

REAL K -THEORIES

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ABSTRACT. The purpose of this short paper is to investigate relations between various real K -theories. In particular, we show how a real projective bundle theorem implies an unexpected relation between Atiyah's KR -theory and the usual equivariant K -theory of real vector bundles. This relation has been used recently in a new computation of the Witt group of real curves [10], Section 4. We also interpret Atiyah's theory as a special case of twisted K -theory.

1. SHORT OVERVIEW OF CLASSICAL TOPOLOGICAL K -THEORY

Since the introduction of algebraic K -theory by Grothendieck [5], various versions of topological K -theories have emerged, essentially due to Atiyah. Historically, as a prototype of “generalized cohomology theory”, Atiyah and Hirzebruch [2] introduced the K -theory of topological complex vector bundles on a compact space X , traditionally written KUX). There is a real analog KOX). When a compact group G is acting continuously on X , Segal [12] defined the equivariant versions $KU_G(X)$ and $KO_G(X)$ by considering vector bundles on X with a group action compatible with the action on X , We simply write $K_G(X)$ in a statement involving either $KU_G(X)$ or $KO_G(X)$.

A more subtle theory was introduced by Atiyah [4]. One considers a space X with involution and complex vector bundles on X , with an involutive antilinear action compatible with the involution on X , Atiyah denoted this theory by $KR(X)$ and, among other things, he showed the interest of this theory in real operator theory and in real Algebraic Geometry. This last point of view was illustrated in recent publications [10] [11].

Finally, there is a less known theory starting from vector bundles which are modules over an algebra bundle, for instance the Clifford bundle associated to a real vector bundle with a non degenerate quadratic form. This is the starting idea for the definition of the so called “twisted

K -theory” with many recent publications in mathematics and physics: see e.g. [6], [13].

As shown some time ago in [7], the general framework for all these theories is the concept of Banach category. For such categories C , one defines not only the usual Grothendieck group $K(C)$ but also “derived functors” $K_i^{top}C$. In the same framework, for any additive functor $\varphi : C \rightarrow C'$ with an adequate continuity condition, one also define “relative groups” $K_i^{top}(\varphi)$ inserted in an exact sequence

$$K_{i+1}^{top}(C) \rightarrow K_{i+1}^{top}(C') \rightarrow K_i^{top}(\varphi) \rightarrow K_i^{top}(C) \rightarrow K_i^{top}(C')$$

For instance, when C is the category of vector bundles on X and C' the category of vector bundles on a closed subspace Y , the groups $K_i^{top}(\varphi)$ are Atiyah-Hirzebruch relative groups¹ $K^{-i}(X, Y)$.

The interest of these relative groups will appear in the next sections.

2. THE REAL PROJECTIVE BUNDLE THEOREMS.

Let G be a compact group acting continuously on a compact space X and let V be an equivariant real vector bundle on X . Let W be an equivariant subbundle of V , $P(V)$ the real projective bundle on X associated to V and $P(W)$ the projective subbundle of $P(V)$. If T is any real vector bundle, we denote by $C^+(T)$ the Clifford bundle associated to T with a positive quadratic form (i.e. a metric). As another definition, we denote by $\mathcal{E}^{C^+(T)}(X)$ the category of vector bundles on X with an action of $C^+(T)$. Finally, if a compact group G is acting on everything, we denote by $\mathcal{E}_G^{C^+(T)}(X)$ the category of vector bundles with an intertwining action of $C^+(T)$ and G .

Theorem 2.1. *The relative group $K_G^{-i}(P(V), P(W))$ is isomorphic to the group K_{i-1} of the forgetful functor*

$$\varphi : \mathcal{E}_G^{C^+(V+1)}(X) \rightarrow \mathcal{E}_G^{C^+(W+1)}(X)$$

Here the notation $T + r$ means in general the vector bundle T with a trivial bundle of rank r added.

Remark 2.2. This theorem is not completely new. For G trivial it has been proved in [7], Corollaire 3.2.2. However, the proof given there cannot be extended to the equivariant case, which is important for the applications we have in mind (see e.g. Example 2.4 below).

¹We use indifferently the notation K_i or K^{-i}

Before giving examples, let us state a second theorem where the notation $C^-(T)$ now means the Clifford bundle of a vector bundle T with a *negative* non degenerate quadratic form (i.e. the opposite of a metric).

Theorem 2.3. *The relative group $K_G^{-i}(P(V + 1), P(W + 1))$ is isomorphic to the group K_{i-1} of the forgetful functor*

$$\Psi : \mathcal{E}_G^{C^-(V)}(X) \rightarrow \mathcal{E}_G^{C^-(W)}(X)$$

Example 2.4. Let us consider the usual real K -theory, so that $K_G = KO_G$. Let $G = \mathbf{Z}/2$ acts on X and on the trivial bundle $V = X \times \mathbf{R}$ by antipode. It is easy to see that the category $\mathcal{E}_G^{C^-(V)}(X)$ is equivalent to the category of Real bundles on X in the sense of Atiyah [4]. Therefore, the K -group of this category is Atiyah's group $KR(X)$. On the other hand, if we choose $W = 0$, we have $P(V + 1) - P(W + 1) = V = X \times \mathbf{R}$, so that the relative group $K_G(P(V + 1), P(W + 1))$ is $KO_G(X \times \mathbf{R})$. Therefore, Theorem 2.3 implies the following exact sequence, apparently unknown:

$$\rightarrow KR(X) \rightarrow KO_G(X) \rightarrow KO_G(X \times \mathbf{R}) \rightarrow KR_{-1}(X) \rightarrow (\mathbf{E})$$

A similar argument with the iterated suspension of X implies the more general exact sequence

$$\rightarrow KR^{-i}(X) \rightarrow KO_G^{-i}(X) \rightarrow KO_G^{-i}(X \times \mathbf{R}) \rightarrow KR^{-i+1}(X) \rightarrow$$

Remark 2.5. A quite different (and more involved) proof of this exact sequence is given in [10], Appendix C.

Remark 2.6. A more concrete description of the group $KO_G(X \times \mathbf{R})$ is to remark that it is isomorphic to the relative group $KO_G(X \times D^1, X \times S^0)$. Therefore, it sits in the middle of an exact sequence

$$\rightarrow KO_G^{-1}(X) \rightarrow KO^{-1}(X) \rightarrow KO_G(X \times \mathbf{R}) \rightarrow KO_G(X) \rightarrow$$

For instance, if $X = Y \times S^0$, with a trivial action of G on Y and the free action on $S^0 \simeq \mathbf{Z}/2$, we have $KO_G(X \times \mathbf{R}) \simeq KO^{-1}(Y)$, $KR(X) \simeq KU(Y)$ and the exact sequence (E) is a reformulation of Bott's exact sequence for the space Y (see e.g. [9] III.5.18):

$$\rightarrow KU(Y) \rightarrow KO(Y) \rightarrow KO^{-1}(Y) \rightarrow KU^1(Y) \rightarrow$$

Example 2.7. Let X be a point, G the trivial group, $V = \mathbf{R}^n$, $W = 0$. In this situation, the group $K_{-1}(\Psi)$ of Theorem 2.3 is inserted in an exact sequence involving K -groups of Clifford algebras:

$$K(C^{n,0}) \rightarrow K(C^{0,0}) \rightarrow K_{-1}(\Psi) \rightarrow K_{-1}(C^{n,0}) = 0$$

In this exact sequence, $C^{p,q}$ denotes in general the Clifford algebra of \mathbf{R}^{p+q} with the quadratic form $-(x_1)^2 - \dots - (x_p)^2 + (x_{p+1})^2 + \dots + (x_{p+q})^2$. Therefore, Theorem 2.3 implies that the reduced K -theory of RP^n is isomorphic to the cokernel of the map $K(C^{n,0}) \rightarrow KC^{0,0}$, a result due to Adams [1]. Note that this result holds in real or complex K -theory, the Clifford algebra being understood over the relevant field.

Example 2.8. Let us consider now the relative group $K_G(P(V+1), P(V))$ which is well known to be the K -group $K_G(V)$ of the Thom space of V . According to Theorem 2.1, it is isomorphic to the K^1 -group of the forgetful functor

$$\mathcal{E}_G^{C^+(V+2)}(X) \rightarrow \mathcal{E}_G^{C^+(V+1)}(X)$$

Using Bott periodicity and the periodicity of Clifford algebras, it is not difficult to show that for $r \geq 0$, this is also the K^r -group of the forgetful functor:

$$\mathcal{E}_G^{C^+(V+r+1)}(X) \rightarrow \mathcal{E}_G^{C^+(V+r)}(X)$$

which is a formulation of Thom's isomorphism theorem in equivariant K -theory [8]. However, we shall not detail this example any further since Thom's isomorphism is needed to prove Theorem 2.1, as we shall see in next Section.

Proof of Theorem 2,3 assuming Theorem 2.1

This part is purely algebraic. According to Theorem 2.1 which we assume, the group $K_G(P(V+1), P(W+1))$ is isomorphic to the K -group of the forgetful functor

$$\mathcal{E}_G^{C^+(V+2)}(X) \rightarrow \mathcal{E}_G^{C^+(W+2)}(X)$$

Note that $C^+(2) = M_2(k)$, where the basic field $k = \mathbf{R}$ or \mathbf{C} and '2' being an abbreviated notation² for k^2 . Therefore, for any real vector bundle T , the Clifford (bundle) algebra $C^+(T+2)$ is isomorphic to a 2×2 matrix algebra over $C^-(T)$, that is

$$C^+(T+2) \simeq C^-(T) \otimes C^+(2),$$

Indeed, we can define a map $T+2 \rightarrow C^-(T) \otimes C^+(2)$ by the formula $(t, v) \mapsto t \otimes e_1 e_2 + 1 \otimes v$, e_1 and e_2 being the basis vectors of k^2 . By the universal property of the Clifford algebra, this map induces an homomorphism $C^+(T+2) \rightarrow C^-(T) \otimes C^+(2)$ which is an isomorphism for dimension reasons. Morita's theorem implies then that the categories

²and also for the trivial bundle of rank 2.

$\mathcal{E}_G^{C^+(T+2)}(T)$ and $\mathcal{E}_G^{C^-(V)}(T)$ are naturally equivalent. As a consequence, the group $K_G(P(V+1), P(W+1))$ is isomorphic to the K^1 -group of the forgetful functor $\mathcal{E}_G^{C^-(V)}(X) \rightarrow \mathcal{E}_G^{C^-(W)}(X)$, which is essentially the formulation of Theorem 2.3.

3. PROOF OF THEOREM 2.1

The main ingredient to prove Theorem 2.1 is Thom's isomorphism theorem in equivariant K -theory. The complex analog is due to Atiyah and states that for a complex G -vector bundle V on a G -space X , the group $KU_G(V)$ is isomorphic to $KU_G(X)$. The general version is more subtle and is proved in [8]. If V is a real G -vector bundle on a G -space X , the group $K_G(V)$ is isomorphic to the Grothendieck group of the forgetful functor $\Psi : \mathcal{E}_G^{C^+(V+1)}(X) \rightarrow \mathcal{E}_G^{C^+(V)}(X)$. In particular; if $B(V)$ (resp. $S(V)$) denotes the ball bundle (resp. the sphere bundle) of V , we have an exact sequence

$$\rightarrow K_G(B(V), S(V)) \rightarrow K_G(B(V)) \rightarrow K_G(S(V)) \rightarrow K_G^1(B(V), S(V)) \rightarrow$$

where $K_G(B(V), S(V))$ is identified with the K -group of Ψ . In order to prove Theorem 2.1 for the simpler case $W = 0$, we have to replace the group G by $G \times \mathbf{Z}/2$, the summand $\mathbf{Z}/2$ acting on V by antipode. In that case, we may identify the group $K_{G \times \mathbf{Z}/2}(S(V))$ with the group $K_G(P(V))$ and the group $K_{G \times \mathbf{Z}/2}(B(V)) = K_{G \times \mathbf{Z}/2}(X)$ with the K -group of the category $\mathcal{E}_G^{C^+(1)}(X)$.

Let us look more closely at the group $K_{G \times \mathbf{Z}/2}(B(V), S(V))$ which is the K -group of the forgetful functor

$$\mathcal{E}_{G \times \mathbf{Z}/2}^{C^+(V+1)}(X) \rightarrow \mathcal{E}_{G \times \mathbf{Z}/2}^{C^+(V)}(X) = \mathcal{E}_G^{C^+(V+1)}$$

Let η represents the antipode action of $\mathbf{Z}/2$ on V ; we use η to "untwist" the action of $G \times \mathbf{Z}/2$ on $C^+(V+1)$. More precisely, by multiplying η with the top generator of $C^+(V+1)$, we show that the category $\mathcal{E}_{G \times \mathbf{Z}/2}^{C^+(V+1)}(X)$ is isomorphic to the product category $\mathcal{E}_{G \times \mathbf{Z}/2}^{C^+(V)}(X) \times \mathcal{E}_{G \times \mathbf{Z}/2}^{C^+(V)}(X)$, that is $\mathcal{E}_G^{C^+(V+1)}(X) \times \mathcal{E}_G^{C^+(V+1)}(X)$. Thanks to this isomorphism, the previous functor may be identified with the one defined by the direct sum:

$$\mathcal{E}_G^{C^+(V+1)}(X) \times \mathcal{E}_G^{C^+(V+1)}(X) \rightarrow \mathcal{E}_G^{C^+(V+1)}(X)$$

We now consider the commutative diagram (up to isomorphism)

$$\begin{array}{ccc}
\mathcal{E}_G^{C+(V+1)}(X) \times \mathcal{E}_G^{C+(V+1)}(X) & \rightarrow & \mathcal{E}_G^{C+(1)}(X) = \mathcal{E}_{G \times \mathbf{Z}/2}(X) \\
\downarrow & & \downarrow \\
\mathcal{E}_G^{C+(V+1)}(X) & \rightarrow & *
\end{array}$$

The horizontal arrows are the algebraic analogs of the map

$$K_{G \times \mathbf{Z}/2}(B(V), S(V)) \rightarrow K_{G \times \mathbf{Z}/2}(B(V)) = K_{G \times \mathbf{Z}/2}(X)$$

Since the K -theory of the first vertical functor is the K -theory of the category $\mathcal{E}_G^{C+(V+1)}(X)$, by taking “homotopy fibers” we see that the group $K_{G \times \mathbf{Z}/2}^{-1}(S(V)) = K_G^{-1}(P(V))$ is isomorphic to the Grothendieck group of the functor

$$\mathcal{E}_G^{C+(V+1)}(X) \rightarrow \mathcal{E}_G^{C+(1)}(X)$$

By considering iterated suspensions of X and using Bott periodicity (and periodicity of the Clifford algebra), we deduce the theorem for $W = 0$. The general case follows from the category diagram

$$\begin{array}{ccc}
\mathcal{E}_G^{C+(V+1)}(X) & \rightarrow & \mathcal{E}_G^{C+(1)}(X) \\
\downarrow & & \downarrow \\
\mathcal{E}_G^{C+(W+1)}(X) & \rightarrow & \mathcal{E}_G^{C+(1)}(X)
\end{array}$$

4. RELATION WITH TWISTED K -THEORY

Let us turn our attention to Atiyah’s KR -theory of a “real” space X , i.e. a space with involution. We denote by G the group $\mathbf{Z}/2$ and assume that G is acting freely on X with quotient space $Y = X/G$. In the case when $X = Y \times G$, Atiyah showed that $KR(X)$ is the usual KU -theory of the space Y [4]. In general, let us call L the real line bundle on Y associated to the covering $X \rightarrow X/G$ and let us consider the Clifford bundle $C^-(L)$ over Y . Note that $C^-(L) \simeq 1+L$ is the trivial complex line bundle $Y \times \mathbf{C}$ when the previous covering is trivial.

Theorem 4.1. *With the above notations, the group $KR(X)$ is naturally isomorphic to the twisted K -theory of Y associated to the Clifford algebra bundle $C^-(L)$.*

Proof. Let E be a Real bundle on X in the sense of Atiyah. Viewed as a real bundle, it is the pull-back of a real bundle F on Y so that $F = E/G$. The complex structure on E does not come from a complex structure on F (since E is not a complex G -bundle) but it induces on the quotient by G a pairing

$$L \otimes F \rightarrow F$$

and we can choose a metric on F such that F becomes a $C^-(L)$ -module. This map $E \mapsto F$ defines the required correspondance. In order to show it induces a K -isomorphism, we may use for instance a Mayer-Vietoris argument since the statement is true when $X = Y \times G$. \square

It is interesting to describe the exact sequence in Example 2.4

$$\rightarrow KR(X) \rightarrow KO_G(X) \rightarrow KO_G(X \times \mathbf{R}) \rightarrow$$

with more familiar terms. The key lemma for this is the following:

Lemma 4.2. *Let L be a real line bundle on a space Y provided with a metric. Then the K -theory of the forgetful functor*

$$\mathcal{E}^{C^-(L)}(Y) \rightarrow \mathcal{E}(Y)$$

is isomorphic to the K -theory of the forgetful functor associated to “positive” Clifford algebras

$$\mathcal{E}^{C^+(L+2)}(Y) \rightarrow \mathcal{E}^{C^+(L+1)}(Y)$$

Proof. We first notice the K -theory equivalence between the sources and the targets. Indeed, if E is vector bundle on Y , $(L+1) \otimes E$ is a $C^+(L+1)$ -module, taking into account that $L \otimes L = 1$ (once a metric is chosen). This correspondance induces an isomorphism between $K(Y)$ and the twisted K -theory associated to the algebra bundle $C^+(L+1)$, using Morita equivalence and a Mayer-Vietoris argument. In the same way we proved above that the categories $\mathcal{E}^{C^-(L)}(Y)$ and $\mathcal{E}^{C^+(L+2)}(Y)$ are equivalent. More precisely, if F is a $C^-(L)$ -module, we associate to it the module $F+F$ over the algebra bundle $C^+(L+2)$. The issue is now to prove that the category diagram

$$\begin{array}{ccc} \mathcal{E}^{C^-(L)}(Y) & \rightarrow & \mathcal{E}(Y) \\ \downarrow & & \downarrow \\ \mathcal{E}^{C^+(L+2)}(Y) & \rightarrow & \mathcal{E}^{C^+(L+1)}(Y) \end{array}$$

induces a commutative diagram between the associated K -groups, the vertical arrows being isomorphisms. This is checked by diagram chasing, the key remark being that if F is a $C^-(L)$ -module, the canonical map $L \otimes F \rightarrow F$ is an isomorphism.

Proposition 4.3. *Let X be a free G -space (with $G = \mathbf{Z}/2$) and $Y = X/G$. Then the general exact sequence*

$$\rightarrow KO_G^{-1}(X \times \mathbf{R}) \rightarrow KR(X) \rightarrow KO_G(X) \rightarrow KO_G(X \times \mathbf{R}) \rightarrow$$

is isomorphic to the exact sequence associated to twisted K -groups

$$\rightarrow K(\Psi) \rightarrow K(\mathcal{E}^{C^-(L)}(Y)) \rightarrow K(\mathcal{E}(Y)) \rightarrow K^1(\Psi) \rightarrow$$

where Ψ is the forgetful functor $\mathcal{E}^{C^-(L)}(Y) \rightarrow \mathcal{E}(Y)$.

Proof. Let L be the real line bundle associated to the covering $X \rightarrow Y$. According to the lemma, the K -theory of Ψ is isomorphic to the K -theory of the functor $\Theta : \mathcal{E}^{C^+(L+2)}(Y) \rightarrow \mathcal{E}^{C^+(L+1)}(Y)$. On the other hand, the K -theory Thom's isomorphism proved in [7] p. 210 shows that $K(\Theta)$ is isomorphic to $KO(L+1) = KO^{-1}(L) = KO_G^{-1}(X \times \mathbf{R})$. \square

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