nu_i = \mathrm{val}_{[y, a_i, b_i]}$ and $(a_2, b_2) \in (a_1, b_1) + C_y(\mathcal{D})$. Since $\mathrm{Loc}(\mathcal{D})$ is affine, for any open affine subset Y_0 of $\mathrm{Loc}(D)$ containing y, $X(\mathcal{D}_{|Y_0})$ is an open affine subset of X containing the centers of ν_1 and ν_2 (by Proposition 3.7). By Remark 2.8 it suffices to show that there exists an open affine subset Y_0 of $\mathrm{Loc}(D)$ containing y such that $\mathcal{C}_{X(\mathcal{D}_{|Y_0})}(\nu_1) \subset \mathcal{C}_{X(\mathcal{D}_{|Y_0})}(\nu_2)$.

By Lemma 3.11 and Proposition 3.13, one thus may assume that $\mathrm{Supp}(\mathcal{D}) \subset \{y\}$ and there is a $\mathbb{T}$-equivariant étale morphism $\theta\colon X \to Z$ where Z is the toric $\mathbb{G}_m \times \mathbb{T}$-variety associated with $(C_y(\mathcal{D}), N)$ and $\mathscr{L}_\infty(\theta)$ maps η_{X,ν_i} to

η_{Z,μ_i} where μ_i is the toric valuation associated with (a_i, b_i). Since $(a_2, b_2) \in (a_1, b_1) + C_y(\mathcal{D})$, one has $\nu_1 \leqslant_{C_y(\mathcal{D})} \nu_2$, thus by Proposition 2.28, η_{Z,μ_2} is a specialization of η_{Z,μ_1}.

By Corollary 2.31 and Theorem 2.30, there exist an extension K of k and a finite sequence $w_1, \ldots, w_r$ of K-wedges on Z such that, denoting by α_i (resp. β_i) the generic arc (resp. the special arc) of w_i, one has $\beta_{i+1} = \alpha_i$ for any $1 \leqslant i \leqslant r-1$, $\alpha_1 = \eta_{Z,\mu_1}$ and $\beta_r = \eta_{Z,\mu_2}$.

By Lemma 3.15, and Proposition 3.13, upon extending K, there exists a K-wedge $\widetilde{w_r}$ on X lifting w_r with special arc $\widetilde{\beta_r} = \eta_{X,\nu_2}$ and generic arc $\widetilde{\alpha_r}$. In particular $\mathscr{L}_\infty(\theta)(\widetilde{\alpha_r}) = \alpha_r$. Applying Lemma 3.15 again, upon extending K, there exists a K-wedge $\widetilde{w_{r-1}}$ on X lifting w_{r-1} with special arc $\widetilde{\beta_{r-1}} = \widetilde{\alpha_r}$ and generic arc $\widetilde{\alpha_{r-1}}$. Continuing in this way, one ends up with the following: upon extending K, there exist a finite sequence $\widetilde{w_1}, \ldots, \widetilde{w_r}$ of K-wedges on X such that, denoting by $\widetilde{\alpha_i}$ (resp. $\widetilde{\beta_i}$) the generic arc (resp. the special arc) of $\widetilde{w_i}$, one has $\widetilde{\beta_{i+1}} = \widetilde{\alpha_i}$ for any $1 \leqslant i \leqslant r-1$, $\widetilde{\beta_r} = \eta_{X,\nu_2}$ and $\mathscr{L}_\infty(\theta)(\widetilde{\alpha_1}) = \eta_{Z,\mu_1}$. Thus η_{X,ν_2} is a specialization of $\widetilde{\alpha_1}$. Since $\mathscr{L}_\infty(\theta)(\widetilde{\alpha_1}) = \eta_{Z,\mu_1}$, still by Proposition 3.13, one has $\widetilde{\alpha_1} \in \mathscr{L}_\infty(X)^{\mathrm{ord}=\nu_1}$. Thus η_{X,ν_2} is a specialization of an element of $\mathscr{L}_\infty(X)^{\mathrm{ord}=\nu_1}$. Therefore one has $\mathcal{C}_X(\nu_2) \subset \mathcal{C}_X(\nu_1)$, as was to be shown.

It remains to show that $\leqslant_{\mathscr{E}} \Rightarrow \leqslant_{\mathrm{mds}}$ holds in general. But on the toroidification, the hypercombinatorial order coincide with the mds order. Now we may apply Proposition 2.9 on order to conclude. ☐

Remark 5.6. We will show later that when the locus Y is a smooth projective curve of positive genus, then one also has $\leqslant_{\mathrm{mds}} = \leqslant_{\mathcal{D}}$.

However, in case Y is the projective line, on the set $\mathrm{DV}(X)_{\mathbb{T}}$, $\leqslant_X$ is in general strictly finer than $\leqslant_{\mathrm{mds}}$, and $\leqslant_{\mathrm{mds}}$ is in general strictly finer than $\leqslant_{\mathcal{D}}$, see Section 8.

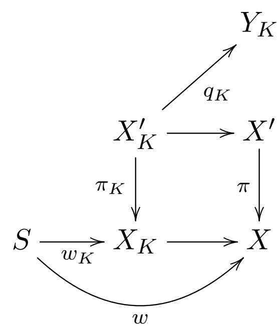
5.2. **Lifting wedges to the toroidification.** Let X be a $\mathbb{T}$-variety of complexity one and locus a projective curve of positive genus. We show that any wedge on X not contained in the singular locus lifts to the toroidification. This is a consequence of the following more general result.

Proposition 5.7. *Let X and X' be algebraic k-varieties. Assume that there exist a proper birational morphism $\pi\colon X' \to X$ and an affine morphism $q\colon X' \to Y$ from X' to a smooth projective algebraic k-curve Y with positive genus. Let U be a non-empty open subset of X such that π induces an isomorphism over U.*

Let K be an extension of k and $w\colon \mathrm{Spec}(K[[t,u]]) \to X$ be a K-wedge on X, whose image is not contained in $X \setminus U$. Then, upon replacing w by the induced L-wedge $\mathrm{Spec}(L[[t,u]]) \to X$, where L/K is an algebraic extension, the wedge w lifts to X', that is, there exists a morphism $w'\colon \mathrm{Spec}(K[[t,u]]) \to X'$ such that $\pi \circ w' = w$.

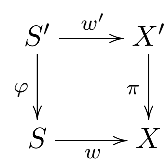
Remark 5.8. The existence of a lifting upon replacing K by an extension is sufficient for our needs. That being said, this restriction in the conclusion was put for the sake of convenience, since basically the same argument as below shows that a lifting exists even without extending K.

Proof. Set $S := \mathrm{Spec}(K[[t,u]])$. Upon replacing K by an algebraic extension, one may assume that K is algebraically closed. Extending the scalars from k to K, one obtains the following commutative diagram



Thus it suffices to show that w_K lifts to X'_K. Note that $\pi_K\colon X'_K \to X_K$ is a proper birational morphism inducing an isomorphism over U_K, q_K is an affine morphism $X'_K \to Y_K$, Y_K is a smooth projective K-curve such that $\rho_g(Y_K) \geqslant 1$ and the image of S by w_K meets U_K. Thus one may assume that $K = k$.

Since S is a noetherian two-dimensional regular scheme, π is an isomorphism over U, the image of S by w meets U, and $\pi\colon X' \to X$ is proper, by [Sha66, Theorem, p 45], there exist a scheme S', a morphism $w'\colon S' \to X'$, and a morphism $\varphi\colon S' \to S$ such that φ is a finite composition of blow-ups at a maximal ideal and the following diagram is commutative:



The exceptional locus E of φ is connected, and since k is algebraically closed, each of its irreducible component is isomorphic to $\mathbb{P}^1_k$. Since Y has positive genus, the morphism $\mathbb{P}^1_k \to Y$ induced by the composition of w' with $q\colon X' \to Y$ is constant. Since q is an affine morphism, there exists an open affine subset V of X' such that $w'^{-1}(V)$ contains E. In particular the image by φ of the closed subset $S' \setminus w'^{-1}(U)$ does not contain the closed point of S. Thus $w'^{-1}(V) = S'$.

Now since φ is proper birational and S is normal, one has $\varphi_*\mathscr{O}_{S'} = \mathscr{O}_S$. In particular one has a factorization $\varphi = \varphi_1 \circ \varphi_2$ where $\varphi_2\colon S' \to \mathrm{Spec}(H^0(S',\mathscr{O}_{S'}))$ is the natural morphism and $\varphi_1\colon \mathrm{Spec}(H^0(S',\mathscr{O}_{S'})) \to S$ is an isomorphism. But since V is affine, the morphism $w'\colon S' \to V \subset X'$ factors as $w' = \psi \circ \varphi_2$ with $\psi\colon \mathrm{Spec}(H^0(S',\mathscr{O}_{S'})) \to V$. Thus $\psi \circ \varphi_1^{-1}$ gives the sought-for lifting of w to X'. $\square$

5.3. **The Nash order in case the locus is a projective curve of positive genus.** As a consequence of the previous sections, we obtain a solution of the generalized Nash problem for normal complexity one $\mathbb{T}$-variety with locus a projective curve of positive genus, in the sense that the Nash order on the set of equivariant valuations is explicitly described by the hypercombinatorial poset structure of Definition 5.1

Theorem 5.9. *Let Y be a smooth projective algebraic curve, $\mathscr{E}$ be a divisorial fan over (Y, N) and $X := X(\mathscr{E})$ be the associated $\mathbb{T}$-variety of complexity one. Assume that $\rho_g(Y) \geqslant 1$. Then on $\mathrm{DV}(X)_{\mathbb{T}}$ one has $\leqslant_{mds} = \leqslant_{\mathscr{E}}$.*

Proof. By Proposition 5.4 it suffices to show that on $\mathrm{DV}(X)_{\mathbb{T}}$ one has $\leqslant_{\mathrm{mds}} \Rightarrow \leqslant_{\mathscr{E}}$. Let ν_1, ν_2 be elements of $\mathrm{DV}(X)_{\mathbb{T}}$ such that $\mathcal{C}_X(\nu_2) \subset \mathcal{C}_X(\nu_1)$. Let $\mathcal{D}$ be a p-divisor of $\mathscr{E}$ such that $X(\mathcal{D})$ contains the centers of ν_1 and ν_2 (see Remark 2.15). One has to show that $\nu_1 \leqslant_{\mathcal{D}} \nu_2$. By Remark 2.8, one has $\mathcal{C}_{X(\mathcal{D})}(\nu_2) \subset \mathcal{C}_{X(\mathcal{D})}(\nu_1)$. One may assume that $X = X(\mathcal{D})$. In case $\mathrm{Loc}(\mathcal{D})$ is affine, the result is given by Proposition 5.4. From now on we assume $\mathrm{Loc}(\mathcal{D}) = Y$. We consider the toroidification $\pi\colon \widetilde{X} \to X$ (see Subsection 3.6). It is a $\mathbb{T}$-equivariant proper birational morphism. In particular we may identify $\mathrm{DV}(X)_{\mathbb{T}}$ and $\mathrm{DV}(\widetilde{X})_{\mathbb{T}}$, the poset structures defined by $\leqslant_{\mathcal{D}}$ and $\leqslant_{\widetilde{\mathcal{D}}}$ coincide, and $\mathscr{L}_\infty(\pi)$ induces a continuous bijection $\mathscr{L}_\infty(\widetilde{X}) \to \mathscr{L}_\infty(X)$.

Since $\mathcal{C}_{X(\mathcal{D})}(\nu_2) \subset \mathcal{C}_{X(\mathcal{D})}(\nu_1)$, by Theorem 2.30 and Corollary 2.31, there exists an extension K/k and a finite sequence of K-wedges $w_1, \ldots, w_r$ on X such that the special arc of w_1 is η_{X,ν_1}, the generic arc of w_r is η_{X,ν_2} and for any $1 \leqslant i \leqslant r-1$ the generic arc of w_i is the special arc of w_{i+1}. Since η_{X,ν_1} and η_{X,ν_2} are fat, the generic arc of any of the w_i's is fat. In particular, the image in X of any of the w_i's is not contained in any proper closed subset of X. On the other hand, $\widetilde{X}$ is equipped with an affine morphism $q\colon \widetilde{X} \to Y$ (Lemma 3.2).

Thus one may apply Proposition 5.7; upon extending K, one obtain K-wedges $\widetilde{w_1}, \ldots, \widetilde{w_r}$ on $\widetilde{X}$ that lift $w_1, \ldots, w_r$. Since $\mathscr{L}_\infty(\pi)\colon \mathscr{L}_\infty(\widetilde{X}) \to \mathscr{L}_\infty(X)$ is a bijection mapping $\eta_{\widetilde{X},\nu_i}$ to η_{X,ν_i} $(i = 1, 2)$, the special arc of $\widetilde{w_1}$ is $\eta_{\widetilde{X},\nu_1}$, the generic arc of $\widetilde{w_r}$ is $\eta_{\widetilde{X},\nu_2}$ and for any $1 \leqslant i \leqslant r-1$ the generic arc of $\widetilde{w_i}$ is the special arc of $\widetilde{w_{i+1}}$. In particular $\eta_{\widetilde{X},\nu_2}$ is a specialization of $\eta_{\widetilde{X},\nu_2}$, which shows that $\mathcal{C}_{\widetilde{X},\nu_2} \subset \mathcal{C}_{\widetilde{X},\nu_1}$. By Proposition 5.4, one has $\nu_1 \leqslant_{\widetilde{\mathcal{D}}} \nu_2$, thus $\nu_1 \leqslant_{\mathcal{D}} \nu_2$, as was to be shown. □

5.4. **The classical Nash problem in case the locus is a projective curve of positive genus.**

Theorem 5.10. *Let Y be a smooth projective algebraic curve, $\mathscr{E}$ be a divisorial fan over (Y, N) and $X := X(\mathscr{E})$ be the associated $\mathbb{T}$-variety of complexity one. Assume that either $\rho_g(Y) \geqslant 1$ or for every $\mathcal{D} \in \mathscr{E}$ the locus $\mathrm{Loc}(\mathcal{D})$ is affine. Then the Nash problem has a positive answer, i.e. the inclusion $\mathrm{Nash}(X) \subset \mathrm{Ess}(X)$ is an equality.*

Proof. By Remark 2.15, one may assume that X is affine and defined by a p-divisor $\mathcal{D}$ whose locus is either affine or a smooth projective algebraic curve of positive genus. By Remark 2.12, one has $\mathrm{Nash}(X) = \mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_{\mathrm{mds}})$. By Proposition 5.4 (in case $\mathrm{Loc}(\mathcal{D})$ is affine) and Theorem 5.9 (in case $\mathrm{Loc}(\mathcal{D})$ is projective with positive genus), on $\mathrm{DV}(X)_{\mathbb{T}}$ one has $\leqslant_{\mathrm{mds}}=\leqslant_{\mathcal{D}}$. Thus $\mathrm{Nash}(X) = \mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_{\mathcal{D}})$. On the other hand, by 4.17, one has $\mathbb{T} - \mathrm{DivEss}(X) \subset \mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_{\mathcal{D}})$. Since the inclusions $\mathrm{Ess}(X) \subset \mathbb{T} - \mathrm{DivEss}(X)$ and $\mathrm{Nash}(X) \subset \mathrm{Ess}(X)$ always hold (see Section 2.1 and Proposition 2.14), one ends up with the conclusion that $\mathrm{Nash}(X) = \mathrm{Ess}(X)$. □

Remark 5.11. A natural general question is whether the bijectivity of the Nash map is invariant by surjective étale morphisms. We thank Shihoko Ishii for pointing out the following. Let $f\colon X \to Y$ be an étale morphism between algebraic variety; let ν be a divisorial valuation on X and μ be the valuation induced by f on Y. Then one can show that if ν is essential, μ is also essential, and if μ is Nash, ν is also Nash. In particular, if the Nash map is bijective for Y, it is also bijective for X (Assuming that f is surjective, it is not clear whether the converse is true). Thus in case $\mathrm{Loc}(\mathcal{D})$ is affine, the result of Theorem 5.10 is also a consequence of Ishii-Kollar's result.

Remark 5.12. In case $\mathrm{Loc}(\mathcal{D})$ is affine, Proposition 5.4 also gives that the inclusion $\mathrm{MinVal}(X) \subset \mathrm{Nash}(X)$ is an equality. In case $\mathrm{Loc}(\mathcal{D})$ is projective of positive genus, this is no longer true in general (see Subsection 7.3 below for an example).

Remark 5.13. Similarly to the toric case, we obtain as a direct consequence of the above argument and Section 2.1 that for the varieties under consideration any divisorially essential valuation is an essential valuation.

6. Terminal valuations and torus actions of complexity one

The aim of this section is to give a combinatorial description of the terminal valuations of a $\mathbb{T}$-variety of complexity one with locus a smooth projective curve of positive genus. One ingredient is the description of terminal toric valuations in [dFD16, §6], and our description (see Theorem 6.16) has analogies with Proposition 6.2 of *op.cit.*.

The main result of *op.cit.* is that on any algebraic variety, any terminal valuation is a Nash valuation. An example is given of a toric Nash valuation which is not terminal, and it is pointed out that any toric Nash valuation is minimal, and that no example of a Nash valuation which is neither terminal nor minimal was known. In the next section, we will use Theorem 6.16 and the results of the previous sections to exhibit examples of non-toroidal $\mathbb{T}$-varieties of complexity one possessing a Nash valuation which is neither terminal nor minimal.

To conclude the introduction of this section, let us point out that the assumption on the genus in Theorem 6.16 is crucial, and that the result fails in the genus zero case (see Section 8). As for the description of the Nash order and of the essential valuations, the description of terminal valuations seems much more challenging in this case.

6.1. **Preliminary results.** Let $\mathscr{E}$ be a divisorial fan over (Y, N), where Y is a smooth projective curve. As before, write $\mathbb{T}$ for the algebraic torus with one-parameter lattice N.

Definition 6.1. The prime $\mathbb{T}$-divisors on the $\mathbb{T}$-variety $X = X(\mathscr{E})$ are divided into two sorts.

- (i) The ones whose restriction of the vanishing order to $\mathbb{C}(X)^{\mathbb{T}} \simeq \mathbb{C}(Y)$ is non-trivial;
- (ii) and the others.

The divisors of type (i) are called the *vertical divisors* while the ones of type (ii) are called the *horizontal divisors*.

In the sequel, we set

$$\deg(\mathscr{E}) := \bigcup_{\mathcal{D}\in\mathscr{E}} \deg(\mathfrak{D})$$

and let $\Sigma := \Sigma(\mathscr{E})$ be the fan generated by the tail cones of the coefficients of the elements of $\mathscr{E}$. Note that $X = X(\mathscr{E})$ is toroidal (i.e isomorphic to its toroidification) if and only if $\deg(\mathscr{E}) = \varnothing$. We also set

$$\mathrm{Ray}(\mathscr{E}) := \{\rho \text{ rays of } \Sigma \mid \rho \cap \deg(\mathscr{E}) = \varnothing\} \text{ and}$$
$$\mathrm{Ver}(\mathscr{E}) := \{(y, v) \in Y \times N \mid v \text{ vertex of } \mathcal{D}_y \text{ for some } \mathcal{D} \in \mathscr{E}\}.$$

For $v \in N_{\mathbb{Q}}$, denote by $\mu(v)$ the number $\inf\{d \in \mathbb{Z}_{>0} \mid dv \in N\}$. Note that associating an element $(y, v) \in \mathrm{Ver}(\mathscr{E})$ with the ray $\rho_{y,v}$ generated in $\mathcal{N}_{y,\mathbb{Q}}$ by $[y, v, 1]$ defines a bijection between $\mathrm{Ver}(\mathscr{E})$ and the rays of $H\Sigma(\mathscr{E})$ not contained in the spine, and that $[y, \mu(v).v, \mu(v)]$ is a primitive generator of $\rho_{y,v}$.

For any ray ρ of Σ, denote by v_ρ its primitive generator. From §3.5, one deduces:

Proposition 6.2. (i) *With an element ρ of $\mathrm{Ray}(\mathscr{E})$ one associates $\mathrm{cent}_{X(\mathscr{E})}(\mathrm{val}_{[\bullet, v_\rho, 0]})$. This defines a bijection between the set $\mathrm{Ray}(\mathscr{E})$ and the set of horizontal prime $\mathbb{T}$-divisors on $X(\mathscr{E})$.*

(ii) *With an element (y, v) of $\mathrm{Ver}(\mathscr{E})$ one associates $\mathrm{cent}_{X(\mathscr{E})}(\mathrm{val}_{[y,\mu(v).v,\mu(v)]})$. This defines a bijection between the set $\mathrm{Ver}(\mathscr{E})$ and the set of vertical prime $\mathbb{T}$-divisors on $X(\mathscr{E})$.*

Remark 6.3. For $\rho \in \mathrm{Ray}(\mathscr{E})$ (respectively $(y, v) \in \mathrm{Ver}(\mathscr{E})$) we denote by D_ρ (respectively $D_{(y,v)}$) the corresponding divisors. Their vanishing orders can be explicitly described as follows. Let $f \in \mathbb{C}(Y) \setminus \{0\}$ and let $m \in M$. Then $\xi = f \otimes \chi^m$ is a homogeneous element of of the function field $\mathbb{C}(X(\mathscr{E}))$ and by §3.4 we have the formulae

$$\mathrm{ord}_{D_\rho}(\xi) = \langle m\,,\, v_\rho\rangle \text{ and } \mathrm{ord}_{D_{(y,v)}}(\xi) = \mu(v)(\mathrm{ord}_y(f) + \langle m\,,\, v\rangle),$$

Therefore the principal divisor associated with ξ is given by the relation

$$\operatorname{div}(\xi) = \sum_{\rho \in \operatorname{Ray}(\mathscr{E})} \langle m\,,\, v_\rho \rangle \cdot D_\rho + \sum_{(y,v)\in \operatorname{Ver}(\mathscr{E})} \mu(v)(\operatorname{ord}_y(f) + \langle m\,,\, v\rangle) \cdot 1 D_{y,v}.$$

For the next proposition, see [PS11, Theorem 3.21].

Proposition 6.4. *Let $K_Y := \sum_{y\in Y} K_{Y,y} \cdot [y]$ be a representative of the canonical class of Y. Then the canonical class of the $\mathbb{T}$-variety $X = X(\mathscr{E})$ is represented by the Weil divisor*

$$K_X = \sum_{(y,v)\in \operatorname{Ver}(\mathscr{E})} (\mu(v)K_{Y,y} + \mu(v) - 1) \cdot D_{(y,v)} - \sum_{\rho\in \operatorname{Ray}(\mathscr{E})} D_\rho.$$

We say that a line bundle on an algebraic variety is *semiample* if a positive power of it is basepoint-free.

Lemma 6.5. *Let $\varphi\colon S \to B$ be a morphism between algebraic varieties. If $\mathcal{L}$ is a semiample line bundle on B, then the line bundle $\varphi^\star\mathcal{L}$ is semiample.*

Proof. Note that the semiample condition on $\mathcal{L}$ is equivalent to the existence of a morphism $\psi\colon B \to B_0$ and an ample line bundle $\mathcal{L}_0$ on B_0 such that $\mathcal{L} = \varphi^\star(\mathcal{L}_0)$. So $\varphi^\star(\mathcal{L}) = \varphi^\star(\psi^\star(\mathcal{L}_0)) = (\psi\circ\varphi)^\star(\mathcal{L}_0)$ is semiample. □

Remark 6.6. Let $\varphi\colon S \to B$ be a dominant morphism between algebraic varieties and D a Cartier divisor on B. Let $(U_i, f_i)_{i\in I}$, where $U_i \subset B$ is a dense open subset and $f_i \in k(B)^\star$, be local data representing D. We recall that the pullback $\varphi^\star(D)$ is defined as the Cartier divisor represented by the local data $(\varphi^{-1}(U_i), f_i \circ \varphi)$.

Notation 6.7. With the same notation as Proposition 6.4 we set

$$\Gamma(\mathscr{E}) := \operatorname{supp}(\mathscr{E}) \cup \operatorname{supp}(K_Y),\ \operatorname{Ver}^+(\mathscr{E}) := \{(y,v) \in \operatorname{Ver}(\mathscr{E}) \mid y \in \Gamma(\mathscr{E})\},$$

$$K_X^+ := \sum_{(y,v)\in \operatorname{Ver}^+(\mathscr{E})} \mu(v)(K_{Y,y}+1) \cdot D_{(y,v)},$$

$$\text{and}\quad K_X^- := -\sum_{(y,v)\in \operatorname{Ver}^+(\mathscr{E})} D_{(y,v)} - \sum_{\rho\in \operatorname{Ray}(\mathscr{E})} D_\rho.$$

Note that we have $K_X = K_X^+ + K_X^-$.

Lemma 6.8. *Assume that $X = X(\mathscr{E})$ is toroidal and that the genus of Y is positive. Then the divisor K_X^+ is Cartier and semiample.*

Proof. Let $q\colon X \to Y$ be the $\mathbb{T}$-invariant dominant morphism induced by the global quotient of X. Let $E = \sum_{y\in Y} a_y \cdot [y]$ be any divisor on Y. Given a local equation α of E on some open subset U of Y, $q^\star\alpha$ is a local equation of $q^\star(E)$ and by Remark 6.3 one has

$$\operatorname{div}(q^\star(\alpha)) = \sum_{\substack{(y,v)\in \operatorname{Ver}(\mathscr{E}) \\ y\in \operatorname{Supp}(\mathscr{E})\cup \operatorname{Supp}(E)}} \mu(v) \operatorname{ord}_y(\alpha) \cdot D_{(y,v)}.$$

Thus

$$q^\star E = \sum_{\substack{(y,v)\in \mathrm{Ver}(\mathscr{E})\\ y\in \mathrm{Supp}(\mathscr{E})\cup \mathrm{Supp}(E)}} \mu(v) a_y \cdot D_{y,v}.$$

Now consider the divisor $E := K_Y + \sum_{y\in\Gamma(\mathscr{E})}[y]$. The above shows that $q^\star(E) = K_X^+$. On the other hand the assumption on the genus of Y implies that E is principal or $\deg(E) > 0$. So E is a Cartier semiample divisor on Y. By Lemma 6.5, K_X^+ is semiample. □

Definition 6.9. Let V be a variety and let $\mathcal{L}$ be a line bundle over V. We say that $\mathcal{L}$ is *nef* if for any (irreducible) complete curve C inside V the intersection $(\mathcal{L}, C)$ is non-negative. Recall that $(\mathcal{L}, C)$ is defined as the degree of any associated divisor of $\kappa^\star\mathcal{L}$, where $\kappa\colon \widetilde{C} \to V$ is the composition of the normalization $\widetilde{C} \to C$ and the inclusion $C \to V$. We define similarly the intersection number and the nef condition for Cartier $\mathbb{Q}$-divisors.

Lemma 6.10. *Let V be a variety and let $\mathcal{L}$ be a line bundle on V. If $\mathcal{L}$ is semiample, then $\mathcal{L}$ is nef.*

Proof. Taking the notation of Definition 6.9, for any complete curve C on V the pullback $\kappa^\star(\mathcal{L})$ is semiample by Lemma 6.5; so $(\mathcal{L}, C) \geqslant 0$. □

6.2. **Minimal models and terminal valuations.** In this section, we let $\mathcal{D}$ be a proper σ-tailed polyhedral divisor over a smooth projective curve Y of genus $\geqslant 1$. We write $X = X(\mathcal{D})$ for the associated $\mathbb{T}$-variety.

Our goal is to describe the *terminal valuations* of X. Let us first recall from [dFD16] the general definition of this notion. For any algebraic variety X, a *relative minimal model over* X is a projective birational morphism $f\colon Z \to X$ such that Z has terminal singularities (in particular Z is normal and its canonical divisor is $\mathbb{Q}$-Cartier) and the canonical divisor of Z is relatively nef over X, that is to say, has positive intersection with any complete curve on Z (see Definition 6.9) which is contracted by the morphism f. The terminal valuations of X are the valuations defined by the codimension one irreducible components of any relative minimal model $f\colon Z \to X$.

Definition 6.11. Let $\tau \subset N_{\mathbb{Q}}$ be a strictly convex polyhedral cone. Recall that τ is *smooth* if it is generated by a subset of a basis of N and that τ_{sing} designates the union of the relative interiors of the non-smooth faces of τ. Let $\Gamma(\tau)$ be the convex hull of $\tau \cap N \setminus \{0\}$ in $N_{\mathbb{Q}}$ and let $\partial_c\Gamma(\tau)$ be the union of the bounded faces of $\Gamma(\tau)$. Following [dFD16, Section 6], we define the set of *terminal points* of τ as the subset

$$\mathrm{Ter}(\tau) = \tau_{\mathrm{sing}} \cap \partial_c\Gamma(\tau) \cap N.$$

Remark 6.12. For any face τ' of τ, one has $\partial_c\Gamma(\tau) \cap \tau' = \partial_c\Gamma(\tau')$. Moreover, if τ is smooth, $\partial_c\Gamma(\tau) \cap N$ is the set of primitive generators of the rays of τ.

Remark 6.13. Using star subdivisions, one can build a triangulation of the polyhedral complex $\partial_c\Gamma(\tau)$ with set of vertices equal to $\partial_c\Gamma(\tau) \cap N$. In this way we define a fan subdivision of τ whose set of primitive generators of rays is exactly $\partial_c\Gamma(\tau) \cap N$, and whose set of primitive generators of rays which are not rays of τ is exactly $\mathrm{Ter}(\tau)$. This construction can be generalized for the polyhedral divisor $\mathcal{D}$, using triangulations on the polyhedral complexes $\partial_c C_y(\mathcal{D})$ that are compatible on $\partial_c\sigma$, where σ is the tail. Again, this may be achieved by suitable star subdivisions.

The toroidal divisorial fan induced by these subdivisions (see Lemma 4.15) will be denoted by $\mathscr{E}$.

Theorem 6.14. *Let $\mathcal{D}$ be a proper polyhedral divisor over a smooth complete curve of positive genus. Let $\mathscr{E}$ be the divisorial fan of Remark 6.13 and $X' = X(\mathscr{E})$. Then the induced proper birational morphism $p\colon X' \to X$ is a relative minimal model over X.*

Proof. Since the Cayley cones of $\mathscr{E}$ are simplicial and $X' = X(\mathscr{E})$ is toroidal, X' has terminal singularities. In order to prove that X' is a minimal model, it suffices to show that $(K_{X'}, C) \geqslant 0$ for any complete curve C on X' contracted by p (note that the latter condition holds in fact for any such curve since X is affine). Let C be such a curve. By Lemmas 6.10 and 6.8, one has $(K^+_{X'}, C) \geqslant 0$ It remains to show that $(K^-_{X'}, C) \geqslant 0$. Let $\beta\colon X' \to Y$ be the quotient map. Note that C is contained in a fiber $\beta^{-1}(\{y\})$; otherwise $p(C)$ would dominate Y via the rational quotient $X \dashrightarrow Y$.

Let (U, φ) be an étale chart of Y around the point y, i.e. the map $\varphi\colon U \to \mathbb{A}^1_k$ is an étale morphism such that $\varphi^{-1}(0) = \{y\}$. Shrinking U if necessary we may assume that $U \cap \mathrm{Supp}(\mathcal{D}') \subset \{y\}$ for any $\mathcal{D}' \in \mathscr{E}$, and that $U \cap \mathrm{Supp}(K_Y) \subset \{y\}$. Set $\mathscr{E}_{|U} = \{\mathcal{D}'_{|U} \,|\, \mathcal{D}' \in \mathscr{E}\}$. Note that the curve C is contained in $W := X(\mathscr{E}_{|U})$, and that it suffices to show that $(K^-_W, C) \geqslant 0$.

Denote by $\mathscr{E}_\varphi$ the divisorial fan over $(\mathbb{A}^1_k, N)$ generated by the p-divisors

$$\{\mathcal{D}'_y \cdot [0] + \sum_{z \in \varphi(U)\setminus\{0\}} \mathrm{Tail}(\mathcal{D}') \cdot [z] \,|\, \mathcal{D}' \in \mathscr{E}\}.$$

From Lemma 3.11 we have a $\mathbb{T}$-equivariant isomorphism $X(\mathscr{E}_{|U}) \simeq U \times_{\mathbb{A}^1} X(\mathscr{E}_\varphi)$. Let

$$\gamma\colon W = X(\mathscr{E}_{|U}) \to X(\mathscr{E}_\varphi)$$

be the corresponding étale morphism. Denote by $\bar{\mathscr{E}}_\varphi$ the divisorial fan over $(\mathbb{A}^1_k, N)$ generated by the p-divisors

$$\mathcal{D}'_y \cdot [0] + \sum_{z \in \mathbb{A}^1_k\setminus\{0\}} \mathrm{Tail}(\mathcal{D}') \cdot [z], \,|\, \mathcal{D}' \in \mathscr{E}\}.$$

Observe that there is a natural open immersion of $X(\mathscr{E}_\varphi)$ into $V := X(\bar{\mathscr{E}}_\varphi)$ and that V is the toric $\mathbb{G}_m \times \mathbb{T}$-variety associated with the fan generated by by $\{C_y(\mathcal{D}') \,|\, \mathcal{D}' \in \mathscr{E}\}$. Note that using Notation 6.7 for $V := X(\bar{\mathscr{E}}_\varphi)$, K^-_V is the canonical class of V. Thus by [dFD16, Section 6] the divisor K^-_V is nef. Let $r \in \mathbb{Z}_{>0}$ such that rK^-_V is Cartier. Since $\varphi^{-1}(0) = \{y\}$ and using the explicit description of γ (see lemma 3.11) we see that $\gamma^\star(rK^-_V) = rK^-_W$.

Furthermore, denoting by $\alpha\colon V \to \mathbb{A}^1_k$ the quotient map for the $\mathbb{T}$-action, the restriction map

$$\beta^{-1}(\{y\}) \to \alpha^{-1}(\{0\}),\ x \mapsto \gamma(x)$$

is an isomorphism. Denoting by C' the image of C by this isomorphism, we thus have $(C', K_V^-) = (C, K_W^-)$ hence $(C, K_W^-) \geqslant 0$ since K_V^- is nef. □

Definition 6.15. We generalize Definition 6.11 in the setting of hypercones. Let $\mathcal{D}$ be a p-divisor with locus a smooth projective curve Y of genus $\geqslant 1$. We set

$$HC(\mathcal{D})_{\text{sing}} = \bigcup_{\theta \in HC(\mathcal{D})^\star_{\text{sing}}} \theta$$

(see Definition 4.4 for the notation $HC(\mathcal{D})^\star_{\text{sing}}$) and $\partial_c\Gamma(\mathcal{D})$ stands for

$$\partial_c\Gamma(\mathcal{D}) := \left(\bigcup_{y\in Y} \{y\} \times \partial_c\Gamma(C_y(\mathcal{D})) \right) / \sim .$$

Theorem 6.16. *Let $\mathcal{D}$ be a p-divisor over a smooth projective curve Y of genus $\geqslant 1$. Then the set of terminal valuations of $X = X(\mathcal{D})$ is given by the formula*

$$\text{Ter}(\mathcal{D}) = HC(\mathcal{D})_{sing} \cap \partial_c\Gamma(\mathcal{D}) \cap \mathcal{N} \subset \mathcal{N}_{\mathbb{Q}}.$$

Proof. This directly follows from Theorem 6.14. and §3.7. □

Remark 6.17. In other words, assuming that $\text{Supp}(\mathcal{D})$ is non-empty (for example assuming that $\text{Loc}(\mathcal{D}) = Y$), the set $\text{Ter}(\mathcal{D})$ is the reunion of the sets $\text{Ter}(C_y(\mathcal{D}))$ where y runs over $\text{Supp}(\mathcal{D})$ and the set of primitive generators of the rays of the tail σ which meet $\deg(\mathcal{D})$. In case $\text{Supp}(\mathcal{D})$ is empty, $\text{Ter}(\mathcal{D}) = \text{Ter}(\sigma)$

7. Some examples

In this section we describe some examples of non-toroidal non-rational $\mathbb{T}$-varieties of complexity one constructed from polyhedral divisors and illustrating our results.

7.1. In [dFD16, Section 6], an example of a Nash toric valuation which is not terminal is given. Here we construct a simple family of $\mathbb{T}$-varieties of complexity one of arbitrarily large dimension with a similar property. Note that the construction does not rely on the aforementioned toric example. Let d be an integer such that $d \geqslant 2$, $N = \mathbb{Z}^d$ and σ be the N-smooth cone generated by the canonical $\mathbb{Z}$-basis of N. Let Y be a smooth projective curve with positive genus and $y_0, \ldots, y_r \in Y$. be a finite set of points of Y. For any $1 \leqslant i \leqslant r$, choose $n_i \in \text{Relint}(\sigma)$, and set $\mathcal{D}_{y_i} := n_i + \sigma$. For $y \in Y \setminus \{y_0, \ldots, y_r\}$, set $\mathcal{D}_y := \sigma$. Thus $\deg(\mathcal{D}) = \sum_{i=0}^r n_i + \sigma$ is contained $\text{Relint}(\sigma)$ Therefore $\mathcal{D}$ is a proper polyhedral divisor over (Y, N) with locus Y. By Theorem 5.9, $\text{Nash}(X(\mathcal{D})) = \{[\bullet, (1, \ldots, 1), 0]\}$. Note that $\widetilde{X}(\mathcal{D})$ is smooth, and that no ray of $HC(\mathcal{D})$ meets $\deg(\mathcal{D})$. Thus by Theorem 6.16,

$X(\mathcal{D})$ has no terminal valuation. Hence $[\bullet,(1,\ldots,1),0]$ is a Nash valuation which is not terminal. Similar examples with more Nash valuations may be obtained; take *e.g.* a subset Δ of the set of faces of σ containing no ray and such that no element of Δ is a face of another element of Δ, and replace each n_i by the convex hull of arbitrary elements of the relative interiors of each face of Δ. The resulting $X(\mathcal{D})$ has a number of Nash valuations equal to the cardinality of Δ, and no one of them is terminal.

7.2. We now give example where the toroidification $\widetilde{X}(\mathcal{D})$ has a Nash valuation which is not a Nash valuation on $X(\mathcal{D})$ (see Remark 4.14).

Take $N=\mathbb{Z}^2$ and let σ be the N-smooth cone in $\mathbb{Q}^2$ spanned by $e_1=(1,0)$ $e_2=(0,1)$. Let Y be a smooth projective curve with positive genus, and $y_1,y_0\in Y$ such that the divisors $2\cdot y_1$ and $2\cdot y_0$. are linearly equivalent. For example, take Y with $\rho_g(Y)=1$, $y_0\in Y$ and y_1 a 2-torsion point of the elliptic curve (Y,y_0).

Set $\mathcal{D}_{y_0}:=[(0,0),(1,-\frac{1}{2})]+\sigma$ and $\mathcal{D}_{y_1}=\{(\frac{1}{2},\frac{1}{2})\}+\sigma$ For $y\in Y\setminus\{y_0,y_1\}$, set $\mathcal{D}_y=\sigma$. Thus

$$\deg(\mathcal{D})=[(\frac{1}{2},\frac{1}{2}),(\frac{3}{2},0)]+\sigma$$

For $m=a\cdot e_0^\vee+b\cdot e_1^\vee\in\sigma^\vee$,

$$\langle m\,,\,\mathcal{D}_{y_0}\rangle=-\frac{b}{2}\quad\text{and}\quad\langle m\,,\,\mathcal{D}_{y_1}\rangle=\frac{a}{2}+\frac{b}{2}$$

thus $\deg(\mathcal{D}(m))=\frac{a}{2}$ and therefore

$$\deg(\mathcal{D}(m))=0\iff a=0\iff\mathcal{D}(m)=\frac{b}{2}(y_1-y_0).$$

Since the divisor $\frac{b}{2}(y_1-y_0)$ is $\mathbb{Q}$-principal, we infer that $\mathcal{D}$ is a proper polyhedral divisor.

The non-trivial Cayley cones are $C_{y_1}(\mathcal{D})=\langle[\bullet,(1,0),0],[\bullet,(0,1),0],[y_1,(1,1),2]\rangle$ which is smooth and $C_{y_0}(\mathcal{D})=\langle[\bullet,(1,0),0],[\bullet,(0,1),0],[y_0,(0,0),1],[y_0,(2,-1),1]\rangle$ which is not smooth but whose any proper face is smooth.

Thus $\mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}}=\mathrm{Relint}(C_{y_0}(\mathcal{D}))\cap\mathcal{N}$ and one may check that

$$\mathrm{Nash}(\widetilde{X})=\mathrm{Min}(\mathrm{DV}(\widetilde{X})^{\mathrm{sing}}_{\mathbb{T}},\leqslant_{\mathcal{D}})=[y_0,(1,0),1]=:\nu_1.$$

On the other hand, since the only faces of σ meeting the degree are σ and the ray generated by $(1,0)$,

$$\mathrm{DV}(X(\mathcal{D}))^{\mathrm{sing}}_{\mathbb{T}}=(\mathrm{Relint}(C_{y_0}(\mathcal{D}))\cup\mathrm{Relint}(\mathbb{Q}_{\geqslant 0}[\bullet,(1,0),0])\cup\mathrm{Relint}(\sigma))\cap\mathcal{N}$$

and one may check that

$$\mathrm{Nash}(X)=\mathrm{Min}(\mathrm{DV}(X)^{\mathrm{sing}}_{\mathbb{T}},\leqslant_{\mathcal{D}})=[\bullet,(1,0),0]=:\nu_2.$$

In particular, ν_1 is a Nash valuation of $\widetilde{X}$ which is not a Nash valuation of X.

One may construct as follows a smooth $\deg(\mathcal{D})$-big and $\deg(\mathcal{D})$-economical star refinement of $C_{y_0}(\mathcal{D})$. For the sake of simplicity, one now identifies $\mathcal{N}_{y_0}$ with $\mathbb{Q}^3$, thus

$$C_{y_0}(\mathcal{D}) = \langle (1,0,0),(0,1,0),(0,0,1),(2,-1,1)\rangle.$$

Then the star subdivision Σ of $C_{y_0}(\mathcal{D})$ with respect to $(0,0,1)$ is a smooth fan and its maximal cones are $\langle (1,0,0),(0,0,1),(2,-1,1)\rangle$ and $\langle (1,0,0),(0,1,0),(0,0,1)\rangle$, meeting along the face $\langle (1,0,0),(0,0,1)\rangle$, which contains ν_1 in its relative interior. One easily sees that Σ is a $\deg(\mathcal{D})$-big and $\deg(\mathcal{D})$-economical star refinement of $C_{y_0}(\mathcal{D})$, and as in section Σ allows to define a divisorial equivariant resolution of $f\colon Z \to X$ such that ν_1 is not f-exceptional. Note that f factors through $g\colon Z \to \widetilde{X}$, but ν_1 is g-exceptional, and the corresponding exceptional component is a curve, contained in $\mathrm{cent}_Z(\nu_2)$ which is a component of codimension 1 of the exceptional locus of f and but is not contained in the exceptional locus of g.

7.3. We now describe a family of examples where $X(\mathcal{D})$ has a Nash valuation which is neither terminal nor minimal. To the best of our knowledge, until now, no such example was known for any algebraic variety (see the discussion in the introduction).

Let d be an integer such that $d \geqslant 2$, $N = \mathbb{Z}^d$ and σ be the N-smooth cone generated by the canonical $\mathbb{Z}$-basis $(\rho_i)_{1\leqslant i\leqslant d}$ of N. Let Y be a smooth projective curve with genus $\geqslant 1$ and $y_0 \in Y$. Set $\mathcal{D}_{y_0} := \{(\frac{1}{2},\dots,\frac{1}{2})\} + \sigma$ and $\mathcal{D}_y = \sigma$ for $y \neq y_0$. Thus $\deg(\mathcal{D}) = \mathcal{D}_{y_0} \subset \mathrm{Relint}(\sigma)$ and $\mathcal{D}$ is a proper polyhedral divisor.

The cone $C_{y_0}(\mathcal{D})$ is the unique non-trivial Cayley cone. For the sake of simplicity, identify $\mathcal{N}_{y_0,\mathbb{Q}}$ with $N_{\mathbb{Q}} \times \mathbb{Q}$ and set $\rho_{d+1} := (0_N, 1)$. Then $C_{y_0}(\mathcal{D})$ is the simplicial cone generated by $\{\rho_i\}_{1\leqslant i\leqslant d} \cup \{\sum_{i=1}^d \rho_i + 2\rho_{d+1}\}$. It is not smooth but any of its proper faces is. Its dual cone is the cone generated by $\{\rho_{d+1}^\vee\} \cup \{2\rho_i^\vee - \rho_{d+1}^\vee\}_{1\leqslant i\leqslant d}$. From this, one easily checks that the set of minimal elements of $C_{y_0}(\mathcal{D})_{\mathrm{sing}} \cap (N \times \mathbb{Z})$ for the combinatorial order is $\{\nu_1 := \sum_{i=1}^{d+1} \rho_i\}$. On the other hand, the only face of σ meeting the degree is σ itself. Set $\nu_0 := \sum_{i=1}^{d+1} \rho_i$. One checks that $\nu_1 - \nu_0 \notin C_{y_0}(\mathcal{D})$. Thus, using our combinatorial interpretation of the Nash order, one sees that

$$\mathrm{Nash}(X) = \{\nu_0, \nu_1\}.$$

Since $C_{y_0}(\mathcal{D})$ is simplicial and has no ray which meets the degree, by Theorem 6.16 X has no terminal valuation.

Now take $f = \sum_{m\in M\cap\sigma^\vee} f_m \cdot \chi^m$ a global regular function on X. For any $m = \sum_{i=1}^d m_i \cdot \rho_i^\vee \in M \cap \sigma^\vee$, one has

$$\nu_0(f_m\chi^m) = \sum_{i=1}^d m_i \quad \text{and} \quad \nu_1(f_m\chi^m) = \mathrm{ord}_{y_0} f_m + \sum_{i=1}^d m_i.$$

But since $f_m \in H^0(Y, \lfloor \frac{1}{2}(\sum_{i=1}^d m_i) \rfloor \cdot [y_0])$, one must have $\mathrm{ord}_{y_0} f_m \leqslant 0$. Thus $\nu_1(f) \leqslant \nu_0(f)$. We infer that the set of minimal valuations of X is reduced to ν_1, and that ν_0 is a Nash valuation on X which is neither terminal nor minimal.

8. The case where the locus is the projective line

Let Y be a smooth algebraic curve, $\mathscr{E}$ be a divisorial fan over (Y, N) and $X := X(\mathscr{E})$ be the associated $\mathbb{T}$-variety of complexity one. Assume that $\mathrm{Loc}(\mathcal{D})$ is affine for any $\mathcal{D} \in \mathscr{E}$ or Y is projective with positive genus. Then any equivariant resolution of X factors through the toroidification (Proposition 4.7). Moreover the poset structures on $\mathrm{DV}(X)_{\mathbb{T}}$ defined by the hypercombinatorial order $\leqslant_{\mathcal{D}}$ and the Nash order $\leqslant_{\mathrm{mds}}$ are the same (Proposition 5.4 and Theorem 5.9), and one has $\mathrm{Nash}(X) = \mathrm{Ess}(X)$ (Proposition 5.4 and Theorem 5.10). On the other hand, if $X := X(\mathcal{D})$ is defined by a polyhedral divisorial with locus $Y \simeq \mathbb{P}^1$, one can see that no one of the above properties hold in general. Even in case X is a surface, though $\mathrm{Nash}(X) = \mathrm{Ess}(X)$ always hold, the first two properties may fail, as the following example shows.

Example 8.1. Consider the affine surfaces $X := \{x_0^3 + x_1^4 + x_2^5 = 0\}$. Let $\sigma := \mathbb{Q}_{\geqslant 0}$ and consider the following σ-tailed polyhedrons: $\mathcal{D}_0 := [5/3, +\infty[$, $\mathcal{D}_1 := [-5/4, +\infty[$ and $\mathcal{D}_\infty := [-2/5, +\infty[$. For $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ set $\mathcal{D}_z = \sigma$. Then $\mathcal{D} := \sum_{z \in \mathbb{P}^1} \mathcal{D}_z \cdot \{z\}$ is a p-divisors with locus $\mathbb{P}^1$, and X endowed the $\mathbb{G}_m$-action $\alpha \cdot (x_0, x_1, x_2) = (\alpha^{20} x_0, \alpha^{15} x_1, \alpha^{12} x_2)$ is isomorphic to $X(\mathcal{D})$ (see [Kru19] for more general computations of polyhedral divisors defining affine trinomial hypersurfaces) One easily checks that

$$\mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_{\mathcal{D}}) = \{[\bullet, 1, 0], [0, 2, 1], [1, -1, 1], [\infty, -1, 3], [\infty, 0, 1]\}.$$

But on the toroidification, the divisor corresponding to $[\bullet, 1, 0]$ is a (-1)-curve. Contracting this curve, one obtains the minimal resolution of X, which is an equivariant resolution of X which does not factor through the toroidification. Using the description of the minimal resolution in terms of a divisorial fan and the fact that the Nash problem holds for surfaces, one deduces that

$$\mathrm{Nash}(X) = \mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_{\mathcal{D}}) \setminus \{[\bullet, 1, 0]\}$$

which shows that $\leqslant_{\mathrm{mds}}$ is strictly finer than $\leqslant_{\mathcal{D}}$. On the other hand, using *e.g.* Proposition 8.4, one has $\mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}}, \leqslant_X) = \{[\infty, -1, 3]\}$, showing that $\leqslant_X$ is strictly finer than $\leqslant_{\mathrm{mds}}$.

Remark 8.2. One may also easily construct examples for which $\leqslant_{\mathrm{mds}}$ is strictly finer than $\leqslant_{\mathcal{D}}$ in any dimension, using toric downgradings and the fact that the Nash problem holds for toric varieties.

Example 8.3 (Johnson-Kollar's threefold)**.** Let σ be the cone of $\mathbb{Q}^2$ generated by $(1, 0)$ and $(1, 10)$. Let $\mathcal{D}_0 = [(1, 0), (1, 1)] + \sigma$, $\mathcal{D}_1 = \{(-\frac{2}{5}, 0)\} + \sigma$

and $\mathcal{D}_\infty = \{(-\frac{1}{2},0)\}+\sigma$. For any $z\in\mathbb{P}^1\setminus\{0,1,\infty\}$, set $\mathcal{D}_z=\sigma$. The polyhedral divisor $\sum_{z\in\mathbb{P}^1}\mathcal{D}_z\otimes\{z\}$ has locus $\mathbb{P}^1$ and degree $[(\frac{1}{10},0),(\frac{1}{10},1)]+\sigma$. It is thus a p-divisor. One can check that the associated $\mathbb{G}_m^2$-variety $X(\mathcal{D})$ is isomorphic to the affine threefold $X=\{x_0x_1=x_2^2+x_3^5\}$, endowed with the restriction of the action of $\mathbb{G}_m^2$ on $\mathbb{A}^4$ given for any $(\alpha,\beta)\in\mathbb{G}_m^2$ by

$$(\alpha,\beta)\cdot x_0=\beta\, x_0, (\alpha,\beta)\cdot x_1=\alpha^{10}\beta^{-1}\, x_1, (\alpha,\beta)\cdot x_2=\alpha^2\, x_2, (\alpha,\beta)\cdot x_3=\alpha^5\, x_3,$$

The threefold X was considered in [JK13] (without the $\mathbb{G}_m^2$-structure) where it was shown that it was a counter-example to the Nash problem. More precisely, translating the results into the hypercombinatorial description of the structure of $\mathbb{G}_m^2$-variety on X, the authors of *op.cit.*showed that X has exactly one Nash valuation, namely $[1,(-1,1),3]$, and exactly one essential valuation which is not Nash, namely $[1,(0,2),1]$.

The previous example is a dramatic illustration of the fact that when the locus is the projective line, the Nash order is finer than the hypercombinatorial order. In this example, one can check that $\mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}},\leqslant_{\mathcal{D}})$ has 17 elements whereas by Johnson-Kollar's result $\mathrm{Min}(\mathrm{DV}(X)_{\mathbb{T}}^{\mathrm{sing}},\leqslant_{\mathrm{mds}})$ is reduced to a singleton. It seems an interesting yet difficult challenge to determine whether the Nash order on the equivariant valuations of a $\mathbb{T}$-variety of complexity one with locus $\mathbb{P}^1$ has a sensible interpretation in terms of the hypercombinatorial description of the variety. Another question, perhaps more tractable, is to obtain a sensible hypercombinatorial description of the sets $\mathbb{T}-\mathrm{DivEss}(X)$ or $\mathbb{T}-\mathrm{Ess}(X)$.

As a first step towards a combinatorial interpretation of the Nash order, let us explain how one can use a construction of Ilten and Manon in [IM19] in order to get an interpretation of the pointwise order. Since one has $\leqslant_X\Rightarrow\leqslant_{\mathrm{mds}}\Rightarrow\leqslant_{\mathcal{D}}$, this gives us two approximations, one by excess and the other by default, of the sought-for interpretation of the Nash order.

First recall the construction of *op.cit.*. Let $\mathcal{D}$ be a p-divisor over $(\mathbb{P}^1,N)$ with locus $\mathbb{P}^1$, tail σ and support $\{y_0,\dots,y_s\}\subset\mathbb{P}^1$ Recall that for $m\in M=N^\vee$ and a polyhedron $\mathcal{P}$ one denotes $\mathrm{Min}_{n\in\mathcal{P}}\langle m\,,\,n\rangle$ by $\mathcal{P}(m)$. Ilten and Manon define an embedding of $X=X(\mathcal{D})$ in a toric variety Z of dimension $\mathrm{rk}(N)+s$ as follows: let (e_i) be the canonical basis of $\mathbb{Z}^s$ and $\mathcal{C}_X$ be the polyhedral cone in $(\mathbb{Z}^s\oplus N)_{\mathbb{Q}}$ defined as the convex hull of

$$\mathcal{T}:=(\mathbb{Q}_{\geqslant0}(-\sum_{i=1}^s e_i)\times C_z(\mathcal{D}_{y_0}))\cup\cup_{i=1}^s(\mathbb{Q}_{\geqslant0}e_i\times C_z(\mathcal{D}_{y_i}))$$

Note that $((v_i),m)\in(\mathbb{Z}^s\oplus M)_{\mathbb{Q}}$ lies in $\mathcal{C}_X^\vee$ if and only if

$$\forall m\in\sigma^\vee,\ 1\leqslant i\leqslant s,\ v_i+\mathcal{D}_{y_i}(m)\geqslant0,\qquad \sum_{i=1}^s v_i\leqslant\mathcal{D}_{y_0}(m),$$

Note also that $\mathcal{T}\cap(\mathbb{Z}^s\times N)$ is naturally identified with a subset $\mathrm{DV}(X)_{\mathbb{T}}^\star$ of the set $\mathrm{DV}(X)_{\mathbb{T}}$ of equivariant divisorial valuation of X. More precisely $\mathrm{DV}(X)_{\mathbb{T}}^\star$ consists of those $\nu\in\mathrm{DV}(X)_{\mathbb{T}}$ which may be represented as $[y_i,\ell,n]$

with $0 \leqslant i \leqslant s$ and $(\ell, n) \in C_{y_i}(\mathcal{D})$. Let $\iota\colon \mathrm{DV}(X)^{\star}_{\mathbb{T}} \to \mathcal{C}_X$ be the natural embedding. Let Z be the toric variety associated with $\mathcal{C}_X$.

The following gives a combinatorial interpretation of the pointwise order on the set $\mathrm{DV}(X)^{\star}_{\mathbb{T}}$. In case $s = 1$, $X = Z$ is toric, and the result is already known by 2.28. Note that by 5.4 and 5.5, $\mathrm{DV}(X)^{\star}_{\mathbb{T}}$ contains $\mathrm{Nash}(X)$.

Proposition 8.4. *Let $\nu, \nu' \in \mathrm{DV}(X)^{\star}_{\mathbb{T}}$. Then $\nu \leqslant_X \nu'$ if and only if $\iota(\nu') \in \iota(\nu) + \mathcal{C}_X$. In other words, identifying $\mathrm{DV}(X)^{\star}_{\mathbb{T}}$ with its image by ι, the pointwise order on $\mathrm{DV}(X)^{\star}_{\mathbb{T}}$ is the restriction of the order $\leqslant_{\mathcal{C}_X}$ (see Definition 2.20).*

Proof. Set $k[\mathbb{Z}^s] = k[z_i^{\pm 1}]_{1 \leqslant i \leqslant s}$ and $k(\mathbb{P}^1) = k(z)$. Without loss of generality, one may assume that $\infty \notin \{y_i\}_{0 \leqslant i \leqslant s}$. The embedding of X into Z is given at the level of regular functions by the surjective morphism of k-algebras $\varphi\colon k[Z] = k[\mathcal{C}_X^{\vee} \cap (\mathbb{Z}^s \oplus M)] \to k[X]$ which maps $z^v\chi^m$ to $\prod_{i=1}^{s} (\frac{z-y_i}{z-y_0})^{v_i} \chi^m$.

Let $z^v\chi^m$ be a monomial in $k[Z]$ and $\nu \in \mathrm{DV}(X)^{\star}_{\mathbb{T}}$. First note that

$$\nu(\varphi(z^v\chi^m)) = \langle (v, m)\, ,\, \iota(\nu) \rangle$$

Indeed, assume that $\nu = [y_i, \ell, n]$ with $1 \leqslant i \leqslant s$. Then

$$\nu(\varphi(z^v\chi^m)) = \ell \,\mathrm{ord}_{y_i} \left(\prod_{j=1}^{s} (\frac{z-y_j}{z-y_0})^{v_j} \right) + \langle m\, ,\, n \rangle = \ell v_i + \langle m\, ,\, n \rangle = \langle (v, m)\, ,\, \iota(\nu) \rangle$$

the last equality being a consequence of the definition of ι.

Now if $\nu = [y_0, \ell, n]$ one has

$$\nu(\varphi(z^v\chi^m)) = \ell \,\mathrm{ord}_{y_0} \left(\prod_{i=1}^{s} (\frac{z-y_i}{z-y_0})^{v_i} \right) + \langle m\, ,\, n \rangle$$

$$= -\ell(\sum_{i=1}^{n} v_i) + \langle m\, ,\, n \rangle = \langle (v, m)\, ,\, \iota(\nu) \rangle\, ,$$

the last equality being again a consequence of the definition of ι.

Now consider $\nu, \nu' \in \mathrm{DV}(X)^{\star}_{\mathbb{T}}$ such that $\nu \leqslant_X \nu'$. In particular for every $z^v\chi^m$ in $k[Z]$ one has $\nu(\varphi(z^v\chi^m)) \leqslant \nu'(\varphi(z^v\chi^m))$, hence $\langle (v, m)\, ,\, \iota(\nu) \rangle \leqslant \langle (v, m)\, ,\, \iota(\nu') \rangle$. This shows $\iota(\nu') \in \iota(\nu) + \mathcal{C}_X$

Assume $\iota(\nu') \in \iota(\nu) + \mathcal{C}_X$. In particular for every $z^v\chi^m$ in $k[Z]$ one has $\nu(\varphi(z^v\chi^m)) \leqslant \nu'(\varphi(z^v\chi^m))$. In order to shows $\nu \leqslant_X \nu'$, it suffices to show that for every semi-homogeneous element $g = f \cdot \chi^m \in k[X]$, one has $\nu(g) \leqslant nu'(g)$ thus it is enough to show that there exists $z^v\chi^m \in k[Z]$ such that $\nu(\varphi(z^v\chi^m)) = \nu(g)$ and $\nu'(\varphi(z^v\chi^m)) = \nu'(g)$.

The following is inspired by the proof of Theorem 5.4 in [IM19]. Let $i, j \in \{0, \ldots, s\}$ such that $\nu = [y_i, \ell, n]$ and $\nu' = [y_j, \ell', n']$ First assume $i, j \geqslant 1$. For $1 \leqslant k \leqslant n$ let $v_k = \mathrm{ord}_{y_k} f$. In particular $\nu(g) = \ell v_i + \langle m\, ,\, n \rangle$ and $\nu'(g) = \ell' v_j + \langle m\, ,\, n' \rangle$. Since $f \cdot \chi^m \in k[X]$, one has $m \in \sigma^{\vee}$ and

$$v_i + \Delta_i(m) \geqslant 0, \quad 0 \leqslant i \leqslant s, \quad \mathrm{ord}_y(f) \geqslant 0, \quad y \notin \{y_0, \ldots, y_n\}$$

And since $\sum_{y\in\mathbb{P}^1}\mathrm{ord}_y(f) = \sum_{i=0}^s v_i + \sum_{y\notin\{y_0,\dots,y_n\}}\mathrm{ord}_y(f) = 0$, one has $\sum_{i=1}^n v_i \leqslant \mathcal{D}_{y_0}(m)$. In particular $z^v\chi^m \in k[Z]$ and clearly $\nu(\varphi(z^v\chi^m)) = \nu(g)$ and $\nu'(\varphi(z^v\chi^m)) = \nu'(g)$.

In case $i = 0$, first notice that $-w = v_0 + \sum_{i=1}^s v_i$ is nonnegative. Pick $k \notin \{i, j\}$ (one may assume $s \geqslant 2$) and let v' be defined by $v'_r = v_r$, $r \neq k$ and $v'_k = v_k + w$. Then one still has

$$v'_i + \Delta_i(m) \geqslant 0, \quad 0 \leqslant i \leqslant s, \quad \sum_{i=1}^n v'_i = -v_0 \leqslant \mathcal{D}_{y_0}(m)$$

thus $z^{v'}\chi^m \in k[Z]$ and the same argument works. □

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