Topology of real algebraic varieties and tropical homology

Ilia Itenberg

August 10, 2018

Abstract

This text is based on the notes of the lectures given by the author at the 5th Franco-Japanese-Vietnamese Symposium on Singularities in Kagoshima.

1 Topology of algebraic hypersurfaces in $\mathbb{R}P^n$

1.1 Hilbert 16-th problem

The questions concerning the topology of real algebraic hypersurfaces in $\mathbb{R}P^n$ were included by D. Hilbert in the 16-th problem of his famous list [10]. By an *algebraic hypersuface* in $\mathbb{R}P^n$ we mean a real homogeneous polynomial in $n+1$ variables which is considered up to multiplication by a non-zero real constant. Such a polynomial has a zero locus $\mathbb{R}X \subset \mathbb{R}P^n$ and a zero locus $\mathbb{C}X \subset \mathbb{C}P^n$. The former zero locus is called the *real part* of the hypersurface. The *degree* of an algebraic hypersurface in $\mathbb{R}P^n$ is the degree of a defining polynomial. All hypersurfaces considered in these notes are *non-singular*, that is, their defining polynomials do not have critical points in $\mathbb{C}^{n+1} \setminus \{0\}$.

Fix positive integers n and d; what kind of topology can have the real part $\mathbb{R}X$ of non-singular hypersurface X of degree d in $\mathbb{R}P^n$? In the case of curves ($n = 2$), this question was solved by A. Harnack [9]. Each connected component of a non-singular curve of degree d in $\mathbb{R}P^2$ is a circle smoothly embedded in $\mathbb{R}P^2$. According to the Harnack theorem, the number of these connected components can take any integer value between 0 (respectively, 1) and $(d-1)(d-2)/2+1$ if d is even (respectively, odd). The upper bound

$$b_0(\mathbb{R}X) \leq \frac{(d-1)(d-2)}{2} + 1,$$

where $b_0(\mathbb{R}X)$ is the number of connected components of the real part $\mathbb{R}X$ a non-singular curve X of degree d in $\mathbb{R}P^2$ is called the *Harnack inequality*. The question about possible topology of the pair $(\mathbb{R}P^2, \mathbb{R}X)$, where X is a non-singular curve of degree d in $\mathbb{R}P^2$, is much more difficult; the answer is known only for $d \leq 7$.

In the case $n = 3$, the answer to the question on possible topology of the real part of a non-singular surface of degree d is known only for $d \leq 4$ (in fact, for non-singular surfaces of degree $d \leq 4$, several finer classifications are available). For non-singular surfaces of degree 5 in $\mathbb{R}P^3$, even the maximal possible number of connected components of the real part is not known (this maximal number is at least 23 and at most 25; see [14], [17]).

1.2 Betti numbers

A natural question, *a priori* simpler than the one formulated above, concerns the possible values of the Betti numbers of the real part of a non-singular hypersurface of degree d in $\mathbb{R}P^n$. In these notes, we consider Betti numbers with $\mathbb{Z}/2\mathbb{Z}$-coefficients: by the p-th Betti number of a topological space Y we mean $b_p(Y) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_p(Y; \mathbb{Z}/2\mathbb{Z})$.

A generalization of the Harnack inequality is the *Smith-Thom inequality* for total Betti numbers (see [8] and [22]): for any algebraic hypersurface X in $\mathbb{R}P^n$, one has

$$\sum_{p=0}^{n-1} b_p(\mathbb{R}X) \leq \sum_{p=0}^{2n-2} b_p(\mathbb{C}X).$$

In fact, the inequality is valid for any (quasi-projective) real algebraic variety of dimension $n-1$ (and even in a more general situation); see, for example, [5] and [27]. A real algebraic variety realizing equality in the Smith-Thom inequality is said to be *maximal*.

Sharp upper bounds for individual Betti numbers of the real part of a non-singular hypersurface of degree d in $\mathbb{R}P^n$ are, in general, difficult to obtain. O. Viro [24] proposed the following conjecture (related to the so-called Ragsdale conjecture for real algebraic curves; see [19]): for any smooth projective real algebraic surface X with simply connected complex point set $\mathbb{C}X$, one has

$$b_1(\mathbb{R}X) \leq h^{1,1}(\mathbb{C}X),$$

where, as before, $\mathbb{R}X$ denotes the real part of X. Counter-examples to the Ragsdale conjecture [11] give rise to counter-examples to the Viro conjecture, in particular, among algebraic surfaces in $\mathbb{R}P^3$ (see [12], [3], [4]). Nevertheless, the Viro conjecture is true for the *primitive T-surfaces* (see [12]), *i.e.*, for algebraic surfaces in $\mathbb{R}P^3$ that are close to non-singular tropical limit.

In these notes, we consider the question on possible values of Betti numbers of real parts of primitive T-hypersurfaces of degree d in $\mathbb{R}P^n$ (see Section 2.1 for the definitions). The following upper bound (adapting the Viro conjecture to the case of primitive T-hypersurfaces in $\mathbb{R}P^n$) was proved very recently by A. Renaudineau and K. Shaw [20] (see Section 4.2 for details): *for any integer $0 \leq p \leq n-1$ and any primitive T-hypersurface X of degree d in $\mathbb{R}P^n$, we have*

$$b_p(\mathbb{R}X) \leq \begin{cases} h^{p,n-1-p}(\mathbb{C}X), & \text{if } p = (n-1)/2, \\ h^{p,n-1-p}(\mathbb{C}X) + 1, & \text{otherwise.} \end{cases}$$

The case $n = 3$ is treated in [12]. A proof of an asymptotic version of the above inequalities for primitive T-hypersurfaces can be found in [15].

2 Combinatorial patchworking

2.1 Construction

The special class of real algebraic hypersurfaces we are interested in is described by the *combinatorial patchworking*. This construction is a particular case of so-called *Viro method*. Let n and d be positive integers, and let $T^n(d)$ be the simplex in $\mathbb{R}^n$ with vertices $(0, 0, \ldots, 0)$, $(0, 0, \ldots, 0, d)$, $(0, \ldots, 0, d, 0)$, $\ldots, (d, 0, \ldots, 0)$. We shorten the notation of $T^n(d)$ to T, when n and d are unambiguous and call $T^n(d)$ the *standard n-simplex of size d*. Take a triangulation τ of T with straight edges and integer vertices (*i.e.*, vertices having integer coordinates). Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates $(i_1, \ldots, i_n)$ is denoted by $\sigma_{i_1,\ldots,i_n}$.

Denote by T_* the union of all symmetric copies of T under reflections and compositions of reflections with respect to coordinate hyperplanes. Extend the triangulation τ to a symmetric triangulation τ_* of T_*, and the distribution of signs $\sigma_{i_1,\ldots,i_n}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate hyperplane we preserve the sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If an n-simplex of the triangulation of T_* has vertices of different signs, select a piece of hyperplane being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by Γ the union of the selected pieces. It is a piecewise-linear hypersurface contained in T_*. It is not a simplicial subcomplex of T_*, but can be deformed by an isotopy preserving τ_* to a subcomplex Υ of the first barycentric subdivision τ_*' of τ_*. Each n-simplex of τ_*' has a unique vertex belonging to τ_*. Denote by τ_*^+ the union of all the n-simplices of τ_*' containing positive vertices of τ_* and by τ_*^- the union of all the remaining n-simplices. The subcomplex Υ is the intersection of τ_*^+ and τ_*^-.

Identify by the symmetry with respect to the origin the faces of T_*. The quotient space $\widetilde{T}$ is homeomorphic to the real projective space $\mathbb{R}P^n$. Denote by $\widetilde{\Gamma}$ the image of Γ in $\widetilde{T}$. (An example for $n = 2$ and $d = 10$ is shown in Figure 1.)

A triangulation τ of T is said to be *convex* if there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the n-simplices of τ.

Theorem 1 (O. Viro) (see [23], [25]) *If τ is convex, there exists a non-singular hypersurface X of degree d in $\mathbb{R}P^n$ and a homeomorphism $\mathbb{R}P^n \to \widetilde{T}$ mapping the real part $\mathbb{R}X$ of X onto $\widetilde{\Gamma}$.*

Consider a family (depending on positive real parameter t) of polynomials

$$P_t(x_0, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in V} \sigma_{i_1,\ldots i_n} t^{-\nu(i_1,\ldots i_n)} x_0^{d-i_1-\ldots-i_n} x_1^{i_1} \ldots x_n^{i_n},$$

where V is the set of vertices of τ and ν is a convex piecewise-linear function certifying that the triangulation τ is convex. If t is big enough, the polynomial P_t is called *Viro polynomial* and defines a hypersurface that satisfies the properties described in Theorem 1. The hypersurface X defined by a Viro polynomial is called a *T-hypersurface*. If the triangulation τ is *primitive* (that is, each n-simplex of τ is of volume $\frac{1}{n!}$), then X is called a *primitive T-hypersurface*.

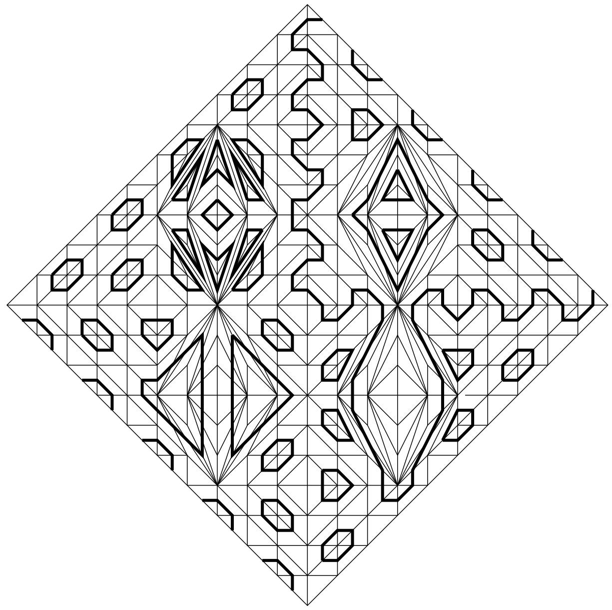

Figure 1: Construction of a counter-example to the Ragsdale conjecture

2.2 Euler characteristic of primitive T-hypersurfaces

A smooth real algebraic variety X is said to be *distinguished* if the Euler characteristic of the real part $\mathbb{R}X$ of X is equal to the signature of the complex point set $\mathbb{C}X$ of X. If X is of even (complex) dimension, then by the signature of $\mathbb{C}X$ we mean the signature of the intersection form in the middle homology; if the dimension of X is odd, we say that the signature of $\mathbb{C}X$ is 0. An example of distinguished real algebraic varieties is provided by primitive T-hypersurfaces in $\mathbb{R}P^n$ (of course, this statement is immediate if n is even).

Theorem 2 (B. Bertrand) *Let n and d be positive integers. Then, any primitive T-hypersurface of degree d in $\mathbb{R}P^n$ is a distinguished real algebraic variety.*

In the case $n = 3$ this statement was proved in [12]. The general statement was proved by B. Bertrand [2]. A simpler proof can be found in [1].

3 Tropical hypersurfaces in $\mathbb{R}^n$

The construction of combinatorial patchworking is directly related to tropical geometry (for tropical reformulations of the combinatorial patchworking, see [6] and [20]). We recall here several basic tropical notions.

3.1 Tropical operations

Tropical geometry can be seen as an algebraic geometry based on so-called *tropical numbers*. The set $\mathbb{T}$ of tropical numbers is the set $\mathbb{R}$ of real numbers completed with negative infinity: $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. Thus, $\mathbb{T}$ has a natural topology and is homeomorphic to a (closed) half-line. The tropical numbers are equipped with two arithmetic operations: *tropical addition* and *tropical multiplication*. The tropical addition is the operation of taking maximum, and the tropical multiplication is the ordinary addition:

$$"a + b" = \max\{a, b\}, \quad "a \cdot b" = a + b.$$

(We use the quotation marks when operations are tropical.) The set $\mathbb{T}$ equipped with these two operations is a *semi-field*, which means that

- $(\mathbb{T}, " + ")$ is a commutative monoid (with the neutral element $-\infty$),
- $(\mathbb{T}^\times, " \cdot ")$, where $\mathbb{T}^\times = \mathbb{T} \setminus \{-\infty\}$, is an abelian group (with the neutral element 0),
- and the tropical multiplication is distributive with respect to the tropical addition:
  $$"a \cdot (b + c)" = "(a \cdot b) + (a \cdot c)"$$
  for any $a, b, c \in \mathbb{T}$.

This semi-field is called *tropical*.

3.2 Maslov's dequantization

The tropical arithmetic operations can be seen as results of deformation of standard addition and multiplication. For any real number $t > 1$, consider the base t logarithmic map

$$\log_t : \mathbb{R}_{\geq 0} \to \mathbb{T}.$$

This map establishes a bijection between the set $\mathbb{R}_{\geq 0}$ of non-negative real numbers and the set $\mathbb{T}$. Thus, $\log_t$ allows one to introduce new arithmetic operations on $\mathbb{T}$:

$$a \oplus_t b = \log_t(t^a + t^b), \quad a \odot_t b = a + b.$$

These operations degenerate to the tropical operations when t tends to $+\infty$.

3.3 Tropical hypersurfaces in $\mathbb{R}^n$

Let n be a positive integer. In the tropical framework, the space $\mathbb{R}^n = (\mathbb{T}^\times)^n$ can be seen as tropical analog of the complex torus $(\mathbb{C}^\times)^n$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Let $V \subset (\mathbb{Z}_{\geq 0})^n$ be a finite set. Consider a *tropical polynomial*

$$p(u_1, \ldots, u_n) = \text{"} \sum_{(i_1,\ldots,i_n)\in V} a_{i_1,\ldots,i_n} u_1^{i_1} \ldots u_n^{i_n} \text{"},$$

where all coefficients $a_{i_1,\ldots,i_n}$ are tropical numbers (assume that all these coefficients are different from $-\infty$). Such a tropical polynomial gives rise to a *tropical polynomial function*

$$f_p : (u_1, \ldots, u_n) \mapsto \max_{(i_1,\ldots,i_n)\in V} \{a_{i_1,\ldots,i_n} + i_1 u_1 + \ldots + i_n u_n\}.$$

This is a convex piecewise-linear function. It is defined on $\mathbb{R}^n = (\mathbb{T}^\times)^n$ and it takes values in $\mathbb{R} = \mathbb{T}^\times$.

Denote by $\mathcal{X}_p$ the *corner locus* in $\mathbb{R}^n$ of the function f_p, that is, the subset of $\mathbb{R}^n$ formed by the points where the function f_p is not locally affine-linear. Figure 2 shows an example of such a corner locus appearing in the case of a tropical polynomial of degree 1 (with three monomials).

The graph of f_p has a natural structure of polyhedral complex, so the corner locus $\mathcal{X}_p$ is also a polyhedral complex. It defines a subdivision Φ_p of $\mathbb{R}^n$. This subdivision is dual to a certain subdivision Ψ_p of the convex hull Δ_p of $V \subset \mathbb{R}^n$; the convex hull Δ_p is called the *Newton polytope* of p; this definition makes sense for any field or semi-field of coefficients. The latter subdivision can be described as follows. For any strata δ of Φ_p, take a point $(u_1^0, \ldots, u_n^0)$ in the relative interior of δ and evaluate at $(u_1^0, \ldots, u_n^0)$ the affine-linear functions $a_{i_1,\ldots,i_n} + i_1 u_1 + \ldots + i_n u_n$, $(i_1, \ldots, i_n) \in V$. The convex hull of the points of V that correspond to the affine-linear functions

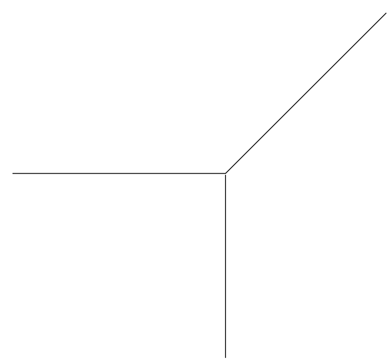

Figure 2: Tropical line

$a_{i_1,\ldots,i_n} + i_1u_1 + \ldots + i_nu_n$ realizing the maximum is the polytope dual to δ. If δ is of dimension ℓ, then the dual polytope is of dimension $n-\ell$. The dual polytopes form a subdivision Ψ_p of Δ_p (see, for example, [6] for details).

The *tropical hypersurface* defined by p in $\mathbb{R}^n$ is the polyhedral complex $\mathcal{X}_p$ whose $(n-1)$-dimensional facets are equipped with positive integer weights equal to integer lengths of the dual segments (the *integer length* of a segment with integer endpoints is the number of its integer points diminished by 1). We use the same notation $\mathcal{X}_p$ for this tropical hypersurface. The *degree* and *Newton polytope* of $\mathcal{X}_p$ are, respectively, the degree and Newton polytope of the defining tropical polynomial p. Figure 3 contains examples of tropical curves of small degrees in $\mathbb{R}^2$ (only the weights different from 1 are indicated); Figure 4 presents examples of dual subdivisions.

The subdivision Ψ_p can be described in a different way. Consider the convex hull in $\mathbb{R}^{n+1}$ of the graph of the function $(i_1, \ldots, i_n) \mapsto -a_{i_1,\ldots,i_n}$ defined on V. The projection to Δ_p of the lower part of the boundary of this convex hull gives the subdivision Ψ_p. In particular, Ψ_p is convex in the sense of Section 2.1: there exists a convex piecewise-linear function $\nu : \Delta_p \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the n-dimensional polytopes of Ψ_p. The tropical polynomial function

$$f_p : (u_1, \ldots, u_n) \mapsto \max_{(i_1,\ldots,i_n)\in V} \{a_{i_1,\ldots,i_n} + i_1u_1 + \ldots + i_nu_n\}$$

is the *Legendre transform* of the function $(i_1, \ldots, i_n) \mapsto -a_{i_1,\ldots,i_n}$ defined on V.

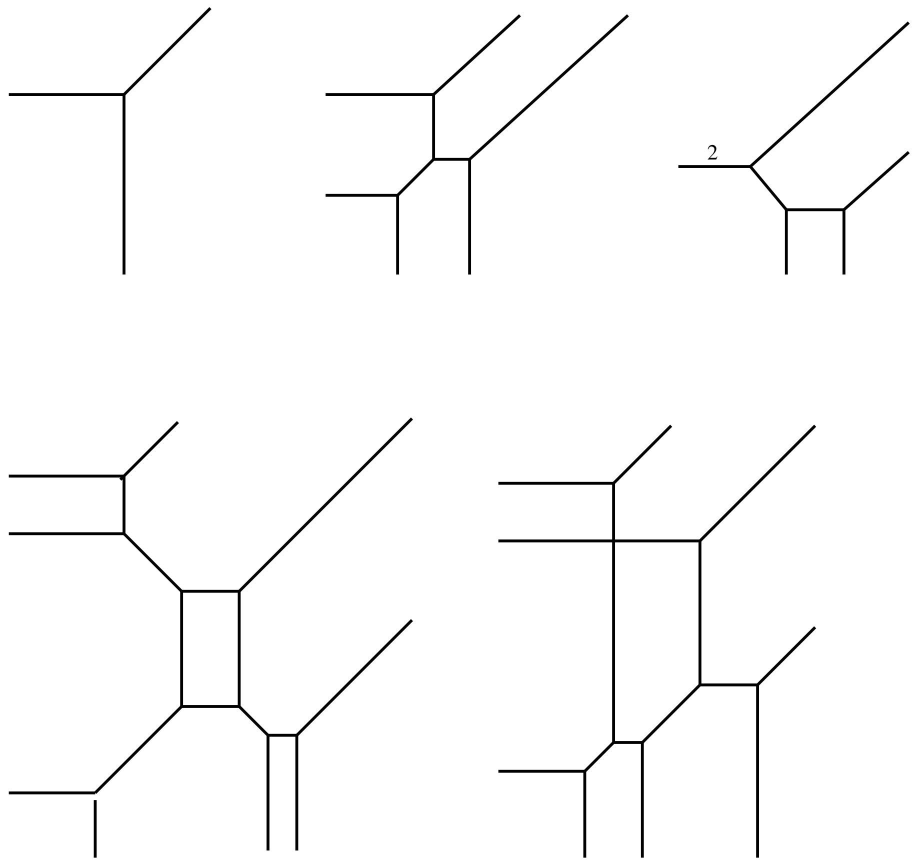

Figure 3: Examples of tropical curves of degrees 1, 2 et 3 in $\mathbb{R}^2$

The tropical hypersurface $\mathcal{X}_p$ is said to be *non-singular*, if the dual subdivision Ψ_p is a primitive triangulation. In this case, the weights of all $(n-1)$-dimensional facets of $\mathcal{X}_p$ are equal to 1, but the latter property does not guarantee that the tropical hypersurface is non-singular.

3.4 Geometric interpretation

Tropical hypersurfaces in $\mathbb{R}^n$ are $(n-1)$-dimensional weighted polyhedral complexes. Any face of such a tropical hypersurface $\mathcal{X}_p$ has rational direction (that is, direction defined over $\mathbb{Q}$). With any face we associate the intersection of the direction of the face with the integral lattice; the intersection is a lattice

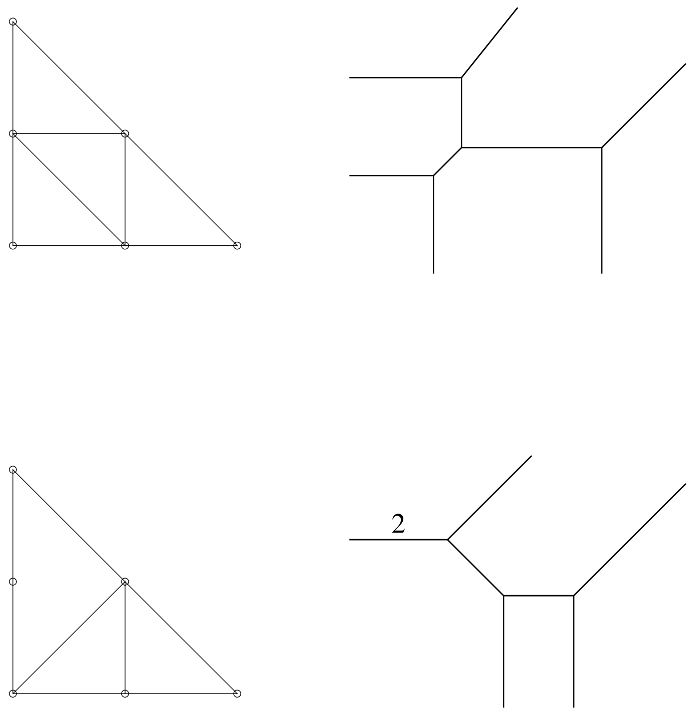

Figure 4: Examples of tropical conics and their dual subdivisions

whose rank coincides with the dimension of the face. Furthermore, at any face δ of $\mathcal{X}_p$ of dimension $n-2$, the following *balancing condition* is satisfied: the vector

$$\sum_{\kappa} w_\kappa v_\kappa$$

belongs to the direction of δ, where the sum is taken over all $(n-1)$-dimensional faces κ of $\mathcal{X}_p$ that are adjacent to δ, the positive integer w_κ is the weight of κ, and v_κ is a vector belonging to the direction of κ such that

- v_κ together with a basis of the lattice associated with δ form a basis of the lattice associated with κ,
- v_κ points towards the $(n-1)$-dimensional half-space determined by κ.

This statement can be deduced, for example, from the duality formulated in Section 3.3. The other way around, it is easy to see that any $(n-1)$-

dimensional finite weighted polyhedral complex in $\mathbb{R}^n$ (the weights are positive integers associated with the faces of dimension $n-1$) satisfying the two conditions above (all faces have rational directions and the balancing condition holds at each face of dimension $n-2$) is a tropical hypersurface, that is, it can be defined by a tropical polynomial.

3.5 Limits of amoebas

Another description of tropical hypersurfaces in $\mathbb{R}^n$ is *via* limits of amoebas. For any real number $t > 1$, consider the base t logarithmic map $\mathrm{Log}_t : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ defined by

$$(z_1, \ldots, z_n) \mapsto (\log_t |z_1|, \ldots, \log_t |z_n|).$$

If Z is an algebraic variety in $(\mathbb{C}^\times)^n$, the *amoeba* (more precisely, t-*amoeba*) of Z is $\mathrm{Log}_t(\mathbb{C}Z) \subset \mathbb{R}^n$, where $\mathbb{C}Z$ is the complex point set of Z. Any tropical hypersurface $\mathcal{X}$ in $\mathbb{R}^n$ can be approximated by a family $\mathrm{Log}_t(\mathbb{C}Z_t)$, $t \to +\infty$, of amoebas of algebraic hypersurfaces having the same Newton polytope as $\mathcal{X}$. Indeed, if

$$p(u_1, \ldots, u_n) = \text{"} \sum_{(i_1,\ldots,i_n)\in V} a_{i_1,\ldots,i_n} u_1^{i_1} \ldots u_n^{i_n} \text{"}$$

is a tropical polynomial defining the tropical hypersurface $\mathcal{X}$, then an approximating family Z_t is given by the Viro polynomials

$$P_t(z_1, \ldots, z_n) = \sum_{(i_1,\ldots,i_n)\in V} t^{a_{i_1,\ldots,i_n}} z_1^{i_1} \ldots z_n^{i_n}.$$

If $t \to +\infty$, the amoebas $\mathrm{Log}_t(\mathbb{C}Z_t)$ converge to $\mathcal{X}$ on any compact subset of $\mathbb{R}^n$ (with respect to the Hausdorff metric on closed subsets, $\mathbb{R}^n$ being equipped with the Euclidean metric). This statement can be viewed as a version of the Mikhalkin-Rullgård theorem (see, [16, 21]). The coefficients of monomials of P_t can be multiplied by arbitrary non-zero complex numbers $\alpha_{i_1,\ldots,i_n}$, this does not change the limit of amoebas of the family: only the powers of t are important for a description of the limit. In Section 3.7, we will deal with real coefficients, and since we will be interested in the real part of the limiting object, we will have to take into account the signs of the coefficients of monomials.

3.6 Hypersurfaces in tropical projective spaces

Let n be a positive integer. The *tropical projective space* $\mathbb{T}P^n$ is defined as the quotient of $\mathbb{T}^{n+1} \setminus \{(-\infty, \ldots, -\infty)\}$ by the equivalence relation

$$(u_0, \ldots, u_n) \sim (u_0 + \lambda, \ldots, u_n + \lambda)$$

for any $(u_0, \ldots, u_n) \in \mathbb{T}^{n+1} \setminus \{(-\infty, \ldots, -\infty)\}$ and any $\lambda \in \mathbb{R} = \mathbb{T}^\times$. The space $\mathbb{T}P^n$ is homeomorphic to n-dimensional simplex. Each point of $\mathbb{T}P^n$ has a *sedentarity*: this is the collection of indices of the coordinates that take value $-\infty$ at this point. All points of an open face of $\mathbb{T}P^n$ have the same sedentarity; the latter is called the sedentarity of the face. Each closed face of $\mathbb{T}P^n$ is naturally identified with the tropical projective space of the corresponding dimension. For any pair (A, B) of disjoint closed faces of $\mathbb{T}P^n$ such that the sum of dimensions of A and B is equal to $n - 1$ (in such a situation, we write $A = B^{\mathrm{op}}$), there is a natural projection

$$\mathrm{pr}_{A,B} : \mathbb{T}P^n \setminus A \to B.$$

One can, of course, describe tropical projective spaces by gluing affine charts: each of the $n + 1$ affine charts of $\mathbb{T}P^n$ is defined by the condition $u_j \neq -\infty$, where j is one of the integers 0, …, n. For example, the tropical projective line $\mathbb{T}P^1$ can be obtained in the following way: take two copies of $\mathbb{T}$ with coordinates $u_1^{(1)}$ and $u_1^{(2)}$, and glue these copies along $\mathbb{T}^\times$ using the identification $u_1^{(2)} = -u_1^{(1)}$. Similarly, the tropical projective space $\mathbb{T}P^n$ can be constructed by gluing $n + 1$ copies of $\mathbb{T}^n$. The intersection of all these $n + 1$ charts (that is, the interior of the simplex $\mathbb{T}P^n$) can be identified with $\mathbb{R}^n$ (once one of the charts is chosen; we always assume that the chosen chart is the one defined by $u_0 \neq -\infty$).

If $x \in \mathbb{T}P^n$ is a point, denote by $\mathcal{D}(x) \subset \mathbb{T}P^n$ the open strata containing x. The tangent space $\mathcal{T}(x)$ of $\mathcal{D}(x)$ at x is identified with $\mathbb{R}^k$, where k is the dimension of $\mathcal{D}(x)$.

Let d be a positive integer, and let $\mathcal{X} \subset \mathbb{R}^n$ be a (non-singular) tropical hypersurface of degree d having the standard n-simplex of size d as Newton polytope. The closure $\overline{\mathcal{X}} \subset \mathbb{T}P^n$ of $\mathcal{X}$ is called *regular* (*non-singular*) *tropical hypersurface of degree* d *in* $\mathbb{T}P^n$. Naturally extending the map $\mathrm{Log}_t : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ to the logarithmic map from $\mathbb{C}P^n$ to $\mathbb{T}P^n$, we can represent $\overline{\mathcal{X}}$ as limit of amoebas of hypersurfaces of degree d in $\mathbb{C}P^n$. These considerations can be generalized to other convex lattices polytopes and the corresponding toric varieties (see, for example, [6]).

3.7 Combinatorial patchworking revisited

A proof of the combinatorial patchworking theorem (Theorem 1) can be obtained along the following lines (see [26] for details). Consider a Viro polynomial

$$P_t(x_0, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in V} \sigma_{i_1,\ldots i_n} t^{-\nu(i_1,\ldots i_n)} x_0^{d-i_1-\ldots-i_n} x_1^{i_1} \ldots x_n^{i_n},$$

where t is sufficiently big (we use the notation of Section 2.1). The statement is proved orthant by orthant, so let us study the zero set of P_t in the open positive orthant of the affine chart $x_0 \neq 0$ with the affine coordinates

$$y_1 = x_1/x_0, \quad \ldots, \quad y_n = x_n/x_0.$$

(The other orthants can be treated using the obvious coordinate changes.) So, from now on, we assume that $y_1 > 0$, ..., $y_n > 0$. The equality $P_t(y_1, \ldots, y_n) = 0$ can be rewritten as

$$P_t^+(y_1, \ldots, y_n) = P_t^-(y_1, \ldots, y_n),$$

where P_t^+ (respectively, P_t^-) is formed by the monomials of P_t that have positive (respectively, negative) coefficients.

When t tends to $+\infty$, the graphs of the polynomials P_t^+ and P_t^- in the logarithmic coordinates

$$u_1 = \log_t y_1, \quad \ldots, \quad u_n = \log_t y_n, \quad u_{n+1} = \log_t y_{n+1}$$

tend, respectively, to the graphs

$$u_{n+1} = p^+(u_1, \ldots, u_n) \quad \text{and} \quad u_{n+1} = p^-(u_1, \ldots, u_n)$$

of the tropical polynomials

$$p^\pm(u_1, \ldots, u_n) = \text{''} \sum_{(i_1,\ldots,i_n)\in V^\pm} -\nu(i_1, \ldots i_n) u_1^{i_1} \ldots u_n^{i_n} \text{''},$$

where V^+ (respectively, V^-) is the set of vertices of τ that have positive (respectively, negative) sign.

The projection that forgets the coordinate u_{n+1} sends the intersection of the graphs of p^+ and p^- to a part $\mathcal{X}_{\text{positive}}$ of the tropical hypersurface $\mathcal{X}$ defined in $\mathbb{R}^n$ (with the coordinates u_1, …, u_n) by the tropical polynomial

$$p(u_1, \ldots, u_n) = \text{"} \sum_{(i_1,\ldots,i_n)\in V} -\nu(i_1, \ldots i_n) u_1^{i_1} \ldots u_n^{i_n} \text{"}.$$

To describe $\mathcal{X}_{\text{positive}}$, recall that each connected component of the complement of $\mathcal{X} \subset \mathbb{R}^n$ corresponds to a vertex of τ. Since the vertices of τ have signs, the connected components of the complement of $\mathcal{X} \subset \mathbb{R}^n$ are divided into two classes: positive and negative. The part $\mathcal{X}_{\text{positive}}$ is the common boundary of the union of positive components and the union of negative components.

Since P_t^+ and P_t^- tend, respectively, to the graphs of the tropical polynomials p^+ and p^-, it can be shown that the projection (that forgets the coordinate u_{n+1}) of the intersection of the graphs of P_t^+ and P_t^- is isotopic (in the stratified sense) to $\mathcal{X}_{\text{positive}}$. Furthermore, under a natural identification of $\mathbb{T}P^n$ and the simplex T, the closure of $\mathcal{X}_{\text{positive}}$ in $\mathbb{T}P^n$ is isotopic (again in the stratified sense) to the intersection of the piecewise-linear hypersurface Γ with the simplex T (see Section 2.1 for notation).

4 Tropical homology

4.1 Definition and properties

This section contains a short introduction to tropical homology. We restrict our attention to the case of regular non-singular tropical hypersurfaces in $\mathbb{T}P^n$. For a general and more detailed presentation of tropical homology and cohomology, the reader is referred to [13] and [6].

Let k be a non-negative integer, and let $\Sigma = \bigcup \varsigma \subset \mathbb{R}^k$ be a rational polyhedral fan (that is, each cone of this fan is generated by a finite collection of integer vectors). For each cone $\varsigma \subset \Sigma$, we denote by $< \varsigma >_{\mathbb{Z}}$ the integral lattice in the vector subspace linearly spanned by ς.

For any non-negative integer p, the group ${}^{\mathbb{Z}}\mathcal{F}_p(\Sigma)$ is the subgroup of $\wedge^p \mathbb{Z}^k$ generated by the elements $v_1 \wedge \cdots \wedge v_p$, where $v_1, \ldots, v_p \in < \varsigma >_{\mathbb{Z}}$ for some cone $\varsigma \in \Sigma$. It is important that all p vectors $v_1, \ldots, v_p$ come from the *same* cone.

Let $\mathcal{X}$ be a regular non-singular tropical hypersurface in $\mathbb{T}P^n$. The polyhedral decomposition of $\mathcal{X}$ into faces gives it a natural cell structure. Let

$x \in \mathcal{X}$ be a point; denote by I its sedentarity, and denote by $|I|$ the number of elements in I. We define $\Sigma(x)$, the *tangent cone to $\mathcal{X}$ at x*, to be the cone in $\mathcal{T}(x) \cong \mathbb{R}^{n-|I|}$ consisting of vectors $r \in \mathcal{T}(x)$ such that $x + \varepsilon r \in \mathcal{X} \cap \mathcal{D}(x)$ for a sufficiently small $\varepsilon > 0$ (depending on r).

For any abelian group G and any non-negative integer p, we define the coefficient group ${}^G\mathcal{F}_p(x)$ to be ${}^{\mathbb{Z}}\mathcal{F}_p(\Sigma(x)) \otimes G$. We will be particularly interested in the cases $G = \mathbb{Q}$ and $G = \mathbb{Z}/2\mathbb{Z}$. For any point x' belonging to the same open strata θ of $\mathcal{X}$ as x, the groups ${}^G\mathcal{F}_p(x')$ and ${}^G\mathcal{F}_p(x)$ are naturally identified; so we use the notation ${}^G\mathcal{F}_p(\theta)$ for these groups. The geometric meaning of the groups ${}^G\mathcal{F}_p(x)$ is as follows. The cone $\Sigma(x)$ can be seen as limit of t-amoebas of a constant family $\mathcal{L}_t = \mathcal{L}$, where $\mathcal{L}$ is a hypersurface defined in $(\mathbb{C}^\times)^{n-|I|}$ by a polynomial whose Newton polytope is dual (in the sense of Section 3.3) to the open face containing x. The monomials of such a defining polynomial are in a natural bijection with the regions of $\mathbb{T}P^n \setminus \mathcal{X}$ that are adjacent to x. Up to monomial coordinate change, the hypersurface $\mathcal{L}$ is a hyperplane in $(\mathbb{C}^\times)^{n-|I|}$ (that is, the complement of a hyperplane arrangement in $\mathbb{C}P^{n-|I|-1}$). The group ${}^G\mathcal{F}_p(x)$ is isomorphic to $H_p(\mathbb{C}\mathcal{L}; G)$.

Let θ_1 and θ_2 be adjacent open faces of $\mathcal{X}$ such that $\dim \theta_2 = \dim \theta_1 - 1$. There are two options: either θ_1 and θ_2 have the same sedentarity, or the sedentarity of θ_2 is equal to the sedentarity of θ_1 increased by 1. In the first case, denote by $\partial_{\theta_1,\theta_2} : {}^G\mathcal{F}_p(\theta_1) \to {}^G\mathcal{F}_p(\theta_2)$ the inclusion map. In the second case, denote by $\partial_{\theta_1,\theta_2} : {}^G\mathcal{F}_p(\theta_1) \to {}^G\mathcal{F}_p(\theta_2)$ the morphism induced by the restriction to θ_1 of the projection $\mathrm{pr}_{\theta_2^{\mathrm{op}},\theta_2}$.

The groups ${}^G\mathcal{F}_p$ can play the role of coefficient system. Considering the singular simplices *compatible with the stratification of $\mathcal{X}$* (that is, the singular simplexes such that the image of each open face of the standard simplex is contained in a face of $\mathcal{X}$) and using the coefficient maps $\partial_{\theta_1,\theta_2} : {}^G\mathcal{F}_p(\theta_1) \to {}^G\mathcal{F}_p(\theta_2)$, we obtain a simplical chain complex $\mathcal{C}(\mathcal{X};\ {}^G\mathcal{F}_p)$ and *tropical homology groups* $H_q(\mathcal{X};\ {}^G\mathcal{F}_p)$, $q = 0, 1, \ldots$.

Consider a family of Viro polynomials

$$P_t(x_0, \ldots, x_n) = \sum_{(i_1,\ldots,i_n) \in V} \alpha_{i_1,\ldots i_n} t^{-\nu(i_1,\ldots i_n)} x_0^{d-i_1-\ldots-i_n} x_1^{i_1} \ldots x_n^{i_n},$$

such that the amoebas of the corresponding hypersurfaces $Z_t \subset \mathbb{C}P^n$ approximate $\mathcal{X}$. A corollary of a more general statement proved in [13] is the following statement: *for any non-negative integers p and q, the dimension (over $\mathbb{Q}$) of the group $H_q(\mathcal{X};\ {}^{\mathbb{Q}}\mathcal{F}_p)$ is equal to the Hodge number $h^{p,q}(Z_t)$ of Z_t for sufficiently big t.*

4.2 Inequalities for Betti numbers

As already mentioned in Section 1.2, in the case of primitive T-hypersurface X in $\mathbb{R}P^n$, there are the following upper bounds for individual Betti numbers $b_p(\mathbb{R}X)$.

Theorem 3 (A. Renaudineau and K. Shaw) *Let n and d be positive integers. Then, for any integer $0 \le p \le n-1$ and any primitive T-hypersurface X of degree d in $\mathbb{R}P^n$, we have*

$$b_p(\mathbb{R}X) \le \begin{cases} h^{p,n-1-p}(\mathbb{C}X), & \textit{if } p = (n - 1)/2, \\ h^{p,n-1-p}(\mathbb{C}X) + 1, & \textit{otherwise.} \end{cases}$$

Remark 4.1 The inequalities of Theorem 3 are false in general (for non-singular real algebraic hypersurfaces in $\mathbb{R}P^n$). For example,

- the real part of a non-singular cubic surface in $\mathbb{R}P^3$ can have two connected components, while the geometric genus $h^{2,0}(Y_3)$ of such a surface Y_3 is zero;
- the real part of a non-singular quartic surface in $\mathbb{R}P^3$ can have 10 connected components, while the geometric genus $h^{2,0}(Y_4)$ of such a surface Y_4 is equal to 1.

Furthermore, the inequalities of Theorem 3 are false even for T-hypersurfaces in $\mathbb{R}P^n$ (without assumption of primitivity). Examples (for the first Betti number in the case of T-surfaces in $\mathbb{R}P^3$) can be found in [12].

The idea of the proof of Theorem 3 is as follows (for details, the reader is referred to [20]). The hypersurface X is produced by the combinatorial patchworking; let

$$P_t(x_0, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in V} \sigma_{i_1,\ldots i_n} t^{-\nu(i_1,\ldots i_n)} x_0^{d-i_1-\ldots-i_n} x_1^{i_1} \ldots x_n^{i_n},$$

be the corresponding family of Viro polynomials, and let $\mathcal{X}$ be the limiting tropical hypersurface of degree d in $\mathbb{T}P^n$. On $\mathcal{X}$, one can introduce a real version $\mathcal{S}_p$ of the coefficient system ${}^{\mathbb{Z}/2\mathbb{Z}}\mathcal{F}_p$, $p = 0,\ 1,\ \ldots$. For each point $x \in \mathcal{X}$, we can choose a hypersurface $\mathcal{L}$ in $(\mathbb{C}^\times)^{n-|I|}$, where I is the sedentarity of x (see Section 4.1), in such a way that a defining polynomial of $\mathcal{L}$ is real and the signs of its coefficients coincide with the signs of the corresponding regions of $\mathbb{T}P^n \setminus \mathcal{X}$ (and, thus, with the signs of the corresponding monomials of P_t). For any non-negative integer p, we put $\mathcal{S}_p(x) = H_p(\mathbb{R}\mathcal{L}; \mathbb{Z}/2\mathbb{Z})$, where $\mathbb{R}\mathcal{L}$ is the real part of $\mathcal{L}$. Thus, $\mathcal{S}_p(x) = 0$ if $p > 0$, and the dimension (over $\mathbb{Z}/2\mathbb{Z}$) of $\mathcal{S}_0(x)$ is equal to the number of connected components of $\mathbb{R}\mathcal{L}$. The definition of the morhisms $\partial_{\theta_1,\theta_2}$ given in Section 4.1 can be easily adapted to the groups $\mathcal{S}_0(x)$, and we obtain a simplicial chain complex $\mathcal{C}(\mathcal{X}, \mathcal{S}_0)$ and homology groups $H_q(\mathcal{X}, \mathcal{S}_0)$ for any non-negative integer q. One has $H_q(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) \simeq H_q(\mathcal{X}; \mathcal{S}_0)$.

For $\mathcal{L}$, or for the corresponding complement of hyperplane arrangement, one can consider the so-called *Viro homomorphisms* bv_p, $p = 0, 1, \ldots$ (see [7] for the definition of these homomorphisms). The complement of any hyperplane arrangement is a maximal variety in the sense of the Smith-Thom inequality (see [18]); thus, the target space of bv_p is $H_p(\mathbb{C}\mathcal{L}; \mathbb{Z}/2\mathbb{Z}) = {}^{\mathbb{Z}/2\mathbb{Z}}\mathcal{F}_p(x)$, see [7] for explanations. The homomorphism bv_0 is defined on $\mathcal{S}_0(x) = H_0(\mathbb{R}\mathcal{L}; \mathbb{Z}/2\mathbb{Z})$, and for each $p \geq 1$, the homomorphism bv_p is defined on the kernel K_p of bv_{p-1}. We obtain a filtration

$$\mathcal{S}_0(x) = K_0 \supset K_1 \supset \ldots \supset K_{n-|I|-1} \supset K_{n-|I|} = 0.$$

The maximality of $\mathcal{L}$ implies that all the homomorphisms bv_p are isomorphisms: for any integer $0 \leq p \leq n - |I| - 1$, one has $K_p/K_{p+1} \simeq {}^{\mathbb{Z}/2\mathbb{Z}}\mathcal{F}_p(x)$.

The above filtration gives rise to a filtration of the simplicial chain complex $\mathcal{C}(\mathcal{X}; \mathcal{S}_0)$ and produce a spectral sequence that calculates the homology groups $H_q(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) \simeq H_q(\mathcal{X}; \mathcal{S}_0)$. The first page of the spectral sequence is formed by the groups $H_q(\mathcal{X}, {}^{\mathbb{Z}/2\mathbb{Z}}\mathcal{F}_p)$. Thus, we obtain the inequalities

$$b_p(\mathbb{R}X) \leq \sum_{q=0}^{n-1} \dim_{\mathbb{Z}/2\mathbb{Z}} H_q(\mathcal{X}, {}^{\mathbb{Z}/2\mathbb{Z}}\mathcal{F}_p).$$

As it was shown by Ch. Arnal, A. Renaudineau and K. Shaw, for any non-negative integers p and q, one has

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_q(\mathcal{X}, {}^{\mathbb{Z}/2\mathbb{Z}}\mathcal{F}_p) = \dim_{\mathbb{Q}} H_q(\mathcal{X}, {}^{\mathbb{Q}}\mathcal{F}_p).$$

Thus, one can use the interpretation given in [13] for tropical homology groups (see Section 4.1) in order to obtain the inequality

$$b_p(\mathbb{R}X) \leq \sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X).$$

It remains to notice that $h^{p,q}(\mathbb{C}X) = 0$ if $p \neq q$ and $p+q \neq n-1$; in addition $h^{p,p}(\mathbb{C}X) = 1$ if $p \neq (n-1)/2$.

References

[1] Ch. Arnal, *Patchwork combinatoire et topologie d'hypersurfaces algébriques réelles*, Master thesis, Université Pierre et Marie Curie, Paris, 2017.

[2] B. Bertrand, *Euler characteristic of primitive T-hypersurfaces and maximal surfaces*, J. Inst. Math. de Jussieu 9 (2010), no. 1.

[3] F. Bihan, *Une sextique de l'espace projectif réel avec un grand nombre d'anses*, Revista Matematica Complutense **14** (2001), no. 2, 439-461.

[4] F. Bihan, *Asymptotic behaviour of Betti numbers of real algebraic surfaces*, Comm. Math. Helv. **78** (2003), 227-244.

[5] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[6] E. Brugallé, I. Itenberg, G. Mikhalkin, and K. Shaw, *Brief introduction to tropical geometry*, Proceedings of the Gökova Geometry-Topology Conference 2014 (GGT), 2015, 1-75.

[7] A. Degtyarev and V. Kharlamov, *Topological properties of real algebraic varieties: Rokhlin's way*, Russian Math. Surveys **55** (2000), no. 4, 735-814.

[8] E. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. AMS **72** (1952), 138–147.

[9] A. Harnack, *Über Vieltheiligkeit der ebenen algebraischen Curven*, Math. Ann. **10** (1876), 189–199.

[10] D. Hilbert, *Mathematische probleme*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1900.

[11] I. Itenberg, *Contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris, Sér. I, Math. **317** (1993), no. 3, 277-282.

[12] I. Itenberg, *Topology of real algebraic T-surfaces*, Rev. Mat. Univ. Complut. Madrid **10** Real algebraic and analytic geometry (Segovia, 1995), 1997, pp. 131-152.

[13] I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov, *Tropical homology*, Math. Annalen, 2018 DOI: 10.1007/s00208-018-1685-9.

[14] I. Itenberg and V. Kharlamov, *Towards the maximal number of components of a nonsingular surface of degree* 5 *in* $\mathbb{R}P^3$, Topology of real algebraic varieties and related topics, volume 173 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1996, pp. 111-118.

[15] I. Itenberg and O. Viro, *Asymptotically maximal real algebraic hypersurfaces of projective space*, Proceedings of Gökova Geometry-Topology Conference 2006 (GGT), 2007, 91-105.

[16] G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology **43** (2004), no. 5, 1035-1065.

[17] S. Orevkov, *Real quintic surface with* 23 *components*, C. R. Acad. Sci. Paris, Sér. I, Math. **333** (2001), no. 2, 115-118.

[18] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften **300**, Springer-Verlag, Berlin, 1992.

[19] V. Ragsdale, *On the arrangement of the real branches of plane algebraic curves*, Amer. J. Math. **28** (1906), 377-404.

[20] A. Renaudineau and K. Shaw, *Bounding the Betti numbers of real hypersurfaces near the tropical limit.* Preprint arXiv:1805.02030, 2018, 1-24.

[21] H. Rullgård, *Stratification des espaces de polynômes de Laurent et la structure de leurs amibes*, C. R. Acad. Sci. Paris, Sér I, Math. **331** (2000), 355-358.

[22] R. Thom, *Sur l'homologie des variétés algébriques réelles*, Diff. and Comb. Top., Symp. in honor of M. Morse, Princeton Univ. Press, 1965, pp. 255–265.

[23] O. Viro, O. Viro, *Gluing of algebraic hypersurfaces, smoothing of singularities and construction of curves*, Proc. Leningrad Int. Topological

Conf. (Leningrad, Aug. 1983), Nauka, Leningrad, 1983, pp. 149–197 (in Russian).

[24] O. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Rus. Math. Surv. **41** (1986), no. 3, 55-82.

[25] O. Viro, *Patchworking real algebraic varieties.* Preprint, Uppsala University, 1994.

[26] O. Viro, *Dequantization of real algebraic geometry on logarithmic paper*, European Congress of Mathematics, Vol. I (Barcelona, 2000), volume 201 of Progr. Math., Birkhäuser, Basel, 2001, pp. 135-146.

[27] G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), no. 1, 53-73.

Institut de Mathématiques de Jussieu - Paris Rive Gauche
Sorbonne Université
4 place Jussieu, 75252 Paris Cedex 5, France
and Département de mathématiques et applications
Ecole Normale Supérieure
45 rue d'Ulm, 75230 Paris Cedex 5, France
E-mail address: ilia.itenberg@imj-prg.fr