Character varieties

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Let k be a field of characteristic 0, Γ be a finitely generated group and G be a reductive affine algebraic group defined over k (see definitions below). The aim of these notes is to study the corresponding character variety $X(\Gamma, G)$ which is by definition the algebraic quotient

$$X(\Gamma, G) = \mathrm{Hom}(\Gamma, G)//G$$

where $\mathrm{Hom}(\Gamma, G)$ is the space of homomorphisms $\rho : \Gamma \to G$ and the action of G is by conjugation: $g.\rho = g\rho g^{-1}$.

We will give a precise definition along the notes and try to answer the following questions:

1. By construction $X(\Gamma, G)$ is an affine variety, what is the algebra of regular functions? Give generators and relations.
2. The character variety is not exactly the set of conjugacy classes of representations, so what are its points? Can we find a favourite conjugacy class of representations in each fiber of the quotient map $\pi : \mathrm{Hom}(\Gamma, G) \to X(\Gamma, G)$?
3. What is the tangent space of $X(\Gamma, G)$ at a point $\pi(\rho)$? When is it smooth?

We will illustrate these general questions with examples.

1 Preamble: conjugation classes of matrices

Before giving a general definition, consider the special case $\Gamma = \mathbb{Z}$ and $G = \mathrm{GL}_n(k)$. We clearly have $\mathrm{Hom}(\mathbb{Z}, \mathrm{GL}_n(k)) = \mathrm{GL}_n(k)$ by the map $\rho \mapsto \rho(1)$. The question reduces to understanding the algebraic quotient of $\mathrm{GL}_n(k)$ acting by conjugation on itself.

The algebra of regular functions on $\mathrm{GL}_n(k)$ is simply

$$k[\mathrm{GL}_n] = k[X_{ij}, \Delta^{-1}] \text{ where } i, j = 1, \ldots, n \text{ and } \Delta = \det(X_{ij}).$$

By definition the character variety $X(\mathbb{Z}, \mathrm{GL}_n(k)) = \mathrm{GL}_n(k)//\mathrm{GL}_n(k)$ is the spectrum of the subalgebra of invariants

$$k[\mathrm{GL}_n//\mathrm{GL}_n] = k[\mathrm{GL}_n]^{\mathrm{GL}_n}.$$

The first question has the following answer:

Proposition 1. *There is an isomorphism of algebras*

$$k[\mathrm{GL}_n]^{\mathrm{GL}_n} = k[c_1, \dots, c_n^{\pm 1}]$$

where $c_0 = 1, c_1 = \mathrm{Tr}(X), \dots, c_n = \Delta$ *are the coefficients of the characteristic polynomial of* $X = (X_{ij})_{i,j=1,\dots,n}$ *given by:*

$$\det(\lambda \mathrm{Id} - X) = \sum_{i=0}^{n} c_{n-i}(X)(-\lambda)^i.$$

We can alternatively replace the generators $c_1, \dots, c_n$ by $t_i = \mathrm{Tr}\, X^i$ for $i \in \mathbb{Z}$. For that, we would like to find all the relations among the t_i's.

Let $V = k^n$ and consider the action of $X \in \mathrm{End}(V)$ on $V^{\otimes(n+1)}$ given by $X(v_0 \otimes \cdots \otimes v_n) = (Xv_0) \otimes \cdots \otimes (Xv_n)$. We also define the representation $\rho : S_{n+1} \to \mathrm{GL}(V^{\otimes(n+1)})$ by setting

$$\rho(\sigma)(v_0 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(0)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

and set $P_{n+1} = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\rho(\sigma)$. As P_{n+1} takes its values in $\Lambda^{n+1}V = 0$, it vanishes identically, giving

$$\mathrm{Tr}(P_{n+1} \circ X) = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\, \mathrm{Tr}(\rho(\sigma) \circ X) = 0$$

Let $l_1(\sigma), \dots, l_{n+1}(\sigma)$ denote the number of cycles of order $1, \dots, n+1$ in σ. As $\mathrm{Tr}(\rho(\sigma) \circ X) = \mathrm{Tr}(X)^{l_1(\sigma)} \cdots \mathrm{Tr}(X^{n+1})^{l_{n+1}(\sigma)}$, we get

Proposition 2. *The elements* $t_1, \dots, t_{n+1} \in k[\mathrm{GL}_n]^{\mathrm{GL}_n}$ *satisfy the Frobenius formula:*

$$\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \prod_{k=1}^{n+1} t_k^{l_k(\sigma)} = 0.$$

We will see later how to generalize this equation to find a generating system of the ideal of relations. To complete the description, we need to invert the determinant function $c_n = \Delta$, and before that, to express it in terms of traces. To this aim, we observe that $P_n \circ X$ is a rank 1 operator multiplied by $\det(X)$, hence $\mathrm{Tr}(P_n \circ X) = \det(X)$. Expanding this expression as above, we express Δ as a polynomial in $t_1, \dots, t_n$ as expected.

Let us answer now the second question: a k-point of $\mathrm{GL}_n(k)//\mathrm{GL}_n(k)$ is a polynomial $\chi \in k[\lambda]$ of degree n with unit leading coefficient and non vanishing constant coefficient. Do there exist a matrix $M \in \mathrm{GL}_n(k)$ such that $\chi = \det(\lambda \mathrm{Id} - M)$, and is this matrix unique up to conjugation? The answer of the first question is yes (take a companion matrix).

The answer of the second question is no, even if the field k is algebraically closed. However in this case, there is up to conjugation a unique diagonalizable matrix having χ as characteristic polynomial. We will see a generalization of this fact later.

2 The fundamental theorem of invariants

Consider now the general case of $X(\Gamma, \mathrm{GL}_n(k))$. By assumption, Γ is finitely generated so that we can take a generating system $\gamma_1, \dots, \gamma_r$. Denoting by F_r the free group of rank r, the generating system gives a presentation of Γ of the form $1 \to R \to F_r \to \Gamma \to 1$.

It is easy to describe the algebra A of functions on $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(k))$:

1. Take a polynomial algebra generated by indeterminates X^l_{ij} for $l = 1, \dots, r$ and $i, j = 1, \dots, n$. Form the matrix $X_l = (X^l_{ij})_{i,j=1,\dots,n}$.
2. Invert $\det(X_l)$ for all $l \in \{1, \dots, r\}$.
3. For any word $w = a_{i_1}^{\epsilon_1} \cdots a_{i_l}^{\epsilon_l} \in R$, take a matrix element of $X_{i_1}^{\epsilon_1} \cdots X_{i_l}^{\epsilon_l} - \mathrm{Id}$ and take the quotient by the ideal generated by all these elements.

There is a tautological representation $\rho_n : \Gamma \to \mathrm{GL}_n(A)$ mapping γ_l to X_l. It has the universal property that for any k-algebra B and representation $\rho : \Gamma \to \mathrm{GL}_n(B)$, there is a unique algebra homomorphism $\phi : A \to B$ such that $\rho = \phi \circ \rho_n$.

By definition, the ring $k[X(\Gamma, \mathrm{GL}_n)]$ is the ring of invariants $A^{\mathrm{GL}_n(k)}$ where $g \in \mathrm{GL}_n(k)$ acts by conjugation on $X_1, \dots, X_r$. Obvious invariants are given for $\gamma \in \Gamma$ by $t_\gamma = \mathrm{Tr}\, \rho_n(\gamma)$ and $\Delta_\gamma = \det \rho_n(\gamma)$.

2.1 Generators

Our first theorem is the following:

Theorem 1. *The elements t_γ for $\gamma \in \Gamma$ and Δ_γ^{-1} generate $k[X(\Gamma, \mathrm{GL}_n(k))]$.*

As explained in the preamble, Δ_γ is a polynomial in t_{γ^l} for $l = 1, \dots, n$, so up to a localization, the t_γ's generate $k[X(\Gamma, \mathrm{GL}_n)]$.

Consider first the following reduction: is obvious from the above construction that the surjection $F_r \to \Gamma$ induces a surjection $k[\mathrm{Hom}(F_r, \mathrm{GL}_n)] \to k[\mathrm{Hom}(\Gamma, \mathrm{GL}_n)]$ hence it is sufficient to consider the case of the free group. In more concrete terms, we would like to show that the $t_\gamma, \Delta_\gamma^{-1}$ for $\gamma \in F_r$ generate $k[\mathrm{GL}_n^r]^{\mathrm{GL}_n(k)}$.

The next reduction consists in replacing here $\mathrm{GL}_n(k)$ by $\mathrm{M}_n(k)$, noticing that the map $k[\mathrm{M}_n(k)] \to k[\mathrm{GL}_n(k)]$ is the localization of the determinant. Hence it is sufficient to show that $k[\mathrm{M}_n(k)^r]^{\mathrm{GL}_n(k)}$ is generated by the t_γ's.

We will deduce it from a standard theorem of representation theory, responsible for the so-called Schur-Weyl duality.

Theorem 2. *Let V be a k-vector space of dimension n and fix an integer m. Recall that we defined a representation $\rho : S_m \to \mathrm{GL}(V^{\otimes m})$. We set $A = \mathrm{Span}\{\rho(\sigma), \sigma \in S_m\} \subset \mathrm{End}(V^{\otimes m})$. We also set $B = \mathrm{Span}\{g \otimes \cdots \otimes g, g \in \mathrm{GL}_n(k)\} \subset \mathrm{End}(V^{\otimes m})$.*

Then A is the centralizer of B and B is the centralizer of A. In formulas

$$A = \mathrm{End}_B(V^{\otimes m}) \text{ and } B = \mathrm{End}_A(V^{\otimes m}).$$

Proof of Theorem 1. Fix V a finite dimensional k-vector space and put $W = \mathrm{End}(V)$. Take $P \in k[W^r]^{\mathrm{GL}(V)}$: we can decompose it into homogeneous parts, that is we write $P = \sum_d P_d$ where $d = (d_1, \ldots, d_r)$ and $P_d(\lambda_1 w_1, \ldots, \lambda_r w_r) = \lambda_1^{d_1} \cdots \lambda_r^{d_r} P(w_1, \ldots, w_r)$. As $\mathrm{GL}(V)$ preserves this decomposition, we can suppose that P is homogeneous of degree d.

We then polarize P with respect to all its variables, let us explain only the case of one variable, that is suppose that $P \in k[W]$ has degree d. Then the polarization of P is the multilinear form $P_{\mathrm{pol}} : W^d \to k$ obtained as the multiple of $\lambda_1 \ldots \lambda_d$ in the expansion of $P(\lambda_1 w_1 + \cdots + \lambda_d w_d)$. We get back P by the formula $d!P(w) = P_{\mathrm{pol}}(w, \ldots, w)$.

Applying this $\mathrm{GL}(V)$-equivariant construction to $P \in k[W^r]$ in each variable, we get a multilinear invariant map $P_{\mathrm{pol}} : W^{d_1} \times \cdots \times W^{d_r} \to k$.

Writing $|d| = d_1 + \cdots + d_r$, we get an element

$$P^{\mathrm{pol}} \in \mathrm{Hom}_{\mathrm{GL}(V)}(W^{\otimes |d|}, k) \simeq (W^{\otimes |d|})^{\mathrm{GL}(V)}.$$

The above isomorphism maps $A_1 \otimes \cdots \otimes A_{|d|}$ on the right to the multilinear map $(B_1, \ldots, B_{|d|}) \mapsto \prod_{i=1}^{|d|} \mathrm{Tr}(A_i B_i)$ (the trace gives a $\mathrm{GL}(V)$-equivariant isomorphism $W \simeq W^*$). Using the isomorphism $\mathrm{End}(V)^{\otimes n} \simeq \mathrm{End}(V^{\otimes n})$ we have finally $P \in \mathrm{End}(V^{\otimes n})^{\mathrm{GL}(V)}$. Thanks to Theorem 2, it follows that P_{pol} is a linear combination of elements of the form $\rho(\sigma)$ for $\sigma \in S_{|d|}$.

We leave to the reader verifing that the homogeneous function corresponding to $\rho(\sigma)$ by this procedure is

$$P_\sigma(w_1, \ldots, w_r) d_1! \cdots d_r! = \mathrm{Tr}(W_{i_1} \ldots W_{i_k}) \mathrm{Tr}(W_{j_1} \cdots W_{j_l}) \cdots \mathrm{Tr}(W_{s_1} \cdots W_{s_t})$$

where $(i_1, \ldots, i_k)(j_1, \ldots, j_l) \cdots (s_1, \ldots, s_t)$ is the decomposition of σ into cycles and $W = (w_1, \ldots, w_1, \ldots, w_r, \ldots, w_r)$ where each w_i is repeated d_i times. It is easy to realize P_σ as a trace function and the theorem follows. $\square$

2.2 Relations

As for the case of matrices, the trace functions t_γ are not algebraically independent. The second fundamental theorem of invariants gives generators for the ideal of invariants, also called syzygies.

Recall the Frobenius formula of Proposition 2: given $\rho : \Gamma \to \mathrm{GL}_n(k)$ and $\gamma_0, \ldots, \gamma_n \in \Gamma$, we have $\mathrm{Tr}\left(P_{n+1} \circ (\rho(\gamma_0) \otimes \cdots \otimes \rho(\gamma_n))\right) = 0$.

Theorem 3. *The ideal of relations among the trace functions $t_\gamma \in k[X(\Gamma, \mathrm{GL}_n)]$ is generated by $t_1 - n$ and the elements*

$$\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) T^\sigma(\gamma_0, \ldots, \gamma_n)$$

where $T^{(i_1, \ldots, i_k)} = t_{\gamma_{i_1} \cdots \gamma_{i_k}}$ and $T^\sigma = \prod_{j=1}^{l} T^{\sigma_j}$ where $\sigma = \sigma_1 \cdots \sigma_l$ is the decomposition of σ into cycles (including the trivial ones).

For instance, if $n=2$ we get that all relations are consequences of

$$t_\alpha t_\beta t_\gamma - t_\alpha t_{\beta\gamma} - t_\beta t_{\alpha\gamma} - t_\gamma t_{\alpha\beta} + t_{\alpha\beta\gamma} + t_{\alpha\gamma\beta} = 0, \quad \alpha,\beta,\gamma \in \Gamma.$$

It is a good exercise to show these relations in the case of $X(\Gamma, \mathrm{SL}_2(k))$ reduce to the relations $t_1 - 2$ and the famous trace relation

$$t_{\alpha\beta} + t_{\alpha\beta^{-1}} = t_\alpha t_\beta \text{ for all } \alpha, \beta \in \Gamma. \tag{1}$$

In order to give concrete applications of this kind of theorem, let us give two illustrations in the case of SL_2.

Corollary 1. *The tangent space of $X(\Gamma, \mathrm{SL}_2(k))$ at the trivial character is given by functions $f : \Gamma \to k$ satisfying the parallelogram identity for all $\alpha, \beta \in \Gamma$*

$$f(\alpha\beta) + f(\alpha\beta^{-1}) = 2f(\alpha) + 2f(\beta).$$

Proof. Put $t_\gamma = 2 + \epsilon f(\gamma) \in k[\epsilon]/(\epsilon^2)$ and the trace equation becomes the parallelogram identity. $\square$

Corollary 2. *Let S be a closed oriented surface of genus g. For any isotopy class of subvariety $\gamma \subset S$ without homotopically trivial component, put $t_\gamma = \prod t_{\gamma_i}$ where $\gamma_1, \ldots, \gamma_r$ are the connected components of γ. Then the trace functions t_γ form a linear basis of $k[X(\pi_1(S), \mathrm{SL}_2)]$.*

Proof. By Theorem 1, the trace functions t_γ generate $k[X(\pi_1(S), \mathrm{SL}_2)]$ as an algebra. Using the trace equation (1), one can replace products by sums hence the trace functions generate $k[X(\pi_1(S), \mathrm{SL}_2)]$ linearly.

Take now $\gamma \in \pi_1(S)$ and represent it by a curve on S with a minimal number of intersection points. Applying the trace relation allows to reduce this number inductively to 0. This corresponds to the generators of the corollary, the second theorem of invariants show that they are linearly independent. $\square$

2.3 Other classical groups

Consider the groups $\mathrm{SO}_n(k) \subset \mathrm{SL}_n(k)$ and $\mathrm{Sp}_{2n}(k) \subset \mathrm{SL}_{2n}(k)$. We may wonder if the character varieties $X(\Gamma, \mathrm{SO}_n(k))$ and $X(\Gamma, \mathrm{Sp}_{2n}(k))$ have a ring of functions generated by trace functions, that is whether the natural map $k[X(\Gamma, \mathrm{SL}_n)] \to k[X(\Gamma, G)]$ is surjective.

Proposition 3. *The above map is surjective for $G = \mathrm{SO}(2n+1)$ and $G = \mathrm{Sp}(2n)$, not for $G = \mathrm{SO}(2n)$.*

We refer to [**?**] for a proof. Let us just give an example of invariant function for $\mathrm{SO}(2n)$ which is not expressible in terms of traces.

If $A \in \mathrm{SO}(2n)$ with $n > 0$, the function $Q(A) = \mathrm{Pfaff}(A - A^T)$ is an invariant function which is not expressible as a polynomial in traces of powers of A. Indeed, even for $X(\mathbb{Z}, \mathrm{SO}(2)) = \mathrm{SO}(2)$, $\mathrm{Tr}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^l \in k[a, b^2]$ whereas $Q(A) = -2b$.

To complete the set of generators for $\mathrm{SO}(2n)$, it is sufficient to add $Q_{\mathrm{pol}}(\gamma_1, \ldots, \gamma_n)$ for all $\gamma_1, \ldots, \gamma_n \in \Gamma$.

3 General properties of character varieties

3.1 Reductive linear algebraic groups

A linear algebraic group G over k is an affine k-variety endowed with a structure of group such that the multiplication $m : G \times G$ and the inversion $i : G \to G$ are regular. By the standard equivalence between affine algebraic varieties and finitely generated k-algebras, it is equivalent to give a finitely generated commutative algebra $k[G]$ together with a coproduct $m^* : k[G] \to k[G] \otimes k[G]$, an antipode $i^* : k[G] \to k[G]$ and a counit $\epsilon : k[G] \to k$ (evaluation at 1_G) satisfying equations dual to the equations defining the group structure.

Let us give these formulas explicitly in the case of the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$.

1. In the case of $\mathbb{G}_a(k) = k$, we have $k[\mathbb{G}_a] = k[t]$, $m^*(t) = t \otimes 1 + 1 \otimes t$, $i^*(t) = -t$ and $\epsilon(t) = 0$.
2. In the case of $\mathbb{G}_m(k) = k^*$, We have $k[\mathbb{G}_m] = k[t, t^{-1}]$, $m^*(t) = t \otimes t$, $i^*(t) = t^{-1}$ and $\epsilon(t) = 1$.

Our main example is GL_n for which we already gave a description of $k[G]$. Other examples include finite groups and Zariski-closed subgroups of $\mathrm{GL}_n(k)$: for instance the classical groups $\mathrm{SL}_n(k), \mathrm{O}_n(k), \mathrm{Sp}_{2n}(k)$ or the group $\mathrm{U}_n(k)$ of unipotent triangular matrices.

To simplify the following discussion, we will suppose that $k = \mathbb{C}$. Then, any linear algebraic group is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$. When looking at representations of G, we will restrict to morphisms of linear algebraic groups $\rho : G \to \mathrm{GL}(V)$, that is regular group homomorphisms. We will say that a representation is irreducible if V has no non-trivial G-stable subspace. It will be called completely reducible if it is isomorphic to a direct sum of irreducible ones.

Before discussing the reductivity assumption, we will explain the following lemma concerning unipotent algebraic groups that is, isomorphic to a subgroup of $\mathrm{U}_n(\mathbb{C})$.

Lemma 1. *Any non-zero finite dimensional representation of a unipotent group contains non-zero fixed points.*

Proof. Let V be a non-zero representation of a unipotent group G and take W a smallest non-zero stable subspace, so that it is irreducible. As G is nilpotent, it has a non-trivial center $Z(G)$ and by Schur Lemma, $g \in Z(G)$ acts on W by multiplication by a scalar $\chi(g)$, yielding a group homomorphism $\chi : Z(G) \to k^*$. As $Z(G) \simeq k^n$, such a morphism must be trivial and the representation factors through a map $G/Z(G) \to \mathrm{GL}(W)$. As $G/Z(G)$ is again nilpotent, we can repeat the argument until we find non-trivial fixed vectors. $\square$

Theorem 4. *The following conditions for an affine algebraic group G over $\mathbb{C}$ are equivalent.*

1. *Any closed normal unipotent subgroup of* G *is trivial.*

2. G *has no closed normal subgroup isomorphic to* $\mathbb{C}$*.*

3. G*, as a Lie group, has a compact subgroup* K *which is Zariski dense.*

4. *Every finite dimensional* G*-module is completely reducible.*

We will say that G is reductive if it satisfies these equivalent properties. We will not spend time on the proof of this theorem, let us comments some of its steps.

Proof. 1. $\implies$ 2. is obvious as $\mathbb{C}$ is unipotent.

2. $\implies$ 3. Deep and difficult.

3. $\implies$ 4. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and μ be the Haar measure on K. Let W be a G-invariant subspace of V and $s : V \to W$ a retraction.

The linear map $R(s) = \int_K \rho(g)s\rho(g)^{-1}d\mu(g)$ is a K-equivariant retraction of V onto W. By Zariski density of K in G, it is also G-invariant. Hence $W' = \ker R(s)$ is a G-invariant complement. By induction on the dimension, it follows that V is completely reducible.

4. $\implies$ 1. Embed G into $\mathrm{GL}(V)$ and take H a closed normal unipotent subgroup of G. By Lemma 1, V^H is non-trivial. As H is normal, V^H is also G-stable. By assumption on G there is a stable G-complement W. As this time $W^H = 0$, we get $W = 0$ by the same lemma. Hence H fixes V pointwise, which implies that H is trivial as $G \subset \mathrm{GL}(V)$.

□

3.2 Invariant theory of affine varieties

Let X be an affine complex variety and G be a reductive group acting on X algebraically. It means that the action mapping $\Phi : G \times X \to X$ defined by $\Phi(g, x) = gx$ is regular, induced by an algebra homomorphism

$$\Phi^* : k[X] \to k[G] \otimes k[X].$$

This implies the following lemma.

Lemma 2. *Consider the algebra* $k[G]$ *as a representation of* G *given by* $(g.f)(x) = f(g^{-1}x)$*. It is a union of finite dimensional* G*-invariant spaces on which* G *acts algebraically.*

Proof. For $f \in k[X]$, we write

$$\Phi^*(f)(g, x) = f(g.x) = \sum \phi_i(g) f_i(x).$$

This formula implies that $g.f = \sum_i \phi(g^{-1}) f_i$, hence the translates of f span a finite dimensional subspace of $k[X]$, proving the lemma. □

Theorem 5. *Let G act algebraically on an affine variety X. Then*

1. *The algebra $k[X]^G$ is finitely generated.*

2. *For any ideal $I \subset k[X]^G$ one has $k[X]I \cap k[X]^G = I$.*

Proof. Denote by $R : k[X] \to k[X]^G$ the unique G-equivariant projection. It is called the Reynolds operator and can be defined for instance by taking a compact subgroup $K \subset G$ and putting

$$R(f) = \int_{g \in K} g f(g^{-1} \cdot) d\mu(g).$$

This formula actually defines a polynomial function by Lemma 2 and shows that R satisfies the extra equation $R(fh) = R(f)h$ for any $h \in k[X]^G$.

Let us admit the first point, which is a very famous theorem of Hilbert, which motivated the notion of Noetherian ring, and show the second point.

Take an ideal $I \subset k[X]^G$, one has clearly $I \subset k[X]I \cap k[X]^G$. Reciprocally, if $f = \sum_i f_i h_i \in k[X]I \cap k[X]^G$ one has $f = R(f) = \sum_i R(f_i h_i) = \sum_i R(f_i) h_i \in I$, proving the second point. $\square$

In the nineteenth century, the abstract notion of quotient did not exist. Algebraic geometers defined the quotient of an affine variety X by a group G by taking generators $f_1, \ldots, f_n$ of $k[X]^G$ and considering the image of the map $F = (f_1, \ldots, f_n) : X \to \mathbb{C}^n$. It is easy to see that this affine variety does not depend on the choice of generators. The abstract definition

$$X//G = \operatorname{Spec} k[X]^G.$$

is almost equivalent. It avoids to choose generators and take into account multiplicities and non reduced subvarieties. For beginners, it is important to have simple examples of algebraic quotients in mind, as those ones.

1. Let $k[\mu_n] = k[\xi]/(\xi^n - 1)$ be the algebraic group of roots of unity. It acts on $\mathbb{A}^1$ by the formula $\xi.t = \xi t$. Its algebra of functions is $k[t]^{\mu_n} = k[t^n]$. Hence, the quotient map is $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $\pi(t) = t^n$.

2. Let $\mathbb{G}_m$ act on $\mathbb{A}^2$ by $t.(x, y) = (tx, t^{-1}y)$. Its algebra of invariant functions is $k[x, y]^{\mathbb{G}_m} = k[xy]$, yielding the quotient map $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ satisfying $\pi(x, y) = xy$. This example is iconic, the fiber of 0 is not a single orbit, but three of them: $\{(0, 0)\}, k^* \times \{0\}, \{0\} \times k^*$. Only one of them is closed.

The most important properties of affine quotients are summed up in the following proposition.

Theorem 6. *Let G be a reductive group acting on an affine variety X. Then*

1. *The map $\pi : X \to X//G$ is surjective.*

2. *If $Z \subset X$ is closed and G-invariant, then $\pi(Z)$ is closed and the map $\pi|_Z : Z \to \pi(Z)$ is the quotient map of Z by G.*

3. *If Z_1, Z_2 are closed G-invariant subsets, then $\pi(Z_1) \cap \pi(Z_2) = \pi(Z_1 \cap Z_2)$.*

4. *For any $x \in X//G$, the fiber $\pi^{-1}(x)$ contains a unique closed orbit O_x. Moreover $\pi(z) = x$ if and only if $O_x \subset \overline{Gz}$.*

Proof. 1. Let x be a point of $X//G$ corresponding to a maximal ideal $I_x \in k[X]^G$. As $k[X]I_x \cap k[X]^G = I_x$ by Theorem 5, the ideal $k[X]I_x$ of $k[X]$ is proper, hence contained in a maximal ideal I_z, giving $z \in \pi^{-1}(x)$. The properties 2. and 3. are easy consequences of Theorem 5, we skip them and refer for instance to [1].

4. Take a point $z \in \pi^{-1}(x)$: then $\overline{Gz} \subset \pi^{-1}(x)$. One can show that Gz is open in $\overline{Gz}$ and its complement is a finite union of orbits of smaller dimension. Repeating the process, we ultimately find a closed orbit in $\overline{Gz} \subset \pi^{-1}(x)$. Suppose we have two distinct closed orbits Gz_1 and Gz_2 in $\pi^{-1}(z)$. From the third point, these closed G-invariant disjoint subsets should have disjoint images, a contradiction. The last property has a similar proof. □

We observe the importance of the closed orbits in invariant theory. There is fortunately a very efficient criterium for determining whether the orbit of a point is closed or not. Let us call a point x *stable* if G_x (the stabilizer of x) is finite and Gx (the orbit of x) is closed.

Theorem 7 (Hilbert-Mumford)**.** *Let G be a reductive group acting on an affine variety X and let x be a point of X. Let O_x be the unique closed orbit adherent to Gx. Then there exists a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)x$ exists and belong to O_x.*

In particular, a point is stable if and only if for any non-trivial 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$, the limit $\lim_{t\mapsto 0} \lambda(t)x$ does not exist.

Proof. We admit the first point, see e.g. [1]. Suppose x is stable and suppose that $\lim_{t\to 0} \lambda(t)x = y$. Then $y \in Gx$, say $y = gx$. As λ is a 1-parameter subgroup one has $\lambda(st)x = \lambda(s)\lambda(t)x$. Letting $t \to 0$ we get $y = \lambda(s)y$ hence $x = g^{-1}\lambda(s)gx$. As G_x is finite, λ has to be trivial, showing that Gx is closed.

Reciprocally if x is such that $\lim_{t\to 0} \lambda(t)x$ never exists, the first point implies that Gx is closed. Moreover G_x has to be finite, unless it would contain a non trivial 1-parameter subgroup which is impossible. □

3.3 Invariant theory of character varieties

Let us apply the tools of the preceding section to the character variety $X(\Gamma, \mathrm{GL}_n(k))$. The main result is the following theorem showing that any fiber of the quotient map $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(k)) \to X(\Gamma, \mathrm{GL}_n(k))$ contains a unique conjugacy class of completely reducible representations. Moreover, the fiber reduces to one orbit of representations if and only if this representation is irreducible.

Theorem 8. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation. Then*

1. *$\mathrm{GL}(V)\rho$ is closed if and only if ρ is completely reducible.*

2. *ρ is stable for the action of $\mathrm{PGL}(V)$ if and only if ρ is irreducible.*

Proof. Up to conjugation, any 1-parameter subgroup can be written in the form

$$\lambda(t) = \begin{pmatrix} t^{a_1} & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & t^{a_n} \end{pmatrix} \quad a_1 \geq \ldots \geq a_n \in \mathbb{Z}$$

We write $\{a_1, \ldots, a_n\} = \{w_1, \ldots, w_r\}$ where $w_1 > \cdots > w_r$ and call the w_i's the weights of λ. We compute that $(\lambda(t)\rho(\gamma)\lambda(t)^{-1})_{ij} = t^{a_i - a_j}\rho(\gamma)_{ij}$. This shows that $\lambda(t).\rho$ converges when $t \to 0$ if and only if $\rho(\gamma)_{ij} = 0$ whenever $a_i < a_j$. This condition is equivalent to the fact that ρ preserves the flag $F = (F_1, \ldots, F_r)$ where F_l is generated by the first $w_1 + \cdots + w_l$ vectors.

Hence, by the above description, ρ is stable if and only if it does not preserve a non-trivial flag, that is if and only if it is irreducible, proving the second point.

Let us show that $\mathrm{GL}(V)\rho$ closed implies ρ completely reducible. Let W be a ρ-invariant subspace and take λ a 1-parameter subgroup which acts by t on W and fixes a complement W'. Then $\lambda(t).\rho$ converges to a representation ρ' which stabilizes W and W' As the orbit of ρ is closed, ρ' is conjugated to ρ, and the conjugating matrix sends W' to a ρ-invariant complement of W, hence ρ is completely reducible.

Suppose now that ρ is completely reducible. Applying Theorem 7, it suffices to show that for any 1-parameter subgroup λ such that $\lambda(t).\rho$ converges to ρ' when $t \to 0$, ρ' is conjugated to ρ. To simplify the proof, suppose that λ has only two weights corresponding to a flag $W \subset V$ so that one has

$$\rho = \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}, \quad \rho' = \lim_{t \to 0} \lambda(t).\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}.$$

This shows that there exists W' stabilized by ρ' such that $V = W \oplus W'$. As ρ is completely reducible and W is G-stable, there exists a G-stable complement W'', hence an element $g \in \mathrm{GL}(V)$ acting trivially on W and V/W and mapping W to W''. Conjugating ρ with g shows that ρ' and ρ are conjugate, ending the proof. □

Let us give the example of $X(\mathbb{Z}^r, \mathrm{GL}_n(k))$. Any r-tuple of commuting matrices can be simultaneously trigonalized: said differently, any representation $\rho : \mathbb{Z}^r \to \mathrm{GL}_n(k)$ can be conjugated to take its values in the standard Borel subgroup of upper-triangular matrices. Denote by T the sub-torus of diagonal matrices. As shown above, the representation ρ is in the same orbit than the representation $\rho' : \mathbb{Z}^r \to T$ consisting in taking out the diagonal part of ρ.

This shows that the map $X(\mathbb{Z}^r, T) \to X(\mathbb{Z}^r, \mathrm{GL}_n(k))$ is surjective. Moreover, this map is invariant by the action of the Weyl group $S_n = N_{\mathrm{GL}_n}(T)/T$ yielding an isomorphism

$$X(\mathbb{Z}^r, \mathrm{GL}_n) = (T^r)//S_n.$$

3.4 The case of reductive groups

There is a generalization of this result replacing $\mathrm{GL}_n(k)$ with any reductive group G. We introduce the notions of irreducibility and completely reducibility so that it works in the same way.

Definition 1. *1. A subgroup $P \subset G$ is called parabolic if G/P it contains a Borel subgroup (maximal closed connected solvable subgroup of G). It is equivalent to the fact that G/P is a projective variety.*

2. a Levi subgroup of P is a connected subgroup L such that $P = R_u(P) \rtimes L$ where $R_u(P)$ is the unipotent radical of P (maximal normal unipotent subgroup). Such subgroups always exist and are reductive.

When $G = \mathrm{GL}(V)$, parabolic subgroups are the stabilizers of flags. Levi subgroup of the parabolic subgroup stabilizing $F = (F_1, \dots, F_r)$ correspond to the stabilizer of a decomposition $E = E_1 \oplus \cdots E_r$ such that $F_l = E_1 \oplus \cdots E_l$ for all $l \leq r$.

Definition 2. *A subgroup $H \subset G$ will be said*

1. *irreducible if it is not contained in any proper parabolic subgroup.*
2. *completely reducible if whenever $H \subset P$, there exists a Levi subgroup $L \subset P$ such that $H \subset L$.*

We observe that in this definition, H can be safely replaced with its Zariski closure. Also, if $\rho : \Gamma \to G$ is a representation, we will say that ρ is IR/CR if $\rho(\Gamma)$ is IR/CR.

The somewhat technical definition of complete reducibility makes sense due to the following characterization:

Theorem 9. *A subgroup $H \subset G$ is completely reducible if and only if its closure $\overline{H}$ is reductive.*

Any Lie group has an adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. It is instructive to compare the properties of ρ and $\mathrm{Ad} \circ \rho$.

Proposition 4. *For any representation $\rho : \Gamma \to G$, one has the following implications:*

$$\begin{array}{ccc} \rho \text{ is irreducible} & \Longrightarrow & \rho \text{ is completely reducible} \\ \Uparrow & & \Downarrow \\ \mathrm{Ad} \circ \rho \text{ is irreducible} & \Longrightarrow & \mathrm{Ad} \circ \rho \text{ is completely reducible} \end{array}$$

Proof. The horizontal implications come directly from the definition. Suppose that ρ is CR: then $\overline{\rho(\Gamma)}$ is reductive. As the image of a reductive group is reductive, it follows that $\overline{\mathrm{Ad} \circ \rho(\Gamma)}$ is reductive and $\mathrm{Ad} \circ \rho$ is CR.

Suppose now that $\mathrm{Ad} \circ \rho$ is irreducible and ρ is not. Let $P \subset G$ be a strict parabolic subgroup containing the image of ρ. Let U be the unipotent radical of P and L be a Levi subgroup of P so that $P = U \rtimes L$.

By Lemma 1, $\mathfrak{g}^U$ is non zero. Writing for $u \in U, l \in L, v \in \mathfrak{g}^U$

$$ul.v = ll^{-1}ul.v = l.v \quad \text{ as } l^{-1}ul \in U$$

we get $lv \in \mathfrak{g}^U$, hence P stabilizes $\mathfrak{g}^U$. As P acts irreducibly on $\mathfrak{g}$ by hypothesis, Schur Lemma implies that $\mathfrak{g}^U = \mathfrak{g}$. Now, it is known that the kernel of $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the center $Z(G)$ whose unipotent radical is trivial. Hence $U = 1$, which is impossible since $P \neq G$. □

When G is semi-simple then $Z(G)$ is finite and the adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ has finite kernel. A representation $\rho : \Gamma \to G$ is then Ad-irreducible if and only if $\rho(\Gamma)$ is Zariski-dense in G. Indeed, the Lie algebra of $\mathrm{Ad}\,\overline{\rho(\Gamma)}$ is an $\mathrm{Ad} \circ \rho$-invariant subspace of $\mathfrak{g}$. Here is the generalization of Theorem 8 in the context of general reductive groups. We refer to [6] for the proof.

Theorem 10. *Let $\rho : \Gamma \to G$ be a homomorphism.*

1. *$G\rho$ is closed if and only if ρ is completely reducible.*
2. *ρ is stable for the action of $G/Z(G)$ if and only if it is irreducible.*

3.5 The case of a non-algebraically closed field

Consider the character variety $X(\Gamma, \mathrm{GL}_n(k))$ where k is no longer algebraically closed (but still has zero characteristic). A representation $\rho : \Gamma \to \mathrm{GL}_n(k)$ will be said absolutely irreducible if it is irreducible when extended to $\mathrm{GL}_n(\overline{k})$ where $\overline{k}$ is an algebraic closure of k. Let us start with a useful lemma.

Lemma 3. *The subset $X^{\mathrm{irr}}(\Gamma, \mathrm{GL}_n(k))$ of characters of irreducible representations is Zariski-open in $X(\Gamma, \mathrm{GL}_n(k))$.*

Proof. This is true for any reductive group, but we give here a simple proof, based on Burnside's theorem on matrix algebras. This claims that (see [5]):

$$\rho : \Gamma \to \mathrm{GL}_n(k) \text{ is absolutely irreducible } \iff \mathrm{Span}\{\rho(\gamma), \gamma \in \Gamma\} = \mathrm{M}_n(k).$$

For any subset $I \subset \Gamma$ of cardinality n^2, the determinant $\Delta_I = \det(\mathrm{Tr}(\rho(\gamma_i \gamma_j)))_{i,j \in I}$ is non zero iff $(\rho(\gamma_i))_{i \in I}$ is a basis of $\mathrm{M}_n(k)$. As $\Delta_I \in k[X(\Gamma, \mathrm{GL}_n)]$, the locus $\bigcup_{I \subset \Gamma} \{\Delta_I \neq 0\}$ is open and equal to $X^{\mathrm{irr}}(\Gamma, \mathrm{GL}_n(k))$, proving the lemma. □

Take now a point $x \in X^{\mathrm{irr}}(\Gamma, \mathrm{GL}_n(k))$. The preceding chapters implies that changing the base field with $\overline{k}$, one can construct a representation $\rho : \Gamma \to \mathrm{GL}_n(\overline{k})$, unique up to conjugation, and mapping to x. This representation satisfies that $\mathrm{Tr}\,\rho(\gamma) \in k$ for all $\gamma \in \Gamma$.

Question 1. *Is it possible to conjugate ρ so that it takes its values in $\mathrm{GL}_n(k)$? If not, what is the smallest extension of k in which ρ can be conjugated?*

There is a nice obstruction class $X(\Gamma, \mathrm{GL}_n(k)) \to \mathrm{Br}(k)$ which measures it. Recall that the Brauer group $\mathrm{Br}(k)$ of k is the group of equivalence classes of central simple algebras over k. Given x and ρ as above, we may form

$$A_x = \mathrm{Span}_k\{\rho(\gamma), \gamma \in \Gamma\} \in \mathrm{M}_n(\overline{k}).$$

It is easy to see that A_x is a central simple k-algebra of dimension n^2 and that it only depends on x as an element of $\mathrm{Br}(k)$. It vanishes in $\mathrm{Br}(k)$ iff it is isomorphic to $\mathrm{M}_n(k)$. Using Skolem-Noether theorem, this implies that $x = \pi(\rho)$ for some $\rho : \Gamma \to \mathrm{GL}_n(k)$.

Exercise 1. *Take $\Gamma = F_2 = \langle \alpha, \beta \rangle$. Show that*

$$k[X(F_2, \mathrm{SL}_2(k))] = k[t_\alpha, t_\beta, t_{\alpha\beta}].$$

Let $x \in X^{\mathrm{irr}}(F_2, \mathrm{SL}_2(k))$ be defined by the equations $t_\alpha = t_\beta = t_{\alpha\beta} = 1$. Is it the character of a representation with values in $\mathrm{SL}_2(\mathbb{Q})$? in $\mathrm{SL}_2(\mathbb{R})$?

4 Tangent space

The aim of this section is to compute the Zariski tangent space of $X(\Gamma, G)$ at a point $[\rho]$ for G a complex reductive group. This tangent space is, in good cases, isomorphic to a cohomology group $H^1(\Gamma, \mathrm{Ad} \circ \rho)$: this cohomological interpretation is responsible for many interesting properties of character varieties.

4.1 Cohomological interpretation of the tangent space

Let us briefly recall the definition of cohomology with twisted coefficients. Let V be a representation of Γ : we write $g.v$ for the action of g on v. We set $C^n(\Gamma, V) = \{f : \Gamma^n \to V\}$ and define $d : C^n(\Gamma, V) \to C^{n+1}(\Gamma, V)$ by the formula

$$(df)(\gamma_0, \ldots, \gamma_n) = \gamma_0.f(\gamma_1, \ldots, \gamma_n) + \sum_{i=1}^{n} (-1)^i f(\gamma_0, \ldots, \gamma_{i-1}\gamma_i, \ldots, \gamma_n) \\ + (-1)^{n+1} f(\gamma_0, \ldots, \gamma_{n-1}).$$

We denote by $H^*(\Gamma, V)$ the cohomology of this complex. We easily check that $H^0(\Gamma, V) = V^\Gamma$, the space of invariants and $H^1(\Gamma, V) = Z^1(\Gamma, V)/B^1(\Gamma, V)$ where $B^1(\Gamma, V)$ is the space of maps $f : \Gamma \to V$ of the form $f(\gamma) = \gamma.v - v$ for some $v \in V$ and

$$Z^1(\Gamma, V) = \{f : \Gamma \to V, f(\gamma_0\gamma_1) = f(\gamma_0) + \gamma_0.f(\gamma_1)\}.$$

Let $\rho : \Gamma \to G$ be a homomorphism. Corresponding dually to $\rho \in \mathrm{Hom}(\Gamma, G)$, there is an algebra morphism $\phi_\rho : k[\mathrm{Hom}(\Gamma, G)] \to k$. This follows from the universal property that we recall now. For any k-algebra B, there is a group structure on $G(B) = \mathrm{Hom}_{\mathrm{alg}}(k[G], B)$ such that

$$\mathrm{Hom}_{\mathrm{alg}}(k[\mathrm{Hom}(\Gamma, G)], B) = \mathrm{Hom}_{\mathrm{gr}}(\Gamma, G(B)).$$

A Zariski tangent vector at ρ is an algebra morphism $\phi_\epsilon : k[\mathrm{Hom}(\Gamma, G)] \to k[\epsilon]/(\epsilon^2)$ which reduces to ϕ_ρ modulo ϵ.

This shows that $T_\rho \mathrm{Hom}(\Gamma, G) = \{\rho_\epsilon : \Gamma \to G(k[\epsilon]/(\epsilon^2)), \rho_\epsilon = \rho \mod \epsilon\}$.

Setting $f : \Gamma \to \mathfrak{g}$ so that $\rho_\epsilon(\gamma) = (1 + \epsilon f(\gamma))\rho(\gamma)$. We denote by $\mathrm{Ad} \circ \rho$ the vector space $\mathfrak{g}$ viewed as a Γ module by the formula $\gamma.v = \mathrm{Ad}(\rho(\gamma)).v$.

A direct computation, left to the reader, shows that ρ_ϵ is a group homomorphism if and only if f satisfies the cocycle condition defining $Z^1(\Gamma, \mathrm{Ad} \circ \rho)$, hence the following proposition holds.

Proposition 5. *For any $\rho : \Gamma \to G$ there is a natural isomorphism*

$$T_\rho \mathrm{Hom}(\Gamma, k) \simeq Z^1(\Gamma, \mathrm{Ad} \circ \rho).$$

In general, the differential $D_\rho \pi : T_\rho \mathrm{Hom}(\Gamma, G) \to X(\Gamma, G)$ induces a linear map $Z^1(\Gamma, \mathrm{Ad} \circ \rho) \to T_{\pi(\rho)} X(\Gamma, G)$.

Define $C_\rho : G \to \mathrm{Hom}(\Gamma, G)$ by $C_\rho(g) = g\rho g^{-1}$: almost by definition of the linear representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, one has the commutative diagram:

$$\begin{tikzcd} \mathfrak{g} = T_1 G \arrow[r, "DC_\rho"] \arrow[d, "\simeq"] & T_\rho \mathrm{Hom}(\Gamma, G) \arrow[d, "\simeq"] \\ C^0(\Gamma, \mathrm{Ad} \circ \rho) \arrow[r, "d"] & C^1(\Gamma, \mathrm{Ad} \circ \rho) \end{tikzcd}$$

As the composition $\pi \circ C_\rho$ is constant, its differential vanishes. Hence $D_\rho \pi$ induces a linear map

$$\Phi_\rho : H^1(\Gamma, \mathrm{Ad} \circ \rho) \to T_{[\rho]} X(\Gamma, G).$$

This is not injective nor surjective in general as shown by the following exercise.

Exercise 2. *Recall that $X(F_2, \mathrm{SL}_2(\mathbb{C})) = \mathbb{C}^3$. Show that at the trivial representation, the map Φ_1 below vanishes:*

$$H^1(F_2, \mathrm{Ad} \circ \rho) = \mathrm{sl}_2(\mathbb{C})^2 \xrightarrow{\Phi_1} T_1 X(F_2, \mathrm{SL}_2(\mathbb{C})) = \mathbb{C}^3.$$

Let us see two more favorable cases:

Theorem 11. *Let $\rho : \Gamma \to G$ be a completely reducible homomorphism and denote by $Z(\rho)$ the centralizer of ρ.*

1. *If $Z(\rho) = Z(G)$ then Φ_ρ is an isomorphism.*

2. *If ρ is a smooth point of* $\mathrm{Hom}(\Gamma, G)$ *then*

$$T_0(H^1(\Gamma, \mathrm{Ad}\circ\rho)//Z(\rho)) \simeq T_{[\rho]}X(\Gamma, G).$$

Proof. We prove only 1. and refer to [6] for a proof of the second point. Both statements follow from Luna's Slice theorem which applies anytime we have a closed orbit in an affine variety. Replacing G by $G/Z(G)$ we can suppose that G acts freely at ρ. Luna's theorem says that there exists a subvariety $S \subset \mathrm{Hom}(\Gamma, G)$ containing ρ such that the map $\Phi : G \times S \to \mathrm{Hom}(\Gamma, G)$ sending (g, ρ) to $g.\rho$ is etale, as the map $\pi|_S : S \to X(\Gamma, G)$.

This implies that $D\Phi$ and $D\pi|_S$ are isomorphisms. Hence $T_\rho S$ is a complement of $B^1(\Gamma, \mathrm{Ad}\circ\rho)$ in $Z^1(\Gamma, \mathrm{Ad}\circ\rho)$, its projection on $H^1(\Gamma, \mathrm{Ad}\circ\rho)$ induces the isomorphism Φ_ρ, proving the theorem. $\square$

4.2 Examples and applications

Let us say that a representation $\rho : \Gamma \to G$ is *good* if it is irreducible and $Z(\rho) = Z(G)$. We denote by $z(G)$ the Lie algebra of $Z(G)$. As shown above, this condition ensures that we have $H^1(\Gamma, \mathrm{Ad}\circ\rho) \simeq T_{[\rho]}X(\Gamma, G)$.

Exercise 3. *Show that for any $\rho : \Gamma \to G$ one has:*

$$\begin{tikzcd} \mathrm{Ad}\circ\rho \textit{ is irreducible} \arrow[r, Rightarrow] & \rho \textit{ is good} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \rho \textit{ is irreducible} \\ & H^0(\Gamma, \mathrm{Ad}\circ\rho) = z(G) & \end{tikzcd}$$

Find counterexamples for the reverse implications.

4.2.1 Free groups

Suppose that $\Gamma = F_r$ so that $X(F_r, G) = G^r//G$. As $F_r = \pi_1(B_r)$ where B_r is a bouquet of r circles which is aspherical, one has $H^*(F_r, \mathrm{Ad}\circ\rho) = H^*(B_r, \mathrm{Ad}\circ\rho)$. In particular, these cohomology groups vanish in degree distinct from $0, 1$ and computing the twisted Euler characteristic gives

$$\dim H^0(B_r, \mathrm{Ad}\circ\rho) - \dim H^1(B_r, \mathrm{Ad}\circ\rho) = \chi(B_r)\dim\mathfrak{g} = (1-r)\dim\mathfrak{g}.$$

If ρ is good, we then have $\dim H^1(F_r, \mathrm{Ad}\circ\rho) - \dim z(G) = \dim T_{[\rho]}X(F_r, G) = (r-1)\dim\mathfrak{g}$. We sum up this discussion in the following proposition:

Proposition 6. *The open set $X^{\mathrm{good}}(F_r, G)$ is smooth of dimension $(r-1)\dim\mathfrak{g} + \dim z(G)$.*

Of course, this also follows from the fact that $X^{\mathrm{good}}(F_r, G)$ is the quotient of an open subset of G^r by a free and proper action of $G/Z(G)$. One can show using Reidemeister torsion that there is an algebraic volume form on $X^{\mathrm{good}}(F_r, G)$, natural in the sense that it is invariant by the action of $\mathrm{Out}(F_r)$ given by $[\phi].[\rho] = [\rho\circ\phi^{-1}]$, see [3].

4.2.2 Surface groups

Suppose that $\Gamma = \pi_1(S)$ where S is a closed compact surface of genus g. As above, S is aspherical, giving $H^*(S, \mathrm{Ad} \circ \rho) = H^*(\Gamma, \mathrm{Ad} \circ \rho)$. If $\rho : \Gamma \to G$ is good then $H^0(\Gamma, \mathrm{Ad} \circ \rho) = z(G)$, and by Poincaré duality, $H^2(\Gamma, \mathrm{Ad} \circ \rho) = z(G)^*$. Hence the argument of the previous proposition repeats.

Suppose moreover that $\mathfrak{g}$ is endowed with an Ad-invariant non-degenerate bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to k$: it induces by cup-product a bilinear map

$$\omega_\rho : H^1(S, \mathrm{Ad} \circ \rho) \times H^1(S, \mathrm{Ad} \circ \rho) \overset{B(\cdot \smile \cdot)}{\to} H^2(S, k) \simeq k.$$

The following is a clebrated theorem of Goldman, see [2]:

Theorem 12. *The open set $X^{\mathrm{good}}(\pi_1(S), G)$ is a smooth symplectic variety of dimension $(2g-2) \dim \mathfrak{g} + 2 \dim z(G)$.*

Again, this symplectic structure is natural in the sense that it is preserved by the group $\mathrm{Out}(\pi_1(S)) = \mathrm{Mod}(S)$. Goldman also showed that this symplectic structure is algebraic in the strong sense: there is a Poisson structure on $k[X(S, G)]$, i.e. a bilinear map $(f, g) \mapsto \{f, g\}$ satisfying the Leibnitz rule and the Jacobi relation which induces the above symplectic structure on each tangent space.

4.2.3 Deformation theory

Let $\rho : \Gamma \to G$ be good representation so that we have $H^1(\Gamma, \mathrm{Ad} \circ \rho) = T_{[\rho]} X(\Gamma, G)$. To determine if $[\rho]$ is a smooth point of $X(\Gamma, G)$ we can try to find a curve tangent to $[\rho]$ and whose first derivative is $[f]$ for $f \in Z^1(\Gamma, \mathrm{Ad} \circ \rho)$. This curve can be constructed by induction on n as a morphism

$$k[\mathrm{Hom}(\Gamma, G)] \to k[\epsilon]/(\epsilon^n)$$

which reduces to $(1 + \epsilon f)\rho$ modulo ϵ^2.

A simple computation shows that one can find g such that $(1 + \epsilon f + \epsilon^2 g)\rho$ is a representation $\Gamma \to G(k[\epsilon]/(\epsilon^3))$ if and only if $[f \smile f] = 0 \in H^2(\Gamma, \mathrm{Ad} \circ \rho)$. When this equation is solved, one faces an infinity of new equations lying again in $H^2(\Gamma, \mathrm{Ad} \circ \rho)$.

One deduces the following useful criterium:

Proposition 7. *If $\rho : \Gamma \to G$ is a good morphism and $H^2(\Gamma, \mathrm{Ad} \circ \rho) = 0$, then $[\rho]$ is a smooth point of $X(\Gamma, G)$.*

4.2.4 An isolated non-reduced character

We construct here a family of examples of representations $\rho : \Gamma \to G$ such that $[\rho]$ is an isolated point of $X(\Gamma, G)$ but $H^1(\Gamma, \mathrm{Ad} \circ \rho) \neq 0$. These examples come from [4].

Let V be a free abelian group of rank r and $\sigma : G \to \mathrm{GL}(V)$ be a representation of a finite group G. We set $\Gamma = V \rtimes G$ and denote by i, L, s the obvious maps shown in the diagram:

$$\begin{tikzcd} 0 \arrow[r, "i"] & V \arrow[r] & \Gamma \arrow[r, "L"'] & G \arrow[l, bend right, "s"'] \arrow[r] & 0 . \end{tikzcd}$$

We consider the representation $\rho = \sigma \circ L : \Gamma \to \mathrm{GL}(V_{\mathbb{C}})$ and wish to prove that it is rigid.

As σ is rigid as a representation of G, any deformation ρ_t of G has to be trivial when restricted to G, hence we can by a conjugation ensure that $\rho_t \circ s = \sigma$ for all t. Write $\rho_t \circ i = \exp(\phi_t)$ where $\phi_t : V \to \mathrm{End}(V_{\mathbb{C}})$ is a path satisfying $\phi_0 = 0$. As ρ_t is a morphism, we get for any $g \in G$, $v \in V$:

$$\phi_t(\sigma(g)(v)) = \sigma(g)\phi_t(v)\sigma(g)^{-1}.$$

Write $\phi_t = t\phi + \frac{1}{2}t^2\psi + o(t^2)$, the above properties imply at first order that $\phi : V \to \mathrm{End}(V_{\mathbb{C}})$ is linear and satisfies $\phi(\sigma(g)v) = \sigma(g)\phi(v)\sigma(g)^{-1}$. Writing $v \cdot w = \phi(v)(w)$, this defines a G-invariant product on V_C.

At second order we find

$$\psi(\sigma(g)v) = \sigma(g)\psi(v)\sigma(g)^{-1}, \quad [\phi(v), \phi(w)] = \psi(v+w) - \psi(v) - \psi(w).$$

The equality on the right is precisely the obstruction described in the previous section, taking values in $H^2(V, \mathrm{End}(V_{\mathbb{C}})) = \mathrm{Hom}(\Lambda^2 V, \mathrm{End}(V_{\mathbb{C}}))$. We notice that this equality relates an antisymmetric expression in (v, w) with a symmetric one, hence both terms must vanish.

To obtain examples, we need to find representations $\sigma : G \to \mathrm{GL}(V)$ such that there exists non-trivial invariant products $\cdot$ on $V_{\mathbb{C}}$ satisfying $u \cdot (v \cdot w) \neq v \cdot (u \cdot w)$ for some $u, v, w \in V$.

Such an example is obtained by taking $V = \{(x, y, z) \in \mathbb{Z}^3, x + y + z = 0\}$ with its natural S_3 action. There is an isomorphism $V = \mathbb{Z}[j]$ where (23) acts by $z \mapsto \overline{z}$ and (123) by $z \mapsto jz$. Up to a scalar the unique invariant product is given by $z \cdot w = \overline{zw}$. We check that $u \cdot (v \cdot w) = \overline{u}vw \neq \overline{v}uw = v \cdot (u \cdot w)$.

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